Noncommutative Geometry and Hirzebruch-Riemann-Roch Formula

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August 31, 2011

Quantum Mechanics vs. General Relativity

Important Problem

- Unify Gravity and quantum mechanics.
- Find a a common mathematical framework for quantum mechanics and general relativity.

NCG Approach

Translate the tools of Riemannian geometry into the Hilbert space formalism of Quantum Mechanics.

Diffeomorphism Invariant Geometry

Setup

- M = smooth manifold.
- Γ = group of diffeomorphisms acting on M.

Facts

1. *If Γ acts freely and properly, then M/Γ is a smooth manifold.*
2. *In general, M/Γ need not even be Hausdorff!!!*

Question

How do study the differential geometry of the action of Γ, when Γ is an arbitrary group of diffeomorphisms?

Crossed-Product Algebra

NCG Approach

Trade the "bad space" M/Γ for its algebra of smooth functions realized as the crossed-product algebra $C_c^\infty(M) \rtimes \Gamma$.

Definition

$$C_c^\infty(M) \rtimes \Gamma := \left\{ \text{finite sums} \sum_{\varphi \in \Gamma} f_\varphi U_\varphi; \ f_\varphi \in C_c^\infty(M) \right\},$$

where the U_φ, $\varphi \in \Gamma$, are formal symbols such that

$$U_\varphi^* = U_\varphi^{-1} = U_{\varphi^{-1}}, \qquad U_\varphi f = (f \circ \varphi^{-1}) U_\varphi.$$

Theorem (Green)

If Γ acts freely and properly, then $C_c^\infty(M/\Gamma) \simeq C_c^\infty(M) \rtimes \Gamma$.

Gel'fand Transform

Theorem (Gel'fand-Naimark)

Any C^*-algebra can be realized as a closed self-adjoint subalgebra of some $\mathcal{L}(\mathcal{H})$.

Theorem (Gel'fand-Naimark)

There is a one-to-one correspondence,

$$\begin{array}{ccc} \{\textit{Locally Compact Spaces}\} & \longleftrightarrow & \{\textit{Commutative } C^*\textit{-algebras}\} \\ X & \longrightarrow & C_0(X) \subset \mathcal{L}\left(L^2(X)\right) \end{array}.$$

Main Ideas and Motivations for NCG

- Use the duality between spaces and algebras to reformulate the main tools of differentiable noncommutative geometry in the Hilbert space formalism of quantum mechanics.
- In this setup the NC algebras at stake are algebras of smooth functions on (ghost) noncommutative manifolds.
- Obtain a framework that allows us to deal with a variety of geometric situations whose "noncommutative natures" prevent us from using classical differential geometry, e.g.,
  - Quantum space-time.
  - Diffeomorphism invariant geometry.

Main Trends in NCG

- Mathematical physics.
- (Higher) index theory.
- Riemann hypothesis.

The Dolbeault Complex

Setup

- (M^{2n}, ω) compact Kälher manifold.
- E (Hermitian) holomorphic vector bundle over M.

$$T_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M,$$

$$T_{1,0}M := \text{Span}\left\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right\}, \quad T_{0,1}M := \text{Span}\left\{\frac{\partial}{\partial \overline{z_1}}, \cdots, \frac{\partial}{\partial \overline{z_n}}\right\}$$

$$\Lambda^{p,q}T^*_{\mathbb{C}}M := \text{Span}\left\{dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z_{k_1}} \wedge \cdots \wedge d\overline{z_{k_q}}\right\}.$$

The Dolbeault Complex

Proposition (Dolbeault)

1. *There exists a unique complex of differential forms,*

$$\overline{\partial} : C^\infty(M, \Lambda^{0,\bullet} T^*_{\mathbb{C}} M) \to C^\infty(M, \Lambda^{0,\bullet+1} T^*_{\mathbb{C}} M), \quad \overline{\partial}^2 = 0,$$

   satisfying Leibniz's rule and such that

$$\overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j} \qquad \forall f \in C^\infty(M).$$

2. *There is a also a Dolbeault complex with coefficients in E,*

$$\overline{\partial}_E : C^\infty(M, \Lambda^{0,\bullet} T^*_{\mathbb{C}} M \otimes E) \to C^\infty(M, \Lambda^{0,\bullet+1} T^*_{\mathbb{C}} M \otimes E), \quad \overline{\partial}_E^2 = 0.$$

The Holomorphic Euler Characteristic

Definition

The cohomology of the Dolbeault complex is denoted $H^{0,\bullet}(M,E)$.

Proposition

1. $H^{0,\bullet}(M,E) \simeq \ker \square_{E|\Lambda^{0,\bullet}}$, *where* $\square_E := \overline{\partial}_E \overline{\partial}_E^* + \overline{\partial}_E^* \overline{\partial}_E$*.*
2. $\dim H^{0,\bullet}(M,E) < \infty$.

Definition

The holomorphic Euler characteristic of E is

$$\chi(M,E) := \sum_{0 \leq q \leq n} (-1)^q \dim H^{0,q}(M,E),$$

Remark

This is an invariant of the holomorphic structures of M and E.

The Hirzebruch-Riemann-Roch Formula

Theorem (Hirzebruch-Riemann-Roch Formula)

We have

$$\chi(M,E) = \int_M \mathrm{Td}\left(R^+\right) \wedge \mathrm{Ch}(F^E),$$

where $\mathrm{Td}\left(R^+\right)$ *and* $\mathrm{Ch}(F^E)$ *are characteristic forms associated to the respective curvatures of* $T_{1,0}M$ *and* E*.*

Sketch of Proof

General arguments, partially involving supersymmetry, show that

$$\begin{aligned}\chi(M,E) &= \dim\ker\left(\overline{\partial}_E + \overline{\partial}_E^*\right)_{|\Lambda^{0,\mathrm{ev}}\otimes E} - \dim\ker\left(\overline{\partial}_E + \overline{\partial}_E^*\right)_{|\Lambda^{0,\mathrm{odd}}\otimes E} \\ &\stackrel{\mathrm{def}}{=} \operatorname{ind}\left(\overline{\partial}_E + \overline{\partial}_E^*\right) \\ &= \operatorname{Tr}\left[(-1)^q e^{-t\Box_E}\right] \qquad \forall t > 0 \\ &= \int_M \operatorname*{FP}_{t\to 0^+} \operatorname{Tr}\left[(-1)^q e^{-t\Box_E}(x,x)\right].\end{aligned}$$

The proof is then completed by using:

Theorem (Atiyah-Bott-Patodi, Gilkey)

$$\operatorname{Tr}\left[(-1)^q e^{-t\Box_E}(x,x)\right] \xrightarrow[t\to 0^+]{} \operatorname{Td}(R^+) \wedge \operatorname{Ch}(F^E).$$

NC Manifolds

Definition

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$, where

- $\mathcal{H}$ is a *super* Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
- $\mathcal{A}$ is an (even) algebra represented by *bounded* operators on $\mathcal{H}$.
- D is a selfadjoint (unbounded) operator such that:
    - D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    - $[D, a]$ is *bounded* for all $a \in \mathcal{A}$.
    - $a(D + i)^{-1}$ is *compact* for all $a \in \mathcal{A}$.

Examples

Example (Dolbeault Spectral Triple)

Let M be a compact Kälher manifold. Then the following is a spectral triple,

$$\left(C^\infty(M), L^2\left(M, \Lambda^{0,\bullet} T^*M\right), \overline{\partial} + \overline{\partial}^*\right),$$

with $L^2\left(M, \Lambda^{0,\bullet} T^*M\right) = L^2\left(M, \Lambda^{0,\mathrm{ev}} T^*M\right) \oplus L^2\left(M, \Lambda^{0,\mathrm{odd}} T^*M\right)$.

Example (Dirac Spectral Triple)

Let M^{2n} be a compact Riemannian *spin* manifold with *spinor bundle* $\$ = \$^+ \oplus \$^-$ and *Dirac operator* $D\!\!\!/$. Then the following is a spectral triple,

$$\left(C^\infty(M), L^2\left(M, \$\right), D\!\!\!/\right),$$

with $L^2(M, \$) = L^2(M, \$^+) \oplus L^2(M, \$^-)$.

Examples from Physics

Standard Model w/ Gravity (Chamseddine-Connes-Marcolli)

The spectral is obtained as a product of the Dirac spectral triple (of dimension 4) and a finite spectral triple,

$$(\mathcal{A}_F, \mathcal{H}_F, D_F),$$

where:

- $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.
- $\mathcal{H}_F$ is a finite dimensional representation of $\mathcal{A}_F$.
- D_F is a matrix whose entries are given by Yukawa parameters.

Loop Quantum Gravity (Aastrup-Grimstrup-Nest)

The spectral triple is obtained as a limit of Dirac spectral triples of dimension $N \uparrow \infty$.

Spectral Triples

Definition

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$, where

- $\mathcal{H}$ is a *super* Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
- $\mathcal{A}$ is an even algebra represented by *bounded* operators on $\mathcal{H}$.
- D is a selfadjoint (unbounded) operator such that:
  - D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
  - $[D, a]$ is *bounded* for all $a \in \mathcal{A}$.
  - $a(D + i)^{-1}$ is *compact* for all $a \in \mathcal{A}$.

Projective Modules

Definition

A *finitely generated projective module* over an algebra $\mathcal{A}$ is a (right-)module of the form,

$$\mathcal{E} = e\mathcal{A}^N, \qquad e \in M_N(\mathcal{A}),\ e^2 = e.$$

Theorem (Serre-Swan)

For $\mathcal{A} = C^\infty(M)$ (with M compact manifold), there is a one-to-one correspondence:

$$\begin{array}{rcl} \{\textit{Vector Bundles over } M\} & \longleftrightarrow & \{\textit{f.g. proj. modules over } C^\infty(M)\} \\ E & \longrightarrow & C^\infty(M, E). \end{array}$$

Index Problem

Setup

- $(\mathcal{A}, \mathcal{H}, D)$ spectral triple, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
- $\mathcal{E} = e\mathcal{A}^N$, $e^2 = e \in M_N(\mathcal{A})$, f.g. projective module over $\mathcal{A}$.

Lemma

The following operator is Fredholm,

$$D_{\mathcal{E}} := e(D \otimes I_N)e : e(\mathcal{H}^+ \otimes \mathbb{C}^N) \longrightarrow e(\mathcal{H}^- \otimes \mathbb{C}^N).$$

Index Problem

Compute ind $D_{\mathcal{E}}$ for any f.g. projective module $\mathcal{E}$ over $\mathcal{A}$.

Example

For $\mathcal{E} = C^\infty(M, E)$ (with E holomorphic vector bundle over M),

$$\text{ind}\left(\overline{\partial} + \overline{\partial}^*\right)_{\mathcal{E}} = \text{ind}\left(\overline{\partial}_E + \overline{\partial}_E^*\right) = \chi(M, E).$$

Cyclic Cohomology

Setup

- $\mathcal{A}$ = unital algebra over $\mathbb{C}$.
- $C^n(\mathcal{A}) = \{(n+1)\text{-linear forms } \varphi : \mathcal{A}^{n+1} \to \mathbb{C}\}$, $n \geq 0$.
- $C^{\mathrm{ev}}(\mathcal{A}) = \oplus_{k\geq 0} C^{2k}(\mathcal{A})$, $C^{\mathrm{odd}}(\mathcal{A}) := \oplus_{k\geq 0} C^{2k+1}(\mathcal{A})$.

Theorem (Connes, Tsygan)

There is a periodic complex,

$$C^{\mathrm{ev}}(\mathcal{A}) \overset{\partial}{\leftrightarrows} C^{\mathrm{odd}}(\mathcal{A}), \qquad \partial^2 = 0,$$
$$\partial C^n(\mathcal{A}) \subset C^{n-1}(\mathcal{A}) \oplus C^{n+1}(\mathcal{A}).$$

Definition

The cohomology of the above complex is called the *(periodic) cyclic cohomology* of $\mathcal{A}$ and is denoted $HC^{\mathrm{ev/odd}}(\mathcal{A})$.

Example

For $\mathcal{A} = C^\infty(M)$, an example of cyclic cocycle is

$$\varphi_{\mathrm{Td}} = (\varphi_0, \varphi_2, \cdots),$$
$$\varphi_{2k}(f^0, f^1, \cdots, f^{2k}) := \frac{1}{(2k)!}\int_M \mathrm{Td}(R^+) \wedge f^0 df^1 \wedge \cdots \wedge f^{2k}.$$

More generally, any even (resp., odd) dimensional closed current defines a cyclic cocycle. In fact, we have

Theorem (Connes)

There are isomorphisms,

$$HC^{\mathrm{ev}/odd}(C^\infty(M)) \simeq H_{\mathrm{ev/odd}}(M, \mathbb{C}),$$

where $H_{\mathrm{ev/odd}}(M, \mathbb{C})$ is the de Rham homology of M.

Pairing with Cyclic Cocyles

Let $\mathcal{M}(\mathcal{A})$ be the class of all f.g. projective modules over $\mathcal{A}$.

Remark

$\mathcal{M}(\mathcal{A})$ is a monoid with respect to the direct sum of modules.

Theorem (Connes)

There is a natural pairing,

$$\langle \cdot, \cdot \rangle : HC^{\mathrm{ev}}(\mathcal{A}) \times \mathcal{M}(\mathcal{A}) \to \mathbb{C}.$$

Example

For $\mathcal{A} = C^\infty(M)$ and $\mathcal{E} = C^\infty(M, E)$,

$$\begin{aligned} \langle \varphi_{\mathrm{Td}}, \mathcal{E} \rangle &= \int_M \mathrm{Td}\left(R^+\right) \wedge \mathrm{Ch}\left(F^E\right) \\ &= \chi(M, E) = \mathrm{ind}\left(\overline{\partial}_E + \overline{\partial}_E^*\right). \end{aligned}$$

The Connes-Chern Character

Setup

1. $(\mathcal{A}, \mathcal{H}, D)$ = spectral triple.
2. There exists $p \geq 1$ such that $\text{Trace}\,|D|^{-p} < \infty$.

Theorem (Connes)

There exists a class $\text{Ch}(\mathcal{A}, D) \in HC^{\text{ev}}(\mathcal{A})$ *such that, for any f.g. projective module* $\mathcal{E}$ *over* $\mathcal{A}$*,*

$$\text{ind}\, D_{\mathcal{E}} = \langle \text{Ch}(\mathcal{A}, D), \mathcal{E} \rangle.$$

Definition

$\text{Ch}(\mathcal{A}, D)$ is called the *Connes-Chern character* of $(\mathcal{A}, \mathcal{H}, D)$.

Remark

The cocycle used by Connes in his original definition of $\text{Ch}(\mathcal{A}, D)$ is difficult to compute in practice, so we need a nicer cocycle.

The CM Cocycle

Theorem (Connes-Moscovici '95)

1. *Under suitable assumptions, the Connes-Chern character is represented by the CM cocycle $\varphi^{\mathrm{CM}} = (\varphi^{\mathrm{CM}}_{2k})$ given by*

$$\varphi^{\mathrm{CM}}_{2k}(a^0, \cdots, a^{2k}) = \sum c_{k,\alpha} ⨍ a^0 [D, a^1]^{[\alpha_1]} \cdots [D, a^{2k}]^{[\alpha_{2k}]} |D|^{-2(|\alpha|+k)},$$

where the $c_{k,\alpha}$ are universal constants, and

$$⨍ T := \mathrm{Res}_{z=0} \, \mathrm{Str} \left[T|D|^{-z} \right], \quad T^{[j]} := \overbrace{[D^2, [D^2, \cdots [D^2, T] \cdots]]}^{j \text{ times}}.$$

2. *For any f.g. projective module $\mathcal{E}$,*

$$\mathrm{ind}\, D_{\mathcal{E}} = \langle \varphi^{\mathrm{CM}}, \mathcal{E} \rangle.$$

Example

For $\left(C^\infty(M), L^2\left(M, \Lambda^{0,*}T^*M\right), \overline{\partial} + \overline{\partial}^*\right)$, the CM cocycle is $\varphi^{\mathrm{CM}} = (\varphi_{2k})_{k\geq 0}$, where

$$\varphi_{2k}(f^0, \cdots, f^{2k}) = \frac{1}{(2k)!}\int_M \mathrm{Td}(R^+) \wedge f^0 df^1 \wedge \cdots \wedge df^{2k}.$$

This allows us to recover the Hirzebruch-Riemann-Roch formula.

Summary

Classical	**NCG**
Manifold M	Spectral triple $(\mathcal{A}, \mathcal{H}, D)$
Vector bundles over M	F.g. projective modules over $\mathcal{A}$
$\text{ind}\left(\overline{\partial}_E + \overline{\partial}_E^*\right)$	$\text{ind}\, D_{\mathcal{E}}$
Differential forms	Cyclic cocycles
Atiyah-Singer Index Formula	Connes-Chern character & CM cocycle
Characteristic classes	Cyclic cohomology for Hopf algebras

CR Manifolds

Definition

A Cauchy-Riemann structure (or CR structure) on an oriented manifold M^{2n+1} is given by the data of:

1. A hyperplane bundle $H \subset TM$.
2. An integrable complex structure J on H, i.e., a section J of $\operatorname{End}_{\mathbb{R}} H$ such that

$$J^2 = -1,$$
$$[T_{1,0}, T_{1,0}] \subset T_{1,0}, \quad T_{1,0} := \ker(J - i) \subset T_{\mathbb{C}} M.$$

Remark

$T_{1,0}$ is the analogue of the holomorphic tangent bundle $T_{1,0}M$ of a complex manifold. It is called the *CR tangent bundle* of M.

Example

Let D be a domain in $\mathbb{C}^{n+1}$. Then $M = \partial D$ carries the CR structure defined by

- $H = T(\partial D) \cap iT(\partial D)$
- $J =$ multiplication by i.

In particular the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ is a CR manifold.

Example

Let (E, h) be a Hermitian line bundle over a complex manifold X. Then the circle bundle,

$$M := \{\xi \in E; h(\xi, \xi) = 1\}$$

carries a natural CR structure which is isomorphic to the complex structure of X.

The Kohn-Rossi Cohomology

Setup

- M^{2n+1} (compact) CR manifold.
- $H \subset TM$ complex hyperplane bundle of M with complex structure J.
- $T_{1,0} := \ker(J - i) \subset T_{\mathbb{C}}M$ CR tangent bundle of M.

Define

$$\Lambda^{0,1} := \left\{\xi \in H^* \otimes \mathbb{C};\ \xi_{|T_{1,0}} = 0\right\},$$

$$\Lambda^{0,q} := \left(\Lambda^{0,1}\right)^q = \left\{\xi_1 \wedge \cdots \wedge \xi_q;\ \xi_j \in \Lambda^{0,q}\right\}.$$

The Kohn-Rossi Cohomology

Proposition (Kohn-Rossi)

There is a complex,

$$\overline{\partial}_b : C^\infty(M, \Lambda^{0,\bullet}) \longrightarrow C^\infty(M, \Lambda^{0,\bullet+1}), \qquad \overline{\partial}_b^2 = 0.$$

Definition

The cohomology of the above complex is called the *Kohn-Rossi cohomology* and is denoted $H_b^{0,\bullet}(M)$.

Example

If $M = \partial D$, then

$$\overline{\partial}_b(u_{|\partial D}) = (\overline{\partial} u)_{|\partial D}.$$

In particular,

$$H_b^{0,0}(\partial D) \simeq \left\{ f_{|\partial D};\ f \in \mathrm{Hol}(D) \right\}.$$

The CR Euler Characteristic

Proposition (Kohn-Rossi, Kohn)

1. $H_b^{0,q}(M) \simeq \ker \Box_{b|\Lambda^{0,q}}$, *where* $\Box_b := \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$.
2. $\dim H_b^{0,q}(M) < \infty$ *for* $1 \leq q \leq n-1$.
3. $\dim H_b^{0,0}(M) = \dim H_b^{0,n}(M) = \infty$.

Definition

The *CR Euler characteristic* of M is

$$\chi_b(M) := \sum_{1 \leq q \leq n-1} (-1)^q \dim H_b^{0,q}(M).$$

Open Problem

Find a geometric expression for $\chi_b(M)$, i.e., reformulate the Hirzebruch-Riemann-Roch formula in CR geometry.

CR Diffeomorphisms

Definition

A *CR diffeomorphism* of M is a diffeomorphism ϕ such that $\phi^* H = H$ and $\phi^* J = J$.

Theorem (Schoen '95)

If M is not CR equivalent to the sphere S^{2n+1} or the Heisenberg group $\mathbb{H}^{2n+1}$, then the group of CR diffeomorphisms is compact for the compact-open topology.

CR-Diffeomorphism Invariant Geometry

Setup

- M^{2n+1} compact (oriented) strictly pseudoconvex CR manifold.
- $\Gamma =$ group of all orientation-preserving CR diffeomorphisms.

Project

Reformulate the Hirzebruch-Riemman-Roch formula in CR-diffeomorphism invariant geometry.

Remark

This amounts to

1. Construct out of $C^\infty(M) \rtimes \Gamma$ and the $\overline{\partial}_b$-complex a spectral triple representing the action of Γ on M.
2. Find a geometric expression for its CM cocycle.
3. Obtain as byproduct a geometric expression for the CR Euler characteristic $\chi_b(M)$.

Biholomorphism Invariant Geometry

Fundamental Problem

Find biholomorphism invariants of strictly pseudoconvex domains in $\mathbb{C}^{n+1}$ or a Stein manifold.

Theorem (Fefferman 70s)

Let $D \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain with boundary ∂D. Then there is a one-to-one correspondence,

$$\left\{ \begin{array}{c} \textit{Biholomorphisms} \\ F : D \to D \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{CR diffeomorphisms} \\ f : \partial D \to \partial D \end{array} \right\}.$$

Consequence

Biholomorphism invariant geometry of D

$$\updownarrow$$

CR-diffeomorphism invariant geometry of ∂D.

Characterization of Strictly Pseudoconvex Boundaries

Open Problem

Determine under which geometric conditions a CR manifold can be realized as the boundary of a strictly pseudoconvex complex domain.

Theorem (Harvey-Lawson, Yau)

1. *A strictly pseudoconvex CR manifold can be realized as the boundary of a strictly pseudoconvex domain with some singular points in the interior.*
2. *The singularities are detected by the $\overline{\partial}_b$-cohomology.*

Consequence

A geometric expression for $\chi_b(M)$ should provide us with a geometric obstruction to being the boundary of a sψc domain.

Spectral Triple of CR-Diffeomorphism Invariant Geometry

First Step

Construct out of $C^\infty(M) \rtimes \Gamma$ and the $\overline{\partial}_b$-complex a spectral triple representing the action of Γ on M.

Assumption

M is not CR equivalent to S^{2n+1} or $\mathbb{H}^{2n+1}$.

Consequence

M carries a Γ-invariant (Levi) metric, i.e., Γ acts isometrically on M with respect to this metric.

Remark

The natural candidate for the spectral triple is

$$\left(C^\infty(M) \rtimes \Gamma, L^2(M, \Lambda^{0,\bullet}), \overline{\partial}_b + \overline{\partial}_b^*\right).$$

However, this is NOT a spectral triple!!!!

Main Obstruction

The operator $\overline{\partial}_b + \overline{\partial}_b^*$ has an infinite dimensional kernel on functions and $(0, n)$-forms, and so $(\overline{\partial}_b + \overline{\partial}_b^* + i)^{-1}$ is not compact.

Szegö Projection

Definition

Let $S : L^2(M, \Lambda^{0,\bullet}) \to L^2(M, \Lambda^{0,\bullet})$ be the orthoprojection onto

$$\left(\ker \overline{\partial}_b \cap L^2(M)\right) \oplus \left(\ker \overline{\partial}_b^* \cap L^2(M, \Lambda^{0,n})\right).$$

Remark

The projection $1 - S$ kills the kernel of $\overline{\partial}_b + \overline{\partial}_b^*$ on functions and $(0, n)$-forms, but it is the identity on other $(0, q)$-forms.

The algebra $C^\infty(M) \rtimes \Gamma$

As Γ acts isometrically on M we get:

Lemma

Γ has a unitary representation $\varphi \to U_\varphi$ in $L^2(M, \Lambda^{0,\bullet})$ such that

$$U_\varphi v(x) := \varphi^* v(x) \qquad \forall v \in L^2(M, \Lambda^{0,\bullet}),$$
$$U_\varphi^* = U_{\varphi^{-1}} = U_\varphi^{-1}, \qquad U_\varphi(fv) = (f \circ \varphi^{-1}) U_\varphi v.$$

Proposition

$C^\infty(M) \rtimes \Gamma$ can be realized as the algebra,

$$\left\{ \text{finite sums} \sum_{\varphi \in \Gamma} f_\varphi U_\varphi; \; f_\varphi \in C^\infty(M) \right\} \subset \mathcal{L}\left(L^2(M, \Lambda^{0,\bullet})\right),$$

where the f_φ act as multiplication operators on $L^2(M, \Lambda^{0,\bullet})$.

Lemma

The projection S is a Γ-invariant operator, i.e.,

$$U_\varphi^* S U_\varphi = S \qquad \forall \varphi \in \Gamma.$$

Definition

$$\begin{aligned}\mathcal{A}_\Gamma :=&(1-S)\left(C^\infty(M) \rtimes \Gamma\right)(1-S)\\ =&\left\{\text{finite sums } \sum_{\varphi\in\Gamma}(1-S)f_\varphi U_\varphi(1-S);\ f_\varphi \in C^\infty(M)\right\}.\end{aligned}$$

Theorem (RP)

1. *The following is a spectral triple,*

$$\left(\mathcal{A}_\Gamma, L^2\left(M, \Lambda^{0,\bullet}\right), \overline{\partial}_b + \overline{\partial}_b^*\right),$$

with $L^2\left(M, \Lambda^{0,\bullet}\right) = L^2\left(M, \Lambda^{0,\mathrm{ev}}\right) \oplus L^2\left(M, \Lambda^{0,\mathrm{odd}}\right)$.

2. $\mathrm{Trace}\left(\left|\overline{\partial}_b + \overline{\partial}_b^*\right|^{-p}\right) < \infty$ *for all* $p > 2n + 2$ *(and hence the Connes-Chern character makes sense).*

Example

$(1-S)$ is an idempotent in $\mathcal{A}_\Gamma$, and

$$\begin{aligned}
\operatorname{ind}_{\overline{\partial}_b+\overline{\partial}_b^*}[1-S] &= \operatorname{ind}(1-S)(\overline{\partial}_b+\overline{\partial}_b^*)(1-S) \\
&= \sum_{1\leq q\leq n-1} (-1)^q \dim\ker(\overline{\partial}_b+\overline{\partial}_b^*)_{|\Lambda^{0,q}} \\
&= \sum_{1\leq q\leq n-1} (-1)^q \dim H_b^{0,q}(M) \\
&= \chi_b(M).
\end{aligned}$$

The CM Cocycle of CR-Diffeo. Invariant Geometry

Second Step

Find a geometric expression for the CM cocycle.

Important Obstacle

One assumption for the existence of CM cocycle fails, and so the CM cocycle doe NOT make sense!!!

Solution

Use the JLO coycle of Jaffe-Lesniewski-Osterwalder.

The JLO Cocycle

Setup

$(\mathcal{A}, \mathcal{H}, D)$ = spectral triple with Trace $|D|^{-p} < \infty$ for some $p \geq 1$.

Theorem (Connes '88)

The Connes-Chern character $\mathrm{Ch}(\mathcal{A}, D)$ *is represented in entire cyclic cohomology by the JLO cocycle(s)* $\varphi_t^{\mathrm{JLO}} = \left(\varphi_{t,2k}^{\mathrm{JLO}}\right)_{k \geq 0}$, $t > 0$, *defined by*

$$\varphi_{t,2k}^{\mathrm{JLO}}(a^0, \cdots, a^{2k}) =$$

$$t^k \int_{\Delta_{2k}} \mathrm{Str}\left\{a^0 e^{-ts_0 D^2}[D, a^1] e^{-ts_1 D^2} \cdots [D, a^{2k}] e^{-ts_{2k} D^2}\right\} ds, \ \ a^j \in \mathcal{A},$$

where $\mathrm{Str} = \mathrm{Tr}_{\mathcal{H}^+} - \mathrm{Tr}_{\mathcal{H}^-}$ *(with* $\mathcal{H} = \mathcal{H}^+ - \mathcal{H}^-$*), and*

$$\Delta_{2k} := \{(s_0, \cdots, s_{2k}) \in \mathbb{R}^{2k+1};\ s_0 + \cdots s_{2k} = 1,\ s_j \geq 0\}.$$

Finite Part of the JLO Cocycle

Theorem (Connes-Moscovici '93)

Assume that, as $t \to 0^+$,

$$\varphi_{t,2k}^{\mathrm{JLO}} = \sum_{\substack{\alpha, l \geq 0 \\ \alpha + l > 0}} t^{-\alpha} (\log^l t) \varphi_{2k}^{(\alpha,l)} + \varphi_{2k}^{(0,0)} + \mathrm{o}(t),$$

where the $\varphi_k^{(\alpha,l)}$ are $2k$-cochains. Then the Connes-Chern character is represented in periodic cyclic cohomology by

$$\mathrm{FP}_{t \to 0^+} \varphi_t^{\mathrm{JLO}} := \left(\varphi_{2k}^{(0,0)} \right)_{k \geq 0}.$$

The JLO Cocycle in CR-Diffeomorphism Geometry

Proposition

For the spectral triple $\left(\mathcal{A}_\Gamma, L^2\left(M, \Lambda^{0,\bullet}\right), \overline{\partial}_b + \overline{\partial}_b^*\right)$:

1. *As* $t \to 0^+$,

$$\varphi^{\mathrm{JLO}}_{t,2k} \sim t^{-(n+k+1)} \sum_{j \geq 0} t^{\frac{j}{2}} \varphi^{(j)}_{2k} + \log t \sum_{l \geq 0} t^l \psi^{(l)}_{2k}.$$

2. *The Connes-Chern character is represented in (periodic) cyclic cohomology by* $\mathrm{FP}_{t \to 0^+} \varphi^{\mathrm{JLO}}_t$.
3. *For any f.g. projective module* $\mathcal{E}$ *over* $\mathcal{A}_\Gamma$,

$$\mathrm{ind}(\overline{\partial}_b + \overline{\partial}_b^*)_{\mathcal{E}} = \langle \mathrm{FP}_{t \to 0^+} \varphi^{\mathrm{JLO}}_t, \mathcal{E} \rangle.$$

The reformulation of the Hirzebruch-Riemann-Roch formula in CR-diffeomorphism invariant geometry boils down to

Project

Find a geometric expression for

$$\underset{t\to 0^+}{\mathrm{FP}}\, t^k \,\mathrm{Str}\left[\left(\int_{\Delta_{2k}} (1-S) f^0 e^{-ts_0\Box_b}[\overline{\partial}_b+\overline{\partial}_b^*, f^1]e^{-ts_1\Box_b}\cdots \right.\right.$$
$$\left.\left.\cdots [\overline{\partial}_b+\overline{\partial}_b^*, f^{2k}]e^{-ts_{2k}\Box_b}U_\varphi ds\right)(x,x)\right],$$

for φ in Γ and $f^0,\cdots,f^{2k}$ in $C^\infty(M)$.

Remark

The asymptotics (and hence its finite part) localizes along the fixed-point set of the diffeomorphism φ.

PSU$(n+1,1)$-Invariant Geometry of S^{2n+1}

Fact

If $M = S^{2n+1}$, then $\Gamma \simeq \mathrm{PSU}(n+1,1)$.

Claim

There is a (twisted) spectral triple representing the $\mathrm{PSU}(n+1,1)$ invariant geometry of S^{2n+1}.

Contact Manifolds

Definition

A *contact structure* on M^{2n+1} is given by the datum of a hyperplane bundle $H \subset TM$ such that $H = \ker \theta$, where θ is a contact form, i.e., $d\theta_{|H}$ is non-degenerate everywhere.

Definition

A *contactomorphism* of a contact manifold (M, H) is a diffeomorphism $\phi : M \to M$ preserving the contact structure, i.e., $\phi^* H = H$.

Contactomorphism Invariant Geometry

Setup

- (M^{2n+1}, H) = oriented contact manifold.
- Γ = group of orientation-preserving contactomorphisms of M.

Claim

There is a spectral triple representing the contactmorphism invariant geometry of M. It uses:

1. *A crossed-product algebra $C_c^\infty(P) \rtimes \Gamma$, where P is a bundle of metrics associated to the contact structure of M.*
2. *A new geometric (hypoelliptic) operator built out of Rumin's contact complex.*

Remark

1. As in the CR case, the CM cocycle does NOT make sense.
2. The finite part of the JLO cocycle does make sense, but it CANNOT be computed explicitly, because Γ is too big.
3. It is expected to make use of Hopf cyclic cohomology to show that the Connes-Chern character can be expressed in terms of a universal Gel'fand-Fuks cohomology class depending only on the dimension of M.