SUSY Gauge Theory

on Squashed Three-Spheres

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Naofumi Hama, Sungjay Lee and KH, arXiv:1102.4716,

Naofumi Hama, Sungjay Lee and KH, JHEP 1103:127,2011, arXiv: 1012.3512,

Sungjay Lee, Jaemo Park and KH, JHEP 1012:079, 2010, arXiv: 1009.0340.

Overview

We computed partition functions (Z) of

3D N=2 SUSY gauge theory on squashed S^3, as functions of

1. coupling constants

. . . “masses”, “FI-couplings”, “Chern-Simons couplings”, . . .

2. axis-length parameters $(\ell, \tilde{\ell})$ of squashed S^3

As metric on squashed S^3 we take the familiar one,

$$ds^2 \;=\; \ell^2(\mu^1\mu^1 + \mu^2\mu^2) + \tilde{\ell}^2\mu^3\mu^3$$

(μ^a : Left-Invariant 1-forms)

as well as that of “hyper-ellipsoid”,

$$ds^2 \;=\; \ell^2(dx_0^2 + dx_1^2) + \tilde{\ell}^2(dx_2^2 + dx_3^2)$$

$$\left(x_0^2 + x_1^2 + x_2^2 + x_3^2 \;=\; 1\right)$$

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→ Z depends on $\tilde{\ell}$ only.

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$$\left(x_0^2 + x_1^2 + x_2^2 + x_3^2 \;=\; 1\right)$$

→ Z depends on both ℓ and $\tilde{\ell}$.

General 3D N=2 SUSY gauge theory consists of

* gauge multiplet : $(A_m, \quad \lambda_\alpha, \quad \sigma, \quad D)$

Spin : 1, 1/2, 0, 0 (aux.)

* matter multiplet : $(\phi \ , \quad \psi_\alpha \ , \quad F) \quad + \quad \text{c.c.}$

Spin : 0, 1/2, 0 (aux.)

Strategy : Localization principle

* Path integral localizes onto “saddle points”

  (= SUSY invariant bosonic field configurations)

* For integration along the directions transverse to saddle point locus,

  Gaussian approx. is exact.

* Saddle points = Coulomb branch vacua

$(\sigma = \text{const.}, \ D \sim \sigma,$ all other fields vanish)

No Higgs branch due to conformal mass.

Partition function reduces to an integral over Coulomb branch $\mathcal{M}_C$.

Schematically

$$Z = \underbrace{\int_{\mathcal{M}_C} [d\sigma]}_{1} \underbrace{\left(\frac{\det \Delta_F}{\det \Delta_B}\right)}_{2} \underbrace{\exp\Big(-S_{\rm cl}\Big)}_{3}$$

1. Integral over Lie algebra of gauge symmetry.

(can be further reduced to Cartan subalgebra)

2. “one-loop determinant” which results from Gaussian integration.

(matter mass parameter enters here)

3. Classical value of Euclidean action

(FI coupling, CS coupling enter here)

Contribution of **gauge** and **matter** multiplets to partition function.

* **gauge multiplet** : (gauge sym: G)

$$\frac{1}{|W|}\int_T d^r\sigma \prod_{\alpha\in\Delta_+} \sinh(\pi b\alpha\cdot\sigma)\sinh(\pi b^{-1}\alpha\cdot\sigma)$$

W : Weyl group

r : rank

T : Cartan subalgebra

Δ_+ : positive root set

* **matter multiplet** : (sitting in the rep. R of G , with R-charge q)

$$\prod_{\rho\in R} s_b(i - iq - \rho\cdot\sigma - M)$$

ρ : weight vector

$$s_b(x) = \prod_{m,n\geq 0} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix} , \quad Q = b + b^{-1}$$

Note:

The expressions for the measure and the determinant appear
as building blocks in structure constants of Liouville or Toda CFTs with coupling b.

(cf. Liouville central charge : $c_L = 1 + 6(b + b^{-1})^2$)

→ "3d version of AGT correspondence"

* For round S^3, $b = 1.$
* For squashed S^3 with (familiar) left-invariant metric, $b = 1.$
* For squashed S^3 with hyper-ellipsoidal metric, we found

$$b = \sqrt{\ell / \tilde{\ell}}$$

3d version of AGT: an example

Recall:

Partition function on S^4 of N=2 SUSY gauge theory	⟺	Liouville or Toda correlation function

$$Z_{SW} = \int d\nu(\alpha)\overline{\mathcal{F}_{m,\alpha}(\tau)}\mathcal{F}_{m,\alpha}(\tau) = \left\langle \prod_i V_{m_i} \right\rangle$$

SW theory	⟷		⟷	Punctured Riemann surface
gauge couplings	⟷	τ	⟷	cpx str.
matter masses	⟷	m	⟷	external Liouville momenta
Coulomb moduli	⟷	α	⟷	Internal Liouville momenta

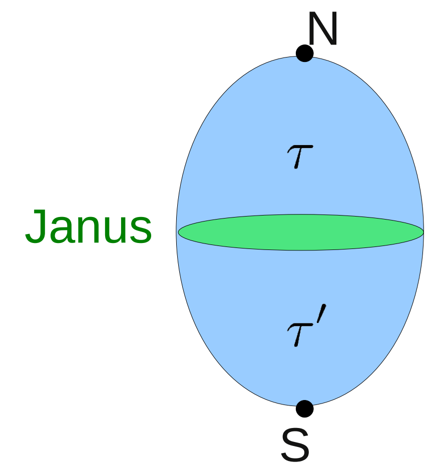
Let us take the example:

SU(2) N=2* SYM theory	⟺	Liouville torus 1pt function

[Drukker-Gaiotto-Gomis]

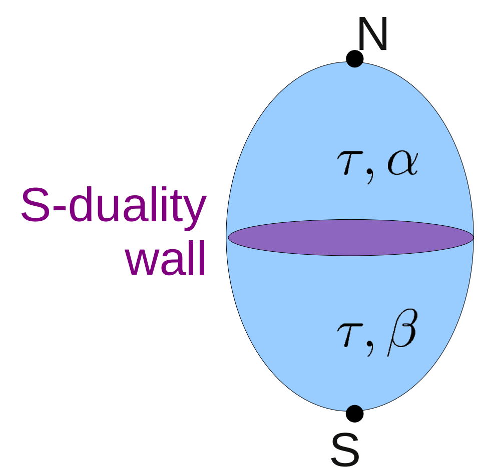
In the presence of a Janus domain wall along the equator S^3 across which the gauge coupling jumps from τ to τ' ,

$$Z_{Janus} = \int d\nu(\alpha)\overline{\mathcal{F}_{m,\alpha}(\tau)}\mathcal{F}_{m,\alpha}(\tau')$$



Set $\tau' = -1/\tau$ and apply the S-duality to the S-hemisphere.

$$Z_{Janus} = \int d\nu(\alpha)\overline{\mathcal{F}_{m,\alpha}(\tau)}\underline{\mathcal{F}_{m,\alpha}(-1/\tau)}$$
$$= \int d\nu(\alpha)\underline{d\nu(\beta)}\overline{\mathcal{F}_{m,\alpha}(\tau)}\underline{S_{\alpha,\beta,m}}\underline{\mathcal{F}_{m,\beta}(\tau)}$$



$S_{\alpha,\beta,m}$ should correspond to the DOF on the "**S-duality wall**".

For N=2* SU(2) SYM, the 3d theory on the wall is a certain N=2 SQED with mass & FI parameters (α, β, m).

[Gaiotto-Witten, Lee-Park-KH]

The partition function of the 3d wall theory:

$$S_{\alpha,\beta,m} = \int_{\mathbb{R}} d\sigma e^{4\pi i\beta\sigma} s_b(-m)$$
$$\times s_b(\tfrac{i}{2} + \tfrac{m}{2} + \alpha + \sigma) s_b(\tfrac{i}{2} + \tfrac{m}{2} + \alpha - \sigma)$$
$$\times s_b(\tfrac{i}{2} + \tfrac{m}{2} - \alpha + \sigma) s_b(\tfrac{i}{2} + \tfrac{m}{2} - \alpha - \sigma)$$

. . . agrees precisely with the S-duality transformation coefficient of torus one-point Virasoro conformal blocks. [Teschner 2003]

Additional comments:

Exact partition function for 3d gauge theories on sphere is
another powerful tool for

* checking various 3d dualities

* determining the R-symmetry of IR superconformal fixed point theory
  ("Z-minimization")

* understanding the O(N^{3/2}) behavior of the DOF
  in multiple M2-brane theory

SUSY on S^3, round and squashed

In order for a curved space to support SUSY, it has to have **Killing spinors**

ε is a Killing spinor $\Longleftrightarrow$ $\nabla_\mu \varepsilon = \gamma_\mu \tilde{\varepsilon}$ for some $\tilde{\varepsilon}$

$\gamma_\mu = e^a_\mu \gamma^a, \quad \gamma^a$: Pauli matrix

On round sphere,

metric: $ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2 + \mu^3\mu^3)$

vielbein: $e^a = \ell\mu^a,$ **LI 1-forms:** $g^{-1}dg = i\mu^a\gamma^a$

there are 4 Killing spinors.

2 of them are constant in the “LI frame”, and satisfy $\nabla_\mu \varepsilon = +\frac{i}{2\ell}\gamma_\mu \varepsilon$

The other 2 are constant in the “RI frame”, and satisfy $\nabla_\mu \varepsilon = -\frac{i}{2\ell}\gamma_\mu \varepsilon$

There are no Killing spinors on squashed S^3's.

We turn on a suitable U(1) background gauge field V_μ , so that there are **charged** Killing spinors, satisfying

$$\nabla_\mu \varepsilon \;=\; (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma^{ab} \mp iV_\mu)\varepsilon \;=\; \gamma_\mu \tilde{\varepsilon} \qquad \text{for some } \tilde{\varepsilon}$$

ε carries U(1) charge ± 1

This U(1) is the **R-symmetry** of 3d N=2 supersymmetry.

EX1. Squashed S^3, with SU(2)_Left-invariant metric

$$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2) + \tilde{\ell}^2\mu^3\mu^3,$$
$$(e^1, e^2, e^3) = (\ell\mu^1, \ell\mu^2, \tilde{\ell}\mu^3)$$

One can show that any constant spinor ε satisfies

$$\frac{i}{2f}\gamma_\mu\varepsilon = \partial_\mu\varepsilon + \frac{1}{4}\gamma^{ab}\omega_\mu^{ab}\varepsilon - \underline{iV_\mu\gamma^3\varepsilon},$$
$$f \equiv \frac{\ell^2}{\tilde{\ell}}, \quad V_\mu \equiv \left(\frac{1}{\tilde{\ell}} - \frac{1}{f}\right)e_\mu^3$$

So, if the background U(1) gauge field V_μ is turned on,

$\varepsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a Killing spinor with U(1) charge +1

$\varepsilon = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a Killing spinor with U(1) charge –1

EX2. Squashed S^3 with hyper-ellipsoidal metric

$$ds^2 = f^2 d\theta^2 + \ell^2 \cos^2\theta d\varphi^2 + \tilde{\ell}^2 \sin^2\theta d\chi^2$$

$$f \equiv \sqrt{\ell^2 \sin^2\theta + \tilde{\ell}^2 \cos^2\theta}$$

$$(e^1, e^2, e^3) = (\ell\cos\theta d\varphi, \tilde{\ell}\sin\theta d\chi, f d\theta)$$

$$\psi_\pm \equiv \begin{pmatrix} e^{\frac{i}{2}(\pm\chi\mp\varphi+\theta)} \\ \mp e^{\frac{i}{2}(\pm\chi\mp\varphi-\theta)} \end{pmatrix}$$ satisfy

$$\frac{i}{2f}\gamma_\mu\psi_\pm = (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma^{ab} \underline{\mp iV_\mu)\psi_\pm}$$

$$V_\mu = \frac{1}{2}\Big(\frac{\ell}{f} - 1\Big)d\varphi - \frac{1}{2}\Big(\frac{\tilde{\ell}}{f} - 1\Big)d\chi$$

SUSY Theories on squashed S^3

For simplicity, lets consider free WZ model. We try

$$\begin{aligned}\delta\phi &= \bar{\epsilon}\psi, \\ \delta\psi &= i\gamma^{\mu}\epsilon\nabla_{\mu}\phi + \bar{\epsilon}F, \\ \delta F &= i\epsilon\gamma^{\mu}\nabla_{\mu}\psi, \\ \mathcal{L} &= \nabla_{\mu}\bar{\phi}\nabla^{\mu}\phi - i\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi + \bar{F}F.\end{aligned}$$

∂_μ of flat space theory was replaced by ∇_μ

We assume $\epsilon, \bar{\epsilon}$ are Killing spinors,

$$\nabla_{\mu}\epsilon \equiv (\partial_{\mu} + \tfrac{1}{4}\omega_{\mu}^{ab}\gamma^{ab} - iV_{\mu})\epsilon = \tfrac{i}{2f}\gamma_{\mu}\epsilon,$$

$$\nabla_{\mu}\bar{\epsilon} \equiv (\partial_{\mu} + \tfrac{1}{4}\omega_{\mu}^{ab}\gamma^{ab} + iV_{\mu})\bar{\epsilon} = \tfrac{i}{2f}\gamma_{\mu}\bar{\epsilon},$$

and check if $\delta\mathcal{L} = 0.$

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∂_μ of flat space theory was replaced by ∇_μ

$$\begin{aligned}\delta\mathcal{L} = & -\underbrace{\epsilon\bar{\psi}}_{\delta\bar{\phi}}\cdot\nabla^\mu\nabla_\mu\phi - \underbrace{i(-i\nabla_\mu\bar{\phi}\bar{\epsilon}\gamma^\mu + \bar{F}\epsilon)}_{\delta\bar{\psi}}\gamma^\nu\nabla_\nu\psi + \underbrace{i\bar{\epsilon}\gamma^\mu\nabla_\mu\psi}_{\delta\bar{F}}\cdot F \\ & -\nabla^\mu\nabla_\mu\bar{\phi}\cdot\underbrace{\bar{\epsilon}\psi}_{\delta\phi} + i\nabla_\mu\bar{\psi}\gamma^\mu\underbrace{(i\gamma^\nu\epsilon\nabla_\nu\phi + \bar{\epsilon}F)}_{\delta\psi} + \bar{F}\cdot\underbrace{i\epsilon\gamma^\mu\nabla_\mu\psi}_{\delta F}.\end{aligned}$$

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$$
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\mathcal{L} &= \nabla_{\mu}\bar{\phi}\nabla^{\mu}\phi - i\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi + \bar{F}F.
\end{aligned}
$$

∂_μ of flat space theory was replaced by ∇_μ

$$
\begin{aligned}
\delta\mathcal{L} = & -\epsilon\bar{\psi}\cdot\nabla^{\mu}\nabla_{\mu}\phi - \nabla_{\mu}\bar{\phi}\bar{\epsilon}\gamma^{\mu}\gamma^{\nu}\nabla_{\nu}\psi \\
& -\nabla^{\mu}\nabla_{\mu}\bar{\phi}\cdot\bar{\epsilon}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\gamma^{\nu}\epsilon\nabla_{\nu}\phi
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∂_μ of flat space theory was replaced by ∇_μ

$$\delta\mathcal{L} = -\epsilon\bar{\psi}\cdot\nabla^\mu\nabla_\mu\phi + \nabla_\nu\nabla_\mu\bar{\phi}\bar{\epsilon}\gamma^\mu\gamma^\nu\psi + \nabla_\mu\bar{\phi}\nabla_\nu\bar{\epsilon}\gamma^\mu\gamma^\nu\psi$$

$$-\nabla^\mu\nabla_\mu\bar{\phi}\cdot\bar{\epsilon}\psi + \bar{\psi}\gamma^\mu\gamma^\nu\epsilon\nabla_\mu\nabla_\nu\phi + \bar{\psi}\gamma^\mu\gamma^\nu\nabla_\mu\epsilon\nabla_\nu\phi$$

$$\nabla_\mu\phi \equiv (\partial_\mu + iqV_\mu)\phi \qquad \nabla_\nu\bar{\epsilon} = -\frac{i}{2f}\bar{\epsilon}\gamma_\nu$$

$$\nabla_\mu\bar{\phi} \equiv (\partial_\mu - iqV_\mu)\bar{\phi} \qquad \nabla_\mu\epsilon = \frac{i}{2f}\gamma_\mu\epsilon$$

SUSY Theories on squashed S^3

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∂_μ of flat space theory was replaced by ∇_μ

$$\delta\mathcal{L} = \tfrac{iq}{2}V_{\mu\nu}\bar{\phi}\bar{\epsilon}\gamma^{\mu\nu}\psi + \tfrac{i}{2f}\nabla_{\mu}\bar{\phi}\bar{\epsilon}\gamma^{\mu}\psi$$

$$+\tfrac{iq}{2}\bar{\psi}\gamma^{\mu\nu}\epsilon V_{\mu\nu}\phi - \tfrac{i}{2f}\bar{\psi}\gamma^{\nu}\epsilon\nabla_{\nu}\phi \qquad \neq 0.$$

$$V_{\mu\nu} \equiv \partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu}, \qquad \nabla_{\mu}\phi \equiv (\partial_{\mu} + iqV_{\mu})\phi \qquad \nabla_{\nu}\bar{\epsilon} = -\tfrac{i}{2f}\bar{\epsilon}\gamma_{\nu}$$

$$\nabla_{\mu}\bar{\phi} \equiv (\partial_{\mu} - iqV_{\mu})\bar{\phi} \qquad \nabla_{\mu}\epsilon = \tfrac{i}{2f}\gamma_{\mu}\epsilon$$

The correct SUSY variation and Lagrangian

for chiral multiplet (ϕ, ψ, F) with U(1) R-charge $(-q, 1-q, 2-q)$ is,

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi, \\
\delta\psi &= i\gamma^{\mu}\epsilon\nabla_{\mu}\phi + \bar{\epsilon}F - \tfrac{q}{f}\epsilon\phi, \\
\delta F &= i\epsilon\gamma^{\mu}\nabla_{\mu}\psi + \tfrac{2q-1}{2f}\epsilon\psi,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} &= \nabla_{\mu}\bar{\phi}\nabla^{\mu}\phi - i\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi + \bar{F}F \\
&\quad - \tfrac{(2q-1)}{2f}\bar{\psi}\psi + \left\{ \tfrac{qR}{4} - \tfrac{q(2q-1)}{2f^2} \right\} \bar{\phi}\phi
\end{aligned}$$

Note:

SUSY δ and Lagrangian $\mathcal{L}$ for the theories on (squashed) S^3 depends explicitly on q (R-charge assignment on matter fields).

Partition function Z also depends on q.

This is made use of in “**Z-minimization**”.

Other Ingredients

Vector multiplet:

$$\delta A_\mu = -\frac{i}{2}(\bar\epsilon\gamma_\mu\lambda - \bar\lambda\gamma_\mu\epsilon),$$

$$\delta\sigma = \frac{1}{2}(\bar\epsilon\lambda - \bar\lambda\epsilon),$$

$$\delta\lambda = \frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma - \frac{1}{f}\sigma\epsilon,$$

$$\delta\bar\lambda = \frac{1}{2}\gamma^{\mu\nu}\bar\epsilon F_{\mu\nu} + D\bar\epsilon - i\gamma^\mu\bar\epsilon D_\mu\sigma + \frac{1}{f}\sigma\bar\epsilon,$$

$$\delta D = -\frac{i}{2}\bar\epsilon\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar\lambda\gamma^\mu\epsilon + \frac{i}{2}\left[\bar\epsilon\lambda + \bar\lambda\epsilon, \sigma\right] - \frac{1}{4f}\bar\epsilon\lambda + \frac{1}{4f}\bar\lambda\epsilon$$

Yang-Mills Lagrangian

$$\mathcal{L}_{\rm YM} \;=\; {\rm Tr}\Bigg(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}\Big(D + \frac{\sigma}{f}\Big)^2 + \frac{i}{2}\bar\lambda\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar\lambda\big[\sigma,\lambda\big] - \frac{1}{4f}\bar\lambda\lambda\Bigg)$$

Gauge-invariant Matter Kinetic Lagrangian

$$\mathcal{L}_{\rm mat} \;=\; D_\mu\bar\phi D^\mu\phi + \bar\phi\sigma^2\phi + i\bar\phi D\phi + \bar F F + \frac{i(2q-1)}{f}\bar\phi\sigma\phi + \left(\frac{qR}{4} - \frac{q(2q-1)}{2f^2}\right)\bar\phi\phi - i\bar\psi\gamma^\mu D_\mu\psi + i\bar\psi\sigma\psi + i\bar\psi\lambda\phi - i\bar\phi\bar\lambda\psi - \frac{(2q-1)}{2f}\bar\psi\psi.$$

See our paper for Chern-Simons, Fayet-Illiopoulos, Superpotential terms.

Calculation of Partition function

Strategy : Localization principle

1. Path integral for partition function localizes onto "**saddle points**", bosonic field configurations satisfying $\delta(\text{fermion}) = 0$

2. $\mathcal{L}_{\text{YM}}, \mathcal{L}_{\text{mat}}$ turn out to be SUSY exact.

   * $\delta(\text{some fermion}) = \mathcal{L}_{\text{YM}} = 0$ at saddle points.

   $$F_{\mu\nu} = D_\mu \sigma = D + \frac{\sigma}{f} = 0.$$

   Saddle points are labelled by constant σ.

   * $\mathcal{L}_{\text{mat}}$ is quadratic in matter fields, so $\phi = F = 0$ at saddle points.

$\mathcal{L}_{\text{YM}}, \mathcal{L}_{\text{mat}}$ can be added to the Lagrangian (with arbitrary coefficients) without changing the value of partition function.

The following “saddle point approximation” gives an **exact** result for partition function.

$$Z = \int d\sigma \left(\frac{\det(\Delta_F)}{\det(\Delta_B)} \right) \exp(-S_{\rm cl})$$

Δ_B, Δ_F are kinetic operators for bosons / fermions which are read from $\mathcal{L}_{\rm YM}, \mathcal{L}_{\rm mat}$ in Gaussian approximation.

We found...

* for the squashed S^3 with SU(2)_Left invariant metric, the determinant is the same as for round S^3.
* for the squashed S^3 with hyper-ellipsoidal metric, the determinant depends on $b = \sqrt{\ell/\tilde{\ell}}$

1-loop determinant, SU(2)_Left invariant case

We notice that

$$e^{a\mu}\partial_\mu \;=\; \Big(\frac{1}{\ell}\mathcal{R}^1, \frac{1}{\ell}\mathcal{R}^2, \frac{1}{\tilde{\ell}}\mathcal{R}^3\Big)$$

$\mathcal{R}^a$: Vector fields generating SU(2)_Right action

Matter determinant.

For simplicity, we consider an electron chiral multiplet of R-charge q which is charged (+1) under an abelian vectormultiplet.
Kinetic operator for boson ϕ and fermion ψ read

$$\Delta_\phi \;=\; \frac{4}{\ell^2}(J^1J^1+J^2J^2)+\frac{4}{\tilde{\ell}^2}\Big(J^3+\frac{q}{2}(1-\frac{\tilde{\ell}}{f})\Big)^2+\sigma^2+\frac{2i(q-1)\sigma}{f}-\frac{q^2}{f^2}+\frac{2q}{f\tilde{\ell}}.$$

$$\Delta_\psi \;=\; \frac{4}{\ell}(S^1J^1+S^2J^2)+\frac{4}{\tilde{\ell}}S^3J^3+\frac{1}{\tilde{\ell}}+\frac{1-q}{f}+2(q-1)\Big(\frac{1}{\tilde{\ell}}-\frac{1}{f}\Big)S^3$$

$$J^a=\tfrac{1}{2i}\mathcal{R}^a,\quad S^a=\tfrac{1}{2}\gamma^a,\quad f=\frac{\ell^2}{\tilde{\ell}}$$

Matter determinant : final form (* after cancellation of many eigenvalues!)

$$\frac{\det\Delta_\psi}{\det\Delta_\phi} = \prod_{n>0}\left(\frac{n+1-q+i\tilde{\ell}\sigma}{n-1+q-i\tilde{\ell}\sigma}\right)^n = s_{b=1}(i-iq-\tilde{\ell}\sigma)$$

Essentially the same as for the round S^3.

n has the meaning $n = 2j+1,$ (orbital angular momentum (j,j))

Degeneration of zeroes and poles is due to unbroken SU(2)_Left.

So, to find the generalization to $b \neq 1$, we need to look for less symmetric squashings.

Vectormultiplet determinant

We decompose the vector multiplet fields into Cartan-Weyl basis, eg

$$\lambda = \sum_i \lambda_i H_i + \sum_{\alpha\in\Delta_+} (\lambda_\alpha E_\alpha + \lambda_{-\alpha} E_{-\alpha})$$

At the saddle point labelled by σ, $(A_\alpha, \lambda_\alpha, \varphi_\alpha)$ acquire **(mass)^2 ~** $(\sigma\cdot\alpha)^2$

φ = (quantum flutuation of the scalar around saddle point σ)

$$\left(\frac{\det\Delta_\lambda}{\det\Delta_{A,\varphi}}\right) = \prod_{\alpha\in\Delta_+}\left(\frac{\det\Delta_{\lambda_\alpha}}{\det\Delta_{A_\alpha,\varphi_\alpha}}\right)^2$$

$\det\Delta_{\lambda_\alpha}$: same as matter fermions.

$\det\Delta_{A_\alpha,\varphi_\alpha}$: complicated, since A_α and φ_α mix.

Calculation of $\det \Delta_{A_\alpha, \varphi_\alpha}$:

First, consider the **4 modes** (with mode-variables x_+, x_-, x_3, x)

$$A_\alpha = x_+ Y_{j,n,m-1}\mu^+ + x_3 Y_{j,n,m}\mu^3 + x_- Y_{j,n,m+1}\mu^-,$$

$$\varphi_\alpha = x Y_{j,n,m}$$

Y_{j,j_L^3,j_R^3} : spherical harmonics, μ^a : LI 1-forms

Then $\Delta_{A_\alpha,\varphi_\alpha}$ mixes these four modes among themselves,
but not with anything else.

Calculation of $\det\Delta_{A_\alpha,\varphi_\alpha}$:

The **4 modes** split into

* 2 longitudinal modes : $A_\alpha \sim d\varphi_\alpha$

* 2 transverse modes : $\varphi_\alpha = d * A_\alpha = 0$

The 2 longitudinal modes have eigenvalues

$$\Delta_{A_\alpha,\varphi_\alpha} = 0$$

(gauge mode)

$$\Delta_{A_\alpha,\varphi_\alpha} = \frac{4j(j+1)-m^2}{\ell^2} + \frac{4m^2}{\tilde{\ell}^2} + (\sigma\cdot\alpha)^2$$

(cancel with the eigenvalues in FP determinant)

Vectormultiplet determinant: final result

$$\int_G d\sigma \left(\frac{\det\Delta_\lambda}{\det\Delta_{A,\varphi}}\right) = \int_G d\sigma \prod_{\alpha\in\Delta_+} \left(\frac{\sinh(\pi\tilde{\ell}\sigma\cdot\alpha)}{\pi\tilde{\ell}\sigma\cdot\alpha}\right)^2$$

$$= \int_T d^r\sigma \prod_{\alpha\in\Delta_+} \sinh^2(\pi\tilde{\ell}\sigma\cdot\alpha)$$

Again, essentially the same as for round S^3.

1-loop determinant, Hyper-ellipsoid case

$$ds^2 = f^2 d\theta^2 + \ell^2 \cos^2\theta d\varphi^2 + \tilde{\ell}^2 \sin^2\theta d\chi^2$$

$$f \equiv \sqrt{\ell^2 \sin^2\theta + \tilde{\ell}^2 \cos^2\theta}$$

$$\epsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{\frac{i}{2}(\chi-\varphi+\theta)} \\ e^{\frac{i}{2}(\chi-\varphi-\theta)} \end{pmatrix}, \quad \bar{\epsilon} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2}(-\chi+\varphi+\theta)} \\ e^{\frac{i}{2}(-\chi+\varphi-\theta)} \end{pmatrix}$$

There is only U(1)xU(1) symmetry.

It is too difficult to find out all the eigenmodes. ⇒ We need a different route.

We recall

Due to SUSY, most of the eigenvalues cancel out between bosons and fermions.

Non-trivial contributions to determinant arise from “**unpaired modes**”.

Matter determinant

for a electron chiral multiplet charged (+1)

under an abelian vectormultiplet.

We take as the regulator Lagrangian

$$\mathcal{L}_{\rm reg} \;=\; \delta_\epsilon \delta_{\bar\epsilon}(\bar\psi\psi - 2i\bar\phi\sigma\phi)$$

= (slightly different from $\mathcal{L}_{\rm mat}$)

and study the spectrum of kinetic operators $\Delta_\phi,\ \Delta_\psi.$

Multiplet structure of eigenmodes: we found

1. $$\Delta_\psi \cdot \Psi \;=\; M\Psi$$

$$\Phi \;\equiv\; \bar{\epsilon}\Psi,$$
$$\longrightarrow \quad \Delta_\phi \cdot \Phi \;=\; M(M-2i\sigma)\Phi$$

2. $$\Delta_\phi \cdot \Phi \;=\; M(M-2i\sigma)\Phi$$

$$\begin{cases} \Psi_1 \;\equiv\; \epsilon\Phi, \\ \Psi_2 \;\equiv\; i\gamma^\mu \epsilon D_\mu \Phi + i\epsilon\sigma\Phi - \frac{q}{f}\epsilon\Phi \end{cases}$$

$$\longrightarrow \quad \begin{pmatrix} \Delta_\psi \cdot \Psi_1 \\ \Delta_\psi \cdot \Psi_2 \end{pmatrix} \;=\; \begin{pmatrix} 2i\sigma & -1 \\ -M(M-2i\sigma) & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

So, one scalar mode $\Phi \;:\; \Delta_\phi = M(M-2i\sigma)$

and two spinor modes $\Psi_1, \Psi_2 \;:\; \Delta_\psi = M, 2i\sigma - M$ form a multiplet.

Nontrivial contributions to determinant arise from

1. unpaired spinor eigenmode

$$\Delta_\psi \cdot \Psi = M\Psi \quad \text{but} \quad \bar{\epsilon}\Psi = 0.$$

. . . contribute M to the enumerator of determinant.

2. missing spinor eigenmode

$$\left.\begin{aligned} \Psi_1 &\equiv \epsilon\Phi, \\ \Psi_2 &\equiv i\gamma^\mu \epsilon D_\mu \Phi + i\epsilon\sigma\Phi - \tfrac{q}{f}\epsilon\Phi \end{aligned}\right\} \quad \text{Linearly dependent.}$$

One can show that if $\Psi_2 = M\Psi_1$, $\begin{cases} \Delta_\psi \cdot \Psi_1 = (2i\sigma - M)\Psi_1 \\ \Delta_\phi \cdot \Phi = M(M - i\sigma)\Phi. \end{cases}$

One can find these cases by solving simple 1st order differential equations.

Matter determinant:

$$\left(\frac{\det\Delta_\psi}{\det\Delta_\phi}\right) = \frac{\prod \text{(unpaired spinor eigenvalues)}}{\prod \text{(missing spinor eigenvalues)}}$$

$$= \prod_{m,n\geq 0} \frac{\frac{m}{\ell} + \frac{n}{\tilde{\ell}} + i\sigma - \frac{q-2}{2}\left(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\right)}{\frac{m}{\ell} + \frac{n}{\tilde{\ell}} - i\sigma + \frac{q}{2}\left(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\right)}$$

$$= s_b\left(\frac{iQ}{2}(1-q) - i\hat{\sigma}\right)$$

where $$Q = b + \frac{1}{b}, \quad b = \sqrt{\ell/\tilde{\ell}}, \quad \hat{\sigma} = \sqrt{\ell\tilde{\ell}}\sigma.$$