Quantum Field Theory and the Analytic S-Matrix

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based on ongoing work in collaboration with

N. Arkani-Hamed, F. Cachazo, J. Trnka,

S. Caron-Huot, and A. Hodges

[arXiv:1012.6032], [arXiv:1012.6030], [arXiv:1011.2477], [arXiv:1008.2958],

[arXiv:1006:1899], [arXiv:0912.4912], [arXiv:0912.3249]

Outline

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Parke & Taylor, Nucl. Phys. B269

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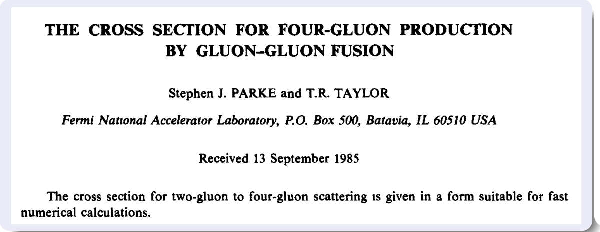
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gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $I = 2$ and replacing the momenta p_3, p_4, p_5, p_6 by $-p_3, -p_4, -p_5, -p_6$.

As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$A_{\binom{0}{2}}(p_1, p_2, p_3, p_4, p_5, p_6) = (\mathscr{D}^\dagger, \mathscr{D}^\dagger_\rho, \mathscr{D}^\dagger_\sigma, \mathscr{D}^\dagger_\tau)_{\binom{0}{2}} \cdot \begin{pmatrix} K & K_\rho & K_\sigma & K_\tau \\ K_\rho & K & K_\tau & K_\sigma \\ K_\sigma & K_\tau & K & K_\rho \\ K_\tau & K_\sigma & K_\rho & K \end{pmatrix} \cdot \begin{pmatrix} \mathscr{D} \\ \mathscr{D}_\rho \\ \mathscr{D}_\sigma \\ \mathscr{D}_\tau \end{pmatrix}_{\binom{0}{2}}, \tag{6}$$

where $\mathscr{D}$, $\mathscr{D}_\rho$, $\mathscr{D}_\sigma$ and $\mathscr{D}_\tau$ are 11-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5 and p_6, and K, K_ρ, K_σ and K_τ are constant 11×11 symmetric matrices. The vectors $\mathscr{D}_\rho$, $\mathscr{D}_\sigma$ and $\mathscr{D}_\tau$ are obtained from the vector $\mathscr{D}$ by the permutations $(p_2 \leftrightarrow p_3)$, $(p_5 \leftrightarrow p_6)$ and $(p_2 \leftrightarrow p_3, p_5 \leftrightarrow p_6)$, respectively, of the momentum variables in $\mathscr{D}$. The individual components of the vector $\mathscr{D}$ represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K, which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N^4(N^2-1)$ and $N^2(N^2-1)$, respectively (N is the number of colors, $N = 3$ for QCD):

$$K = \tfrac{1}{8} g^8 N^4 (N^2-1) K^{(4)} + \tfrac{1}{2} g^8 N^2 (N^2-1) K^{(2)} . \tag{7}$$

Here g denotes the gauge coupling constant. The matrices $K^{(4)}$ and $K^{(2)}$ are given in table 1. The vector $\mathscr{D}$ is related to the thirty-three diagrams $D^{\mathrm{G}}(I = 1-33)$ for two-gluon to four-scalar scattering, eleven diagrams $D^{\mathrm{F}}(I = 1-11)$ for two-fermion to four-scalar scattering and sixteen diagrams $D^{\mathrm{S}}(I = 1-16)$ for two-scalar to four-scalar scattering, in the following way:

$$\mathscr{D}_0 = \frac{2s_{14}}{\sqrt{|s_{15}s_{45}s_{16}s_{46}|\, s_{23}s_{56}}} \{ t_{123}^2 C^{\mathrm{G}} \cdot D_0^{\mathrm{G}} - 4 s_{14} t_{123} E(p_5+p_6, p_6) C^{\mathrm{F}} \cdot D_0^{\mathrm{F}} - 2 s_{14} G(p_5+p_6, p_5+p_6) C^{\mathrm{S}} \cdot D_0^{\mathrm{S}} \},$$

$$\mathscr{D}_2 = \frac{s_{56}}{s_{23}} C^{\mathrm{G}} \cdot D_2^{\mathrm{G}}, \tag{8}$$

where the constant matrices $C^{\mathrm{G}}(11 \times 33)$, $C^{\mathrm{F}}(11 \times 11)$ and $C^{\mathrm{S}}(11 \times 16)$ are given in table 2. The Lorentz invariants s_{ij} and t_{ijk} are defined as $s_{ij} = (p_i + p_j)^2$, $t_{ijk} = (p_i + p_j + p_k)^2$ and the complex functions E and G are given by

$$E(p_i, p_j) = \tfrac{1}{2}\{(p_1 p_4)(p_i p_j) - (p_1 p_i)(p_j p_4) - (p_1 p_j)(p_i p_4) + i\varepsilon_{\mu\nu\rho\lambda} p_1^\mu p_i^\nu p_j^\rho p_4^\lambda\}/(p_1 p_4),$$

$$G(p_i, p_j) = E(p_i, p_5) E(p_j, p_6), \tag{9}$$

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TABLE 1
Matrices $K(I,J)[I=1\text{-}11, J=1\text{-}11]$.

Matrix $K^{(4)}$

8	4	−2	2	−1	2	0	1	0	0	−1
4	8	−1	1	−1	0	2	1	0	1	−1
−2	−1	8	4	4	1	1	2	2	1	2
2	1	4	8	2	−1	−1	4	1	1	1
−1	−1	4	2	8	1	2	4	−2	−1	4
2	0	1	−1	1	8	4	−1	0	1	0
0	2	1	−1	2	4	8	−2	0	0	0
1	1	2	4	4	−1	−2	8	−1	−1	2
0	0	2	1	−2	0	0	−1	8	4	−2
0	1	1	1	−1	1	0	−1	4	8	−1
−1	−1	2	1	4	0	0	2	−2	−1	8

Matrix $K^{(2)}$

0	0	0	0	0	0	0	0	3	3	−3
0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	0	0	0
3	3	0	0	0	3	3	0	0	0	0
3	3	0	0	0	3	3	0	0	0	0
−3	0	0	0	0	0	0	0	0	0	0

Matrix $K_\rho^{(4)}$

0	0	0	0	1	1	0	1	1	0	−1
0	0	0	0	2	0	1	1	2	1	−2
0	0	0	0	0	1	1	1	0	1	1
0	0	0	0	1	0	0	2	0	1	0
1	2	0	1	0	1	2	2	0	0	2
1	0	1	0	1	4	2	0	0	0	−1
0	1	1	0	2	2	4	0	0	0	−2
1	1	1	2	2	0	0	4	0	0	0
1	2	0	0	0	0	0	0	0	2	−1
0	1	1	1	0	0	0	0	2	4	0
−1	−2	1	0	2	−1	−2	0	−1	0	4

Matrix $K_\rho^{(2)}$

3	3	0	3	0	0	0	3	0	0	0
3	3	0	0	0	0	0	0	0	0	0
0	0	3	0	0	3	0	0	3	3	0
3	0	0	0	0	3	0	0	0	0	0
0	0	0	0	0	0	0	0	3	3	0
0	0	3	3	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	0	3	0
3	0	0	0	0	0	0	0	3	0	0
0	0	3	0	3	3	0	3	0	0	0
0	0	3	0	3	3	3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Matrix $K_\sigma^{(4)}$

4	2	0	2	0	1	0	1	0	0	0
2	4	0	1	0	0	1	1	0	1	0
0	0	4	2	2	1	1	1	2	1	0
2	1	2	0	1	2	1	0	1	0	0
0	0	2	1	0	0	0	0	4	2	2
1	0	1	2	0	0	0	0	1	2	0
0	1	1	1	0	0	0	0	2	4	0
1	1	1	0	0	0	0	0	2	0	1
0	0	2	1	4	1	2	2	0	0	−4
0	1	1	0	2	2	4	0	0	0	−2
0	0	0	0	2	0	0	1	−4	−2	4

Matrix $K_\sigma^{(2)}$

0	0	0	0	0	0	0	0	3	0	−3
0	0	0	0	3	0	0	3	0	0	0
0	0	0	0	0	0	3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	3	0	0	0	0	3	0	0	0	0
0	0	0	0	0	3	3	0	0	0	−3
0	0	3	0	3	3	3	0	0	0	0
0	3	0	0	0	0	0	3	0	0	0
3	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	3	0
−3	0	0	0	0	−3	0	0	0	0	0

Matrix $K_\tau^{(4)}$

0	1	−1	−1	1	1	0	1	2	0	0
1	0	−2	−1	2	0	1	1	4	2	0
−1	−2	0	0	0	1	1	1	−1	1	0
−1	−1	0	1	0	2	1	0	1	−1	0
1	2	0	0	1	−1	−1	0	−2	2	1
1	0	1	2	−1	0	1	−2	2	4	−1
0	1	1	1	−1	1	0	−1	4	8	−1
1	1	1	0	0	−2	−1	0	2	−2	0
2	4	−1	1	−2	2	4	2	1	0	−2
0	2	1	−1	2	4	8	−2	0	0	0
0	0	0	0	1	−1	−1	0	−2	0	2

Matrix $K_\tau^{(2)}$

3	3	0	0	0	3	3	0	0	0	0
3	3	0	0	0	3	3	0	0	0	0
0	0	3	3	3	0	0	3	0	0	0
0	0	3	3	3	0	0	3	0	0	3
0	0	3	3	3	0	0	3	0	0	0
3	3	0	0	0	3	3	0	0	0	−3
3	3	0	0	0	3	3	0	0	0	0
0	0	3	3	3	0	0	3	0	0	0
0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	3	3	0
0	0	0	3	0	−3	0	0	0	0	0

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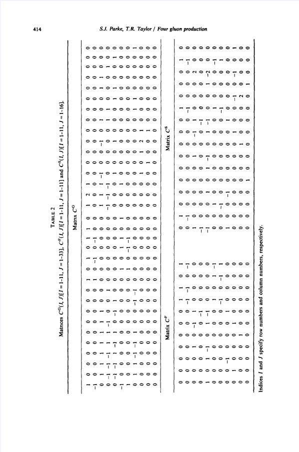

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S.J. Parke, T.R. Taylor / Four gluon production 415

where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt}=1$. For the future use, we define one more function,

$$F(p_i,p_j)=\{(p_1p_4)(p_ip_j)+(p_1p_i)(p_jp_4)-(p_1p_j)(p_ip_4)\}/(p_1p_4)\,. \tag{10}$$

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_4, p_5, p_6.

The diagrams D_2^G are listed below:

$$D_2^G(1)=\frac{\delta_2}{s_{14}s_{25}s_{36}}\{[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)]-[(p_2-p_5)(p_3+p_6)]$$
$$\times[(p_1-p_4)(p_3-p_6)]+[(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)]\}\,,$$

$$D_2^G(2)=\frac{1}{s_{25}s_{36}}\{2E(p_2-p_5,p_3-p_6)-2E(p_3-p_6,p_2-p_5)+\delta_2[(p_2-p_5)(p_3-p_6)]\}\,,$$

$$D_2^G(3)=\frac{4}{s_{25}s_{36}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2,p_3)$$
$$-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)$$
$$-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3)$$
$$+[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)$$
$$-[p_1(p_2-p_5)]E(p_3-p_6,p_3+p_6)-[p_4(p_3-p_6)]E(p_2+p_5,p_2-p_5)$$
$$+\delta_2[p_1(p_2-p_5)][p_4(p_3-p_6)]\}\,,$$

$$D_2^G(4)=\frac{-2}{s_{36}t_{125}}\{E(p_3-p_6,p_3+p_6)-\delta_2[p_4(p_3-p_6)]\}\,,$$

$$D_2^G(5)=\frac{-2}{s_{25}t_{125}}\{E(p_2+p_5,p_2-p_5)-\delta_2[p_1(p_2-p_5)]\}\,,$$

$$D_2^G(6)=\frac{\delta_2}{t_{125}}\,,$$

$$D_2^G(7)=\frac{4}{s_{12}s_{36}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2,p_3)$$
$$-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)-[p_4(p_3-p_6)]E(p_2,p_2-p_5)\}\,,$$

$$D_2^G(8)=\frac{4}{s_{34}s_{25}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2,p_3)$$
$$-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3)-[p_1(p_2-p_5)]E(p_3-p_6,p_3)\}\,,$$

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416 *S.J. Parke, T.R. Taylor / Four gluon production*

$$D_2^G(9) = \frac{4}{s_{15}s_{36}t_{125}}\{[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3) - [(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6) + [p_4(p_3-p_6)]E(p_5,p_2-p_5)\},$$

$$D_2^G(10) = \frac{4}{s_{25}s_{46}t_{125}}\{[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6) - [(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6) + [p_1(p_2-p_5)]E(p_3-p_6,p_6)\},$$

$$D_2^G(11) = \frac{\delta_2}{s_{36}t_{124}}[s_{35}-s_{56}+s_{36}],$$

$$D_2^G(12) = \frac{-\delta_2}{s_{36}t_{145}}[s_{23}-s_{26}-s_{36}],$$

$$D_2^G(13) = \frac{\delta_2}{s_{14}s_{36}t_{124}}[s_{12}-s_{24}][s_{35}-s_{56}+s_{36}],$$

$$D_2^G(14) = \frac{\delta_2}{s_{14}s_{36}t_{145}}[s_{15}-s_{45}][s_{23}-s_{26}-s_{36}],$$

$$D_2^G(15) = \frac{\delta_2}{s_{14}s_{36}}(p_1-p_4)(p_3-p_6),$$

$$D_2^G(16) = \frac{-4}{s_{12}s_{36}t_{124}}[s_{35}-s_{56}+s_{36}]E(p_2,p_2),$$

$$D_2^G(17) = \frac{4}{s_{36}s_{45}t_{145}}[s_{23}-s_{26}-s_{36}]E(p_5,p_5),$$

$$D_2^G(18) = \frac{-4}{s_{12}s_{36}s_{45}}[2(p_1+p_2)(p_3-p_6)-s_{36}]E(p_2,p_5),$$

$$D_2^G(19) = \frac{-2}{s_{12}s_{36}}E(p_2,p_3-p_6),$$

$$D_2^G(20) = \frac{2}{s_{36}s_{45}}E(p_3-p_6,p_5),$$

$$D_2^G(21) = \frac{-4}{s_{25}s_{34}t_{134}}[s_{26}-s_{56}+s_{25}]E(p_3,p_3),$$

$$D_2^G(22) = \frac{4}{s_{16}s_{25}t_{146}}[s_{23}-s_{35}-s_{25}]E(p_6,p_6),$$

$$D_2^G(23) = \frac{4}{s_{16}s_{25}s_{34}}[2(p_1+p_6)(p_2-p_5)+s_{25}]E(p_6,p_3),$$

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$$D_2^G(24) = \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3)\,,$$

$$D_2^G(25) = \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5)\,,$$

$$D_2^G(26) = \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5)\,,$$

$$D_2^G(27) = \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6)\,,$$

$$D_2^G(28) = \frac{2}{s_{15}t_{125}} E(p_5, p_2 - p_5)\,,$$

$$D_2^G(29) = \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3)\,,$$

$$D_2^G(30) = \frac{4}{s_{12}s_{34}t_{125}} [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6) - t_{125}] E(p_2, p_3)\,,$$

$$D_2^G(31) = \frac{4}{s_{12}s_{46}t_{125}} [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) + t_{125}] E(p_2, p_6)\,,$$

$$D_2^G(32) = \frac{4}{s_{15}s_{34}t_{125}} [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) + t_{125}] E(p_5, p_3)\,,$$

$$D_2^G(33) = \frac{4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) - t_{125}] E(p_5, p_6)\,, \qquad (11)$$

where $\delta_2 = 1$.

The diagrams D_0^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the functions $E(p_i, p_j)$ by $G(p_i, p_j)$.

The diagrams D_0^F are listed below:

$$D_0^F(1) = \frac{4}{s_{15}s_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) + [F(p_6, p_3) + s_{34}]E(p_5, p_5)\}\,,$$

$$D_0^F(2) = \frac{-4}{s_{16}s_{25}s_{34}} \{[F(p_6, p_2) + \tfrac{1}{2}s_{16}]E(p_3, p_5) + [F(p_2, p_3) + \tfrac{1}{2}s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_2, p_5)\}\,,$$

$$D_0^F(3) = \frac{4}{s_{15}s_{36}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) - [F(p_3, p_6) - \tfrac{1}{2}s_{36} - \tfrac{1}{2}s_{34} + \tfrac{1}{2}s_{46}]E(p_5, p_5)\}\,,$$

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418 *S.J. Parke, T.R. Taylor / Four gluon production*

$$D_0^{\mathrm{F}}(4) = \frac{4}{s_{25}s_{34}t_{125}}\{F(p_2,p_3)E(p_5,p_5) - F(p_5,p_3)E(p_2,p_5)$$
$$+[F(p_5,p_2) - \tfrac{1}{2}s_{25} - \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{15}]E(p_3,p_5)\},$$

$$D_0^{\mathrm{F}}(5) = \frac{2}{s_{16}s_{25}t_{146}}[s_{35} - s_{23} + s_{25}]E(p_6,p_5),$$

$$D_0^{\mathrm{F}}(6) = \frac{2}{s_{25}s_{34}t_{134}}[s_{56} - s_{26} - s_{25}]E(p_3,p_5),$$

$$D_0^{\mathrm{F}}(7) = \frac{4}{s_{25}s_{36}t_{125}}\{[F(p_5,p_2) - \tfrac{1}{2}s_{25} - \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{15}]E(p_3,p_5)$$
$$+[F(p_2,p_3) + \tfrac{1}{4}t_{125}]E(p_5,p_5) - [F(p_5,p_3) + \tfrac{1}{4}t_{125}]E(p_2,p_5)\},$$

$$D_0^{\mathrm{F}}(8) = \frac{1}{s_{14}s_{36}}E(p_3 - p_6, p_5),$$

$$D_0^{\mathrm{F}}(9) = \frac{2}{s_{14}s_{36}t_{124}}[s_{35} - s_{56} + s_{36}]E(p_2,p_5),$$

$$D_0^{\mathrm{F}}(10) = \frac{2}{s_{14}s_{36}t_{145}}[s_{23} - s_{26} - s_{36}]E(p_5,p_5),$$

$$D_0^{\mathrm{F}}(11) = \frac{1}{2s_{14}s_{25}s_{36}}\{[s_{23} + s_{35} - s_{26} - s_{56}]E(p_2 - p_5, p_5)$$
$$-[s_{23} + s_{26} - s_{35} - s_{56}]E(p_3 - p_6, p_5) - [s_{23} + s_{56} - s_{35} - s_{26}]E(p_2 + p_5, p_5)\}. \quad (12)$$

The diagrams D_0^{S} are listed below:

$$D_0^{\mathrm{S}}(1) = \frac{1}{s_{25}s_{36}t_{125}}[s_{34} - s_{46} + s_{36}][s_{12} - s_{15} - s_{25}],$$

$$D_0^{\mathrm{S}}(2) = \frac{1}{s_{14}s_{36}t_{124}}[s_{12} - s_{24} - s_{14}][s_{35} - s_{56} + s_{36}],$$

$$D_0^{\mathrm{S}}(3) = \frac{1}{s_{14}s_{36}t_{145}}[s_{15} - s_{45} + s_{14}][s_{23} - s_{26} - s_{36}],$$

$$D_0^{\mathrm{S}}(4) = \frac{1}{s_{15}s_{36}t_{125}}[s_{15} + s_{25} - s_{12}][s_{34} - s_{46} + s_{36}],$$

$$D_0^{\mathrm{S}}(5) = \frac{1}{s_{15}s_{34}t_{156}}[s_{56} - s_{15} - s_{16}][s_{23} - s_{24} - s_{34}],$$

$$D_0^{\mathrm{S}}(6) = \frac{1}{s_{15}s_{34}t_{125}}[s_{46} - s_{34} - s_{36}][s_{12} - s_{25} - s_{15}],$$

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$$D_0^S(7) = \frac{1}{s_{25}s_{34}t_{125}}[s_{36} - s_{46} + s_{34}][s_{12} - s_{15} - s_{25}]\,,$$

$$D_0^S(8) = \frac{1}{s_{16}s_{25}t_{146}}[s_{25} + s_{35} - s_{23}][s_{14} - s_{46} + s_{16}]\,,$$

$$D_0^S(9) = \frac{1}{s_{25}s_{34}t_{134}}[s_{14} + s_{34} - s_{13}][s_{26} - s_{56} + s_{25}]\,,$$

$$D_0^S(10) = \frac{1}{s_{25}s_{36}}(p_2 - p_5)(p_3 - p_6)\,,$$

$$D_0^S(11) = \frac{1}{s_{14}s_{36}}(p_1 - p_4)(p_3 - p_6)\,,$$

$$D_0^S(12) = \frac{1}{s_{16}s_{25}}(p_6 - p_1)(p_2 - p_5)\,,$$

$$D_0^S(13) = \frac{1}{s_{15}s_{34}}(p_5 - p_1)(p_3 - p_4)\,,$$

$$D_0^S(14) = \frac{1}{s_{25}s_{34}}(p_2 - p_5)(p_3 - p_4)\,,$$

$$\begin{aligned} D_0^S(15) = \frac{1}{s_{14}s_{25}s_{36}}\{&[(p_2 + p_5)(p_3 - p_6)][(p_1 - p_4)(p_2 - p_5)] \\ &+ [(p_2 - p_5)(p_3 - p_6)][(p_1 - p_4)(p_3 + p_6)] \\ &+ [(p_1 + p_4)(p_2 - p_5)][(p_1 - p_4)(p_3 - p_6)]\}\,, \end{aligned}$$

$$\begin{aligned} D_0^S(16) = \frac{2}{s_{16}s_{34}s_{25}}\{&[(p_2 - p_5)(p_3 + p_4)][(p_1 - p_6)(p_3 - p_4)] \\ &+ [(p_1 + p_6)(p_3 - p_4)][(p_1 - p_6)(p_2 - p_5)] \\ &+ [(p_1 - p_6)(p_2 + p_5)][(p_3 - p_4)(p_2 - p_5)]\}\,. \end{aligned} \tag{13}$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11)-(13). The result is substituted to eq. (8) to obtain the vectors $\mathscr{D}_0$ and $\mathscr{D}_2$. After generating the vectors $\mathscr{D}_{0_\rho}$, $\mathscr{D}_{0_\sigma}$, $\mathscr{D}_{0_\tau}$, $\mathscr{D}_{2_\rho}$, $\mathscr{D}_{2_\sigma}$ and $\mathscr{D}_{2_\tau}$ by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_0 and A_2. Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi-gluon amplitudes are tested by checking the gauge invariance. Due to the specifics

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of our calculation, the most powerful test does not rely on the gauge symmetry, but on the appropriate permutation symmetries. The function $A_0(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1, p_2, p_3) and separately, (p_4, p_5, p_6), whereas the function $A_2(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under the permutations of (p_1, p_2, p_3, p_4) and separately, (p_5, p_6). This test is extremely powerful, because the required permutation symmetries are hidden in our supersymmetry relations, eqs. (1) and (3), and in the structure of amplitudes involving different species of particles. Another, very important test relies on the absence of the double poles of the form $(s_{ij})^{-2}$ in the cross section, as required by general arguments based on the helicity conservation. Further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the two goes to three cross section [3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

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References

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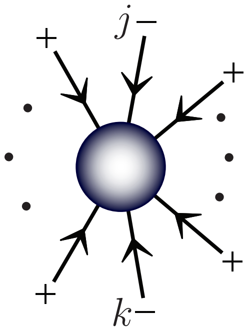

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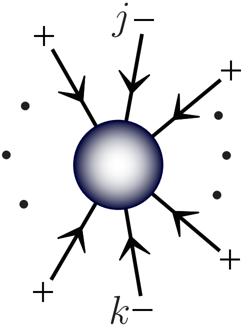

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We compute the even part of the planar two-loop MHV amplitude in $\mathcal{N}=4$ supersymmetric Yang-Mills theory, for an arbitrary number of external particles. The answer is expressed as a sum of conformal integrals.

C. Vergu, arXiv:0908.2394

complexity of the computations. It has also been useful to use the results for the cuts already computed when computing the coefficients of integrals detected by new cuts. In this way, one can insure the consistency of results from different cuts and reduce the number of unknowns at the same time.

Let us make a further comment about our computation procedure. The conformal integrals with pentagon loops have numerators containing the loop momenta in combinations like $(k+l)^2$, where l is the loop momentum and k is an external on-shell momentum. If the propagator with momentum l is cut then, on that cut, one cannot distinguish between $(k+l)^2$ and $2k\cdot l$. However, it is easy to see that one can choose to cut another propagator and in that case this ambiguity does not arise and the numerator factor is uniquely defined.

IV. RESULTS

We use dual variable notation (see Ref. [48]) for the integrals. The external dual variables are listed in clockwise direction. To the left loop we associate the dual variable x_p and to the right loop we associate the dual variable x_q. We use the notation $x_{ij} \equiv x_i - x_j$.

We introduce the following notation which will be useful in the following

$$\begin{bmatrix} a & b & c & \cdots \\ a' & b' & c' & \cdots \end{bmatrix} = x^2_{aa'} x^2_{bb'} x^2_{cc'} \cdots \pm (\text{permutations of } \{a', b', c', \ldots\}). \tag{6}$$

The sign $\pm$ above takes into account the signature of the permutation of $\{a', b', c', \ldots\}$. It is easy to show that

$$\begin{bmatrix} a & b & c & \cdots \\ a' & b' & c' & \cdots \end{bmatrix} = \det_{\substack{i\in\{a,b,c,\cdots\} \\ j\in\{a',b',c',\cdots\}}} x^2_{ij}. \tag{7}$$

For some topologies, the expansion of the $[\ ]$ symbol yields terms that would cancel propagators. For those cases we make the convention that all the terms that would cancel propagators are absent. In fact, as we will see, terms that would cancel propagators of the double pentagon topologies naturally yield coefficients for some of the topologies with a smaller number of propagators.

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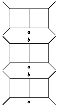
A. Double box topologies

In the case of the double box topologies the massive legs attached to the vertices incident with the common edge have to be a sum of at least three massless momenta. The cases where these massive legs are the sum of two massless momenta are treated separately in the subsection. IV A 7. This distinction only arises for the double box topologies.

1. No legs attached

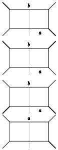


$$\frac{1}{2}\left(x^2_{a,a+2}\right)^2 x^2_{a-1,a+1} \tag{8}$$

$$\frac{1}{4}\left(x^2_{ab}\right)^2 x^2_{a-1,a+1} \tag{9}$$

$$-\frac{1}{4}x^2_{ab}\left(x^2_{a,b-1}x^2_{a-1,b}-x^2_{ab}x^2_{a-1,b-1}\right)^2 \tag{10}$$

2. One massless leg attached



$$\frac{1}{4}\left(x^2_{a,b+1}x^2_{a+1,b}-x^2_{ab}x^2_{a+1,b+1}\right)x^2_{a+2,b} \tag{11}$$

$$\frac{1}{4}\Big(-x^2_{a-1,b}x^2_{a,a+2}x^2_{a+1,b}+x^2_{a-1,a+2}x^2_{ab}x^2_{a+1,b}- \\ -x^2_{a-1,a+1}x^2_{ab}x^2_{a+2,b}\Big) \tag{12}$$

$$-\frac{1}{4}x^2_{ab}x^2_{a+1,b}x^2_{b-1,b+1} \tag{13}$$

$$-\frac{1}{4}x^2_{a-3,a}x^2_{a-2,a}x^2_{a-1,a+1} \tag{14}$$

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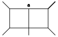
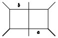
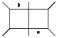


$$\frac{1}{4}\left(x^2_{a-4,a-1}x^2_{a-3,a} - 2x^2_{a-4,a}x^2_{a-3,a-1}\right)x^2_{a-2,a} \tag{15}$$

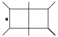
3. Two massless legs attached



$$\frac{1}{4}\Big(x^2_{a,b+2}x^2_{a+1,b}x^2_{b-1,b+1} - x^2_{a,b+1}x^2_{a+1,b}x^2_{b-1,b+2} + \\ + x^2_{a,b+1}x^2_{a+1,b-1}x^2_{b,b+2}\Big) \tag{16}$$

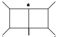


$$\frac{1}{4}\Big(-x^2_{a-1,b-1}x^2_{a,b+1}x^2_{a+1,b} + x^2_{a-1,b-1}x^2_{ab}x^2_{a+1,b+1} - \\ - x^2_{a-1,a+1}x^2_{ab}x^2_{b-1,b+1}\Big) \tag{17}$$

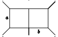


$$\frac{1}{4}\Big(x^2_{a-2,a+3}x^2_{a-1,a+1}x^2_{a,a+2} - 2x^2_{a-2,a+2}x^2_{a-1,a+1}x^2_{a,a+3} + \\ + x^2_{a-2,a+1}x^2_{a-1,a+2}x^2_{a,a+3} - x^2_{a-2,a}x^2_{a-1,a+2}x^2_{a+1,a+3}\Big) \tag{18}$$

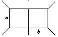
4. One massive leg attached



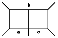
$$\frac{1}{4}x^2_{a-2,a}x^2_{a-1,a+1}x^2_{a,a+2} \tag{19}$$



$$\frac{1}{4}\left(x^2_{a-1,a+1}x^2_{a,b-1}x^2_{a+1,b} - x^2_{a-1,a+1}x^2_{ab}x^2_{a+1,b-1}\right) \tag{20}$$



$$0 \tag{21}$$



$$\frac{1}{4}\Big(x^2_{ac}x^2_{a+1,b}x^2_{b,c-1} - x^2_{ab}x^2_{a+1,c}x^2_{b,c-1} - \\ - x^2_{a,c-1}x^2_{a+1,b}x^2_{bc} + x^2_{ab}x^2_{a+1,c-1}x^2_{bc}\Big) \tag{22}$$

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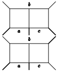
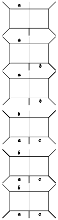


$$0 \qquad (23)$$

$$0 \qquad (24)$$

5. One massless leg and one massive leg attached



$$-\frac{1}{4}x^2_{a-2,a}x^2_{a-1,a+2}x^2_{a+1,a+3} \qquad (25)$$

$$0 \qquad (26)$$

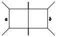
$$\frac{1}{4}x^2_{a-2,a}\left(x^2_{a-1,b}x^2_{a+1,b-1}-x^2_{a-1,b-1}x^2_{a+1,b}\right) \qquad (27)$$

$$\frac{1}{4}\Big(-x^2_{ac}x^2_{a+1,b}x^2_{b+1,c-1}+x^2_{ab}x^2_{a+1,c}x^2_{b+1,c-1}+ + x^2_{a,c-1}x^2_{a+1,b}x^2_{b+1,c}-x^2_{ab}x^2_{a+1,c-1}x^2_{b+1,c}\Big) \qquad (28)$$

$$0 \qquad (29)$$

$$0 \qquad (30)$$

6. Two massive legs attached



$$0 \qquad (31)$$

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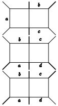
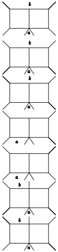


$$0 \quad (32)$$

$$0 \quad (33)$$

$$0 \quad (34)$$

7. Extra double boxes



$$\frac{1}{4}\Big(-x^2_{a-2,b}x^2_{a-1,a+2}x^2_{a+1,b}+x^2_{a-2,a+2}x^2_{a-1,b}x^2_{a+1,b}+ \\ +x^2_{a-2,b}x^2_{a-1,a+1}x^2_{a+2,b}-x^2_{a-2,a+1}x^2_{a-1,b}x^2_{a+2,b}\Big) \quad (35)$$

$$-\frac{1}{4}\begin{bmatrix} a+1 & b-1 & b \\ b & b+1 & a-1 \end{bmatrix} \quad (36)$$

$$0 \quad (37)$$

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & a+2 \\ a+2 & a+3 & a-2 \end{bmatrix} \quad (38)$$

$$\frac{1}{4}\left(x^2_{a-3,a+1}x^2_{a-2,a+2}-x^2_{a-3,a+2}x^2_{a-2,a+1}\right)x^2_{a,a+2} \quad (39)$$

$$-\frac{1}{4}\begin{bmatrix} a+1 & b-1 & b \\ b+1 & b+2 & a-1 \end{bmatrix} \quad (40)$$

$$0 \quad (41)$$

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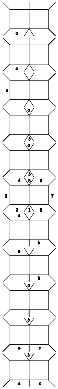

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & a+2 \\ a+3 & a+4 & a-2 \end{bmatrix} \tag{42}$$

$$\frac{1}{4}\left(x^2_{a-3,a+3}x^2_{a-2,a+1} - x^2_{a-3,a+1}x^2_{a-2,a+3}\right) x^2_{a,a+2} \tag{43}$$

$$-\frac{1}{4}\begin{bmatrix} a-1 & a & a+1 \\ a+3 & a-4 & a-3 \end{bmatrix} \tag{44}$$

$$0 \tag{45}$$

$$0 \tag{46}$$

$$-\frac{1}{2}\begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix} \tag{47}$$

$$0 \tag{48}$$

$$-\frac{1}{4}\begin{bmatrix} a-2 & a-1 & a \\ a+2 & b-1 & b \end{bmatrix} \tag{49}$$

$$-\frac{1}{4}\begin{bmatrix} a-3 & a-2 & a-1 \\ a+1 & a+2 & a+3 \end{bmatrix} \tag{50}$$

$$0 \tag{51}$$

$$0 \tag{52}$$

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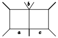
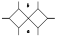
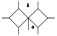


$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 \\ b+1 & c-1 & c \end{bmatrix} \tag{53}$$

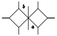
B. Kissing double-box topologies



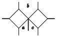
$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b & b+1 & a-1 & a \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b & b+1 \\ a-1 & a \end{bmatrix}= \frac{1}{4}\Big(x^2_{a-1,b+1}x^2_{a+1,b-1}\left(x^2_{ab}\right)^2-x^2_{a-1,b-1}x^2_{a+1,b+1}\left(x^2_{ab}\right)^2+ +x^2_{a-1,a+1}x^2_{b-1,b+1}\left(x^2_{ab}\right)^2-x^2_{a-1,b}x^2_{a,b+1}x^2_{a+1,b-1}x^2_{ab}- -x^2_{a-1,b+1}x^2_{a,b-1}x^2_{a+1,b}x^2_{ab}+x^2_{a-1,b-1}x^2_{a,b+1}x^2_{a+1,b}x^2_{ab}+ +x^2_{a-1,b}x^2_{a,b-1}x^2_{a+1,b+1}x^2_{ab}\Big) \tag{54}$$



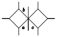
$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b \\ b & b+1 & a-1 & a \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a-1 & a \\ b & b+1 \end{bmatrix}\begin{bmatrix} a+1 & a+2 \\ b-1 & b \end{bmatrix} \tag{55}$$



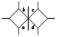
$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b \\ b+1 & b+2 & a-1 & a \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a+1 & a+2 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b+1 & b+2 \\ a-1 & a \end{bmatrix} \tag{56}$$



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b & b+1 & c-1 & c \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b & b+1 \\ c-1 & c \end{bmatrix} \tag{57}$$



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & c \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b+1 & b+2 \\ c-1 & c \end{bmatrix} \tag{58}$$



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ c & c+1 & d-1 & d \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} c & c+1 \\ d-1 & d \end{bmatrix} \tag{59}$$

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C. Box-Pentagon topologies

1. No legs attached

$$\frac{1}{4}x^2_{ab}x^2_{a+1,q}\left(x^2_{a,b+1}x^2_{a-1,b}-x^2_{ab}x^2_{a-1,b+1}\right) \tag{60}$$

$$\frac{1}{2}x^2_{a,a+2}x^2_{a+1,q}\left(x^2_{a-1,a+2}x^2_{a,a+3}-x^2_{a-1,a+3}x^2_{a,a+2}\right) \tag{61}$$

2. One massless leg attached

$$\frac{1}{4}\left(x^2_{a-1,b+1}x^2_{ab}-x^2_{a-1,b}x^2_{a,b+1}\right)\left(x^2_{a+1,q}x^2_{a+2,b}-x^2_{a+1,b}x^2_{a+2,q}\right) \tag{62}$$

$$\frac{1}{4}x^2_{a-1,b}\left(x^2_{ab}x^2_{a+1,q}x^2_{b-1,b+1}+x^2_{a,b+1}x^2_{a+1,b}x^2_{b-1,q}-x^2_{ab}x^2_{a+1,b+1}x^2_{b-1,q}\right) \tag{63}$$

$$\begin{aligned}\frac{1}{4}\Big(&x^2_{a-4,a}x^2_{a-3,a}x^2_{a-2,q}x^2_{a-1,a+1}-x^2_{a-4,a+1}x^2_{a-3,a}x^2_{a-2,a}x^2_{a-1,q}+\\&+2x^2_{a-4,a}x^2_{a-3,a+1}x^2_{a-2,a}x^2_{a-1,q}-x^2_{a-4,a}x^2_{a-3,a}x^2_{a-2,a+1}x^2_{a-1,q}\Big)\end{aligned} \tag{64}$$

3. One massive leg attached

$$0 \tag{65}$$

$$\frac{1}{4}\left(x^2_{aq}x^2_{a+1,b}-x^2_{ab}x^2_{a+1,q}\right)\left(x^2_{bc}x^2_{b+1,c-1}-x^2_{b,c-1}x^2_{b+1,c}\right) \tag{66}$$

$$\frac{1}{4}x^2_{a-1,a+1}x^2_{aq}\left(x^2_{a+1,b-1}x^2_{a+2,b}-x^2_{a+1,b}x^2_{a+2,b-1}\right) \tag{67}$$

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4. One massless, one massive leg attached

$$0 \tag{68}$$

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b & b+1 \\ b+2 & c-1 & c & q \end{bmatrix}. \tag{69}$$

Note that in the previous formula we suppress the terms containing $x^2_{b+1,q}$ which would otherwise cancel a propagator of the underlying topology. When expanded out, the expression above has 12 terms.

$$-\frac{1}{4}\begin{bmatrix} a-2 & a-1 & a & a+1 \\ a+2 & b-1 & b & q \end{bmatrix}. \tag{70}$$

In the previous formula we suppress the terms containing $x^2_{a+1,q}$ which would otherwise cancel a propagator of the underlying topology.

5. Two massless legs attached

$$\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b+1 & b+2 & a-1 & q \end{bmatrix} \tag{71}$$

In the previous formula we suppress the terms containing $x^2_{a+1,q}$ which would otherwise cancel a propagator of the underlying topology.

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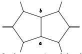
$$\frac{1}{4}\begin{bmatrix} a-2 & a-1 & a & a+1 \\ a+2 & a+3 & a-3 & q \end{bmatrix} - \frac{1}{4}\begin{bmatrix} a-1 & a & a+1 & a+2 \\ a+3 & a-3 & a-2 & q \end{bmatrix} =$$
$$\frac{1}{4}\Big(-x^2_{a-3,a+1}x^2_{a-2,a+3}x^2_{a-1,q}x^2_{a,a+2} + x^2_{a-3,a-1}x^2_{a-2,a+3}x^2_{a+1,q}x^2_{a,a+2} -$$
$$- x^2_{a-3,a+2}x^2_{a-2,a+1}x^2_{a-1,q}x^2_{a,a+3} + 2x^2_{a-3,a+1}x^2_{a-2,a+2}x^2_{a-1,q}x^2_{a,a+3} +$$
$$+ x^2_{a-3,a+1}x^2_{a-2,a+3}x^2_{a-1,a+2}x^2_{aq} + x^2_{a-3,a+2}x^2_{a-2,a+1}x^2_{a-1,a+3}x^2_{aq} -$$
$$- 2x^2_{a-3,a+1}x^2_{a-2,a+2}x^2_{a-1,a+3}x^2_{aq} + x^2_{a-3,a+2}x^2_{a-2,a}x^2_{a-1,q}x^2_{a+1,a+3} -$$
$$- 2x^2_{a-3,a}x^2_{a-2,a+2}x^2_{a-1,q}x^2_{a+1,a+3} + 2x^2_{a-3,a-1}x^2_{a-2,a+2}x^2_{aq}x^2_{a+1,a+3} -$$
$$- x^2_{a-3,a}x^2_{a-2,a+3}x^2_{a-1,a+2}x^2_{a+1,q} - x^2_{a-3,a+2}x^2_{a-2,a}x^2_{a-1,a+3}x^2_{a+1,q} +$$
$$+2x^2_{a-3,a}x^2_{a-2,a+2}x^2_{a-1,a+3}x^2_{a+1,q} - 2x^2_{a-3,a-1}x^2_{a-2,a+2}x^2_{a,a+3}x^2_{a+1,q}\Big). \tag{72}$$

We have written down this formula to emphasize how nontrivial it is. We suppress the terms containing $x^2_{a-2,q}$ and $x^2_{a+2,q}$, respectively. These terms would otherwise cancel a propagator of the underlying topology. We will see below that the box-pentagon topologies with massless legs attached to the vertices of the edge common to both loops can in fact be seen to originate in double-pentagon topologies, by cancelling some propagators.

D. Double pentagon topologies

1. No legs attached



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b & p \\ b & b+1 & a-1 & a & q \end{bmatrix} \tag{73}$$

In the expansion of the above formula we drop terms that would cancel propagators (in this case, the terms containing x^2_{ap}, x^2_{aq}, x^2_{bp}, x^2_{bq}, or x^2_{pq}). This expression has 6 terms when expanded.

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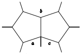
2. One massless leg attached

$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b & p \\ b & b+1 & a-1 & a & q \end{bmatrix} \tag{74}$$

In the formula above we drop terms that would cancel propagators (in this case, the terms are x^2_{bp}, x^2_{bq} and x^2_{pq}). This expression has 15 terms when expanded.

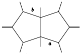
3. One massive leg attached



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b & p \\ b & b+1 & c-1 & c & q \end{bmatrix} \tag{75}$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x^2_{bp}, x^2_{bq} or x^2_{pq}). This expression has 16 terms when expanded.

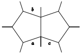
4. Two massless legs attached



$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b & p \\ b+1 & b+2 & a-1 & a & q \end{bmatrix} \tag{76}$$

In the formula we drop terms that would cancel propagators (in this case, the terms containing x^2_{pq}). This expression has 64 terms when expanded.

5. One massless, one massive leg attached



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b & p \\ b+1 & b+2 & c-1 & c & q \end{bmatrix} \tag{77}$$

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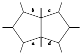
In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{pq}^2). This expression has 78 terms when expanded.

6. Two massive legs attached



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b & p \\ c & c+1 & d-1 & d & q \end{bmatrix} \tag{78}$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{pq}^2). When expanded, the above expression contains 96 terms. The number of conformal dressings is 160 (the number of coefficients unrelated by symmetries is lower).

E. Assembly of the result

As explained in Sec. II, for the MHV amplitudes the ratio between the ℓ-loop amplitude and the tree-level amplitude can be written as a sum between parity even and parity odd contributions

$$M_n^{(\ell)} = M_n^{(\ell),\text{even}} + M_n^{(\ell),\text{odd}}. \tag{79}$$

Then, the even part can be written

$$M_n^{(2),\text{even}} = -\pi^{-D} e^{2\gamma\epsilon} \int d^D x_p d^D x_q \sum_{\sigma} \sum_{i \in \text{Topologies}} s_i c_i I_i, \tag{80}$$

where the first sum runs over cyclic and anti-cyclic permutations of the external legs, the second sum runs over all the topologies, s_i is a symmetry factor associated to topology i, c_i is the numerator of the topology i, as listed in Sec. IV and I_i is the denominator or the product of propagators in the topology i.

Apart from the parity odd part which we have not computed, there is also a contribution which is not detectable from four-dimensional cuts, denoted by $M^{(2),\mu}$. This part of the result is such that its integrand vanishes in four dimensions, but the integral itself can give contributions to the divergent and finite parts. In Ref. [32], for $n = 6$ case, this part of the result was found to be closely related to $\mathcal{O}(\epsilon)$ contributions at one loop, $M^{(1),\mu}$.

Based on previous computations we expect that the odd part and the μ integrals will not be needed in order to compare with the Wilson loop results. The odd parts could be

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computed by using the leading singularity method (see Ref. [33] and also [34, 35]) or the technique of maximal cuts of Ref. [31]. In order to compute $M^{(2),\mu}$, one would have to compute D-dimensional cuts. In practice this is done by computing the cuts of $\mathcal{N}=1$ super-Yang-Mills in ten dimensions, dimensionally reduced to D dimensions.

V. DISCUSSION

In this paper we computed the even part of the two-loop planar MHV scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills. The answer can be expressed in terms of a finite (and relatively small) number of two-loop pseudo-conformal integrals.

A computation of these integrals in dimensional regularization through the finite parts (of order $\mathcal{O}(\epsilon^0)$) would be very interesting and would allow a comparison with the results of Ref. [28], where the corresponding Wilson loop computation was performed.

However, a computation of these integrals seems to be rather difficult. In Ref. [28] the Wilson loop result was expressed in terms of some master integrals called: "hard", "curtain", "cross", "Y" and "factorized cross." These master integrals depend on whether some momenta are zero, massless or massive (this is similar to the situation for scattering amplitudes; in that case also, the value of the integral depends on whether the external legs are massive or massless).

It is interesting to note that for the Wilson loop computation, there are no new master integrals beyond nine sides (this number arises by considering the "hard" integral where all the momenta Q_1, Q_2 and Q_3 are massive). For the scattering amplitude, however, new integrals appear until twelve points, as shown in this paper. It would be interesting to get a deeper understanding of this "mismatch."

The results presented in this paper hint that a different organization of the result may be possible. For example, the coefficients written down using the square brackets symbols can be assembled over a common denominator whose topology is that of a double pentagon. Sometimes, the coefficient of a given topology needs to be split into two contributions which get assembled into different double pentagon topologies (see Eq. (72) for an example).

It is also noteworthy that part of the kissing double boxes coefficient neatly combines with a double pentagon topology after multiplying the numerator and denominator by x_{pq}^2, while the remaining part has a factorized form. This factorized form is a product of "one-mass"

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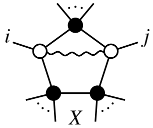

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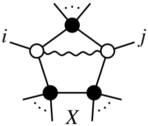
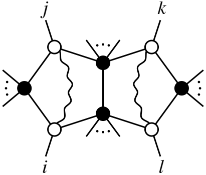


$$+\ \frac{1}{2}\sum_{i<j<k<l<i}$$



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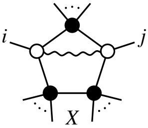

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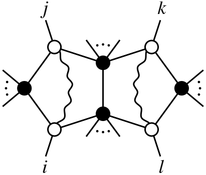
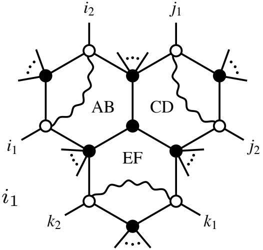
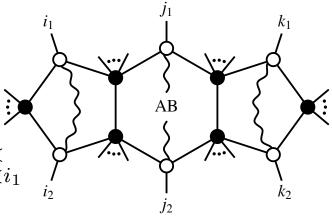


$$+ \frac{1}{3}\sum_{\substack{i_1\leq i_2<j_1\leq\\ \leq j_2<k_1\leq k_2<i_1}}$$



$$+ \frac{1}{2}\sum_{\substack{i_1\leq j_1<k_1<\\ <k_2\leq j_2<i_2<i_1}}$$



$$\left. \right\}$$

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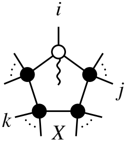
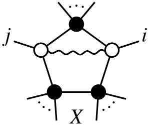
$$\left\{ \sum_{i<j<i} R[1\; i\; i{+}1\; j\; j{+}1] + \sum_{i<j<k<i} \right.$$



$$\times R\,[i\; j\; j{+}1\; k\; k{+}1] + \sum_{i<j<i}$$



$$\left. \times \mathcal{A}^{\text{tree}}_{\text{NMHV}}(j, j{+}1, \ldots, i{-}1, i) \right.$$

Pushing Computational Boundaries for NMHV Amplitudes

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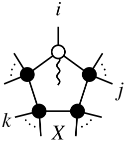

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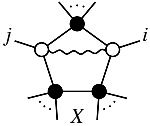
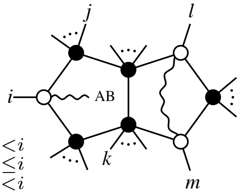
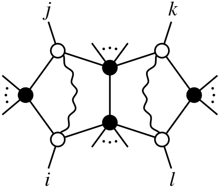


$$\times \mathcal{A}^{\text{tree}}_{\text{NMHV}}(j, j{+}1, \ldots, i{-}1, i)$$

$$+ \sum_{\substack{i<j<l<m\leq k<i \\ i<j<k<l<m\leq i \\ i\leq l<m\leq j<k<i}}$$



$$\times R\,[i\; j\; j{+}1\; k\; k{+}1] + \frac{1}{2}\sum_{i<j<k<l<i}$$



$$\times \left\{ \begin{array}{c} \mathcal{A}^{\text{tree}}_{\text{NMHV}}(j, \ldots, k;\; l, \ldots, i) \\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(i, \ldots, j) \\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(k, \ldots, l) \end{array} \right\} \right\}$$

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An n-point scattering amplitude is specified by listing each particle's:

- momentum,
- helicity
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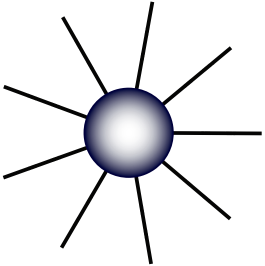

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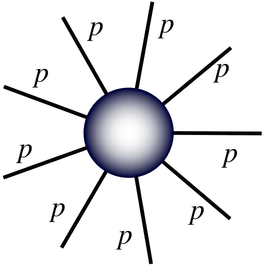

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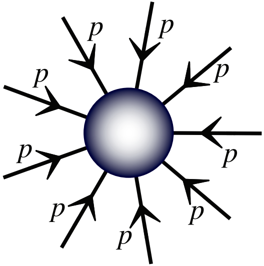

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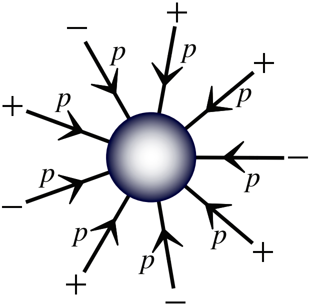

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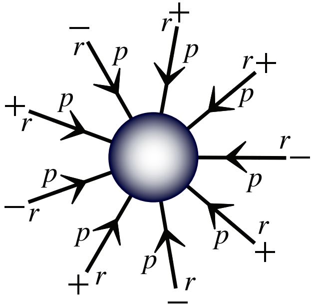

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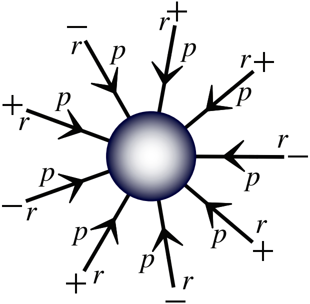

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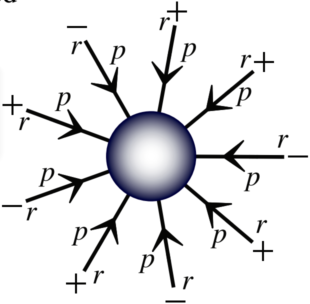

Colour-ordered partial amplitudes

$$A_n(\{p_a\}) = \sum \mathrm{Tr}(T^{a_1}\cdots T^{a_n})\mathcal{A}_n(p_{a_1},\ldots,p_{a_n})$$

e.g. $\mathcal{A}_9(1^+,2^+,3^-,4^+,5^-,6^+,7^-,8^+,9^-)$

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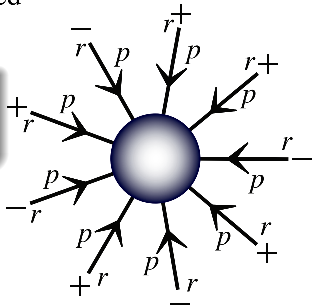

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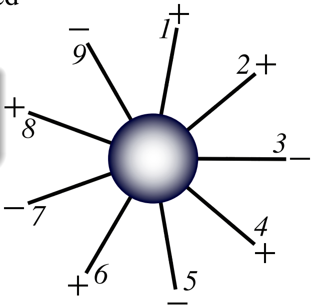

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Useful Lorentz-invariant scalars:

$$\langle ab\rangle \equiv \begin{vmatrix} \lambda_a^1 & \lambda_b^1 \\ \lambda_a^2 & \lambda_b^2 \end{vmatrix}, \qquad [ab] \equiv \begin{vmatrix} \widetilde\lambda_a^{\dot 1} & \widetilde\lambda_b^{\dot 1} \\ \widetilde\lambda_a^{\dot 2} & \widetilde\lambda_b^{\dot 2} \end{vmatrix}$$

$$(p_a+p_b)^2 = \langle ab\rangle[ba] \equiv s_{ab}, \qquad \langle a|(b+\ldots+c)|d] \equiv \langle a|\Big(b\rangle[b+\ldots+c\rangle[c\Big)|d].$$

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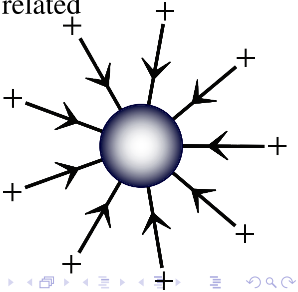

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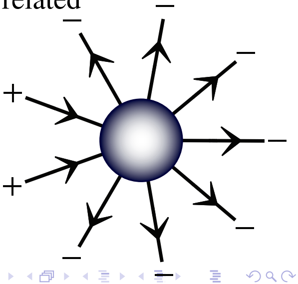

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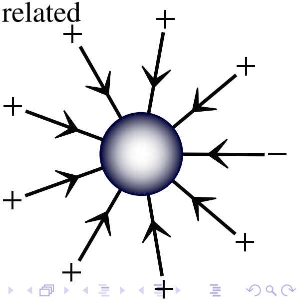

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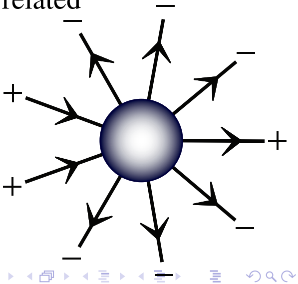

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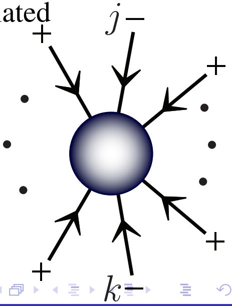

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Tree amplitudes are entirely fixed by analyticity.

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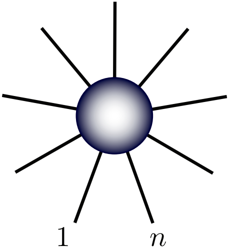
$$\lambda_1 \mapsto \widehat{\lambda}_1 \equiv \lambda_1 + z\lambda_n$$



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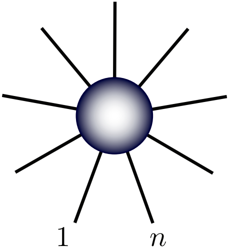

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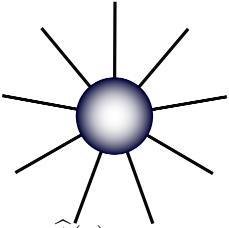

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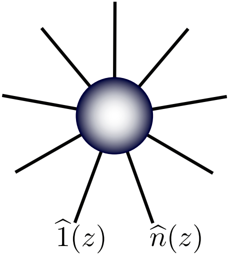

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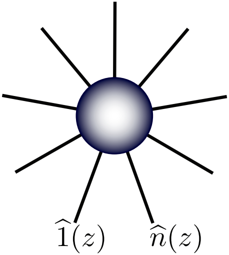

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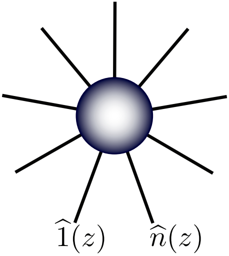

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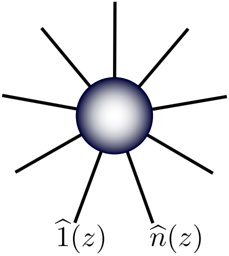

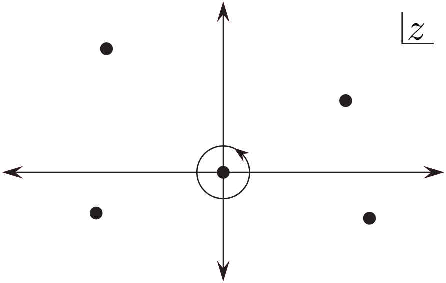

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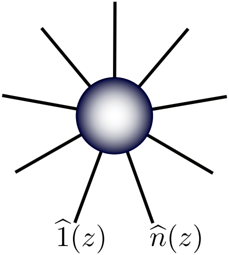

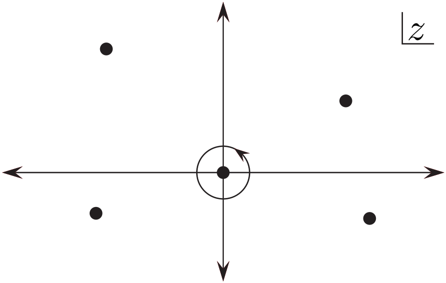

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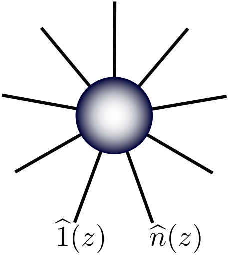

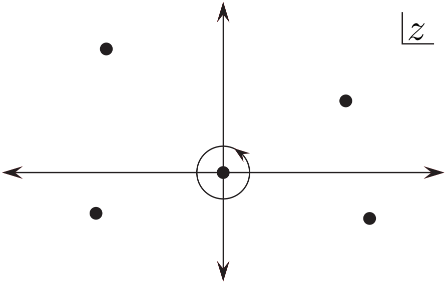

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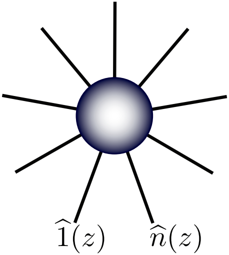

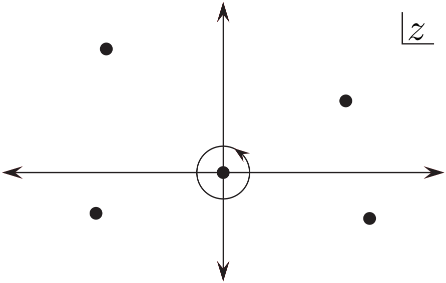

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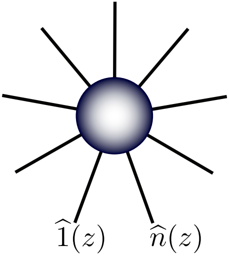

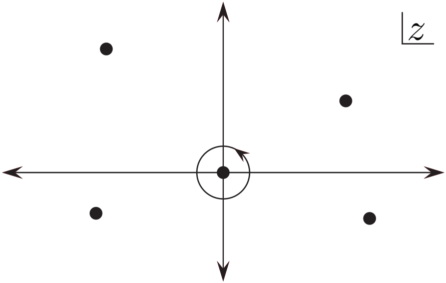

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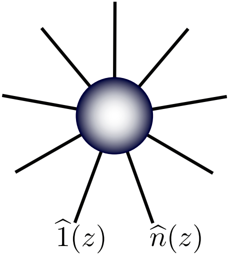

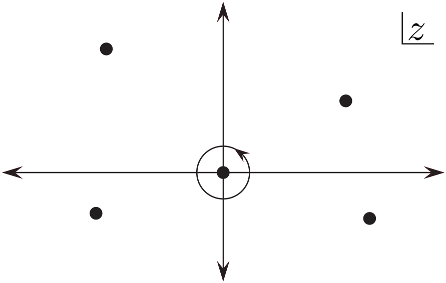

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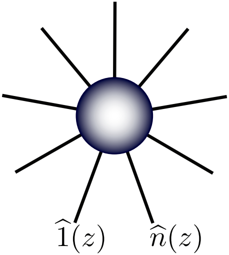

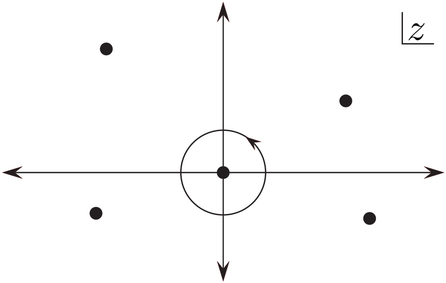

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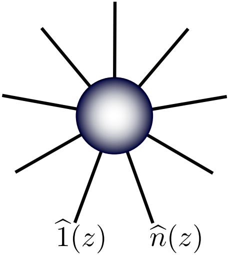

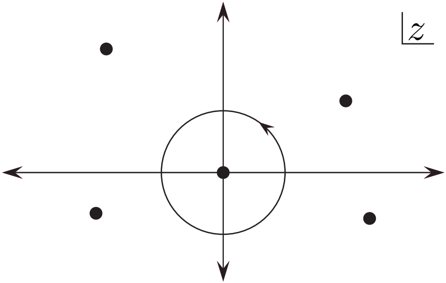

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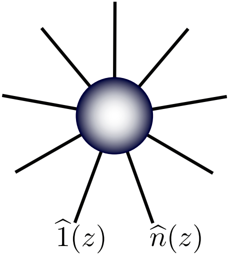

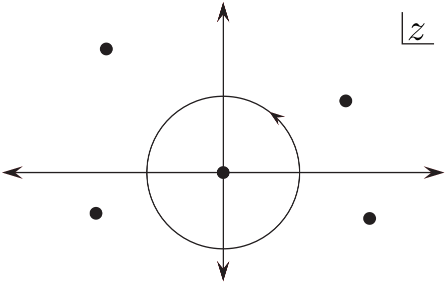

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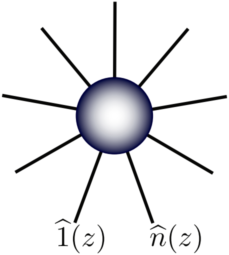

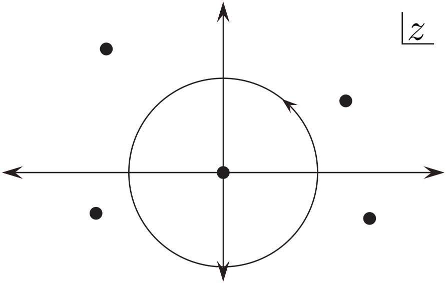

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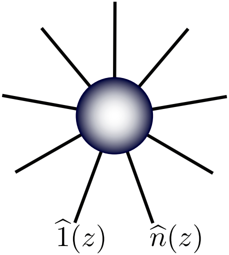

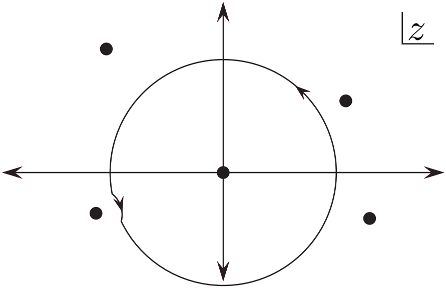

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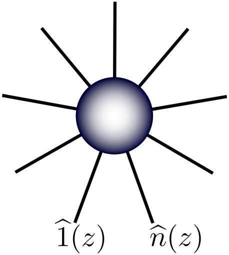

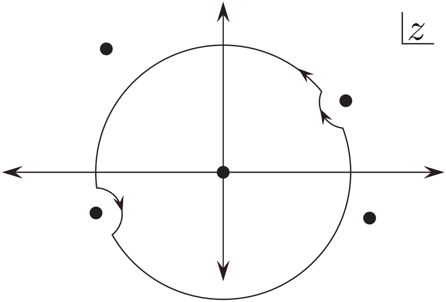

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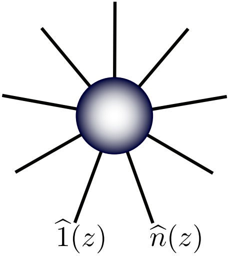

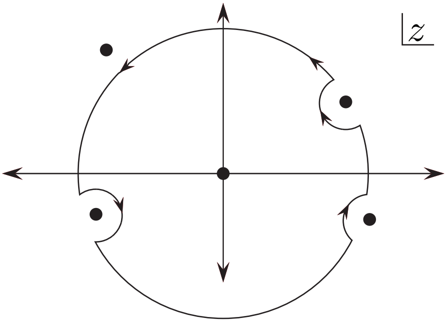

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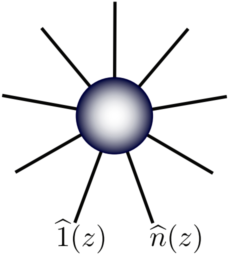

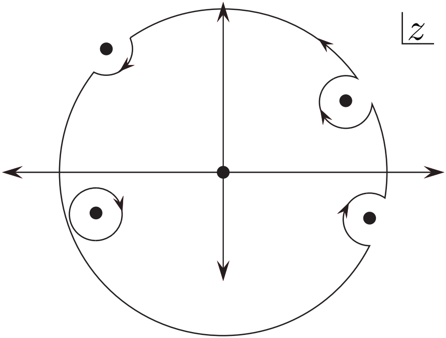

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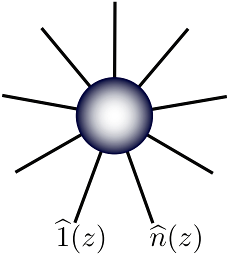

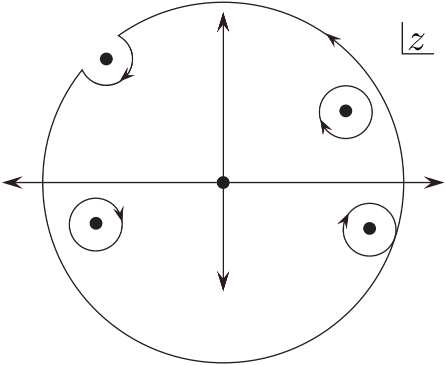

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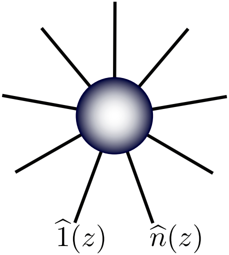

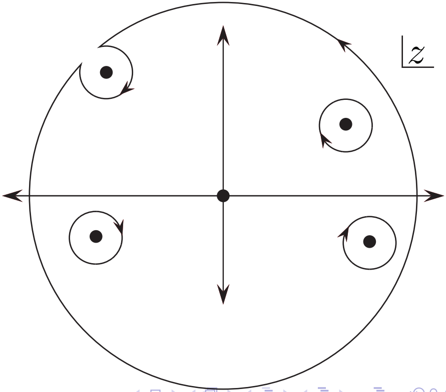

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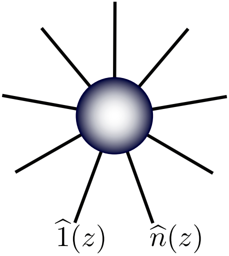

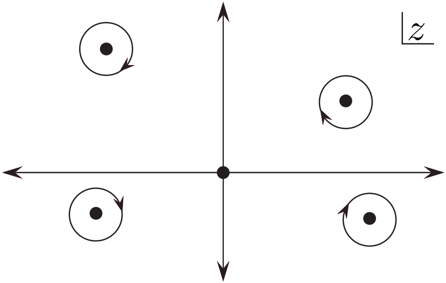

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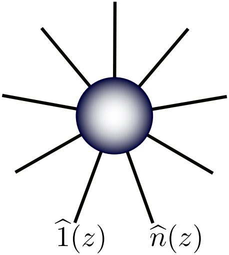

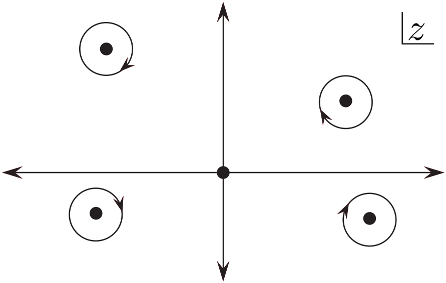

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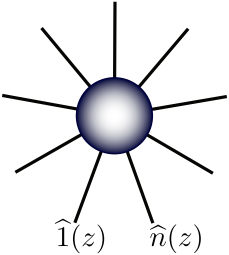

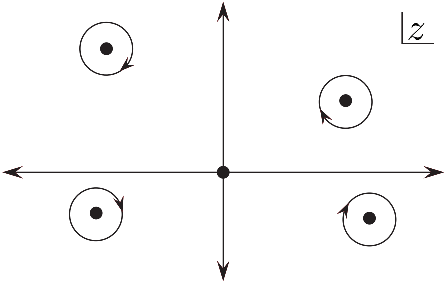

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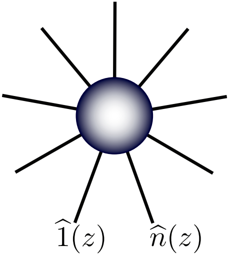

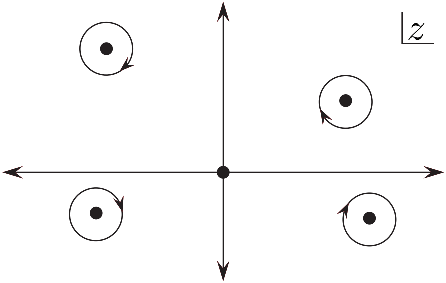

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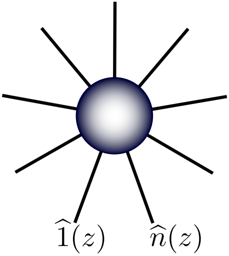

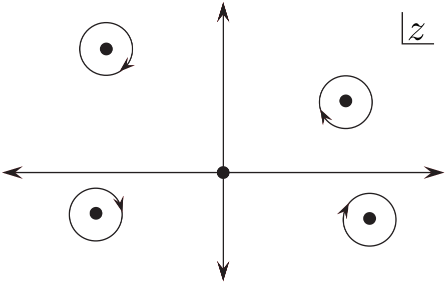

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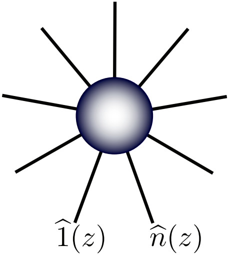

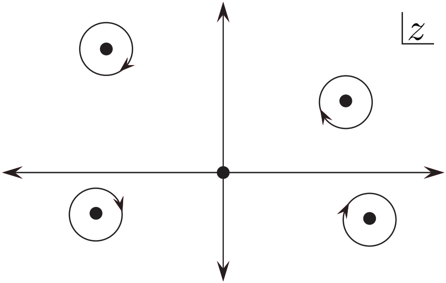

Analytic S-Matrix Redux: Tree-Level Recursion Relations

Tree amplitudes are entirely fixed by analyticity.

Consider the simplest deformation of any amplitude: $\mathcal{A}_n \mapsto \widehat{\mathcal{A}}_n(z)$
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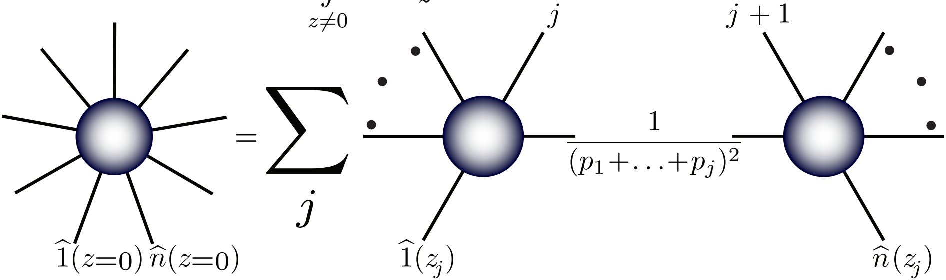

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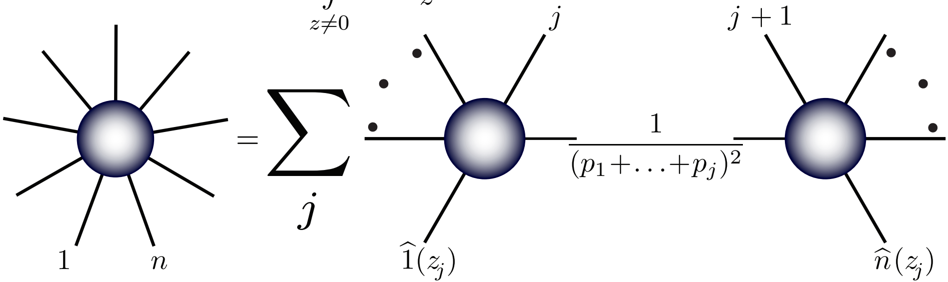

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For 8-point N^2MHV, there are 74 linearly-independent 40-term identities connecting the different BCFW formulae.

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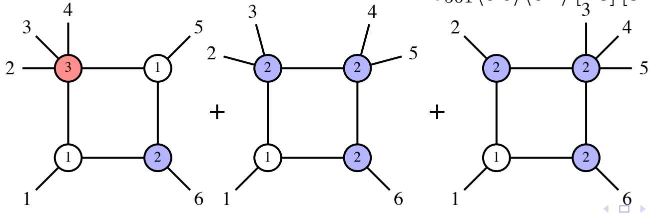

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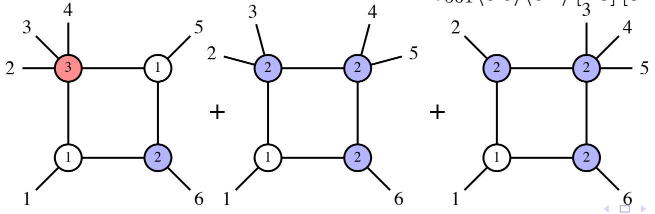

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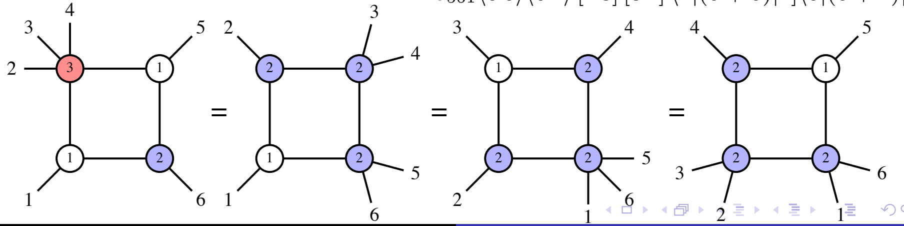

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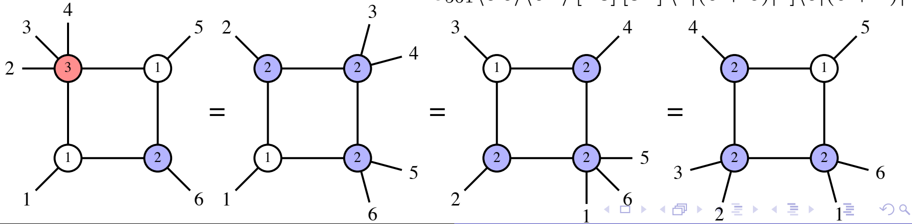

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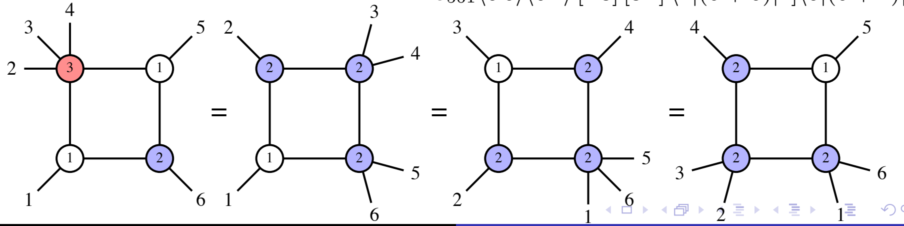

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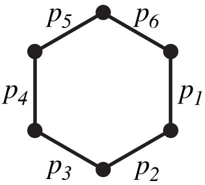

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- $p_a \equiv x_{a+1} - x_a$
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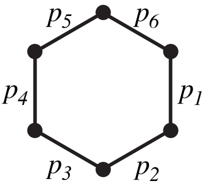
- combined with the ordinary-space superconformal invariance, scattering amplitudes are invariant under an infinite-dimensional Yangian symmetry.



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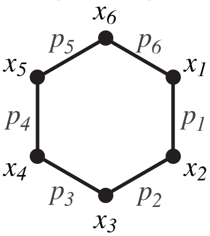

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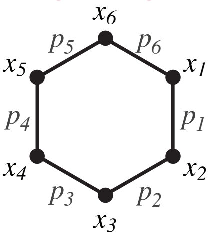

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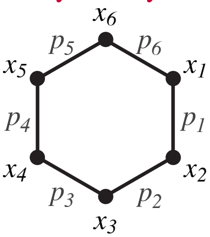

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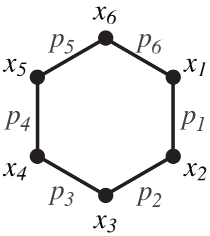

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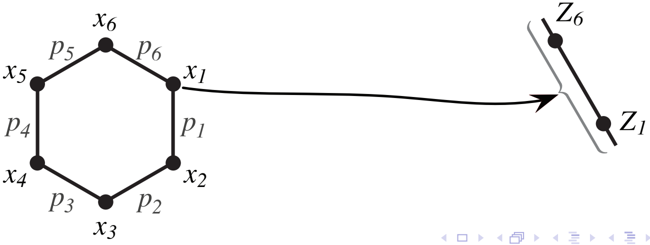

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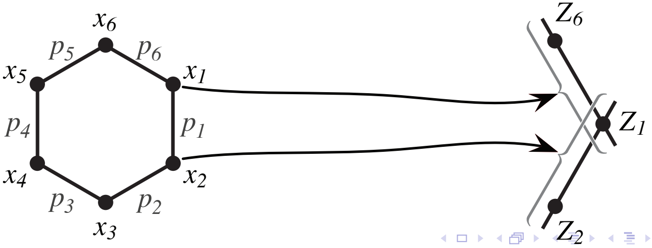

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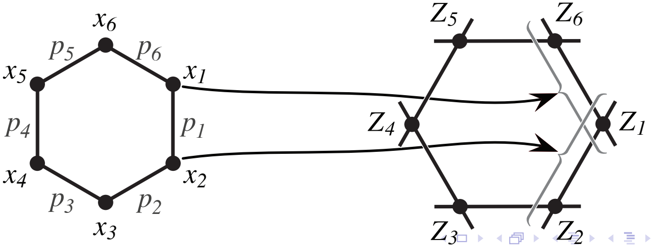

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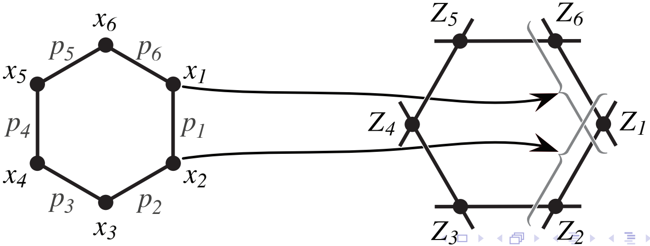

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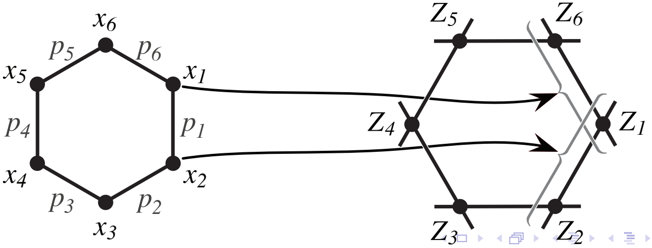

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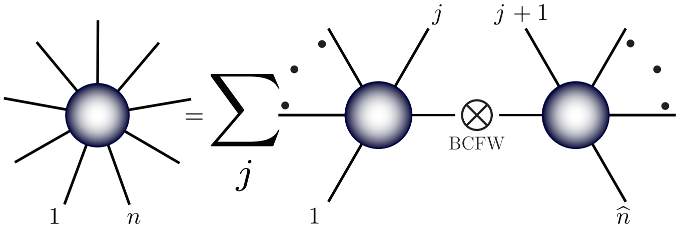

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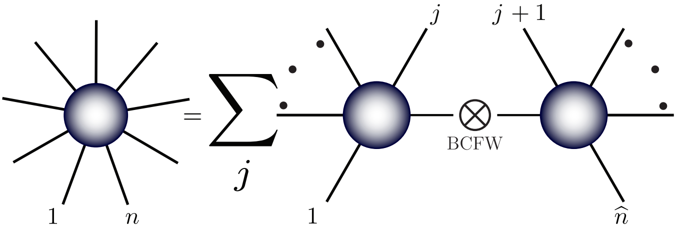

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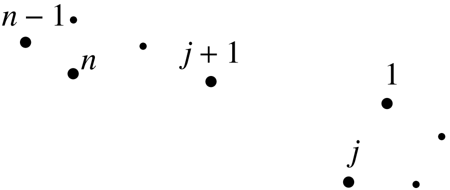

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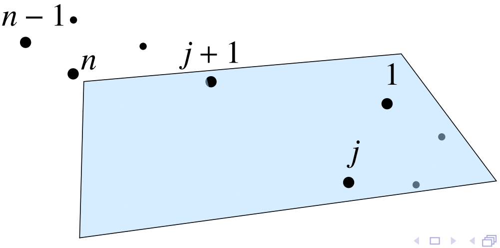

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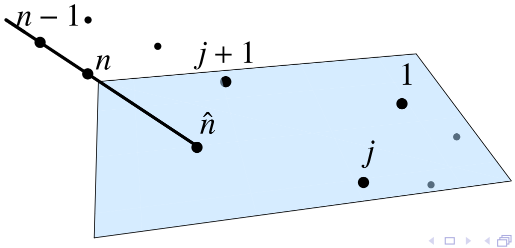

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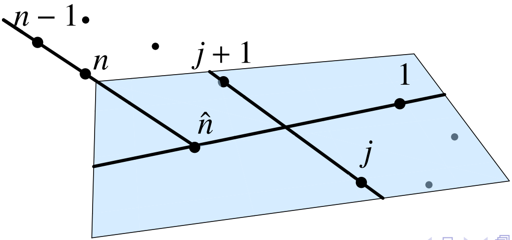

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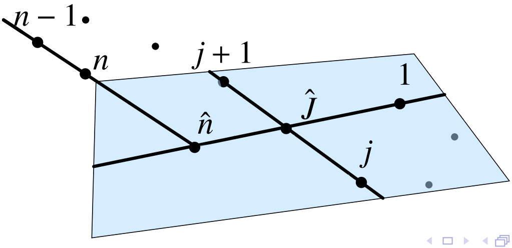

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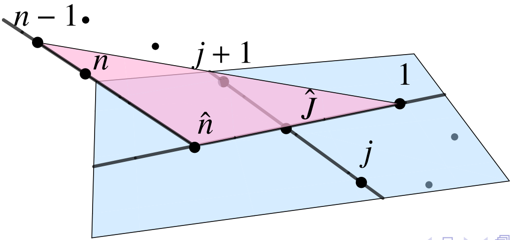

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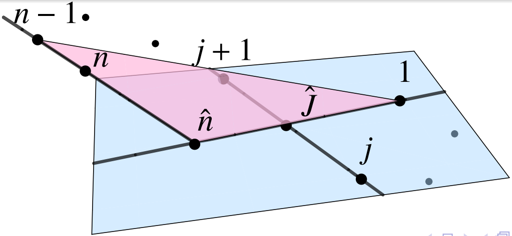

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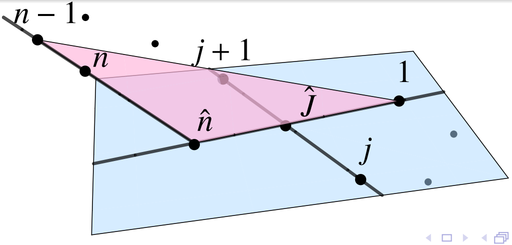

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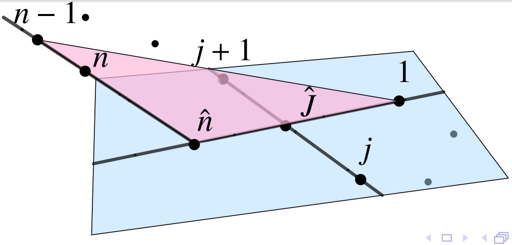

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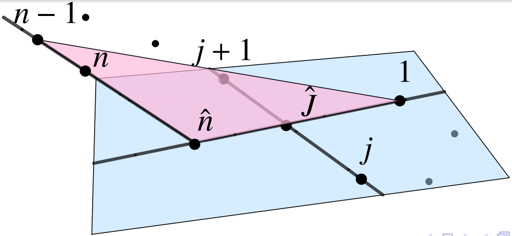

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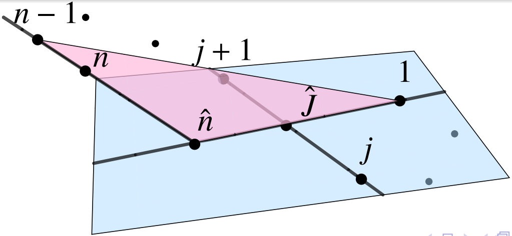

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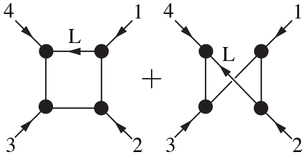
In a general theory, there is no naturally well-defined way to combine disparate Feynman loop integrals:



$$= \begin{cases} \displaystyle\int d^4\ell_1 \frac{(p_1+p_2)^2(p_2+p_3)^2}{\ell_1^2(\ell_1-p_1)^2(\ell_1-p_1-p_2)^2(\ell_1+p_4)^2} \\ \displaystyle\int d^4\ell_2 \frac{(p_1+p_2)^2(p_2+p_3)^2}{\ell_2^2(\ell_2-p_2)^2(\ell_2-p_1-p_2)^2(\ell_2+p_4)^2} \end{cases}$$

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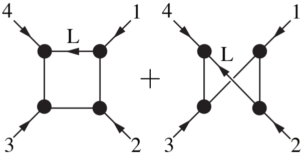

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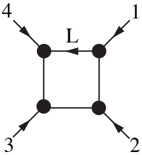

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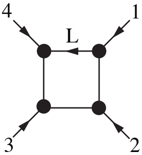

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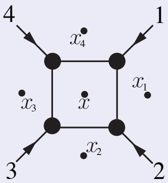

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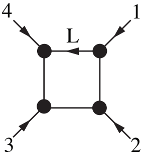

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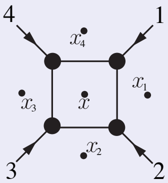

$$= \int d^4x \frac{(x_1-x_3)^2(x_2-x_4)^2}{(x-x_1)^2(x-x_2)^2(x-x_3)^2(x-x_4)^2}$$

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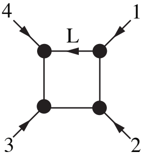

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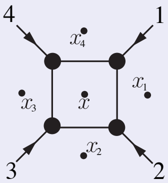

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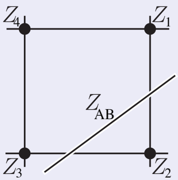

Integrals over Lines in Momentum-Twistor Space

Integration over all x corresponds to the integration over all lines $(Z_A Z_B)$ in momentum-twistor space.

$$\int d^4x \Longleftrightarrow \int \frac{d^4 Z_A d^4 Z_B}{\mathrm{vol}\,(GL_2) \times \langle \lambda_A \lambda_B \rangle^4} \equiv \int_{AB}$$

The propagators are

$$(x - x_1)^2 \Longleftrightarrow \langle AB\,12 \rangle \qquad (x - x_2)^2 \Longleftrightarrow \langle AB\,23 \rangle \qquad etc.$$

and the integral becomes

$$\int_{AB} \frac{\langle 12\,34 \rangle^2}{\langle AB\,12 \rangle \langle AB\,23 \rangle \langle AB\,34 \rangle \langle AB\,41 \rangle}$$

The Origin of Loop Amplitudes: Forward Limits

Let us reconsider the BCFW deformation for momentum-twistors:

$$Z_n \mapsto Z_n + zZ_{n-1}.$$

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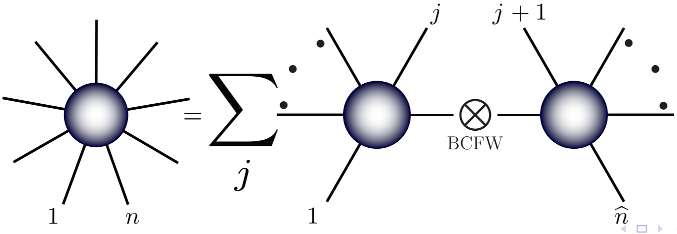

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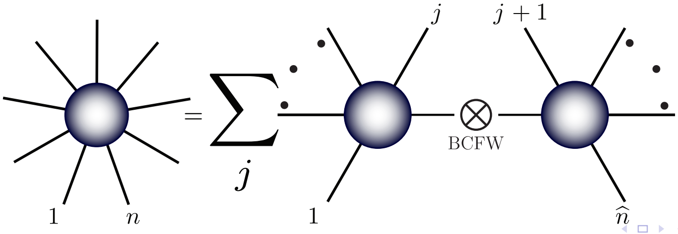

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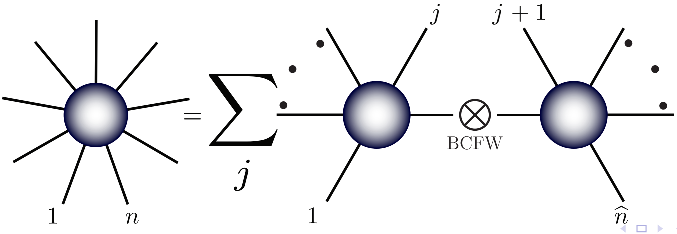

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Let us reconsider the BCFW deformation for momentum-twistors:

$$Z_n \mapsto Z_n + z Z_{n-1}.$$

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- The new terms come from cutting a propagator: $\langle AB\, \widehat{n}\, 1 \rangle = 0$.

$$\mathcal{A}^{(m)}_{n,\ell} = \sum_{\substack{\text{partitions}\\ \text{of } n,m,\ell}} \mathcal{A}^{(m_L)}_{n_L,\ell_L}(1,\ldots,j,\widehat{J}) \underset{\text{BCFW}}{\bigotimes} \mathcal{A}^{(m_R)}_{n_R,\ell_R}(\widehat{J}, j+1,\ldots,n-1,\widehat{n})$$

$$\widehat{J} \equiv (j\, j{+}1) \bigcap (n{-}1\, n\, 1)$$
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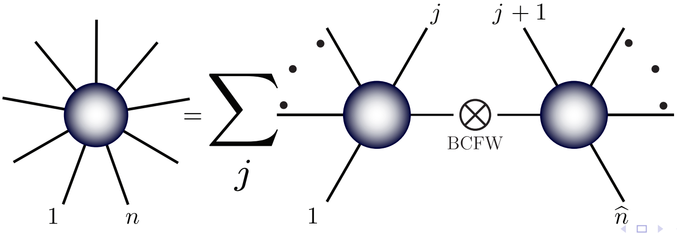

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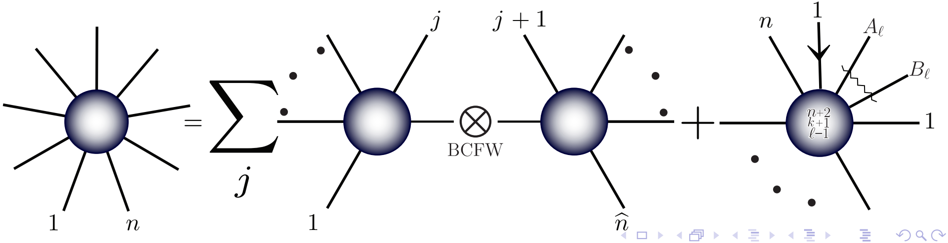

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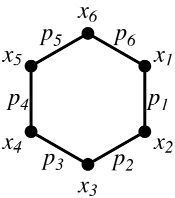

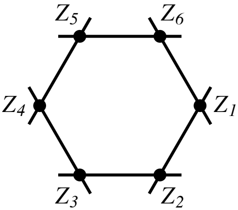

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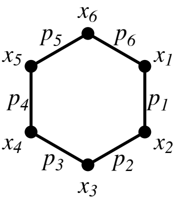

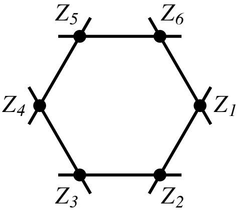

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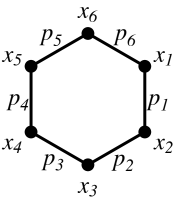

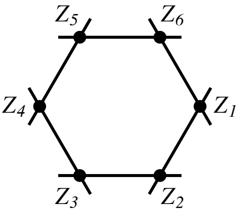

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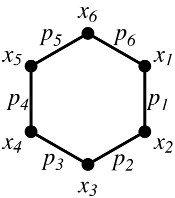

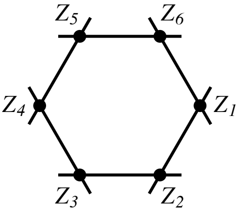

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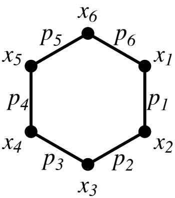

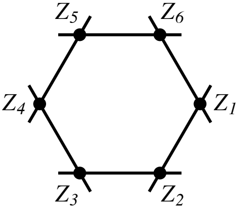

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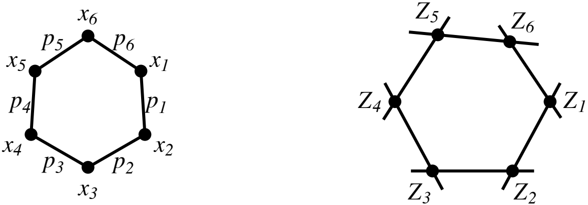

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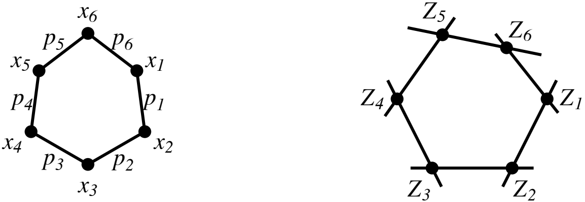

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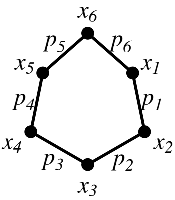

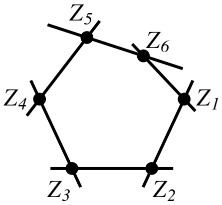

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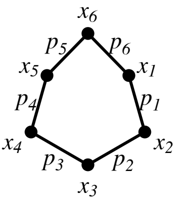

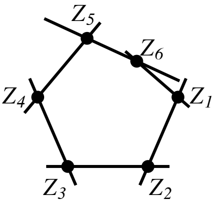

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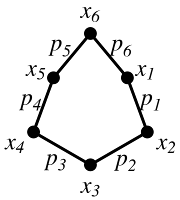

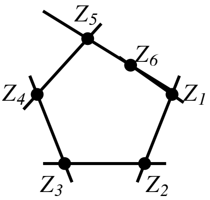

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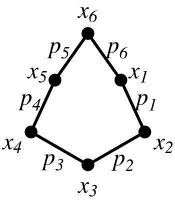

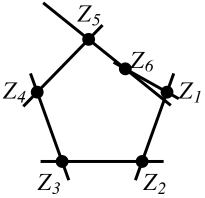

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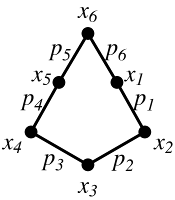

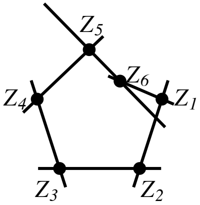

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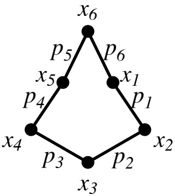

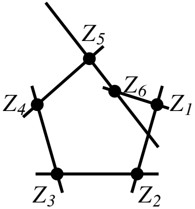

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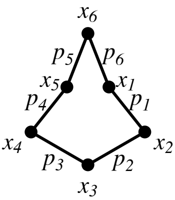

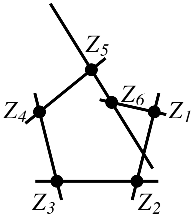

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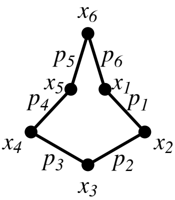

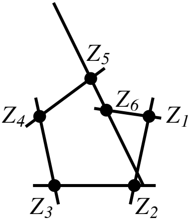

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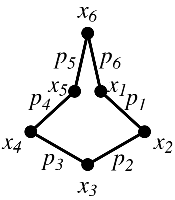

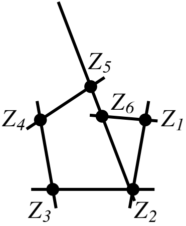

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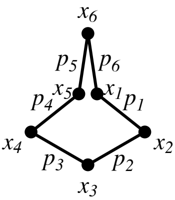

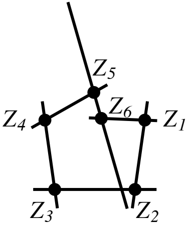

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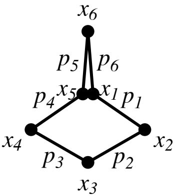

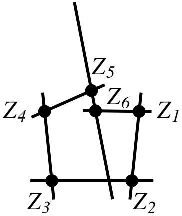

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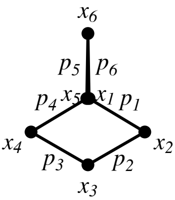

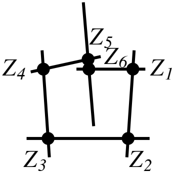

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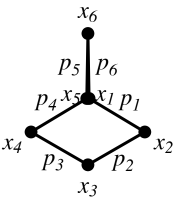

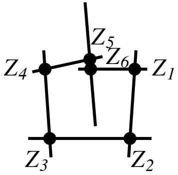

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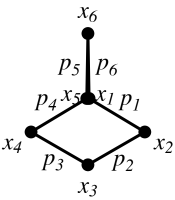

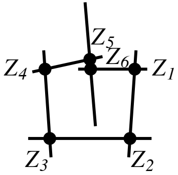

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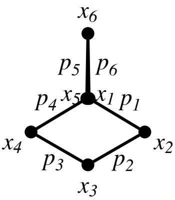

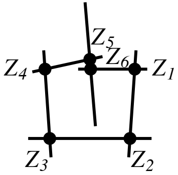

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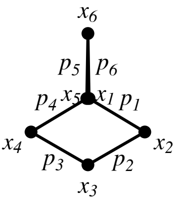

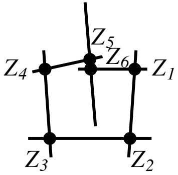

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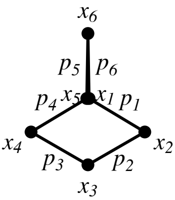

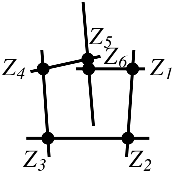

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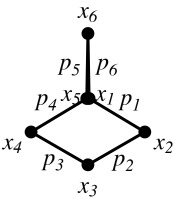

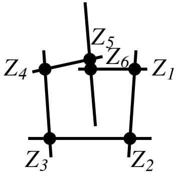

Exempli Gratia: BCFW Form of MHV Loop Amplitudes

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$$K[abc;xyz] \equiv \frac{\langle AB\,(a\,b\,c)\bigcap(x\,y\,z)\rangle^2}{\langle AB\,ab\rangle\langle AB\,bc\rangle\langle AB\,ca\rangle\langle AB\,xy\rangle\langle AB\,yz\rangle\langle AB\,zx\rangle}$$

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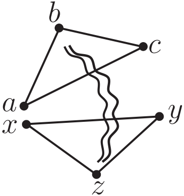

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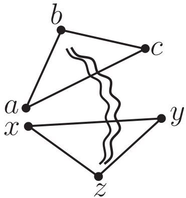
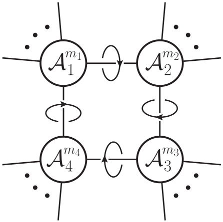


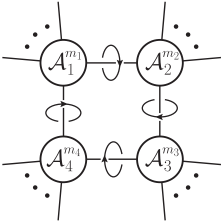
$$= d\log\left(\frac{\langle AB\,ab\rangle}{\langle AB\,bc\rangle}\right)\wedge d\log\left(\frac{\langle AB\,bc\rangle}{\langle AB\,ca\rangle}\right)\wedge d\log\left(\frac{\langle AB\,xy\rangle}{\langle AB\,yz\rangle}\right)\wedge d\log\left(\frac{\langle AB\,yz\rangle}{\langle AB\,zx\rangle}\right)$$

Sewing Together Tree Amplitudes in $\mathcal{N} = 4$



Sewing Together Tree Amplitudes in $\mathcal{N} = 4$

Two-Mass-Easy Schubert Problem



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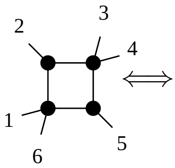
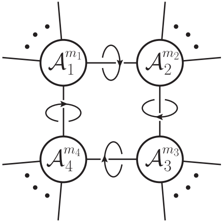


$$\Longleftrightarrow \int\limits_{AB} \frac{\langle 123\,5\rangle\langle 2\,345\rangle}{\langle AB\,12\rangle\langle AB\,23\rangle\langle AB\,45\rangle\langle AB\,56\rangle},$$



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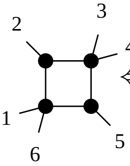

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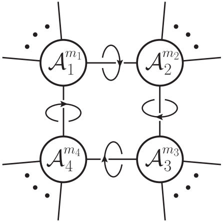
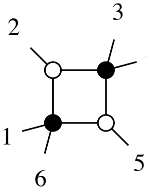
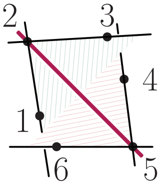
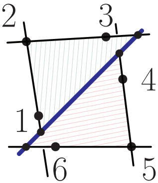




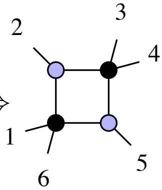
$\Longleftrightarrow$



$$(AB)=(25)$$

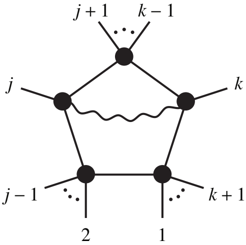


$\Longleftrightarrow$



$$(AB)=(123)\bigcap(456)$$

Finite, 'Nice' Integrals in Momentum Twistor Space



$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

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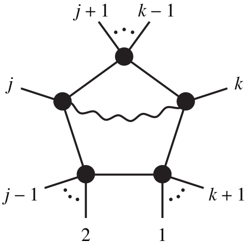

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$$= \quad \mathrm{Li}_2(1-u_1)$$

$$u_1 \equiv \frac{\langle k\,k{+}1\,1\,2\rangle\langle j{-}1\,j\,k{-}1\,k\rangle}{\langle k\,k{+}1\,j{-}1\,j\rangle\langle 1\,2\,k{-}1\,k\rangle}$$

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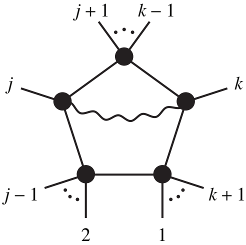

$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

$$= \quad \mathrm{Li}_2(1-u_1)+\mathrm{Li}_2(1-u_2)$$

$$u_1 \equiv \frac{\langle k\,k{+}1\,1\,2\rangle\langle j{-}1\,j\,k{-}1\,k\rangle}{\langle k\,k{+}1\,j{-}1\,j\rangle\langle 1\,2\,k{-}1\,k\rangle} \qquad u_2 \equiv \frac{\langle j\,j{+}1\,k\,k{+}1\rangle\langle 1\,2\,j{-}1\,j\rangle}{\langle j\,j{+}1\,1\,2\rangle\langle k\,k{+}1\,j{-}1\,j\rangle}$$

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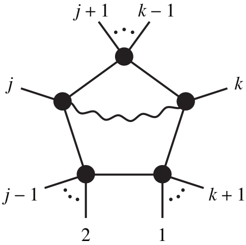


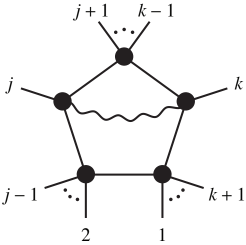
$$\int_{AB} \frac{\langle AB(j{-}1\, j\, j{+}1) \bigcap (k{-}1\, k\, k{+}1)\rangle \langle 1\, 2\, j\, k\rangle}{\langle AB\, 12\rangle \langle AB\, j{-}1\, j\rangle \langle AB\, j\, j{+}1\rangle \langle AB\, k{-}1\, k\rangle \langle AB\, k\, k{+}1\rangle}$$

$$= \quad \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) - \mathrm{Li}_2(1-u_3)$$

$$u_1 \equiv \frac{\langle k\, k{+}1\, 1\, 2\rangle \langle j{-}1\, j\, k{-}1\, k\rangle}{\langle k\, k{+}1\, j{-}1\, j\rangle \langle 1\, 2\, k{-}1\, k\rangle} \qquad u_2 \equiv \frac{\langle j\, j{+}1\, k\, k{+}1\rangle \langle 1\, 2\, j{-}1\, j\rangle}{\langle j\, j{+}1\, 1\, 2\rangle \langle k\, k{+}1\, j{-}1\, j\rangle}$$

$$u_3 \equiv \frac{\langle k\, k{+}1\, 1\, 2\rangle \langle j\, j{+}1\, k{-}1\, k\rangle}{\langle k\, k{+}1\, j\, j{+}1\rangle \langle 1\, 2\, k{-}1\, k\rangle}$$

Finite, 'Nice' Integrals in Momentum Twistor Space



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$$= \quad \mathrm{Li}_2(1-u_1)+\mathrm{Li}_2(1-u_2)-\mathrm{Li}_2(1-u_3) - \mathrm{Li}_2(1-u_4)$$

$$u_1 \equiv \frac{\langle k\,k{+}1\,1\,2\rangle\langle j{-}1\,j\,k{-}1\,k\rangle}{\langle k\,k{+}1\,j{-}1\,j\rangle\langle 1\,2\,k{-}1\,k\rangle} \qquad u_2 \equiv \frac{\langle j\,j{+}1\,k\,k{+}1\rangle\langle 1\,2\,j{-}1\,j\rangle}{\langle j\,j{+}1\,1\,2\rangle\langle k\,k{+}1\,j{-}1\,j\rangle}$$

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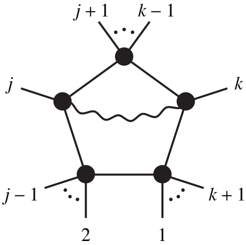

$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

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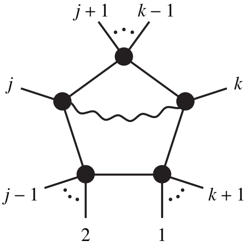
Finite, 'Nice' Integrals in Momentum Twistor Space



$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

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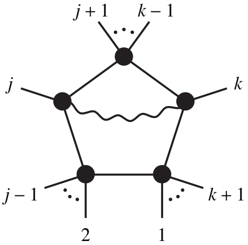
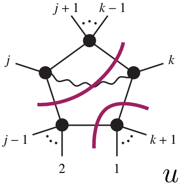
$$u_3 \equiv \frac{\langle k\,k{+}1\,1\,2\rangle\langle j\,j{+}1\,k{-}1\,k\rangle}{\langle k\,k{+}1\,j\,j{+}1\rangle\langle 1\,2\,k{-}1\,k\rangle} \qquad u_4 \equiv \frac{\langle j\,j{+}1\,k{-}1\,k\rangle\langle 1\,2\,j{-}1\,j\rangle}{\langle j\,j{+}1\,1\,2\rangle\langle k{-}1\,k\,j{-}1\,j\rangle}$$

Finite, 'Nice' Integrals in Momentum Twistor Space



$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

$$= \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) - \mathrm{Li}_2(1-u_3) - \mathrm{Li}_2(1-u_4) + \mathrm{Li}_2(1-u_5) + \log(u_1)\log(u_2)$$



$$u_2 \equiv \frac{\langle j\,j{+}1\,k\,k{+}1\rangle\langle 1\,2\,j{-}1\,j\rangle}{\langle j\,j{+}1\,1\,2\rangle\langle k\,k{+}1\,j{-}1\,j\rangle}$$

$$u_5 \equiv \frac{\langle j\,j{+}1\,k{-}1\,k\rangle\langle k\,k{+}1\,j{-}1\,j\rangle}{\langle j\,j{+}1\,k\,k{+}1\rangle\langle k{-}1\,k\,j{-}1\,j\rangle}$$

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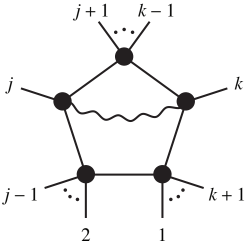
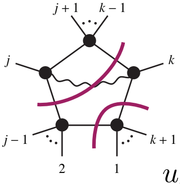
$$u_4 \equiv \frac{\langle j\,j{+}1\,k{-}1\,k\rangle\langle 1\,2\,j{-}1\,j\rangle}{\langle j\,j{+}1\,1\,2\rangle\langle k{-}1\,k\,j{-}1\,j\rangle}$$

Finite, 'Nice' Integrals in Momentum Twistor Space

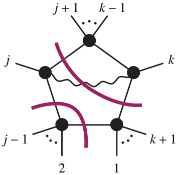


$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

$$= \mathrm{Li}_2(1-u_1)+\mathrm{Li}_2(1-u_2)-\mathrm{Li}_2(1-u_3) - \mathrm{Li}_2(1-u_4)+\mathrm{Li}_2(1-u_5)+\log(u_1)\log(u_2)$$



$$u_5 \equiv \frac{\langle j\,j{+}1\,k{-}1\,k\rangle\langle k\,k{+}1\,j{-}1\,j\rangle}{\langle j\,j{+}1\,k\,k{+}1\rangle\langle k{-}1\,k\,j{-}1\,j\rangle}$$



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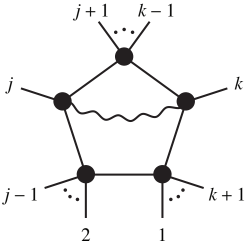
$$u_4 \equiv \frac{\langle j\,j{+}1\,k{-}1\,k\rangle\langle 1\,2\,j{-}1\,j\rangle}{\langle j\,j{+}1\,1\,2\rangle\langle k{-}1\,k\,j{-}1\,j\rangle}$$

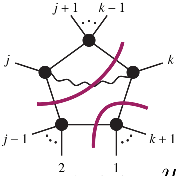
Finite, 'Nice' Integrals in Momentum Twistor Space

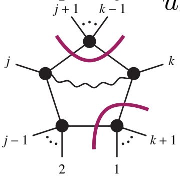


$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

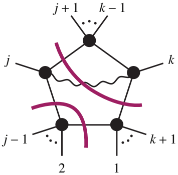
$$= \mathrm{Li}_2(1-u_1)+\mathrm{Li}_2(1-u_2)-\mathrm{Li}_2(1-u_3)$$

$$-\mathrm{Li}_2(1-u_4)+\mathrm{Li}_2(1-u_5)+\log(u_1)\log(u_2)$$



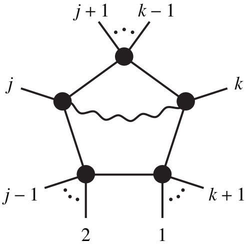


$$u_5 \equiv \frac{\langle j\,j{+}1\,k{-}1\,k\rangle\langle k\,k{+}1\,j{-}1\,j\rangle}{\langle j\,j{+}1\,k\,k{+}1\rangle\langle k{-}1\,k\,j{-}1\,j\rangle}$$



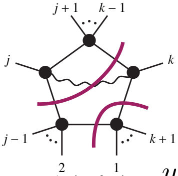
$$u_4 \equiv \frac{\langle j\,j{+}1\,k{-}1\,k\rangle\langle 1\,2\,j{-}1\,j\rangle}{\langle j\,j{+}1\,1\,2\rangle\langle k{-}1\,k\,j{-}1\,j\rangle}$$

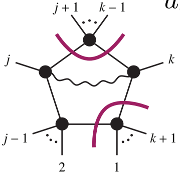
Finite, 'Nice' Integrals in Momentum Twistor Space



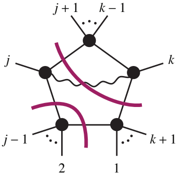
$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

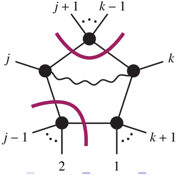
$$= \mathrm{Li}_2(1-u_1)+\mathrm{Li}_2(1-u_2)-\mathrm{Li}_2(1-u_3) - \mathrm{Li}_2(1-u_4)+\mathrm{Li}_2(1-u_5)+\log(u_1)\log(u_2)$$





$$u_5 \equiv \frac{\langle j\,j{+}1\,k{-}1\,k\rangle\langle k\,k{+}1\,j{-}1\,j\rangle}{\langle j\,j{+}1\,k\,k{+}1\rangle\langle k{-}1\,k\,j{-}1\,j\rangle}$$





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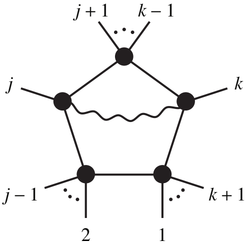
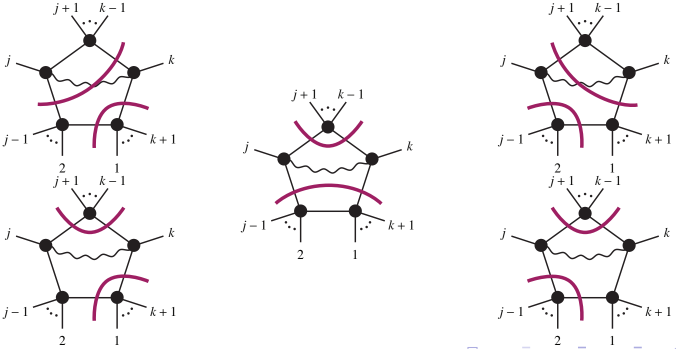


$$\int_{AB} \frac{\langle AB(j{-}1\, j\, j{+}1) \bigcap (k{-}1\, k\, k{+}1)\rangle\langle 1\, 2\, j\, k\rangle}{\langle AB\, 12\rangle\langle AB\, j{-}1\, j\rangle\langle AB\, j\, j{+}1\rangle\langle AB\, k{-}1\, k\rangle\langle AB\, k\, k{+}1\rangle}$$

$$= \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) - \mathrm{Li}_2(1-u_3) - \mathrm{Li}_2(1-u_4) + \mathrm{Li}_2(1-u_5) + \log(u_1)\log(u_2)$$



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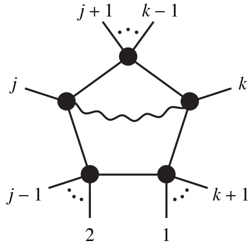

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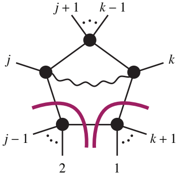

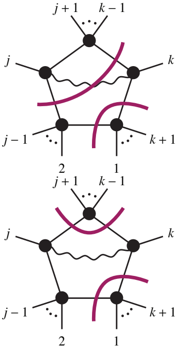

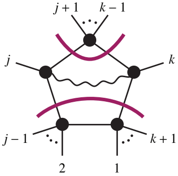

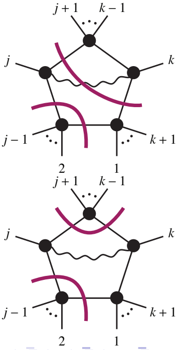

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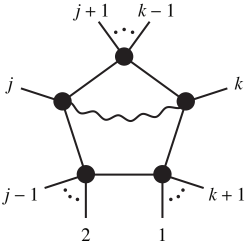

$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

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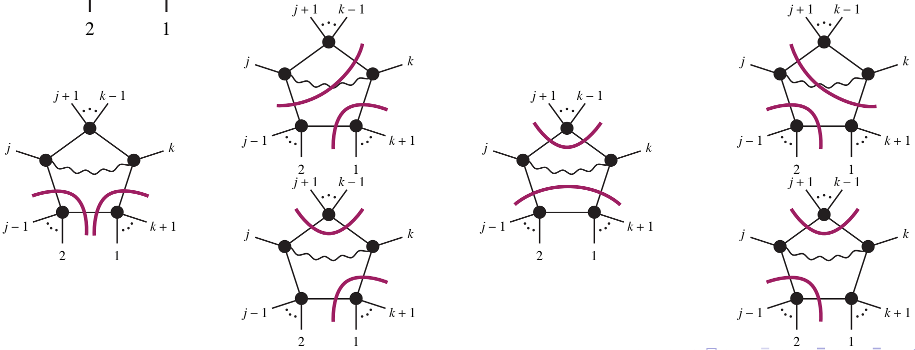

Following the Logic of Leading Singularities

These finite integrals are dramatically nicer than the more familiar **scalar** boxes. Moreover, they suggest a natural 'guess' for MHV amplitudes [arXiv:1012.6032]:

$$\mathcal{A}_n^{(2)}(\ldots,j^-,\ldots,k^-,\ldots) = \frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

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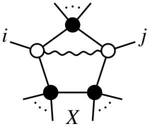

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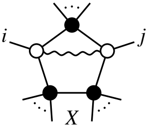
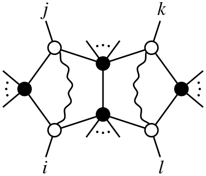


$$+\quad \frac{1}{2}\sum_{i<j<k<l<i}$$



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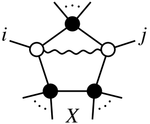
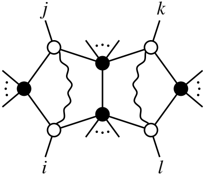
$$\mathcal{A}_n^{(2)}(\ldots, j^-, \ldots, k^-, \ldots) = \frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

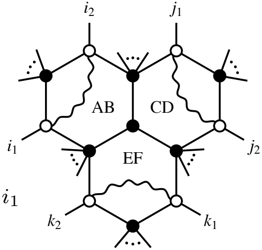
$$\times \Big\{ 1 + \sum_{i<j<i}$$



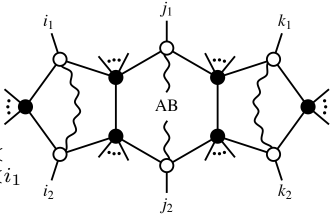
$$+ \frac{1}{2}\sum_{i<j<k<l<i}$$



$$+ \frac{1}{3}\sum_{\substack{i_1\leq i_2<j_1\leq \\ \leq j_2<k_1\leq k_2<i_1}}$$



$$+ \frac{1}{2}\sum_{\substack{i_1\leq j_1<k_1< \\ <k_2\leq j_2<i_2<i_1}}$$



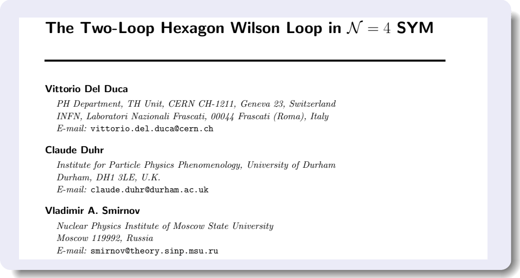
$$\Big\}$$

More Evidence for Underlying Elegance

Last year, Del Duca, Duhr, and Smirnov found an analytic formula for the 2-loop, 6-point MHV amplitude:

Del Duca, Duhr, & Smirnov arXiv:1003.1702

- dimensionally-regulating thousands of separately-divergent integrals
- expressed in terms of 18 pages of Goncharov polylogarithms



The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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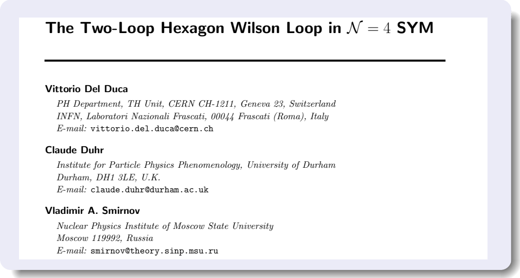

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$$\begin{aligned}
G\left(-a,a^2,a^2,1;1\right) =\; & H\left(-1,-1,-1,-1;\frac{1}{a}\right) + H\left(-1,-1,0,-1;\frac{1}{a}\right) \\
& - H\left(-1,-1,0,1;\frac{1}{a}\right) + H\left(-1,0,-1,-1;\frac{1}{a}\right) - H\left(-1,0,-1,1;\frac{1}{a}\right) \\
& - H\left(-1,0,1,-1;\frac{1}{a}\right) + H\left(-1,0,1,1;\frac{1}{a}\right) - H\left(0,-1,-1,-1;\frac{1}{a}\right) \\
& - H\left(0,-1,-1,1;\frac{1}{a}\right) - 2H\left(0,-1,0,-1;\frac{1}{a}\right) + 2H\left(0,-1,0,1;\frac{1}{a}\right) \\
& - H\left(0,-1,1,-1;\frac{1}{a}\right) + H\left(0,-1,1,1;\frac{1}{a}\right) - 4H\left(0,0,-1,-1;\frac{1}{a}\right) \\
& + 4H\left(0,0,-1,1;\frac{1}{a}\right) + 4H\left(0,0,1,-1;\frac{1}{a}\right) - 4H\left(0,0,1,1;\frac{1}{a}\right) \\
& - H\left(0,1,-1,-1;\frac{1}{a}\right) + H\left(0,1,-1,1;\frac{1}{a}\right) + H\left(0,1,1,-1;\frac{1}{a}\right) \\
& - H\left(0,1,1,1;\frac{1}{a}\right) - 2H\left(1,0,-1,-1;\frac{1}{a}\right) - 4H\left(1,0,0,-1;\frac{1}{a}\right) \\
& + 4H\left(1,0,0,1;\frac{1}{a}\right) - 2H\left(1,1,0,-1;\frac{1}{a}\right) + H\left(1,1,0,1;\frac{1}{a}\right)
\end{aligned} \tag{G.242}$$

$$\begin{aligned}
G\left(a,a^2,a^2,1;1\right) =\; & -H\left(-1,-1,0,-1;\frac{1}{a}\right) + 2H\left(-1,-1,0,1;\frac{1}{a}\right) \\
& + 4H\left(-1,0,0,-1;\frac{1}{a}\right) - 4H\left(-1,0,0,1;\frac{1}{a}\right) + 2H\left(-1,0,1,1;\frac{1}{a}\right) \\
& + H\left(0,-1,-1,-1;\frac{1}{a}\right) - H\left(0,-1,-1,1;\frac{1}{a}\right) - H\left(0,-1,1,-1;\frac{1}{a}\right) \\
& + H\left(0,-1,1,1;\frac{1}{a}\right) - 4H\left(0,0,-1,-1;\frac{1}{a}\right) + 4H\left(0,0,-1,1;\frac{1}{a}\right) \\
& + 4H\left(0,0,1,-1;\frac{1}{a}\right) - 4H\left(0,0,1,1;\frac{1}{a}\right) - H\left(0,1,-1,-1;\frac{1}{a}\right) \\
& + H\left(0,1,-1,1;\frac{1}{a}\right) + 2H\left(0,1,0,-1;\frac{1}{a}\right) - 2H\left(0,1,0,1;\frac{1}{a}\right) \\
& + H\left(0,1,1,-1;\frac{1}{a}\right) + H\left(0,1,1,1;\frac{1}{a}\right) - H\left(1,0,-1,-1;\frac{1}{a}\right) \\
& + H\left(1,0,-1,1;\frac{1}{a}\right) + H\left(1,0,1,-1;\frac{1}{a}\right) - H\left(1,0,1,1;\frac{1}{a}\right) \\
& + H\left(1,1,0,-1;\frac{1}{a}\right) - H\left(1,1,0,1;\frac{1}{a}\right) + H\left(1,1,1,1;\frac{1}{a}\right)
\end{aligned} \tag{G.243}$$

H. The analytic expression of the remainder function

In this appendix we present the full analytic expression of the remainder function. The result is also available in electronic form from www.arXiv.org. Using the notation introduced in Eqs. (3.23) and (5.7), the full expression reads,

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Del Duca, Duhr, & Smirnov arXiv:1003.1702

$$
\begin{aligned}
&R^{(2)}_{6,WL}(u_1,u_2,u_3) = \qquad\qquad \text{(H.1)}\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+\\
&G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+\\
&G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)+\\
&G\left(0,\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)-\\
&\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-\\
&\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+
\end{aligned}
$$

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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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Del Duca, Duhr, & Smirnov arXiv:1003.1702

$$\begin{aligned}
&\frac14 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,1;1\right)-\frac14 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+\\
&\frac14 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1,0;1\right)-\frac14 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)+\\
&\frac14 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)-\frac14 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\\
&\frac14 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},1;1\right)+\\
&\frac14 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)+\\
&\frac12 G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac12 G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-\\
&\frac14 G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac14 G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac14 G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\\
&\frac14 G\left(\frac{1}{u_1},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac14 G\left(\frac{1}{1-u_2},1,\frac{1}{u_1},0;1\right)+\\
&\frac12 G\left(\frac{1}{1-u_2},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\frac14 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,1;1\right)-\\
&\frac14 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+\frac14 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1,0;1\right)-\\
&\frac14 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)+\frac14 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)-\\
&\frac14 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\\
&\frac14 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},1;1\right)+\\
&\frac14 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+\\
&\frac12 G\left(\frac{1}{u_2},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_3};1\right)+\frac12 G\left(\frac{1}{u_2},0,0,\frac{1}{u_2+u_3};1\right)-\\
&\frac14 G\left(\frac{1}{u_2},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac14 G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac14 G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\\
&\frac14 G\left(\frac{1}{u_2},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac14 G\left(\frac{1}{1-u_3},1,\frac{1}{u_2},0;1\right)+\\
&\frac12 G\left(\frac{1}{1-u_3},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\frac14 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right)-\\
&\frac14 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac14 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right)-\\
&\frac14 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)+\frac14 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)-\\
&\frac14 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-\\
&\frac14 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+\\
&\frac14 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-
\end{aligned}$$

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$$
\begin{aligned}
&G\left(\frac{1}{u_3},0,0,\frac{1}{u_2};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_1+u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{132};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{312};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{123},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{132},0;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,v_{132},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{213},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{213},\frac{1}{1-u_2};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{312},\frac{1}{1-u_3};1\right)+\mathcal{G}\left(0,0,v_{321},0;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{132};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{231};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{321};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},0;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1},0;1\right)-
\end{aligned}
$$

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$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{123},1,\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{132},0,\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{213},0,\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{231},0,\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{231},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},0;1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},0;1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{321},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,0,v_{132};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},0,\frac{1}{1-u_1},v_{123};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},0,\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},0,v_{123};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},0,v_{132};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-$$
$$\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},1;1\right)+$$

More Evidence for Underlying Elegance

Last year, Del Duca, Duhr, and Smirnov found an analytic formula for the 2-loop, 6-point MHV amplitude:

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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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Del Duca, Duhr, & Smirnov arXiv:1003.1702

$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{u_2-1}{u_1+u_2-1},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{u_2-1}{u_1+u_2-1},\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{u_3},0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},0,0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},0,\tfrac{1}{1-u_1};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},1,0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},1,\tfrac{1}{1-u_1};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},0,0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},0,\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},1,0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},1,\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1},0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,0,v_{213};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,0,v_{231};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},0,\tfrac{1}{1-u_2},v_{213};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},0,\tfrac{1}{1-u_2},v_{231};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{213},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{213},\tfrac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{231},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{231},\tfrac{1}{1-u_2};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},0,v_{213};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},0,v_{231};1\right)-\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213};1\right)-\\
&\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213},\tfrac{1}{1-u_2};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231},\tfrac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},0,1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},0,\tfrac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},1,0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{u_1},0;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{1-u_2},0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{1-u_2},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{u_3-1}{u_2+u_3-1},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{u_3-1}{u_2+u_3-1},\tfrac{1}{1-u_2};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},0,0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},0,\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},1,0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},1,\tfrac{1}{1-u_2};1\right)+
\end{aligned}
$$

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$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},\tfrac{1}{1-u_2},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},\tfrac{1}{1-u_2},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},0,0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},0,\tfrac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},1,0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},1,\tfrac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},\tfrac{1}{1-u_2},0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},\tfrac{1}{1-u_2},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,0,v_{312};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,0,v_{321};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},0,\tfrac{1}{1-u_3},v_{312};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},0,\tfrac{1}{1-u_3},v_{321};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{312},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{312},\tfrac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{321},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{321},\tfrac{1}{1-u_3};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},0,v_{312};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},0,v_{321};1\right)-\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{312};1\right)-\\
&\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{321};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{312},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{312},\tfrac{1}{1-u_3};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{321},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{321},\tfrac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},0,1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},0,\tfrac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},1,0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{1}{u_2},0;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{1}{1-u_3},0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{1}{1-u_3},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{u_1-1}{u_1+u_3-1},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{u_1-1}{u_1+u_3-1},\tfrac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},0,0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},0,\tfrac{1}{1-u_3};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},1,0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},1,\tfrac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},\tfrac{1}{1-u_3},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},\tfrac{1}{1-u_3},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},0,0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},0,\tfrac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},1,0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},1,\tfrac{1}{1-u_3};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},\tfrac{1}{1-u_3},0;1\right)-
\end{aligned}
$$

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$$
\begin{aligned}
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{123},0,1,\frac{1}{1-u_1};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{123},0,\frac{1}{1-u_1},1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{123},1,0,\frac{1}{1-u_1};1\right)-\\
&\tfrac{5}{4}\mathcal{G}\left(v_{123},1,1,\frac{1}{1-u_1};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},0;1\right)-\tfrac{5}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},1;1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},0,1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,0;1\right)-\\
&\tfrac{5}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(v_{132},1,1,\frac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1,1;1\right)-\tfrac{1}{4}\mathcal{G}\left(v_{213},1,1,\frac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1,1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{231},0,1,\frac{1}{1-u_2};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{231},0,\frac{1}{1-u_2},1;1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{231},1,0,\frac{1}{1-u_2};1\right)-\tfrac{5}{4}\mathcal{G}\left(v_{231},1,1,\frac{1}{1-u_2};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},0;1\right)-\\
&\tfrac{5}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},0,1;1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,0;1\right)-\tfrac{5}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2},1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{312},0,1,\frac{1}{1-u_3};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{312},0,\frac{1}{1-u_3},1;1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{312},1,0,\frac{1}{1-u_3};1\right)-\tfrac{5}{4}\mathcal{G}\left(v_{312},1,1,\frac{1}{1-u_3};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},0;1\right)-\\
&\tfrac{5}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},0,1;1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,0;1\right)-\tfrac{5}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,1;1\right)+\tfrac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\\
&\tfrac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(v_{321},1,1,\frac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1,1;1\right)-\tfrac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\tfrac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\tfrac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\tfrac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\tfrac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+\\
&\tfrac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\tfrac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\tfrac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\tfrac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\\
&\tfrac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\tfrac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\\
&\tfrac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\tfrac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1};1\right)H\left(0;u_1\right)+
\end{aligned}
$$

More Evidence for Underlying Elegance

Last year, Del Duca, Duhr, and Smirnov found an analytic formula for the 2-loop, 6-point MHV amplitude:

- dimensionally-regulating thousands of separately-divergent integrals
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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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Del Duca, Duhr, & Smirnov arXiv:1003.1702

$$
\begin{aligned}
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_2},\tfrac{u_3-1}{u_2+u_3-1},1;1\right)H(0;u_1)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_2},\tfrac{u_3-1}{u_2+u_3-1},\tfrac{1}{1-u_2};1\right)H(0;u_1)+\tfrac{1}{2}G\left(\tfrac{1}{u_2},0,\tfrac{1}{u_1};1\right)H(0;u_1)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_2},0,\tfrac{1}{u_1+u_2};1\right)H(0;u_1)+\tfrac{1}{4}G\left(\tfrac{1}{u_2},\tfrac{1}{u_1},\tfrac{1}{u_1+u_2};1\right)H(0;u_1)+\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_3},\tfrac{u_1-1}{u_1+u_3-1},0;1\right)H(0;u_1)+\tfrac{1}{4}G\left(\tfrac{1}{1-u_3},\tfrac{u_1-1}{u_1+u_3-1},\tfrac{1}{1-u_3};1\right)H(0;u_1)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_3},\tfrac{u_1-1}{u_1+u_3-1},\tfrac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+\tfrac{1}{2}G\left(\tfrac{1}{u_3},0,\tfrac{1}{u_1};1\right)H(0;u_1)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_3},0,\tfrac{1}{u_1+u_3};1\right)H(0;u_1)+\tfrac{1}{4}G\left(\tfrac{1}{u_3},\tfrac{1}{u_1},\tfrac{1}{u_1+u_3};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_1},v_{123};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_1},v_{132};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_2},v_{213};1\right)H(0;u_1)-\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_2},v_{231};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_3},v_{312};1\right)H(0;u_1)-\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_3},v_{321};1\right)H(0;u_1)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{u_1};1\right)H(0;u_1)-\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{1-u_2};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{1-u_3};1\right)H(0;u_1)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(0,v_{132},\tfrac{1}{1-u_1};1\right)H(0;u_1)-\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2};1\right)H(0;u_1)+\tfrac{1}{2}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{123};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{132};1\right)H(0;u_1)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123};1\right)H(0;u_1)+\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},1;1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},1;1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{213};1\right)H(0;u_1)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{231};1\right)H(0;u_1)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213};1\right)H(0;u_1)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231};1\right)H(0;u_1)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},1;1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{u_1};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{1-u_2};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},0;1\right)H(0;u_1)+\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},\tfrac{1}{1-u_2};1\right)H(0;u_1)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},0;1\right)H(0;u_1)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{231},\tfrac{1}{1-u_2};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{312};1\right)H(0;u_1)-
\end{aligned}
$$

More Evidence for Underlying Elegance

Last year, Del Duca, Duhr, and Smirnov found an analytic formula for the 2-loop, 6-point MHV amplitude:

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$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},0,v_{321};1\right)H(0;u_1)+\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{312};1\right)H(0;u_1)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},\tfrac{1}{1-u_3},v_{321};1\right)H(0;u_1)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},0;1\right)H(0;u_1)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{1}{1-u_3};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312},\tfrac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},0;1\right)H(0;u_1)+\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312},\tfrac{1}{1-u_3};1\right)H(0;u_1)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},0;1\right)H(0;u_1)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321},\tfrac{1}{1-u_3};1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(v_{123},1,\tfrac{1}{1-u_1};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(v_{123},\tfrac{1}{1-u_1},1;1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(v_{132},1,\tfrac{1}{1-u_1};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(v_{132},\tfrac{1}{1-u_1},1;1\right)H(0;u_1)+\\
&\tfrac{1}{4}\mathcal{G}\left(v_{213},1,\tfrac{1}{1-u_2};1\right)H(0;u_1)+\tfrac{1}{4}\mathcal{G}\left(v_{213},\tfrac{1}{1-u_2},1;1\right)H(0;u_1)-\\
&\tfrac{3}{4}\mathcal{G}\left(v_{231},1,\tfrac{1}{1-u_2};1\right)H(0;u_1)-\tfrac{3}{4}\mathcal{G}\left(v_{231},\tfrac{1}{1-u_2},1;1\right)H(0;u_1)+\\
&\tfrac{3}{4}\mathcal{G}\left(v_{312},1,\tfrac{1}{1-u_3};1\right)H(0;u_1)+\tfrac{3}{4}\mathcal{G}\left(v_{312},\tfrac{1}{1-u_3},1;1\right)H(0;u_1)-\\
&\tfrac{1}{4}\mathcal{G}\left(v_{321},1,\tfrac{1}{1-u_3};1\right)H(0;u_1)-\tfrac{1}{4}\mathcal{G}\left(v_{321},\tfrac{1}{1-u_3},1;1\right)H(0;u_1)-\\
&\tfrac{1}{4}G\left(0,\tfrac{1}{u_1},\tfrac{1}{u_1+u_2};1\right)H(0;u_2)-\tfrac{3}{4}G\left(0,\tfrac{1}{u_2},\tfrac{1}{u_1+u_2};1\right)H(0;u_2)-\\
&\tfrac{3}{4}G\left(0,\tfrac{1}{u_2},\tfrac{1}{u_2+u_3};1\right)H(0;u_2)-\tfrac{1}{4}G\left(0,\tfrac{u_2-1}{u_1+u_2-1},\tfrac{1}{1-u_1};1\right)H(0;u_2)-\\
&\tfrac{1}{4}G\left(0,\tfrac{1}{u_3},\tfrac{1}{u_2+u_3};1\right)H(0;u_2)+\tfrac{1}{4}G\left(0,\tfrac{u_1-1}{u_1+u_3-1},\tfrac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_1},\tfrac{u_2-1}{u_1+u_2-1},0;1\right)H(0;u_2)+\tfrac{1}{4}G\left(\tfrac{1}{1-u_1},\tfrac{u_2-1}{u_1+u_2-1},\tfrac{1}{1-u_1};1\right)H(0;u_2)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_1},\tfrac{u_2-1}{u_1+u_2-1},\tfrac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\tfrac{1}{2}G\left(\tfrac{1}{u_1},0,\tfrac{1}{u_2};1\right)H(0;u_2)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_1},0,\tfrac{1}{u_1+u_2};1\right)H(0;u_2)+\tfrac{1}{4}G\left(\tfrac{1}{u_1},\tfrac{1}{u_2},\tfrac{1}{u_1+u_2};1\right)H(0;u_2)-\\
&\tfrac{3}{4}G\left(\tfrac{1}{u_2},0,\tfrac{1}{u_1+u_2};1\right)H(0;u_2)-\tfrac{3}{4}G\left(\tfrac{1}{u_2},0,\tfrac{1}{u_2+u_3};1\right)H(0;u_2)+\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_2},\tfrac{1}{u_1},\tfrac{1}{u_1+u_2};1\right)H(0;u_2)+\tfrac{1}{2}G\left(\tfrac{1}{u_2},\tfrac{1}{u_2},\tfrac{1}{u_1+u_2};1\right)H(0;u_2)+\\
&\tfrac{1}{2}G\left(\tfrac{1}{u_2},\tfrac{1}{u_2},\tfrac{1}{u_2+u_3};1\right)H(0;u_2)+\tfrac{1}{4}G\left(\tfrac{1}{u_2},\tfrac{1}{u_3},\tfrac{1}{u_2+u_3};1\right)H(0;u_2)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_3},1,\tfrac{1}{u_2};1\right)H(0;u_2)+\tfrac{1}{4}G\left(\tfrac{1}{1-u_3},\tfrac{u_1-1}{u_1+u_3-1},1;1\right)H(0;u_2)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_3},\tfrac{u_1-1}{u_1+u_3-1},\tfrac{1}{1-u_3};1\right)H(0;u_2)+\tfrac{1}{2}G\left(\tfrac{1}{u_3},0,\tfrac{1}{u_2};1\right)H(0;u_2)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_3},0,\tfrac{1}{u_2+u_3};1\right)H(0;u_2)+\tfrac{1}{4}G\left(\tfrac{1}{u_3},\tfrac{1}{u_2},\tfrac{1}{u_2+u_3};1\right)H(0;u_2)+
\end{aligned}
$$

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$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\tfrac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0;1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H(0;u_2)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_2)+\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_2)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_2)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_2)+\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\tfrac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_2)+\tfrac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_2)-
\end{aligned}
$$

More Evidence for Underlying Elegance

Last year, Del Duca, Duhr, and Smirnov found an analytic formula for the 2-loop, 6-point MHV amplitude:

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- expressed in terms of 18 pages of Goncharov polylogarithms

The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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Del Duca, Duhr, & Smirnov arXiv:1003.1702

$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(v_{132},1,\tfrac{1}{1-u_1};1\right)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(v_{132},\tfrac{1}{1-u_1},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(v_{213},1,\tfrac{1}{1-u_2};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(v_{213},\tfrac{1}{1-u_2},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(v_{231},1,\tfrac{1}{1-u_2};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(v_{231},\tfrac{1}{1-u_2},1;1\right)H(0;u_2)-\\
&\tfrac{3}{4}\mathcal{G}\left(v_{312},1,\tfrac{1}{1-u_3};1\right)H(0;u_2)-\tfrac{3}{4}\mathcal{G}\left(v_{312},\tfrac{1}{1-u_3},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}\mathcal{G}\left(v_{321},1,\tfrac{1}{1-u_3};1\right)H(0;u_2)+\tfrac{1}{4}\mathcal{G}\left(v_{321},\tfrac{1}{1-u_3},1;1\right)H(0;u_2)+\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_1},\tfrac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+\tfrac{1}{4}G\left(\tfrac{1}{u_2},\tfrac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_3},\tfrac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},u_{312};1\right)H(0;u_1)H(0;u_2)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{312};1\right)H(0;u_1)H(0;u_2)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_3},v_{321};1\right)H(0;u_1)H(0;u_2)+\tfrac{5}{24}\pi^2H(0;u_1)H(0;u_2)-\\
&\tfrac{1}{4}G\left(0,\tfrac{1}{u_1},\tfrac{1}{u_1+u_3};1\right)H(0;u_3)-\tfrac{1}{4}G\left(0,\tfrac{1}{u_2},\tfrac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\tfrac{1}{4}G\left(0,\tfrac{u_2-1}{u_1+u_2-1},\tfrac{1}{1-u_1};1\right)H(0;u_3)-\tfrac{3}{4}G\left(0,\tfrac{1}{u_3},\tfrac{1}{u_1+u_3};1\right)H(0;u_3)-\\
&\tfrac{3}{4}G\left(0,\tfrac{1}{u_3},\tfrac{1}{u_2+u_3};1\right)H(0;u_3)-\tfrac{1}{4}G\left(0,\tfrac{u_3-1}{u_2+u_3-1},\tfrac{1}{1-u_2};1\right)H(0;u_3)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_1},1,\tfrac{1}{u_3};1\right)H(0;u_3)+\tfrac{1}{4}G\left(\tfrac{1}{1-u_1},\tfrac{u_2-1}{u_1+u_2-1},1;1\right)H(0;u_3)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_1},\tfrac{u_2-1}{u_1+u_2-1},\tfrac{1}{1-u_1};1\right)H(0;u_3)+\tfrac{1}{2}G\left(\tfrac{1}{u_1},0,\tfrac{1}{u_3};1\right)H(0;u_3)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_1},0,\tfrac{1}{u_1+u_3};1\right)H(0;u_3)+\tfrac{1}{4}G\left(\tfrac{1}{u_1},\tfrac{1}{u_3},\tfrac{1}{u_1+u_3};1\right)H(0;u_3)+\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_2},\tfrac{u_3-1}{u_2+u_3-1},0;1\right)H(0;u_3)+\tfrac{1}{4}G\left(\tfrac{1}{1-u_2},\tfrac{u_3-1}{u_2+u_3-1},\tfrac{1}{1-u_2};1\right)H(0;u_3)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{1-u_2},\tfrac{u_3-1}{u_2+u_3-1},\tfrac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)+\tfrac{1}{2}G\left(\tfrac{1}{u_2},0,\tfrac{1}{u_3};1\right)H(0;u_3)-\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_2},0,\tfrac{1}{u_2+u_3};1\right)H(0;u_3)+\tfrac{1}{4}G\left(\tfrac{1}{u_2},\tfrac{1}{u_3},\tfrac{1}{u_2+u_3};1\right)H(0;u_3)-\\
&\tfrac{3}{4}G\left(\tfrac{1}{u_3},0,\tfrac{1}{u_1+u_3};1\right)H(0;u_3)-\tfrac{3}{4}G\left(\tfrac{1}{u_3},0,\tfrac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\tfrac{1}{4}G\left(\tfrac{1}{u_3},\tfrac{1}{u_1},\tfrac{1}{u_1+u_3};1\right)H(0;u_3)+\tfrac{1}{4}G\left(\tfrac{1}{u_3},\tfrac{1}{u_2},\tfrac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\tfrac{1}{2}G\left(\tfrac{1}{u_3},\tfrac{1}{u_3},\tfrac{1}{u_1+u_3};1\right)H(0;u_3)+\tfrac{1}{2}G\left(\tfrac{1}{u_3},\tfrac{1}{u_3},\tfrac{1}{u_2+u_3};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_1},v_{132};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_2},v_{213};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(0,\tfrac{1}{1-u_2},v_{231};1\right)H(0;u_3)+
\end{aligned}
$$

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$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)-\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+\tfrac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_3)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1;1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H(0;u_3)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H(0;u_3)+\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_3)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_3)+\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0;1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H(0;u_3)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H(0;u_3)+\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_3)+\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_3)+\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1;1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1;1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_3)-\\
&\tfrac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_3)-\tfrac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_3)+\\
&\tfrac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_3)+\tfrac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_3)-\\
&\tfrac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_3)-\tfrac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_3)+
\end{aligned}
$$

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$$\begin{aligned}
&\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+\\
&\frac{5}{24}\pi^2H\left(0;u_1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\frac{5}{24}\pi^2H\left(0;u_2\right)H\left(0;u_3\right)+\\
&3H\left(0;u_2\right)H\left(0,0;u_1\right)H\left(0;u_3\right)+3H\left(0;u_1\right)H\left(0,0;u_2\right)H\left(0;u_3\right)+\\
&\frac{1}{4}H\left(0;u_2\right)H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H\left(0;u_3\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0,1;\left(u_1+u_3\right)\right)H\left(0;u_3\right)+\\
&\frac{1}{4}H\left(0;u_1\right)H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H\left(0;u_3\right)+\frac{1}{2}H\left(0;u_2\right)H\left(0,1;\left(u_2+u_3\right)\right)H\left(0;u_3\right)+\\
&\frac{3}{4}H\left(0;u_2\right)H\left(1,0;u_1\right)H\left(0;u_3\right)+\frac{3}{4}H\left(0;u_1\right)H\left(1,0;u_2\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0,0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0,0;u_1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0,0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0,0;u_1\right)-\frac{23}{24}\pi^2H\left(0,0;u_1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0,0;u_2\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0,0;u_2\right)-\\
&\frac{25}{4}H\left(0,0;u_1\right)H\left(0,0;u_2\right)-\frac{23}{24}\pi^2H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0,0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0,0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0,0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0,0;u_3\right)+3H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,0;u_3\right)-\\
&\frac{25}{4}H\left(0,0;u_1\right)H\left(0,0;u_3\right)-\frac{25}{4}H\left(0,0;u_2\right)H\left(0,0;u_3\right)-\frac{23}{24}\pi^2H\left(0,0;u_3\right)+\frac{1}{12}\pi^2H\left(0,1;u_1\right)+\\
&\frac{1}{12}\pi^2H\left(0,1;u_2\right)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,1;\left(u_1+u_2\right)\right)+
\end{aligned}$$

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Del Duca, Duhr, & Smirnov arXiv:1003.1702

$$
\begin{aligned}
&\frac{1}{12}\pi^2 H(0,1;(u_1+u_2)) + \frac{1}{12}\pi^2 H(0,1;u_3) + \frac{1}{4}H(0;u_1)\,H(0;u_2)\,H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right) - \\
&\frac{1}{24}\pi^2 H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right) + \frac{1}{12}\pi^2 H(0,1;(u_1+u_3)) - \frac{1}{24}\pi^2 H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right) + \\
&\frac{1}{12}\pi^2 H(0,1;(u_2+u_3)) - \frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right) H(1,0;u_1) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right) H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right) H(1,0;u_1) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right) H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right) H(1,0;u_1) + \\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right) H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right) H(1,0;u_1) - \\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right) H(1,0;u_1) - \frac{3}{4}H(0,0;u_2)\,H(1,0;u_1) - \frac{3}{4}H(0,0;u_3)\,H(1,0;u_1) + \\
&\frac{1}{4}H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right) H(1,0;u_1) - \frac{1}{3}\pi^2 H(1,0;u_1) - \frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right) H(1,0;u_2) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right) H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right) H(1,0;u_2) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right) H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right) H(1,0;u_2) + \\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right) H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right) H(1,0;u_2) - \\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right) H(1,0;u_2) - \frac{3}{4}H(0,0;u_1)\,H(1,0;u_2) - \frac{3}{4}H(0,0;u_3)\,H(1,0;u_2) + \\
&\frac{1}{4}H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right) H(1,0;u_2) - \frac{1}{4}H(1,0;u_1)\,H(1,0;u_2) - \frac{1}{3}\pi^2 H(1,0;u_2) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right) H(1,0;u_3) - \frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right) H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right) H(1,0;u_3) + \frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right) H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right) H(1,0;u_3) - \frac{1}{3}\pi^2 H(1,0;u_3) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right) H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right) H(1,0;u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right) H(1,0;u_3) + \\
&\frac{3}{4}H(0;u_1)\,H(0;u_2)\,H(1,0;u_3) - \frac{3}{4}H(0,0;u_1)\,H(1,0;u_3) - \frac{3}{4}H(0,0;u_2)\,H(1,0;u_3) + \\
&\frac{1}{4}H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right) H(1,0;u_3) - \frac{1}{4}H(1,0;u_1)\,H(1,0;u_3) - \frac{1}{4}H(1,0;u_2)\,H(1,0;u_3) + \\
&\frac{1}{24}\pi^2 H(1,1;u_1) + \frac{1}{24}\pi^2 H(1,1;u_2) + \frac{1}{24}\pi^2 H(1,1;u_3) + \frac{1}{2}H(0;u_2)\,H(0,0,0;u_1) + \\
&\frac{1}{2}H(0;u_3)\,H(0,0,0;u_2) + \frac{1}{2}H(0;u_1)\,H(0,0,0;u_3) - \frac{1}{2}H(0;u_2)\,H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right) - \\
&\frac{1}{2}H(0;u_3)\,H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right) - H(0;u_1)\,H(0,0,1;(u_1+u_2)) - \\
&H(0;u_2)\,H(0,0,1;(u_1+u_2)) - \frac{1}{2}H(0;u_1)\,H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right) -
\end{aligned}
$$

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- dimensionally-regulating thousands of separately-divergent integrals
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$$
\begin{aligned}
&\frac{1}{2}H(0;u_2)\,H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-H(0;u_1)\,H(0,0,1;(u_1+u_3))-\\
&H(0;u_3)\,H(0,0,1;(u_1+u_3))-\frac{1}{2}H(0;u_1)\,H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\\
&\frac{1}{2}H(0;u_3)\,H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-H(0;u_2)\,H(0,0,1;(u_2+u_3))-\\
&H(0;u_3)\,H(0,0,1;(u_2+u_3))-\frac{1}{2}H(0;u_2)\,H(0,1,0;u_1)-\frac{1}{2}H(0;u_3)\,H(0,1,0;u_2)-\\
&\frac{1}{2}H(0;u_1)\,H(0,1,0;u_3)+\frac{1}{4}H(0;u_2)\,H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\\
&\frac{1}{4}H(0;u_3)\,H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H(0;u_1)\,H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\\
&\frac{1}{4}H(0;u_2)\,H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)\,H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\\
&\frac{1}{4}H(0;u_3)\,H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(0;u_2)\,H(1,0,0;u_1)-\frac{1}{2}H(0;u_3)\,H(1,0,0;u_1)-\\
&\frac{1}{2}H(0;u_1)\,H(1,0,0;u_2)+\frac{1}{2}H(0;u_3)\,H(1,0,0;u_2)+\frac{1}{2}H(0;u_1)\,H(1,0,0;u_3)-\\
&\frac{1}{2}H(0;u_2)\,H(1,0,0;u_3)-\frac{1}{4}H(0;u_3)\,H\left(1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-\\
&\frac{1}{4}H(0;u_2)\,H\left(1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)\,H\left(1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\\
&7H(0,0,0,0;u_1)-7H(0,0,0,0;u_2)-7H(0,0,0,0;u_3)+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\\
&3H(0,0,0,1;(u_1+u_2))+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+3H(0,0,0,1;(u_1+u_3))+\\
&\frac{3}{2}H\left(0,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+3H(0,0,0,1;(u_2+u_3))+\frac{9}{4}H(0,0,1,0;u_1)+\\
&\frac{9}{4}H(0,0,1,0;u_2)+\frac{9}{4}H(0,0,1,0;u_3)-\frac{1}{2}H(0,1,0,0;u_1)-\frac{1}{2}H(0,1,0,0;u_2)-\\
&\frac{1}{2}H(0,1,0,0;u_3)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&\frac{1}{2}H\left(0,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+H(0,1,1,0;u_1)+H(0,1,1,0;u_2)+H(0,1,1,0;u_3)-\\
&\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\\
&\frac{1}{4}H\left(0,1,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+H\left(1,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+H\left(1,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&H\left(1,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+2H(1,0,1,0;u_1)+2H(1,0,1,0;u_2)+2H(1,0,1,0;u_3)+\\
&\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&\frac{1}{4}H\left(1,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(1,1,1,0;u_1)+\frac{1}{2}H(1,1,1,0;u_2)+\frac{1}{2}H(1,1,1,0;u_3)-\\
&\frac{1}{24}\pi^2H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{u_{123}}\right)-\frac{1}{24}\pi^2H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{u_{231}}\right)-\frac{1}{24}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{u_{312}}\right)+\\
&\frac{1}{8}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{123}}\right)-\frac{1}{8}\pi^2H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{24}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-
\end{aligned}
$$

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$$
\begin{aligned}
&\tfrac{1}{24}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\tfrac{1}{v_{132}}\right)-\tfrac{1}{24}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\tfrac{1}{v_{213}}\right)+\tfrac{1}{24}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\tfrac{1}{v_{213}}\right)-\\
&\tfrac{1}{8}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{8}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{8}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\tfrac{1}{v_{312}}\right)-\\
&\tfrac{1}{8}\pi^2 H(0;u_2)\,\mathcal{H}\left(1;\tfrac{1}{v_{312}}\right)+\tfrac{1}{24}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\tfrac{1}{v_{321}}\right)-\tfrac{1}{24}\pi^2 H(0;u_2)\,\mathcal{H}\left(1;\tfrac{1}{v_{321}}\right)-\\
&\tfrac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{u_{123}}\right)-\tfrac{1}{4}H(1,0;u_2)\,\mathcal{H}\left(0,1;\tfrac{1}{u_{123}}\right)+\tfrac{1}{24}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{u_{123}}\right)+\\
&\tfrac{1}{24}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{u_{231}}\right)-\tfrac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{u_{231}}\right)-\tfrac{1}{4}H(1,0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{u_{231}}\right)-\\
&\tfrac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\tfrac{1}{u_{312}}\right)-\tfrac{1}{4}H(1,0;u_1)\,\mathcal{H}\left(0,1;\tfrac{1}{u_{312}}\right)+\tfrac{1}{24}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{u_{312}}\right)-\\
&\tfrac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{123}}\right)+\tfrac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{123}}\right)+\\
&\tfrac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{123}}\right)+\tfrac{1}{6}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{v_{123}}\right)-\tfrac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{132}}\right)+\\
&\tfrac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{6}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{v_{132}}\right)-\\
&\tfrac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{213}}\right)+\tfrac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{213}}\right)+\\
&\tfrac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{213}}\right)+\tfrac{1}{6}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{v_{213}}\right)-\tfrac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{231}}\right)+\\
&\tfrac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{6}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{v_{231}}\right)-\\
&\tfrac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{312}}\right)+\tfrac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{312}}\right)+\\
&\tfrac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{312}}\right)+\tfrac{1}{6}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{v_{312}}\right)-\tfrac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{321}}\right)+\\
&\tfrac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{321}}\right)+\tfrac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\tfrac{1}{v_{321}}\right)+\tfrac{1}{6}\pi^2\mathcal{H}\left(0,1;\tfrac{1}{v_{321}}\right)-\\
&\tfrac{1}{2}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{123}}\right)+\tfrac{1}{2}H(0,0;u_2)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{123}}\right)+\\
&\tfrac{1}{2}H(0,0;u_3)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{123}}\right)+\tfrac{11}{24}\pi^2\mathcal{H}\left(1,1;\tfrac{1}{v_{123}}\right)-\tfrac{1}{24}\pi^2\mathcal{H}\left(1,1;\tfrac{1}{v_{132}}\right)-\\
&\tfrac{1}{24}\pi^2\mathcal{H}\left(1,1;\tfrac{1}{v_{213}}\right)-\tfrac{1}{2}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{2}H(0,0;u_1)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{231}}\right)+\\
&\tfrac{1}{2}H(0,0;u_3)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{231}}\right)+\tfrac{11}{24}\pi^2\mathcal{H}\left(1,1;\tfrac{1}{v_{231}}\right)-\tfrac{1}{2}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{312}}\right)+\\
&\tfrac{1}{2}H(0,0;u_1)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{312}}\right)+\tfrac{1}{2}H(0,0;u_2)\,\mathcal{H}\left(1,1;\tfrac{1}{v_{312}}\right)+\tfrac{11}{24}\pi^2\mathcal{H}\left(1,1;\tfrac{1}{v_{312}}\right)-\\
&\tfrac{1}{24}\pi^2\mathcal{H}\left(1,1;\tfrac{1}{v_{321}}\right)+\tfrac{1}{2}H(0;u_2)\,\mathcal{H}\left(0,0,1;\tfrac{1}{u_{123}}\right)+\tfrac{1}{2}H(0;u_3)\,\mathcal{H}\left(0,0,1;\tfrac{1}{u_{123}}\right)+\\
&\tfrac{1}{2}H(0;u_1)\,\mathcal{H}\left(0,0,1;\tfrac{1}{u_{231}}\right)+\tfrac{1}{2}H(0;u_3)\,\mathcal{H}\left(0,0,1;\tfrac{1}{u_{231}}\right)+\tfrac{1}{2}H(0;u_1)\,\mathcal{H}\left(0,0,1;\tfrac{1}{u_{312}}\right)+\\
&\tfrac{1}{2}H(0;u_2)\,\mathcal{H}\left(0,0,1;\tfrac{1}{u_{312}}\right)+\tfrac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\tfrac{1}{u_{123}}\right)+\tfrac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\tfrac{1}{u_{231}}\right)+
\end{aligned}
$$

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\begin{aligned}
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-\\
&\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+\\
&\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{231}}\right)+\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-\\
&\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\\
&\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+\\
&\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+H(0;u_2)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-H(0;u_3)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-\\
&H(0;u_1)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H(0;u_3)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H(0;u_1)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-\\
&H(0;u_2)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{123}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{231}}\right)-\\
&\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{321}}\right)-\\
&\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{312}}\right)-\\
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&\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{132}}\right)+\zeta_3H(0;u_1)+\zeta_3H(0;u_2)+\zeta_3H(0;u_3)+\\
&\frac{5}{2}\zeta_3H(1;u_1)+\frac{5}{2}\zeta_3H(1;u_2)+\frac{5}{2}\zeta_3H(1;u_3)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{123}}\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{231}}\right)+\\
&\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{312}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right)+\\
&\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{132}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+\\
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\end{aligned}
$$

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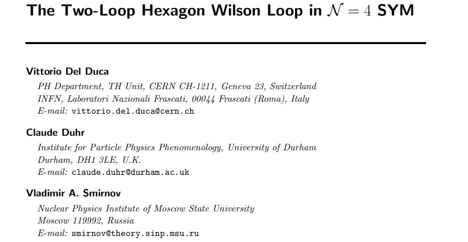

The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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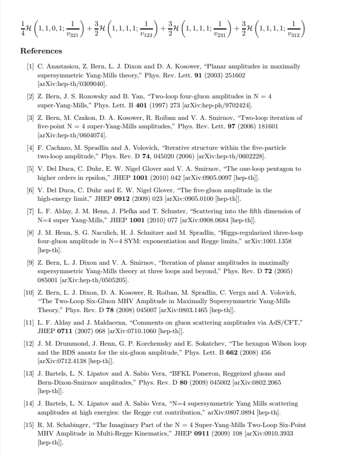
Del Duca, Duhr, & Smirnov arXiv:1003.1702



$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{321}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{231}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{312}}\right)$$

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Classical Polylogarithms for Amplitudes and Wilson Loops

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[1] *Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2] *Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

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Goncharov, Spradlin, Vergu, & Volovich, arXiv:1006.5703

$$R(u_1,u_2,u_3)=\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) -\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{J^4}{24}+\chi\frac{\pi^2}{12}\left(J^2+\zeta(2)\right).$$

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Forward Looking Comments

- Do there exist alternative, *e.g.* purely geometric ways of characterizing the full S-Matrix?
- How can we systematically regulate and perform momentum-twistor loop integration?
  - Can we perform these integrals analytically at the outset?
  - Are there connections to the leading-singularity programme? connections to 'symbols' & mixed Tate motives?
  - Can the integrals coming from BCFW recursion be done directly?
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