Exact results in 3d gauge theories and M-theory

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Based on

arXiv:1101.0557, Y.I. and S. Yokoyama,
arXiv:1102.0621, Y.I., D. Yokoyama, and S.Yokoyama

Part 1: Brief summary

AdS_4/CFT_3 duality

CFT_3

3d conformal field theory

A 3d supersymmetric Chern-Simons theory with matter fields

AdS_4

M-theory in $AdS_4 x M_7$

AdS_4 : anti-de Sitter space (non-compact 4d space)

M_7: 7d compact space

dual

(equivalent)

Weak coupling limit

Strongly coupled large N gauge theory

Classical gravity

This is a strong-weak duality.

We can use weakly coupled M-theory (classical gravity) to obtain non-trivial results for the strongly coupled gauge theory.

Gauge theory $\Leftarrow$ Classical gravity

This duality has not been proved, and it is important to collect non-trivial evidences.

In this talk, we focus on the symmetry of the systems.

Gravity side

isometry G of M_7 is a symmetry of the system.

Gauge theory side

we have an internal symmetry H which keep the action S[ψ] invariant.

In general, only a part of G is manifest on the gauge theory side.

$$\boxed{\text{symmetry H} \quad \subset \quad \text{isometry G}}$$

In this case, the duality implies that the true symmetry of the systems is G, and the manifest symmetry H of the gauge theory is enhanced to G.

Simple example: D. Martelli and J. Sparks, arXiv:0909.2036

A certain CS theory w/
U(N)xU(N) gauge group

M-theory in $AdS_4 x V^{5,2}$

Symmetry of S[ψ]

$SU(2)_F \times U(1)_m \times U(1)_R$

$V^{5,2}=SO(5)/SO(3)$

Isometry = $SO(5) \times U(1)_R$

Symmetry enhancement

$SU(2)_F \times U(1)_m \rightarrow SO(5)$

is predicted by AdS/CFT.

$SU(2)_F$: rotates elementary fields
$U(1)_m$: magnetic charge of monopoles.

The enhanced symmetry SO(5) mixes elementary fields and solitonic objects.

This symmetry enhancement is non-perturbative.

We want to perform a non-trivial check of this symmetry enhancement by computing an index on the gauge theory side.

$$I(z_1, z_2, x) = \mathrm{tr}[z_1^{F_1} z_2^{F_2} \mathcal{O}(x)]$$

F_1: $SU(2)_F$ Cartan generator
F_2: $U(1)_m$ generator

SO(5) contains elements exchanging F_1 and F_2.

If the symmetry is actually enhanced to SO(5), the index should satisfy the relation

$$I(z_1, z_2, x) = I(z_2, z_1, x)$$

We can compute the index as the path integral

$$I(z_1, z_2, x) = \int \mathcal{D}\Psi e^{-S[\Psi]}$$

We define the theory on $S^2 \mathrm{x} S^1$, and the parameters z_1 and z_2 are introduced as holonomies around S^1.

In general, it is difficult to perform the path integral.

The theory we consider here is supersymmetric, and we can use localization to perform the path integral exactly.

Localization theorem:

If the system has a fermionic symmetry δ, and $\delta S[\psi]=\delta^2 V[\psi]=0$, the path integral

$$\int \mathcal{D}\Psi e^{-S[\Psi]-t\delta V[\Psi]}$$

Does not depend on the deformation parameter t, and in the $t \to \infty$ limit, the integral localizes at the fixed points of δ^2.

We can rewrite the integral as a summation over fixed points.

(Duistermaat-Heckman formula)

We derived a general formula for the index which is applicable to N=2 supersymmetric theories even when fields have large anomalous dimensions.

arXiv:1101.0557, Y.I. and S. Yokoyama

(In the derivation of the formula, we did not use the DH formula. We perform the path integral directly after deforming the theory by δ-exact term. This is easy in the case of 3d theory. This may not be the ``orthodox'' localization.)

Result: N=1 case (Gauge group is U(N)xU(N))

In the case of U(1)xU(1) gauge group, we can perform the integral completely.

$$\begin{aligned} I(z_1, z_2, x) = \mathrm{PE}\big[&f(z_1 z_2, x) + f(z_1 z_2^{-1}, x) \\ &+ f(z_1^{-1} z_2, x) + f(z_1^{-1} z_2^{-1}, x) \\ &+ 2f(1, x)\big]. \end{aligned}$$

where

$$f(q, x) = \frac{q x^{\frac{2}{3}} - q^{-1} x^{\frac{4}{3}}}{1 - x^2}$$

$$\mathrm{PE}\, g(z_1, z_2, x) \equiv \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} g(z_1^m, z_2^m, x^m)\right]$$

This is indeed symmetric under $z_1 \Leftrightarrow z_2$.

This strongly suggests the symmetry enhancement.

Large N limit

We also computed the index for large N gauge group as a series expansion with respect to x.

We confirmed that it is symmetric under $z_1 \Leftrightarrow z_2$ at least for the first few terms in the expansion.

arXiv:1102.0621, Y.I., D. Yokoyama, and S. Yokoyama

I'm going to explain details in part II.

Introduction

Duality

-equivalence of two theories, which usually look very different .

$$\text{Theory 1} \quad \xLeftrightarrow{\text{dual}} \quad \text{Theory 2}$$

Many dualities are proposed in string/M-theory.
In general it is difficult to prove them.

How can we obtain non-trivial evidences?

If we could determine the energy spectrum (eigenvalues) of two systems and could show that they agree, it would provide a strong evidence for the duality.

Instead of treating eigenvalues directly, it is convenient to use the partition function

$$Z(\beta) = \mathrm{tr} \exp(-\beta H).$$

If the partition function of two theories agree as functions of β

$$Z_1(\beta) = Z_2(\beta),$$

two theories have the same energy spectra.

We can generalize the partition function by introducing ``chemical potentials'' μ_k for generators Q_k of global symmetries of the systems.

$$Z(\beta,\mu_k) = \mathrm{tr}\,\exp(-\beta H - \mu_k Q_k)$$

This gives more information than the original function $Z(\beta)$.

In general, it is difficult (impossible) to compute these partition functions analytically due to large quantum corrections.

In supersymmetric theories, however, we can compute the partition function exactly if we tune the chemical potentials appropriately.

Such partition functions have in general the form

$$I(\beta,\mu_k) = \mathrm{tr}\,[(-1)^F \exp(-\beta H - \mu_k Q_k)].$$

Due to the factor $(-1)^F$, bosonic and fermionic contributions partially cancel each other, and quantum corrections become milder. This kind of quantities are called ``indices.''

Recently, we derived a general formula for the superconformal index for 3d N=2 gauge theories.

In this talk, I will explain how we can derive the formula, and then use it to obtain a non-trivial evidence for AdS_4/CFT_3 duality.

- AdS_4/CFT_3
- Localization and Index
- A non-trivial check of duality
- Summary

AdS_4/CFT_3

AdS_4/CFT_3 is a duality between a 3d Chern-Simons theory and M-theory in a background $AdS_4 x M_7$

Quiver Chern-Simons theory on S^2 $\Longleftrightarrow$ M-theory in $AdS_4 x M_7$

This is a strong-weak duality.

We consider strongly coupled gauge theories and weakly coupled M-theory.

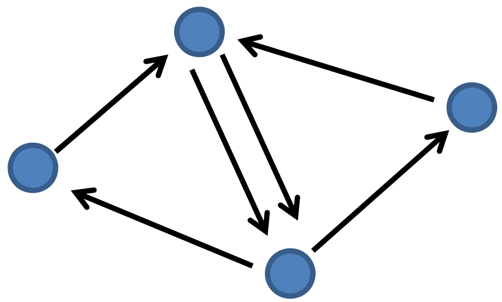
Gauge theory side

N=2 superconformal Quiver Chern-Simons theory

Gauge group : $G = U(N)^n = \prod_{A=1}^{n} U(N)_A$

Matter fields : bi-fundamental representations

Described by a quiver diagram



circles: U(N) gauge groups
(vector multiplets)

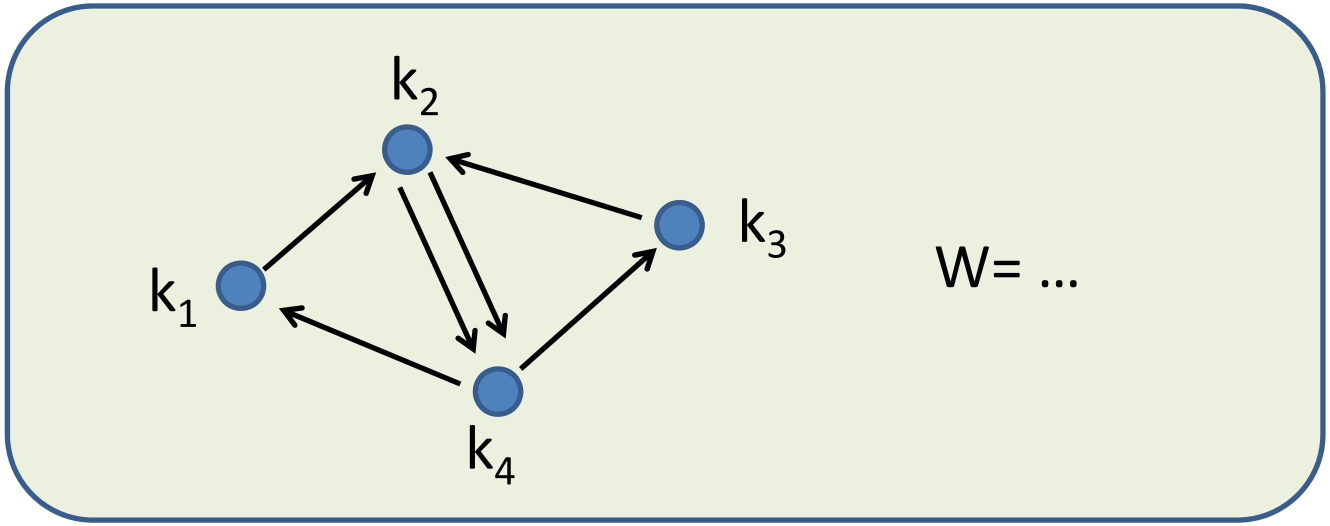
arrows: bi-fundamental rep.
(chiral multiplets)

For each U(N) gauge group we can introduce the Chern-Simons coupling

$$S_{\mathrm{CS}} = \sum_{A=1}^{n} \frac{k_A}{4\pi} \int \left(A_A dA_A - \frac{2i}{3} A_A A_A A_A \right)$$

$k_A \in \mathbf{Z}$ are called Chern-Simons levels.

The theory is specified by giving a quiver diagram, Chern-Simons levels, and the superpotential.



Gravity side

The gravity side of the duality is M-theory in $AdS_4 x M_7$.

AdS_4: 4-dim anti-de Sitter space

M_7: 7-dim Sasaki-Einstein space

In this talk we consider only weakly coupled M-theory = 11-dim supergravity

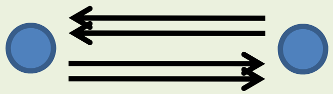
For various Sasaki-Einstein manifolds M_7, dual Quiver Chern-Simons theories are proposed.

<u>The simplest example of AdS_4/CFT_3</u>

O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, arXiv:0806.1218

ABJM model

A_1, A_2

+k  -k

B_1, B_2

$W = \mathrm{tr}\,(A_1B_1A_2B_2 - A_1B_2A_2B_1)$

dual

$\Longleftrightarrow$

M-theory in $AdS_4 \times S^7/Z_k$

The ABJM model has N=6 supersymmetry

Since the discovery of the ABJM model, many examples of AdS_4/CFT_3 duality have been proposed.

In particular, in the case with N=2 supersymmetry, there exists a simple prescription to obtain dual quiver CS theory for a large class of M_7 (toric Sasaki-Einstein manifold). (brane tilings)

In many cases, however, only agreement of the moduli space (vacuum structure) has been confirmed.

If we can compute the superconformal index

$$I(x, h_i) = \mathrm{tr}\left[(-1)^F q^{D-R-J} x^{R+2J} \prod_i h_i^{F_i}\right]$$

on both sides, we can obtain information about excitations.

The interpretation of trace on each side of the duality is as follows.

<u>Gauge theory side</u>
The trace is taken over all states in the Hilbert space of the gauge theory defined in S^2.

<u>Gravity side</u>
The trace is taken over all Kaluza-Klein excitations In M_7.
Because we consider weak coupling limit on this side, this computation is straightforward (but tedious).

For the ABJM model and its dual theory, the superconformal index has been computed, and the complete agreement has been confirmed.

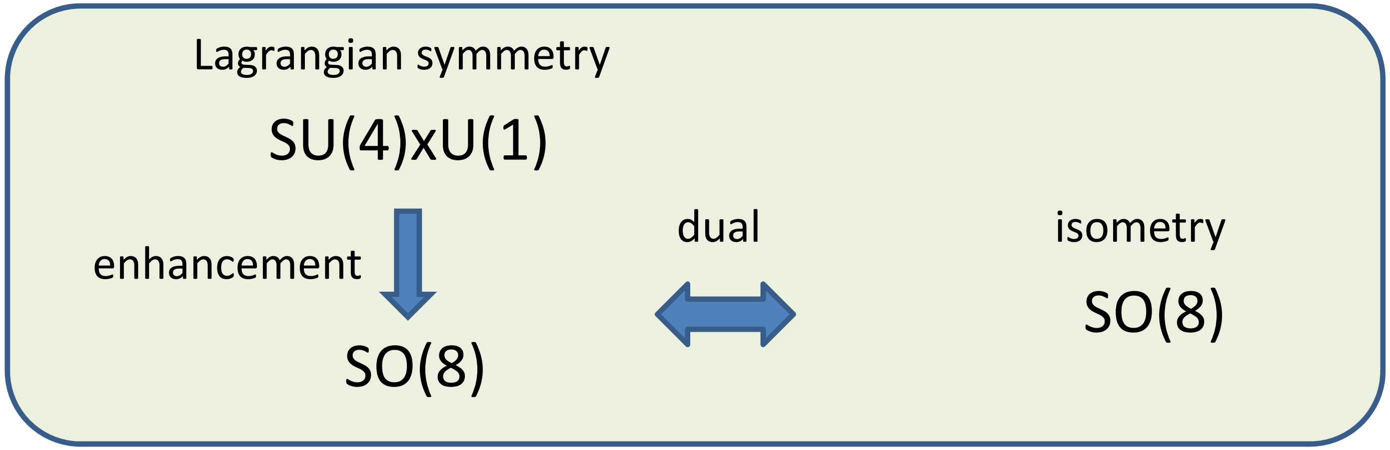
J. Bhattacharya and S.Minwalla, arXiv:0806.3251[hep-th]
S. Kim, arXiv:0903.4172[hep-th]

This fact means that there exists one-to-one correspondence between spectra of BPS states on both sides of the duality.

This one-to-one correspondence is highly non-trivial.

When k=1, the dual geometry is S^7, and the KK excitations belong to representations of SO(8).
On the other hand, the manifest global symmetry of ABJM Lagrangian is SU(4)xU(1), a subgroup of SO(8).

To obtain SO(8) representations on the gauge theory side, we need to combine monopole operators and operators consisting of elementary fields.



This was extended later to N=4 susy case.

J. Choi, S. Lee, and J. Song, arXiv:0811.2855
Y.I. and S. Yokoyama, arXiv:0908.0988

If we can extend the same analysis to N=2 case, it provides a strong evidence of the duality for a large class of dual pairs.

Difficulty:

When $N \geqq 3$, the R-symmetry is non-Abelian, and the R-charge is protected from the quantum corrections.

In the case of N=2 theory, the R-symmetry is SO(2)=U(1), and thus the R-charge can change continuously.
The R-charge are not protected even for BPS operators.

→ large anomalous dimension

Until recent, there was no way to treat such theories with large quantum corrections.

Recently, a formula of S^3-partition function was derived for theories with such quantum corrections.

S^3 partition function: $Z = \int \mathcal{D}\Phi e^{-S[\Phi]}$

D. L. Jafferis, arXiv:1012.3210
N. Hama, K. Hosomichi, and S. Lee, arXiv:1012.3512

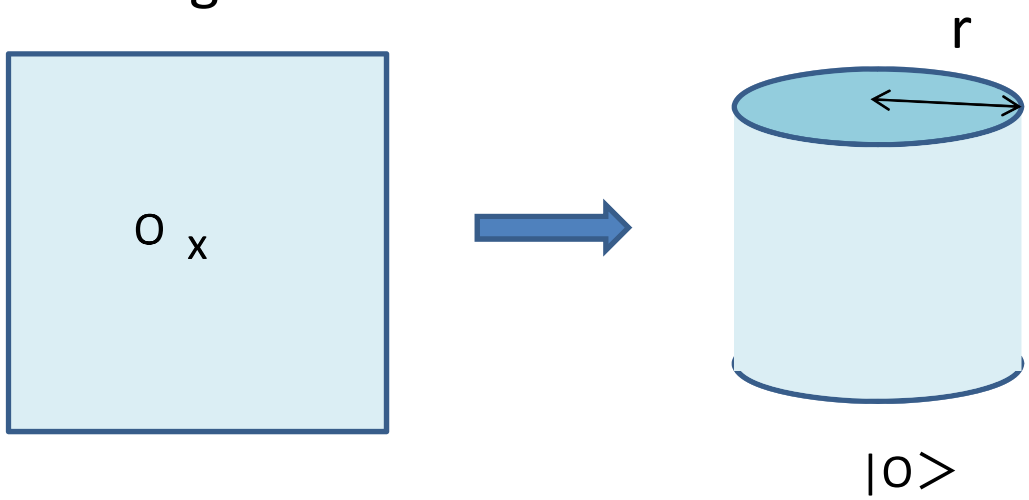
This result was obtained by using ``localization.''

S. Yokoyama and I showed in arXiv:1101.0557 that the same method can be used for the superconformal index.

Theory in S^2x(time)

The index can be written in a path integral form of the theory in $S^2 x S^1$.

The theory on S^2x(time) is obtained from the theory on R^3 by a Weyl rescaling.



With this map, the operator O is mapped into the corresponding state |O>.

The dilatation D becomes the time translation (Hamiltonian).

3-dim N=2 superconformal symmetry

We consider N=2 superconformal field theories in 3d.

3-dim N=2 superconformal group --- OSp(2|4).

Bosonic subgroup --- $SO(2)_R \times Sp(4,R) = U(1)_R \times SO(3,2)$

Cartan generators

D: dilatation (time translation)

J: 3rd component of the spin

R: R-charge

In general, we also have flavor symmetries, which commute with superconformal group.

F_i: Cartan generators of flavor symmetries.

3d N=2 superconformal algebra contains eight supercharges.

$$Q_1, \quad Q_2, \quad Q_3, \quad Q_4, \quad \overline{Q}_1, \quad \overline{Q}_2, \quad \overline{Q}_3, \quad \overline{Q}_4.$$

In the following, we only use ``anti-holomorphic'' supercharges with R-charge +1.

The parameter $\overline{\epsilon}$ for $\overline{Q}$ satisfies the Killing equation.

$$D_m \overline{\epsilon} = \gamma_m (\textbf{arbitrary spinor})$$

In the flat R^3, there are four linearly independent solutions.

$$\overline{\epsilon} = \overline{\xi} + x^m \gamma_m \overline{\zeta}$$

SUSY corresponding to $\overline{\xi}$ and $\overline{\zeta}$ are called rigid and conformal susy.

Transformation laws (anti-holomorphic part)

3d N=2 theory contains vector and chiral multiplets.

Vector multiplets $(A_m, \sigma, \lambda, D)$

$$\delta\sigma = (\overline{\epsilon}\lambda), \quad \delta A_\mu = -i(\overline{\epsilon}\gamma_\mu\lambda), \quad \delta\lambda = 0,$$

$$\delta D = i(\overline{\epsilon}\gamma^\mu D_\mu\lambda) + i(\overline{\epsilon}[\sigma, \lambda]) + \frac{i}{3}(D_\mu\overline{\epsilon}\gamma^\mu\lambda),$$

$$\delta\overline{\lambda} = -\frac{i}{2}\gamma^{\mu\nu}\overline{\epsilon}F_{\mu\nu} - \gamma^\mu\overline{\epsilon}D_\mu\sigma + iD\overline{\epsilon} - \frac{2}{3}\gamma^\mu D_\mu\overline{\epsilon}\sigma.$$

Chiral multiplets with weight Δ (ϕ, ψ, F)

$$\delta\phi^\dagger = \sqrt{2}(\overline{\epsilon}\overline{\psi}), \quad \delta\phi = 0, \quad \delta\overline{\psi} = \sqrt{2}i\epsilon F^\dagger, \quad \delta F^\dagger = 0,$$

$$\delta\psi = \sqrt{2}\overline{\epsilon}\sigma\phi - \sqrt{2}\gamma^\mu\overline{\epsilon}D_\mu\phi - \frac{2\sqrt{2}}{3}\Delta\phi\gamma^\mu D_\mu\overline{\epsilon},$$

$$\delta F = \sqrt{2}i(\overline{\epsilon}\gamma^\mu D_\mu\psi) + \sqrt{2}i(\overline{\epsilon}\sigma\psi) + 2i(\overline{\epsilon}\overline{\lambda})\phi + \frac{2\sqrt{2}i}{3}\left(\Delta - \frac{1}{2}\right)(D_\mu\overline{\epsilon}\gamma^\mu\psi).$$

Localization and Index

Localization

Let us consider path integral

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]}$$

Usually, it is difficult to perform the path integral analytically.

Let us assume the existence of the fermionic symmetry Q, and consider the following deformation

$$Z(t) = \int \mathcal{D}\Phi e^{-S[\Phi]-tQV}$$

If $Q^2V=0$, this does not depend on the parameter t.

$$\frac{d}{dt}Z(t) = \int \mathcal{D}\Phi(-QV)e^{-S[\Phi]-tQV}$$
$$= \int \mathcal{D}\Phi - Q\left[Ve^{-S[\Phi]-tQV}\right] = 0$$

If we can find appropreate Q and V, we can perform the path integral in the weak coupling limit $t \to \infty$.

In $N=2$ theory, we can use one of the supercharges as Q.

The superconformal index is defined by

$$I(x, h_i) = \mathrm{tr}\left[(-1)^F q^{D-R-J} x^{R+2J} h_i^{F_i}\right]$$
$$= \mathrm{tr}\left[(-1)^F q^D \mathcal{O}\right]$$

$q^D = e^{-\beta D}$ Time translation

$\mathcal{O} = q^{-R-J} x^{R+2J} z_i^{F_i}$ Chemical potentials

One can easily show that this does not depend on q, and only states saturating the BPS bound

$$\{Q, Q^\dagger\} = D - R - J \geq 0 \qquad (Q \equiv \overline{Q}_1)$$

contribute to the index.

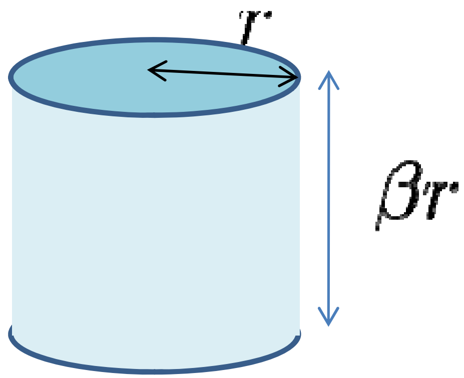
$$q\frac{\partial I}{\partial q} = \mathrm{tr}\left[(-1)^F (QQ^\dagger + Q^\dagger Q)(\cdots)\right]$$
$$= \mathrm{tr}\left[(-1)^F (-Q^\dagger Q + Q^\dagger Q)(\cdots)\right] = 0$$

To use the localization, we rewrite the definition of the index in the path integral form.

The trace is realized by compactifying the time (radial) direction.

I can be expressed as

$$I(x, h_i) = \int \mathcal{D}\Phi e^{-S[\Phi]}$$



The insertion of the operator O is introduced as the boundary conditions around S^1

$$\mathcal{O}\Phi(x^3 + \beta r) = \Phi(x^3)$$

Deformation terms

$$S \to S + t\,\overline{Q}_1 V$$

For vector multiplets, we use the following V

$$V = \overline{Q}_2 \left(-\frac{1}{4} \mathrm{tr} \overline{\lambda\lambda} \right)$$

Then $\overline{Q}_1 V$ is given by

$$\overline{Q}_1 V = \frac{1}{4} F_{mn} F^{mn} + \frac{i}{2} \gamma^{mnp} F_{mn} D_p \sigma + \frac{1}{2} D_m \sigma D^m \sigma - \frac{1}{2} DD + \frac{1}{2r^2} \sigma\sigma$$
$$- (\overline{\lambda} \gamma^m D_m \lambda) - (\overline{\lambda}[\sigma, \lambda]) - \frac{1}{2r} (\overline{\lambda} \gamma^3 \lambda) + \frac{1}{r} \left[\frac{i}{2} \gamma^{3pq} \sigma F_{pq} + \sigma D^3 \sigma \right]$$

This contains kinetic terms for vector multiplets.
In the weak coupling limit t→∞ the path integral reduces to Gaussian integral around saddle points.

Saddle points

Saddle points are given by

$$A = A_m dx^m = a dx^3 + m\frac{1-\cos\theta}{2}d\phi,$$
$$D = 0, \quad \sigma = \frac{m}{2r}, \quad \lambda = 0.$$

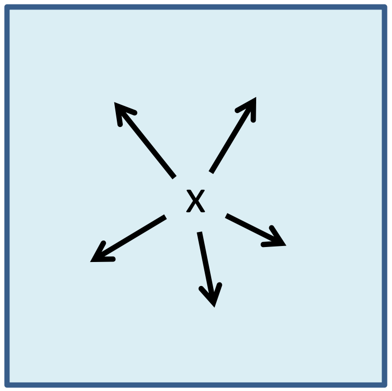
Labeled by

a --- holonomy around S^1 (flat direction) $\qquad a = \oint_{\mathbf{S}^1} A$

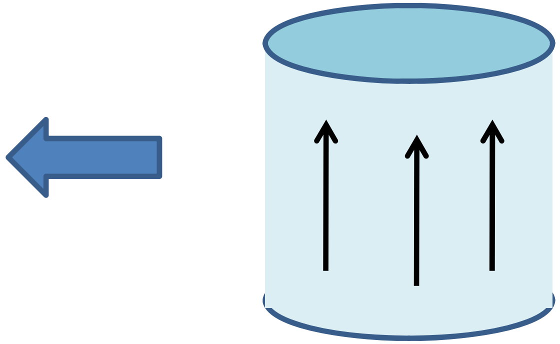
m --- GNO monopole charge. (quantized) $\qquad m = \frac{1}{2\pi}\oint_{\mathbf{S}^2} F$

Both a and m take values in Lie algebra of the Cartan subgroup of the gauge group G.

By the Weyl rescaling from $S^2 xR$ to R^3, states with magnetic flux are mapped to local operators with magnetic charge. Such operators are called monopole operators.



Monopole operator



Magnetic flux along S^1

GNO monopole charges

The monopoles arising here are GNO monopoles.

For U(N) gauge theory, the GNO monopoles are Dirac monopoles for the Cartan subgroup $U(1)^N$, whose charge is specified by N integers.

$$\frac{1}{2\pi}\oint_{\mathbf{S}^2} F = \mathrm{diag}(m_1, \ldots, m_N)$$

We can change the order of the N components of the magnetic charge by Weyl reflections. We always arrange the components in discending order.

They are not conserved charges. Only the sum of N charges (trace) is conserved .

$$m = \frac{1}{2\pi}\oint_{\mathbf{S}^2} \mathrm{tr}F = m_1 + \cdots + m_N .$$

This is often called a topological charge.

For quiver gauge theories with gauge group $U(N)^n$, monopole charge is specified by nN integers, and we can define n topological charges.

Correspondingly, we have n global U(1) symmetries. (One of them is decoupled.)

Deformation terms for chiral multiplets

For a chiral multiplet with Weyl Weight Δ, we adopt

$$V = \overline{Q}_2 \left(-\frac{1}{2} \phi^\dagger F \right)$$

and then the deformation Lagrangian is

$$\overline{Q}_1 V = -\phi^\dagger D_m D^m \phi + \phi^\dagger \sigma\sigma\phi + \phi^\dagger D \phi - \frac{1}{r} (2\Delta - 1) \phi^\dagger D_3 \phi + \frac{1}{r^2} \Delta(1-\Delta) \phi^\dagger \phi - F^\dagger F$$
$$- (\overline{\psi} \gamma^m D_m \psi) - (\overline{\psi} \sigma \psi) - \frac{1}{r} \left(\Delta - \frac{1}{2} \right) (\overline{\psi} \gamma_3 \psi) - \sqrt{2} \phi^\dagger (\lambda \psi) - \sqrt{2} (\overline{\psi} \overline{\lambda}) \phi$$

There is no flat direction for chiral multiplets.

In the large t limit, the path integral for chiral multiplets completely reduces to Gaussian integrals.

Formula for the index

Gaussian integral gives

$$\int \mathcal{D}\Phi e^{-S[\Phi]} = \sum_m \int [da] e^{-S_0(a,m)} Z_{\text{vector}}(a,m) Z_{\text{chiral}}(a,m)$$

where $\sum_m \int [da]$ represents summation over saddle points.

$$Z_{\text{vec}}(a,m) = \prod_{\alpha \in G, \alpha(m)\neq 0} x^{-\frac{1}{2}|\alpha(m)|} \left(1 - e^{i\alpha(a)} x^{|\alpha(m)|}\right),$$

$$Z_{\text{ch}}(a,m) = \prod_{\rho \in R} e^{-\frac{i}{2}|\rho(m)|\rho(a)} x^{\frac{1}{2}(1-\Delta)|\rho(m)|} \prod_i h_i^{-\frac{1}{2}|\rho(m)|F_i}$$
$$\times \frac{\prod_{k=0}^{\infty}(1 - e^{-i\rho(a)} x^{|\rho(m)|+2-\Delta+2k} \prod_i h_i^{-F_i})}{\prod_{k=0}^{\infty}(1 - e^{i\rho(a)} x^{|\rho(m)|+\Delta+2k} \prod_i h_i^{F_i})}.$$

($h_i \equiv z_i$)

Large N limit

In the large N limit, it is convenient to decompose monopole charges into three parts.

(5,3,3,2,0,0,0,-1,-3,-4) $\longrightarrow$ (5,3,3,2) + (0,0,0) + (-1,-3,-4)

Correspondingly, the index factorizes into three parts:

$$I = I^{(+)} I^{(0)} I^{(-)}$$

$I^{(+)}$ ($I^{(-)}$) includes contribution of only positive(negative)-charge monopoles.

$I^{(0)}$ is perturbative factor which does not contain monopole contributions.

A non-trivial check of AdS/CFT

Simple example D. Martelli and J. Sparks, arXiv:0909.2036

Gravity side

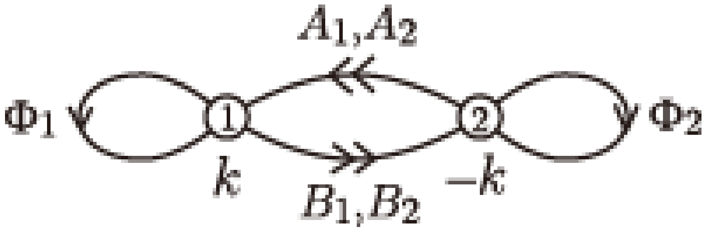
$$V^{5,2} = SO(5)/SO(3)$$

Homogeneous Sasaki-Einstein manifold

Isometry = SO(5) x $U(1)_R$ (non-toric)

Gauge theory side

Dual Chern-Simons theory (We consider k=1 case)



$$W = \mathrm{tr}(\Phi_1^3 - \epsilon^{ij}\Phi_1 A_i B_j + \epsilon^{ji} B_j A_i \Phi_2 - \Phi_2^3).$$

Manifest global symmetry = SU(2) x $U(1)_{top}$ x $U(1)_R$

We want to confirm the symmetry enhancement by the monopole operators.

$$SU(2) \times U(1)_{top} \rightarrow SO(5)$$

On the gauge theory side, we define F_1 and F_2 as follows.

	Φ_1	Φ_2	A_1	A_2	B_1	B_2
Δ	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
F_1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
F_2	0	0	0	0	0	0

F_2 is a topological charge (monopole charge), which does not act on elementary fields.

SU(2) symmetry

F_1 is the Cartan generator of the manifest SU(2) global symmetry.
Due to the SU(2) symmetry, the spectrum is symmetric under $F_1 \to -F_1$.
This implies that the index satisfies

$$I(x,z_1,z_2)=I(x,1/z_1,z_2).$$

Charge conjugation

The charge conjugation flips the sign of the monopole charge F_2.
Due to the charge conjugation symmetry index satisfies

$$I(x,z_1,z_2)=I(x,z_1,1/z_2).$$

Weyl reflection

If the symmetry is enhanced to SO(5), the index should be invariant under Weyl reflections of SO(5). This require the index to satisfy

$$I(x,z_1,z_2)=I(x,z_2,z_1).$$

We want to confirm that the index actually has this symmetry.

Perturbative factor

$$I^{(0)} = 1 + 2x^{2/3} + (\chi_1(z_1) + 4)x^{4/3} + (6 + 2\chi_1(z_1))x^2 + \cdots ,$$

Where χ is the SU(2) character

$$\chi_s(z) = \frac{z^{s+1} - z^{-s}}{z-1} = z^s + z^{s-1} + \cdots + z^{-s}.$$

This does not include monopole contributions, and is independent of z_2.

Monopole contributions

U(N)$_1$	U(N)$_2$	
{1}	{1}	$x^{2/3}\chi_{\frac{1}{2}}(z_1)z_2^{1/2}+x^{4/3}\chi_{\frac{1}{2}}(z_1)z_2^{1/2}+x^2\chi_{\frac{3}{2}}(z_1)z_2^{1/2}+\cdots$
{2}	{2}	$x^{4/3}\chi_1(z_1)z_2+x^2(\chi_1(z_1)-1)z_2+\cdots$
{1,1}	{1,1}	$x^{4/3}\chi_1(z_1)z_2+x^2(\chi_1(z_1)+1)z_2+\cdots$
{3}	{3}	$x^2(\chi_{\frac{3}{2}}(z_1))z_2^{3/2}+\cdots$
{2,1}	{2,1}	$x^2(\chi_{\frac{1}{2}}(z_1)+\chi_{\frac{3}{2}}(z_1))z_2^{3/2}+\cdots$
+) {1,1,1}	{1,1,1}	$x^2(\chi_{\frac{3}{2}}(z_1))z_2^{3/2}+\cdots$

$$I^{(+)}=1+x^{2/3}\chi_{\frac{1}{2}}(z_1)z_2^{1/2}+x^{4/3}\left(\chi_{\frac{1}{2}}(z_1)z_2^{1/2}+2\chi_1(z_1)z_2\right)$$
$$+x^2\left(\chi_{\frac{3}{2}}(z_1)z_2^{1/2}+2\chi_1(z_1)z_2+\left(3\chi_{\frac{3}{2}}(z_1)+\chi_{\frac{1}{2}}(z_1)\right)z_2^{3/2}\right)+\cdots$$

The charge conjugation gives

$$I^{(-)}(x, z_1, z_2) = I^{(+)}(x, z_1, z_2^{-1}),$$

Combining all factors, we obtain the complete index

$$\begin{aligned}
I =& I^{(+)} I^{(0)} I^{(-)} \\
=& 1 + x^{2/3}\left(\chi_{\frac{1}{2}}(z_1)\chi_{\frac{1}{2}}(z_2) + 2\right) + x^{4/3}\left(2\chi_1(z_1)\chi_1(z_2) + 3\chi_{\frac{1}{2}}(z_1)\chi_{\frac{1}{2}}(z_2) + 5\right) \\
&+ x^2\Big(3\chi_{\frac{3}{2}}(z_1)\chi_{\frac{3}{2}}(z_2) + \chi_{\frac{3}{2}}(z_1)\chi_{\frac{1}{2}}(z_2) + \chi_{\frac{1}{2}}(z_1)\chi_{\frac{3}{2}}(z_2) \\
&\qquad + 6\chi_1(z_1)\chi_1(z_2) + 8\chi_{\frac{1}{2}}(z_1)\chi_{\frac{1}{2}}(z_2) + 10\Big) + \cdots
\end{aligned}$$

This is invariant under the exchange of z_1 and z_2.
This fact strongly suggests the symmetry enhancement to SO(5).

We also performed similar analysis for the following dual pairs.

Quiver CS theories		Dual manifolds	
G	Lagrangian symmetry	manifold	isometry
$U(N)^4$	SU(2)xU(1)xU(1)xU(1)	Q^{111}	SU(2)xSU(2)xSU(2)xU(1)
$U(N)^4$	SU(2)xSU(2)xU(1)xU(1)	Q^{222}	SU(2)xSU(2)xSU(2)xU(1)
$U(N)^3$	SU(3)xU(1)xU(1)	M^{111}	SU(3)xSU(2)xU(1)
$U(N)^2$	SU(2)xU(1)xSU(2)	N^{010}	SU(3)xSU(2)

In all cases, the symmetry of the index is consistent with the symmetry enhancement predicted by AdS/CFT.

Summary

We derived a general formula for the superconformal index.

By using it, we computed the index for several large N quiver Chern-Simons theories which are proposed as dual theories of M-theory in $AdS_4 x M_7$.

The obtained indices are consistent with the isometry of the internal space M_7.
(Index is invariant under the Weyl group of the isometry.)

Open questions

Comparison to the index computed on the gravity side.

S. Cheon, H. C. Kim, and S. Kim, arXiv:11011101[hep-th]

Analytic proof of the agreement for N(size of gauge group)$\geqq$2.

A technical difficulty for chiral theories. (large N limit)

Relations to 2d and 4d theories.

Y. I., arXiv:1104.4482

Gravity dual of the generalized index.

A. Kapustin, and B. Willett, arXiv:1106.2484