Holographic Entanglement Entropy

(Introduction)

(with H. Casini, M. Huerta, J. Hung, A. Sinha, M. Smolkin & A. Yale)

1. **Entanglement Entropy**
   - condensed matter
   - quantum information
   - black hole microphysics

2. **AdS/CFT correspondence**
   (gauge/gravity duality)
   - string theory
   - quantum gravity

3. **Holographic Entanglement Entropy**
   - proposal by Ryu & Takayanagi (2006)

4. **Two Recent Developments:**
   - precise connection between EE and central charges
   - derivation of holographic EE for special geometries

5. **Summary:**
   - holographic EE provides framework where we can learn about properties of both EE and quantum gravity
**Entanglement Entropy**

- What is entanglement entropy?
  
  A very general tool; divide a quantum system into two parts and use entropy as a measure of correlations between subsystems.

- In QFT, typically introduce a (smooth) boundary or entangling surface $\Sigma$ which divides the space into two separate regions.

- Integrate out degrees of freedom in the “outside” region.

- Remaining degrees of freedom are described by a density matrix $\rho_A$.

  
  → calculate von Neumann entropy: $S_{EE} = -Tr[\rho_A \log \rho_A]$
Entanglement Entropy

- remaining dof are described by a density matrix $\rho_A$

\[ S_{EE} = -Tr[\rho_A \log \rho_A] \]  

- result is UV divergent!
- must regulate calculation: $\delta = \text{short-distance cut-off}$

\[ S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots \]

- $d = \text{spacetime dimension}$

- careful analysis reveals geometric structure, eg, $S = \tilde{c}_0 \frac{A_\Sigma}{\delta^{d-2}} + \cdots$
Entanglement Entropy

- remaining dof are described by a density matrix $\rho_A$

  - calculate von Neumann entropy: $S_{EE} = -Tr[\rho_A \log \rho_A]$

  - must regulate calculation: $\delta =$ short-distance cut-off

  \[
  S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots
  \]

  - leading coefficients sensitive to details of regulator, eg, $\delta \to 2\delta$

  - find universal information characterizing underlying QFT in subleading terms, eg,

  \[
  S = \cdots + c_d \log \left( \frac{R}{\delta} \right) + \cdots
  \]
More general comments on **Entanglement Entropy**:

- nonlocal quantity which is (at best) very difficult to measure
  - no accepted experimental procedure

- in condensed matter theory: diagnostic to characterize quantum critical points or topological phases (eg, quantum hall fluids)

- in quantum information theory: useful measure of quantum entanglement (a computational resource)

- **black hole microphysics**: leading term obeys “area law” $S \approx c_0 \frac{A_{\Sigma}}{\delta^{d-2}}$
  - suggested as origin of black hole entropy (eg, $\delta \approx \ell_P$)

  (Bombelli, Koul, Lee & Sorkin `86; Srednicki; Frolov & Novikov; Callan & Wilczek; Susskind; . . . . )

- recently considered in **AdS/CFT correspondence**

  (Ryu & Takayanagi `06)
AdS/CFT correspondence:

anti-de Sitter space \[\text{quantum gravity}\]
- negative cosmological constant
- d+1 spacetime dimensions

conformal field theory \[\text{quantum field theory}\]
- no scale (at quantum level)
- d spacetime dimensions

holography

Favorite example:

Type IIb superstrings on \(\text{AdS}_5 \times S^5\)
with RR flux \(N_c\)

\((3+1)\)-dimensional \(\mathcal{N}=4\) \(\text{SU}(N_c)\) super-Yang-Mills

(Maldacena `97)
AdS/CFT correspondence:

anti-de Sitter space ↔ conformal field theory

quantum gravity
• negative cosmological constant
• d+1 spacetime dimensions

quantum field theory
• no scale (at quantum level)
• d spacetime dimensions

holography

classical gravity with small curvatures → large central charge \((N_c \rightarrow \infty)\) → strong coupling \((\lambda \rightarrow \infty)\)
anti-de Sitter space:
maximally symmetric geometry with negative curvature

\[ R \sim - \frac{1}{L^2} \]

(simplest) solution of Einstein’s equations with negative \( \Lambda \):

\[ R_{ab} = - \frac{d}{L^2} \]

\[ ds^2 = \frac{r^2}{L^2} (-dt^2 + d\vec{x}^2) + \frac{L^2}{r^2} dr^2 \]
anti-de Sitter space:
maximally symmetric geometry with negative curvature

Boundary (d dimensions)

Bulk (d+1 dimensions)

Quantum Gravity

CFT

\[ ds^2 = \frac{r^2}{L^2} (-dt^2 + d\bar{x}^2) + \frac{L^2}{r^2} dr^2 \]
**anti-de Sitter space:**

maximally symmetric geometry with negative curvature

\[ R \sim -\frac{1}{L^2} \]

\[ r = \infty \]

\[ E \sim r/L^2 \]

"redshift": proper distances get smaller for small \( r \)

\[ ds^2 = \frac{r^2}{L^2} \left( -dt^2 + d\vec{x}^2 \right) + \frac{L^2}{r^2} dr^2 \]

boundary/CFT metric
anti-de Sitter space:
maximally symmetric geometry with negative curvature

$R \sim -\frac{1}{L^2}$

Boundary (d dimensions)

$r = \infty$

$E \sim \frac{r}{L^2}$

$T_{Hawking} = \frac{r_H}{\pi L^2} = T_{CFT}$

AdS Black Hole = CFT Thermal State
anti-de Sitter space:
maximally symmetric geometry with negative curvature

\[ R \sim -\frac{1}{L^2} \]

Boundary
(d dimensions)

Horizon
\[ r = \infty \]

\[ E \sim \frac{r}{L^2} \]

Hawking temperature
\[ T_{\text{Hawking}} = \frac{r_H}{\pi L^2} = T_{\text{CFT}} \]

Horizon Entropy = Entropy of Thermal State
\[ S = \frac{A_{\text{horizon}}}{4G_N} \]
Holographic Entanglement Entropy:

(Ryu & Takayanagi `06)

$S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N}$

looks like BH entropy!

AdS boundary

AdS bulk spacetime

$\Sigma$

$A$

$B$

$V$

$r = \infty$

conformal field theory

gravitational potential/redshift
Holographic Entanglement Entropy:

\[ S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N} \]

**Conjecture**

Extensive consistency tests:

1) leading contribution yields “area law”

\[ S = \tilde{c}_0 \frac{A_{\Sigma}}{\delta^{d-2}} + \cdots \]

\[ \cdots \text{(more tests)} \cdots \]

6) for general even \( d \), connection to central charges of CFT

(Hung, RCM & Smolkin, arXiv:1101.5813)

7) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RCM, arXiv:1102.044)

(see also: RCM & Sinha, arXiv:1011.5819)
6) for general even d, connection to central charges of CFT

(Hung, RCM & Smolkin, arXiv:1101.5813)

- trace anomaly in CFT (with even d) defines central charges

\[ \langle T_{\mu}^{\mu} \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 \]

\[ I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \]

- universal/logarithmic contribution to entanglement entropy determined by central charges using trace anomaly, eg,

\[ S_{uni} = \log(R/\delta) \frac{1}{2\pi} \int_{\Sigma} d^2x \sqrt{\hbar} \left[ c \left( C^{ijkl} \tilde{g}_{ik} \tilde{g}_{jl} - K_{a}^{i} b K_{b}^{j} a + \frac{1}{2} K_{a}^{i} a K_{b}^{j} b \right) - a \mathcal{R} \right] \]

(Solodukhin)

- R&T proposal for holographic EE reproduce precisely this result

- extends to certain higher curvature theories (eg, GB gravity)

\[ S = \min_{\partial V = \Sigma} \frac{2\pi}{\ell_p^3} \int_{V} d^3x \sqrt{\hbar} \left[ 1 + \lambda L^2 \mathcal{R} \right] \]
7) derivation of holographic EE for spherical entangling surfaces

   (Casini, Huerta & RCM, arXiv:1102.044)
   (see also: RCM & Sinha, arXiv:1011.5819)

- holographic translation for standard calculation of EE is difficult
- new calculation for special case:

  CFT in d-dim. flat space and choose $\Sigma = S^{d-2}$ with radius $R$

  by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $R \sim 1/R^2$ and $T=1/2\pi R$

  $S_{EE} = S_{thermal}$
7) derivation of holographic EE for spherical entangling surfaces

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• holographic translation for standard calculation of EE is difficult
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$$S_{EE} = S_{thermal} = S_{horizon}$$

thermal bath in CFT = black hole in AdS

• can calculate holographic EE for any bulk gravity theory

universal contributions:

$$S = \cdots + (-)^{d-1}4 a^*_d \log (2R/\delta) + \cdots \text{ for even } d$$

$$\cdots + (-)^{d-1} 2\pi a^*_d + \cdots \text{ for odd } d$$
Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories.

- Holographic entanglement entropy is part of an interesting dialogue has opened between string theorists and physicists in a variety of fields (e.g., condensed matter, nuclear physics, ...).

- Potential to learn new lessons about general properties of entanglement entropy that have application beyond the context of AdS/CFT correspondence.

- Potential to learn new lessons about general properties of quantum gravity or string theory.

Lots to explore!
Holographic Entanglement Entropy

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Holographic Entanglement Entropy:

\[ S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N} \]

looks like BH entropy!

(Ryu & Takayanagi `06)
Holographic Entanglement Entropy:

(Ryu & Takayanagi `06)

AdS boundary

AdS bulk spacetime

(d + 1) dimensional

$S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N}$

$r = \infty$

$d$ dimensional boundary CFT

gravitational potential/redshift

(\(d - 1\) dimensional)

technicalities

looks like BH entropy!
Holographic Entanglement Entropy:

![Diagram of holographic entanglement entropy](image)

- $S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N} = \infty!!$

- “UV divergence” because area integral extends to $r = \infty$

(Ryu & Takayanagi `06)
Holographic Entanglement Entropy:

\[ S(A) = \text{ext} \left( V \sim A \right) \frac{A_V}{4G_N} \sim \frac{L^{d-1}}{G_N} \frac{A_{\Sigma}}{\delta^{d-2}} + \cdots \]

- "UV divergence" because area integral extends to \( r = \infty \)
- finite result by stopping radial integral at large radius: \( r = R_0 \)
- short-distance cut-off in boundary theory: \( \delta = L^2/R_0 \)
Holographic Entanglement Entropy:

\[ S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N} \simeq \frac{L^{d-1}}{G_N} \frac{A_{\Sigma}}{\delta^{d-2}} + \cdots \]

AdS boundary

cut-off surface

central charge (counts dof) \( \left( \frac{L}{\ell_{\text{Planck}}} \right)^{d-1} \)

"Area Law"
Holographic Entanglement Entropy:

\[ S(A) \sim c_0 \left( \frac{R}{\delta} \right)^{d-2} + c_2 \left( \frac{R}{\delta} \right)^{d-4} + \cdots \]

\[ \begin{cases} 
+ c_{d-2} \log \left( \frac{R}{\delta} \right) + \cdots & (d \text{ even}) \\
+ c_{d-2} + \cdots & (d \text{ odd})
\end{cases} \]

cut-off: \( \delta = \frac{L^2}{R_0} \)
Holographic Entanglement Entropy:

\[ S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N} \]

**conjecture**

Extensive consistency tests:

1) leading contribution yields “area law”

2) recover known results of Calabrese & Cardy for \( d=2 \) CFT

\[ S \approx \frac{L^{d-1}}{G_N} \frac{A_\Sigma}{\delta^{d-2}} + \cdots \]

(also result for thermal ensemble)

\( C = \) circumference
Holographic Entanglement Entropy:

\[ S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N} \]

conjecture

Extensive consistency tests:

1) leading contribution yields “area law”

2) recover known results of Calabrese & Cardy for \( d=2 \) CFT

\[ S = \frac{c}{3} \log \left( \frac{C}{\pi \delta} \sin \frac{\pi \ell}{C} \right) \]

(also result for thermal ensemble)

3) \( S(A) = S(\bar{A}) \) in a pure state

\( A \) and \( \bar{A} \) both yield same bulk surface \( V \)

(not pure state \( \rightarrow \) horizon in bulk; \( S(A) \neq S(\bar{A}) \) for thermal state)
Holographic Entanglement Entropy:

\[ S(A) = \lim_{V \sim A} \frac{A_V}{4G_N} \]

conjecture

Extensive consistency tests:

4) Entropy of eternal black hole =

entanglement entropy of boundary CFT & thermofield double

(Headrick)
Holographic Entanglement Entropy:

\[ S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N} \]

conjecture

Extensive consistency tests:

4) Entropy of eternal black hole = entanglement entropy of boundary CFT & thermofield double

\[
S(A \cup B) + S(A \cap B) \leq S(A) + S(B)
\]

5) sub-additivity: (Headrick & Takayanagi)

[“all” other inequalities: Hayden, Headrick & Maloney]
Holographic Entanglement Entropy:

\[ S(A) = \text{ext} \left( \frac{A_V}{4G_N} \right) \]

Extensive consistency tests:

4) Entropy of eternal black hole = entanglement entropy of boundary CFT & thermofield double
   (Headrick)

5) sub-additivity: \( S(A \cup B) + S(A \cap B) \leq S(A) + S(B) \)
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Central charges and trace anomaly:

\[ \langle T_{\mu}^{\mu} \rangle = -\frac{c}{12} R \]

\[ \langle T_{\mu}^{\mu} \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 \]

\[ I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \]

- in higher (even) dimensions, number of central charges grows

\[ \langle T_{\mu}^{\mu} \rangle = \sum B_i (\text{Weyl invariants})_i - 2(-)^{d/2} A (\text{Euler density})_d \]

(Deser & Schwimmer)

- universal contribution to entanglement entropy determined using trace anomaly (for even \(d\))

(Holzhey, Larsen & Wilczek; Calabrese & Cardy; Takayanagi & Ryu; Schwimmer & Theisen)

\[ S_{\text{univ}} = \log \left( \frac{R}{\delta} \right) 2\pi \int_{\Sigma} d^{d-2}x \sqrt{\hbar} \frac{\partial \langle T_{\lambda}^{\lambda} \rangle}{\partial R_{\mu\nu}^{\rho\sigma}} \hat{e}_{\mu\nu} \hat{e}_{\rho\sigma} \]

(RCM & Sinha)

- partial result! needs rotational symmetry on entangling surface \(\Sigma\)
Central charges and trace anomaly:

\[ \langle T^\mu_\mu \rangle = -\frac{c}{12} R \]

\[ \langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 \]

\[ I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \]

- in higher (even) dimensions, numbers of central charges grows
- universal contribution to entanglement entropy determined using trace anomaly (for even \( d \))

\[ d=2: \quad S = \frac{c}{3} \log \left( \frac{C}{\pi \delta \sin \frac{\pi \ell}{C}} \right) \quad \text{(Holzhey, Larsen & Wilczek; Calabrese & Cardy)} \]

\[ d=4: \quad S_{uni} = \log(R/\delta) \frac{1}{2\pi^2} \int_\Sigma \frac{d^2x}{\sqrt{h}} \left[ c \left( C^{ijkl} \tilde{g}_{ik} \tilde{g}_{jl} - K^a_{ib} K^a_{b} + \frac{1}{2} K^a_{ia} K^a_{ib} \right) - a \mathcal{R} \right] \]

corrections for general (smooth) \( \Sigma \) 

(Solodukhin)
Central charges and trace anomaly:

\[ \langle T_{\mu}^{\mu} \rangle = -\frac{c}{12} R \]

\[ \langle T_{\mu}^{\mu} \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 \]

\[ I_4 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \]

- in higher dimensions, numbers of central charges grows
- universal contribution to entanglement entropy determined using trace anomaly (for even \( d \))
- central charges identified in AdS/CFT using holographic trace anomaly: \(^{\text{(Henningson & Skenderis)}}\)
  - e.g., for (boundary) \( d=4 \): \[ a = c = \frac{\pi^2 L^3}{\ell_P^3} \]
- for general \( d \), central charges \( \propto (L/\ell_P)^{d-1} \)
- for Einstein gravity, all central charges equal for any \( d \)
- distinguishing central charges requires higher curvature gravity
Holographic Entanglement Entropy:

• consider more general gravity theory in AdS:

\[ I = \int d^{d+1}x \sqrt{-g} \mathcal{L}(g^{ab}, R^{ab}_{\; cd}, \nabla_e R^{ab}_{\; cd}, \ldots; \text{matter}) \]

• how do we evaluate holographic entanglement entropy?

  take direction from tests of R&T prescription
Holographic Entanglement Entropy:

\[ S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N} \]  

conjecture

Extensive consistency tests:

4) Entropy of eternal black hole = entanglement entropy of boundary CFT & thermofield double

(Headrick)

5) sub-additivity:  
\[ S(A \cup B) + S(A \cap B) \leq S(A) + S(B) \]  

(Headrick & Takayanagi)

[other inequalities: Hayden, Headrick & Maloney]
Holographic Entanglement Entropy:

• consider more general gravity theory in AdS:

\[ I = \int d^{d+1}x \sqrt{-g} \mathcal{L}(g^{ab}, R^{ab}_{\,cd}, \nabla_e R^{ab}_{\,cd}, \ldots, \text{matter}) \]

• natural conjecture: extremize Wald’s entropy formula

\[ S = -2\pi \int d^{d-1}x \sqrt{h} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}^{\rho\sigma}} \hat{\varepsilon}^{\mu\nu} \hat{\varepsilon}_{\rho\sigma} \]

• focus on universal term for \( d=4 \):

\[ S_{uni} = \log(R/\delta) \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[ C^{ijkl} \tilde{g}_{ik} \tilde{g}_{jl} - K^i_b K^i_a + \frac{1}{2} K^i_a K^i_b \right] - a \mathcal{R} \]

• holographic calculation following above conjecture yields

\[ S_{uni} = \log(R/\delta) \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[ C^{ijkl} \tilde{g}_{ik} \tilde{g}_{jl} - K^i_b K^i_a + \frac{1}{2} K^i_a K^i_b \right] - a \mathcal{R} \]

→ conjecture wrong 😞

(Hung, Myers & Smolkin)

(Solodukhin)
Holographic Entanglement Entropy:

• consider more general gravity theory in AdS:

\[ I = \int d^{d+1}x \sqrt{-g} \mathcal{L}(g^{ab}, R^{ab}_{\ cd}, \nabla_e R^{ab}_{\ cd}, \ldots, \text{matter}) \]

• natural conjecture: extremize Wald’s entropy formula

\[ S = -2\pi \int d^{d-1}x \sqrt{\hat{h}} \frac{\partial \mathcal{L}}{\partial R^{\mu\nu}_{\ \ \rho\sigma}} \hat{\varepsilon}^{\mu\nu} \hat{\varepsilon}_{\rho\sigma} \]

• holographic calculation following above conjecture yields

\[ S_{uni} = \log(R/\delta) \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[ a \left( C^{ijkl} \tilde{\gamma}_{ik}^\perp \tilde{\gamma}_{jl}^\perp - K_{a}^{\ i} K_{b}^{\ a} + \frac{1}{2} K_{a}^{\ i} K_{b}^{\ i} \right) - a \mathcal{R} \right] \]

• triumph of R&T prescription in Einstein gravity!! (c = a)
• for general gravity action, conjecture is wrong
• there is nothing wrong with Wald’s formula!!

→ to proceed further, focus on special gravity actions
Holographic Entanglement Entropy:

- consider special case of Gauss-Bonnet gravity:

\[
I = \frac{1}{2\ell_p^3} \int d^5 x \sqrt{-g} \left[ \frac{12}{L^2} + R + L^2 \frac{\lambda}{2} \left( R^{abcd} R_{abcd} - 4 R_{ab} R^{ab} + R^2 \right) \right]
\]

4d Euler density

- higher curvature but eom are still **second order!!** (Lovelock)

- studied in detail for stringy gravity in 1980’s

  (Zwiebach; Boulware & Deser; Wheeler; Myers & Simon; . . . .)

- interest recently in AdS/CFT studies – a toy model with \( c \neq a \)

  (eg, Brigante, Liu, Myers, Shenker, Yaida, de Boer, Kulaxizi, Parnachev, Camanho, Edelstein, Buchel, Sinha, Paulos, Escobedo, Smolkin, Cremonini, Hofman, . . . .)

- black hole entropy:

\[
S_{JM} = \frac{2\pi}{\ell_p^3} \int d^3 x \sqrt{h} \left[ 1 + \lambda L^2 R \right]
\]

- not precisely same as Wald entropy – agree when \( K^i_{ab} \) vanish
Holographic Entanglement Entropy: (Hung, Myers & Smolkin)

- consider special case of Gauss-Bonnet gravity:

\[
I = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left[ \frac{12}{L^2} + R + L^2 \frac{\lambda}{2} (R^{abcd} R_{abcd} - 4R_{ab}R^{ab} + R^2) \right]
\]

4d Euler density

- second conjecture: extremize JM entropy formula

\[
S_{JM} = \frac{2\pi}{\ell_p^3} \int d^3x \sqrt{h} \left[ 1 + \lambda L^2 R \right]
\]

- again consider universal term for d=4: (Solodukhin)

\[
S_{univ} = \log(\ell/\delta) \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[ c \left( C^{ijkl} \tilde{g}_{ik} \tilde{g}_{jl} - K_a^i K_b^j + \frac{1}{2} K_a^i K_b^j \right) - a R \right]
\]

- holographic calculation following above conjecture yields

\[
S_{univ} = \log(\ell/\delta) \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[ c \left( C^{ijkl} \tilde{g}_{ik} \tilde{g}_{jl} - K_a^i K_b^j + \frac{1}{2} K_a^i K_b^j \right) - a R \right]
\]

passes nontrivial test 😊
Holographic Entanglement Entropy: (Hung, Myers & Smolkin)

• consider special case of Gauss-Bonnet gravity:

\[ l = \frac{1}{2\ell^3_p} \int d^5x \sqrt{-g} \left[ \frac{12}{L^2} + R + L^2 \frac{\lambda}{2} \left( R^{abcd} R_{abcd} - 4R_{ab}R^{ab} + R^2 \right) \right] \]

4d Euler density

• second conjecture: extremize JM entropy formula

\[ S_{JM} = \frac{2\pi}{\ell^3_p} \int d^3x \sqrt{h} \left[ 1 + \lambda L^2 \mathcal{R} \right] \]

- reproduces universal term for any smooth surface in d=4
- partial results for d=6 (geometries with rotational symmetry; found new curvature corrections when \( K^i_{ab} = 0 \))
- extends to general Lovelock theories for d\( \geq 6 \)

- still no general result for completely general gravity action - with sufficient symmetry, Wald entropy seems correct
- curious instability to adding handles for \( \lambda > 0 \) (Ogawa & Takayanagi)
Holographic Entanglement Entropy: \( S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N} \) \text{(conjecture)}

Extensive consistency tests:

4) Entropy of eternal black hole = entanglement entropy of boundary CFT & thermofield double \text{(Headrick)}

5) strong subadditivity: \( S(A \cup B) + S(A \cap B) \leq S(A) + S(B) \) \text{(Headrick & Takayanagi)}

6) for general even \( d \), connection to central charges of CFT \text{(Hung, RCM & Smolkin, arXiv:1101.5813)}

7) derivation of holographic EE for spherical entangling surfaces \text{(Casini, Huerta & RCM, arXiv:1102.044)}
\text{(see also: RCM & Sinha, arXiv:1011.5819)}
Calculating Entanglement Entropy:

\[ S_{EE} = -Tr \left[ \rho_A \log \rho_A \right] \]

- “standard” approach relies on replica trick and calculating Renyi entropy first and taking \( n \to 1 \) limit

\[
S_n = \frac{1}{1-n} \log Tr \left[ \rho^n_A \right] \quad S_{EE} = \lim_{n \to 1} S_n
\]

- replica trick involves path integral of QFT in singular \( n \)-fold cover of background spacetime

- problematic in holographic framework
  - produce singularity in dual gravity description
    (resolved by quantum gravity/string theory?)
    (Fursaev; Headrick)

- need another calculation with simpler holographic translation
Calculating Entanglement Entropy: (Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose with radius R
- entanglement entropy: \( S_{EE} = -Tr \left[ \rho_A \log \rho_A \right] \)

- density matrix \( \rho_A \) describes physics in entire causal domain \( \mathcal{D} \)
- conformal mapping: \( \mathcal{D} \rightarrow \mathcal{H} = R \times H^{d-1} \)
General result for any CFT (Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose $S^{d-2}$ with radius $R$
  
  \[ S_{EE} = -Tr \left[ \rho_A \log \rho_A \right] \]

- conformal mapping: $\mathcal{D} \rightarrow \mathcal{H} = R \times H^{d-1}$
  
  \[
  t = R \frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)} \\
  r = R \frac{\sinh u}{\cosh u + \cosh(\tau/R)}
  \]

  curvature scale: $1/R$  
  temperature: $T=1/2\pi R$ !!

- for CFT: $\rho_{thermal} = U \rho_A U^{-1}$
  \[ S_{EE} = S_{thermal} \]
General result for any CFT

- take CFT in d-dim. flat space and choose $S^{d-2}$ with radius $R$
  - entanglement entropy: $S_{EE} = -Tr [\rho_A \log \rho_A]$
  - by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $R \sim 1/R^2$ and $T=1/2\pi R$

- $S_{EE} = S_{thermal}$

AdS/CFT correspondence:

- thermal bath in CFT = black hole in AdS
  - $S_{EE} = S_{thermal} = S_{horizon}$
- only need to find appropriate black hole
  - topological BH with hyperbolic horizon which intersects A on AdS boundary

(Casini, Huerta & RCM)

(Aminneborg et al; Emparan; Mann; . . .)
\[ S_{EE} = S_{\text{thermal}} = S_{\text{horizon}} \]

- desired “black hole” is a hyperbolic foliation of empty AdS space

\[ ds^2 = \frac{L^2}{\zeta^2} (dz^2 - dt^2 + d\bar{x}^2) d\tau^2 + \rho^2 d\Sigma_{d-1}^2 \]

\[ \rightarrow T = \frac{1}{2\pi R} \]

- “Rindler coordinates” of AdS space
\[ S_{EE} = S_{thermal} = S_{horizon} \]

- desired “black hole” is a hyperbolic foliation of empty AdS space

\[
ds^2 = \frac{L^2 \, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2} \, d\tau^2 + \rho^2 \, d\Sigma_{d-1}^2 \quad \rightarrow \quad T = \frac{1}{2\pi R}
\]

- apply Wald’s formula (for any gravity theory) for horizon entropy:

\[
S = -2\pi \int d^{d-1}x \sqrt{\hbar} \, \frac{\partial L}{\partial R_{\mu\nu}^{\rho\sigma}} \, \hat{\epsilon}^{\mu\nu} \hat{\epsilon}_{\rho\sigma}
\]

\[
= \frac{2\pi}{\pi^{d/2} \Gamma (d/2)} \, \frac{a_d^*}{R^{d-1}} \, V(H^{d-1})
\]

(RCM & Sinha)

where \( a_d^* \) = central charge for “A-type trace anomaly”

for even \( d \)

\[ \text{= entanglement entropy defines effective central charge} \]

for odd \( d \)
\[ S_{EE} = S_{thermal} = S_{horizon} \]

- desired "black hole" is a hyperbolic foliation of empty AdS space

\[ ds^2 = \frac{L^2 \, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2} \, d\tau^2 + \rho^2 \, d\Sigma_{d-1}^{d-1} \quad \Rightarrow \quad T = \frac{1}{2\pi R} \]

- apply Wald’s formula (for any gravity theory) for horizon entropy:

\[ S = \frac{2\pi}{\pi^{d/2}} \Gamma \left( \frac{d}{2} \right) \frac{a_d^*}{R^{d-1}} \, V \left( H^{d-1} \right) \]

intersection with standard regulator surface: \( z_{min} = \delta \)

\[ S = a_d^* \frac{4\pi^{\frac{d-3}{2}}}{(d-2)\Gamma \left( \frac{d-1}{2} \right)} \left( \frac{R}{\delta} \right)^{d-2} + \cdots \]

\[ ds^2 = R^2 \left[ \frac{du^2}{1 + u^2} + u^2 \, d\Omega_{d-2}^{d-2} \right] \]

"area law" for d-dimensional
\[ S_{EE} = S_{\text{thermal}} = S_{\text{horizon}} \]

- desired “black hole” is a hyperbolic foliation of empty AdS space

\[
ds^2 = \frac{L^2 \, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2} \, d\tau^2 + \rho^2 \, d\Sigma_{d-1}^2 \quad \rightarrow \quad T = \frac{1}{2\pi R}
\]

- apply Wald’s formula (for any gravity theory) for horizon entropy:

\[
S = \frac{2\pi}{\pi^{d/2} \Gamma (d/2)} \frac{a_d^*}{R^{d-1}} \, V (H^{d-1})
\]

\[
ds^2 = R^2 \left[ \frac{du^2}{1 + u^2} + u^2 \, d\Omega_{d-2}^2 \right]
\]

universal contributions:

\[
S = \cdots + (-)^{\frac{d}{2} - 1} 4 \, a_d^* \, \log \left( \frac{2R}{\delta} \right) + \cdots \text{ for even } d
\]

\[
\cdots + (-)^{\frac{d-1}{2}} 2\pi \, a_d^* \quad + \quad \cdots \quad \text{ for odd } d
\]

- discussion extends to case with background \( R^{1,d-1} \rightarrow R \times S^{d-1} \)
Holographic Renyi entropy:

• turn to Renyi entropy (close cousin of entanglement entropy)

\[ S_n = \frac{1}{1 - n} \log \text{Tr} [\rho_A^n] \]

\[ S_{EE} = \lim_{n \to 1} S_n \]

• universal contribution (for even $d$)

\[ S_n = \cdots + \text{constant} \times \log \left( \frac{R}{\delta} \right) + \cdots \]
Holographic Renyi entropy:

- turn to Renyi entropy (close cousin of entanglement entropy)

\[ S_n = \frac{1}{1 - n} \log Tr \left[ \rho_A^n \right] \]

\[ S_{EE} = \lim_{n \to 1} S_n \]

- universal contribution (for even d)

\[ d=2: \quad S_n = \cdots + \frac{c}{6} \left( 1 + \frac{1}{n} \right) \log \left( \frac{R}{\delta} \right) + \cdots \]

(Calabrese & Cardy)

- (almost) no calculations for d > 2

(Hung, RCM, Smolkin & Yale)
Holographic Renyi entropy:

- turn to Renyi entropy (close cousin of entanglement entropy)

$$S_n = \frac{1}{1 - n} \log \text{Tr} [\rho^n_A] \quad \quad S_{EE} = \lim_{n \to 1} S_n$$

- “standard” calculation involves singular n-fold cover of spacetime problematic for translation to dual AdS gravity

- our previous derivation lead to thermal density matrix

$$\rho_A = U^{-1} \frac{e^{-H/T_0}}{\text{Tr} \left[ e^{-H/T_0} \right]} U \quad \text{with} \quad T_0 = \frac{1}{2\pi R}$$

$$\text{Tr} [\rho^n_A] = \frac{\text{Tr} \left[ e^{-nH/T_0} \right]}{\text{Tr} \left[ e^{-H/T_0} \right]^n}$$

partition function at new temperature, $T = T_0/n$
Holographic Renyi entropy:

- turn to Renyi entropy (close cousin of entanglement entropy)

\[ S_n = \frac{1}{1 - n} \log \text{Tr} [\rho_A^n] \]

\[ S_{EE} = \lim_{n \to 1} S_n \]

- “standard” calculation involves singular n-fold cover of spacetime problematic for translation to dual AdS gravity

- with bit more work, find convenient formula:

\[ S_n = \frac{n}{n - 1} \frac{1}{T_0} \int_{T_0/n}^{T_0} S(T) dT \quad \text{where} \quad T_0 = \frac{1}{2\pi R} \]

Renyi entropy for spherical \( \Sigma \)

thermal entropy on hyperbolic space \( \mathbb{H}^{d-1} \)

- in holographic framework, need to know topological black hole solutions for arbitrary temperature
Holographic Renyi entropy:

- Renyi entropy of CFT for spherical entangling surface:

\[
S_n = \frac{n}{n-1} \frac{1}{T_0} \int_{T_0/n}^{T_0} S(T) dT \quad \text{where} \quad T_0 = \frac{1}{2\pi R}
\]

- need to know topological black holes for arbitrary temperature

- focus on gravity theories where we can calculate: Einstein, Gauss-Bonnet, Lovelock, quasi-topological, ….

- for example, with GB gravity and (boundary) d=4:

\[
S_n = \frac{n}{n-1} \frac{V(H^3)}{4\pi} \frac{3c-a}{3a-c} (1-x^2) \left[ (5a-c)x^2 - (13a-5c) + 4a \frac{2ax^2 - (a-c)}{(3a-c)x^2 - (a-c)} \right]
\]

where

\[
0 = x^3 - \frac{3a-c}{5a-c} \left( \frac{x^2}{n} + x \right) + \frac{1}{n} \frac{a-c}{5a-c}
\]
• no elegant result as was found for d=2 CFT, ie, $S_n$ depends on both central charges and dependence on $n$ does not factor out
• further work (with quasi-topological gravity) shows the universal coefficient depends on more data from the boundary CFT than central charges appearing in the trace anomaly (eg, $t_4$)

• preliminary work indicates positivity of Renyi entropies may constrain gravitational couplings in higher curvature models

GB gravity: $\frac{C_T}{a^* \alpha}$

constraint from demanding $S_\infty > 0$

constraints from demanding boundary theory is causal
Conclusions:

• AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories.

• Holographic entanglement entropy is part of an interesting dialogue has opened between string theorists and physicists in a variety of fields (eg, condensed matter, nuclear physics, . . . )

• Potential to learn lessons about issues in boundary theory eg, readily calculate Renyi entropies for wide class of theories in higher dimensions.

• Potential to learn lessons about issues in bulk gravity theory eg, holographic entanglement entropy may give new insight into quantum gravity or emergent spacetime (eg, van Raamsdonk).

Lots to explore!