Holographic Entanglement Entropy
(Introduction)

(with H. Casini, M. Huerta, J. Hung, A. Sinha, M. Smolkin & A. Yale)

(arXiv:1101.5813, arXiv:1102.0440, arXiv:1109.0???)

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

1. Entanglement Entropy

- condensed matter
- quantum information
- black hole microphysics

2. AdS/CFT correspondence

(gauge/gravity duality)

- string theory
- quantum gravity

3. Holographic Entanglement Entropy

- proposal by Ryu & Takayanagi (2006)

4. Two Recent Developments:

- precise connection between EE and central charges
- derivation of holographic EE for special geometries

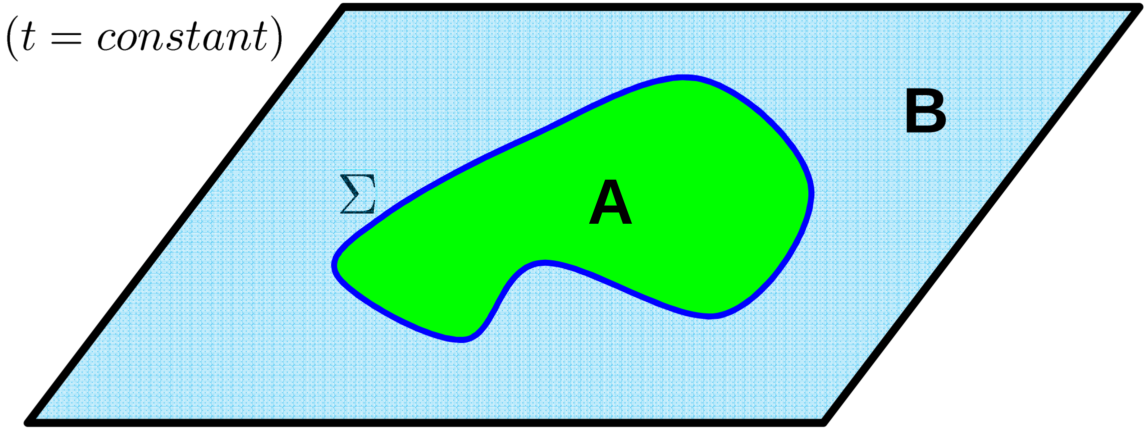
5. Summary:

- holographic EE provides framework where we can learn about properties of both EE and quantum gravity

Entanglement Entropy

- what is entanglement entropy?
  very general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions
- integrate out degrees of freedom in "outside" region
- remaining dof are described by a density matrix ρ_A

⟶ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



Entanglement Entropy

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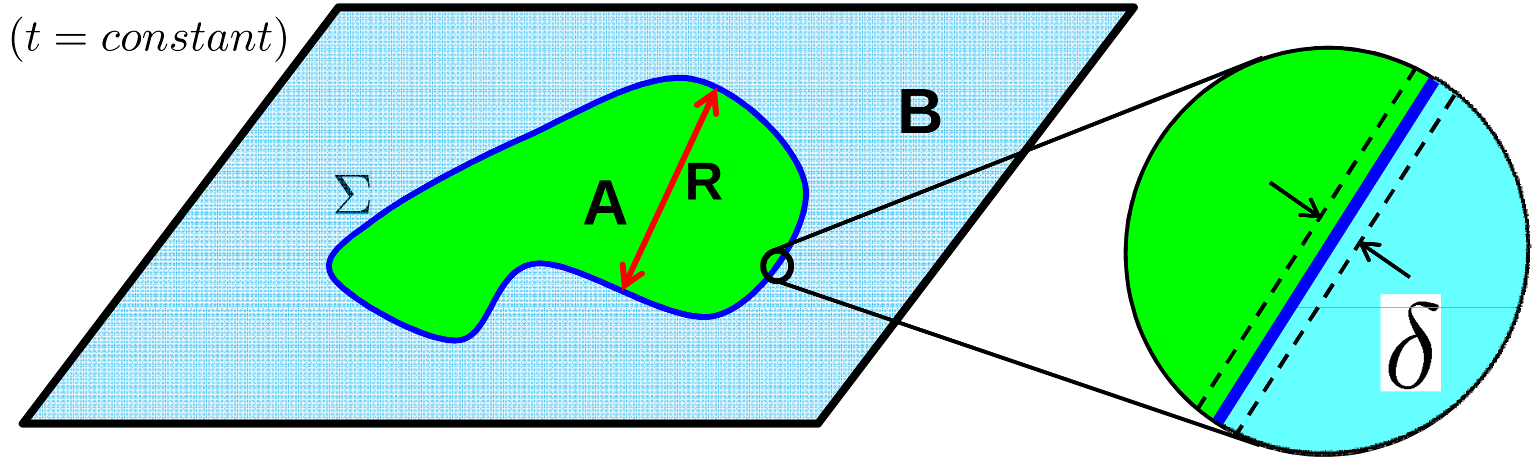

• result is UV divergent!

• must regulate calculation: δ = short-distance cut-off

$$S \; = \; c_0 \frac{R^{d-2}}{\delta^{d-2}} \; + \; c_2 \frac{R^{d-4}}{\delta^{d-4}} \; + \; \cdots$$

d = spacetime dimension

• careful analysis reveals geometric structure, eg, $S = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

Entanglement Entropy

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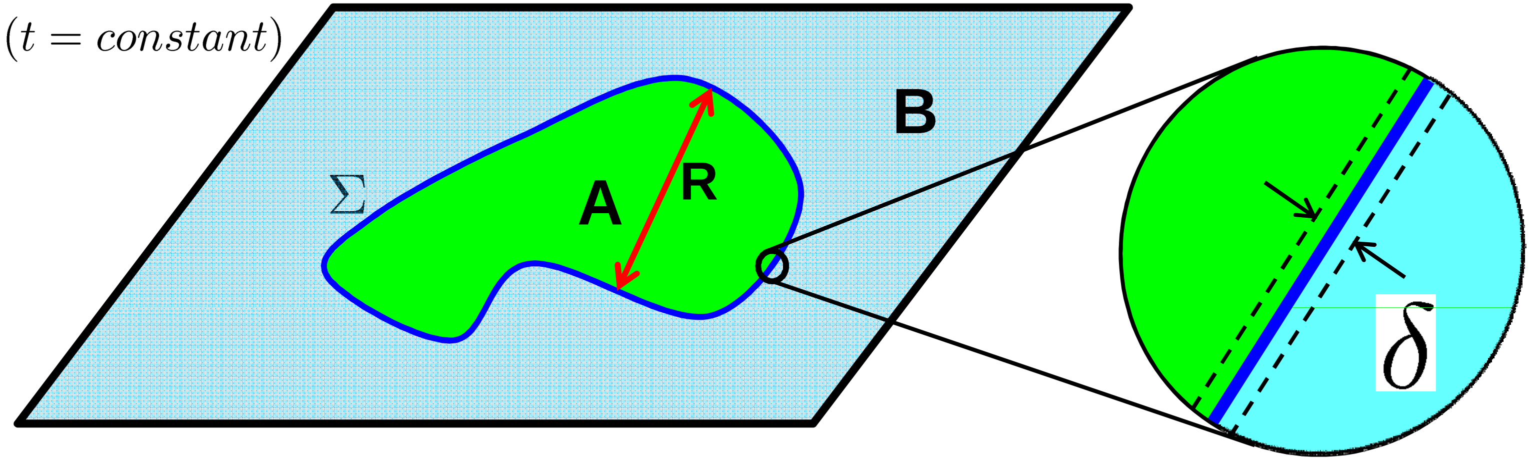

- must regulate calculation: δ = short-distance cut-off

$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots$$

d = spacetime dimension

- leading coefficients sensitive to details of regulator, eg, $\delta \to 2\delta$
- find universal information characterizing underlying QFT in subleading terms, eg,

$$S = \cdots + c_d \log(R/\delta) + \cdots$$

More general comments on **Entanglement Entropy:**

- nonlocal quantity which is (at best) very difficult to measure

  $\longrightarrow$ no accepted experimental procedure

- in condensed matter theory: diagnostic to characterize quantum critical points or topological phases (eg, quantum hall fluids)

- in quantum information theory: useful measure of quantum entanglement (a computational resource)

- black hole microphysics: leading term obeys "area law" $S \simeq c_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}}$

  $\longrightarrow$ suggested as origin of black hole entropy (eg, $\delta \simeq \ell_P$)

(Bombelli, Koul, Lee & Sorkin `86; Srednicki; Frolov & Novikov; Callan & Wilczek; Susskind; . . . .)

- recently considered in AdS/CFT correspondence

(Ryu & Takayanagi `06)

AdS/CFT correspondence:

anti-de Sitter space ⟺ conformal field theory

quantum gravity
- negative cosmological constant
- d+1 spacetime dimensions

⟷ **holography**

quantum field theory
- no scale (at quantum level)
- d spacetime dimensions
- **no gravity!**

Favorite example:

Type IIb superstrings on $AdS_5 \times S^5$ with RR flux N_c ⟺ (3+1)-dimensional $\mathcal{N}$=4 SU(N_c) super-Yang-Mills

(Maldacena `97)

AdS/CFT correspondence:

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↔ **holography**

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↑

classical gravity
with small curvatures

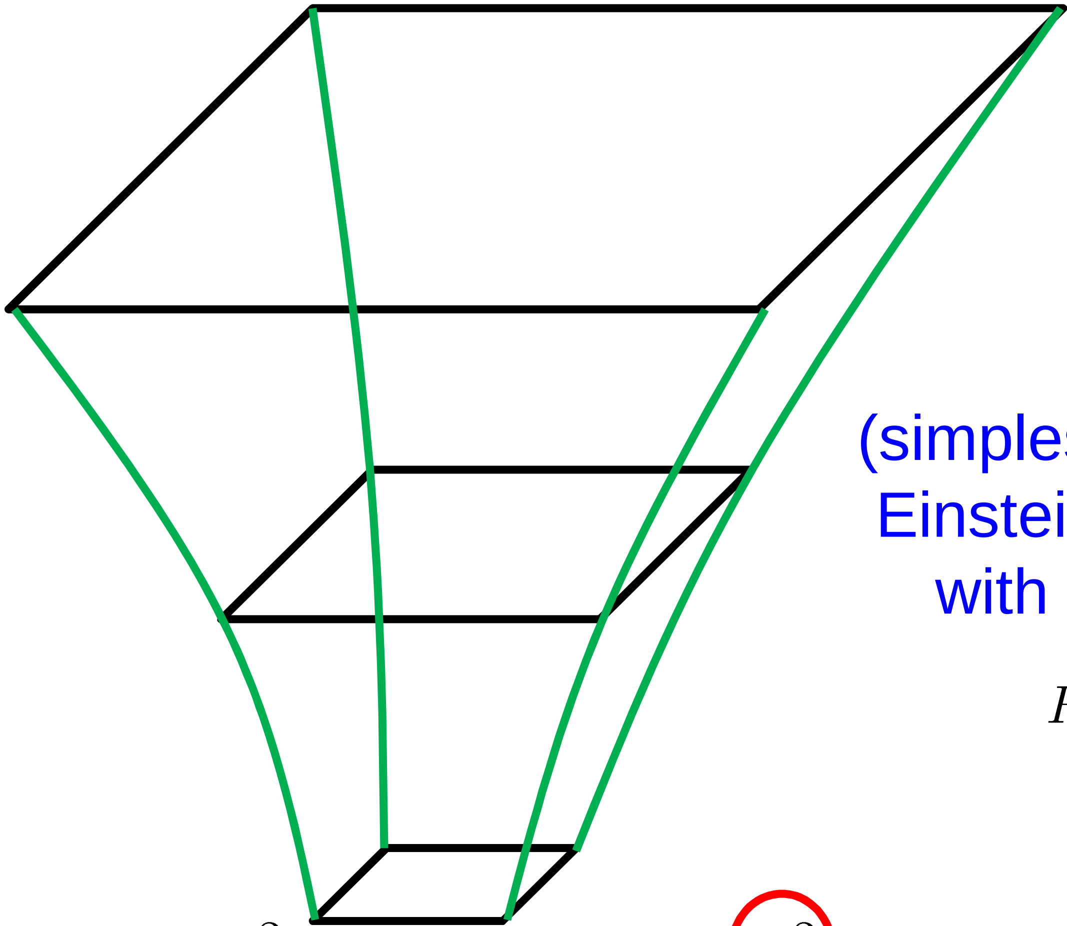
→ large central charge ($N_c \to \infty$)

→ strong coupling ($\lambda \to \infty$)

anti-de Sitter space:

$$R \sim -\frac{1}{L^2}$$

maximally symmetric geometry with negative curvature



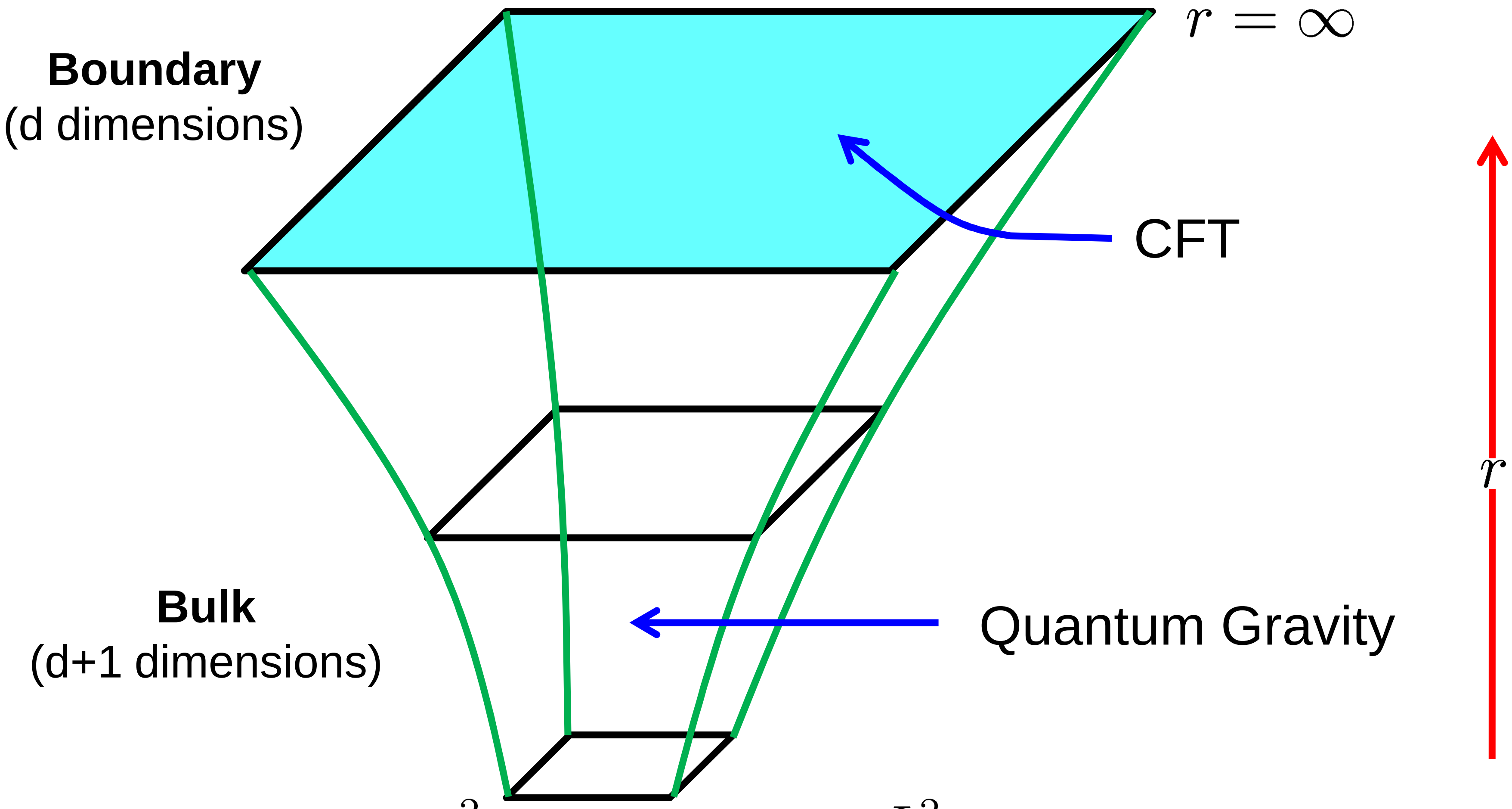
(simplest) solution of Einstein's equations with negative Λ:

$$R_{ab} = -\frac{d}{L^2}$$

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}dr^2$$

anti-de Sitter space:

maximally symmetric geometry with negative curvature



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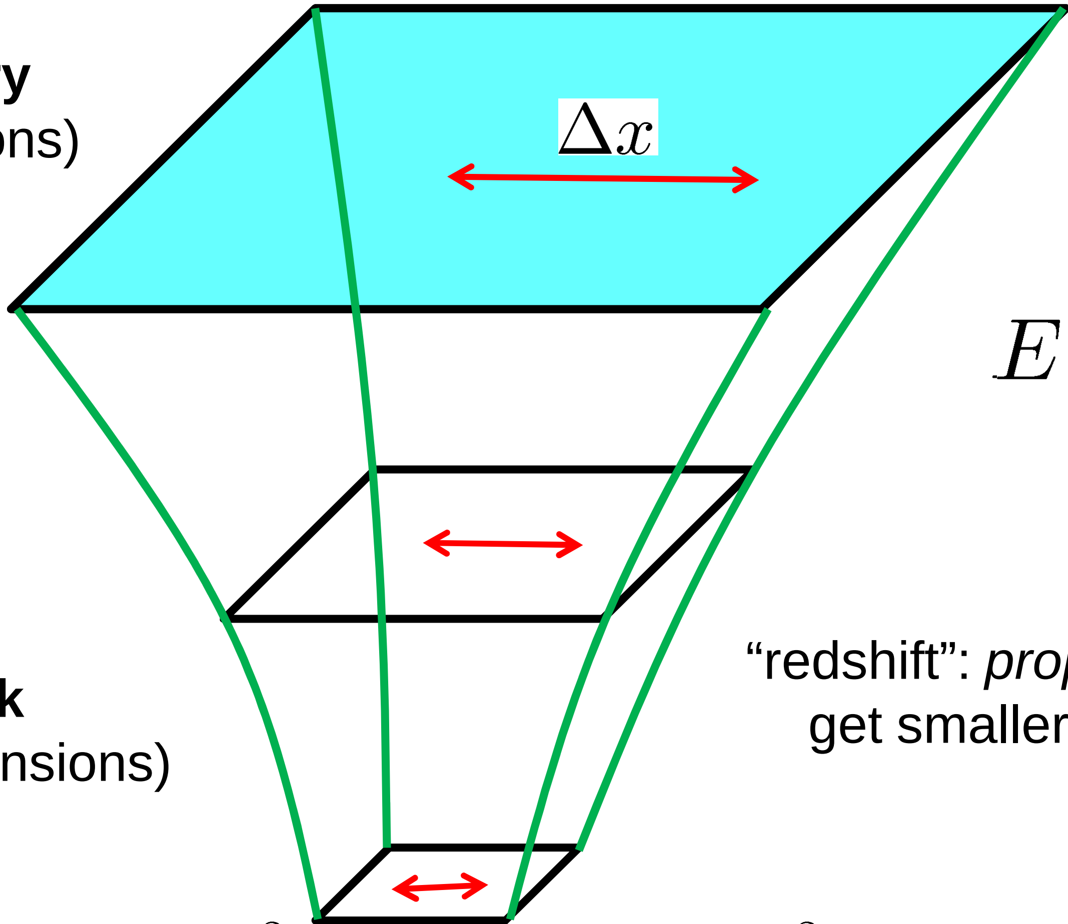
maximally symmetric geometry with negative curvature

Boundary
(d dimensions)

$r = \infty$



$E \sim r/L^2$

Bulk
(d+1 dimensions)

"redshift": *proper* distances get smaller for small r

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}dr^2$$

boundary/CFT metric

anti-de Sitter space:

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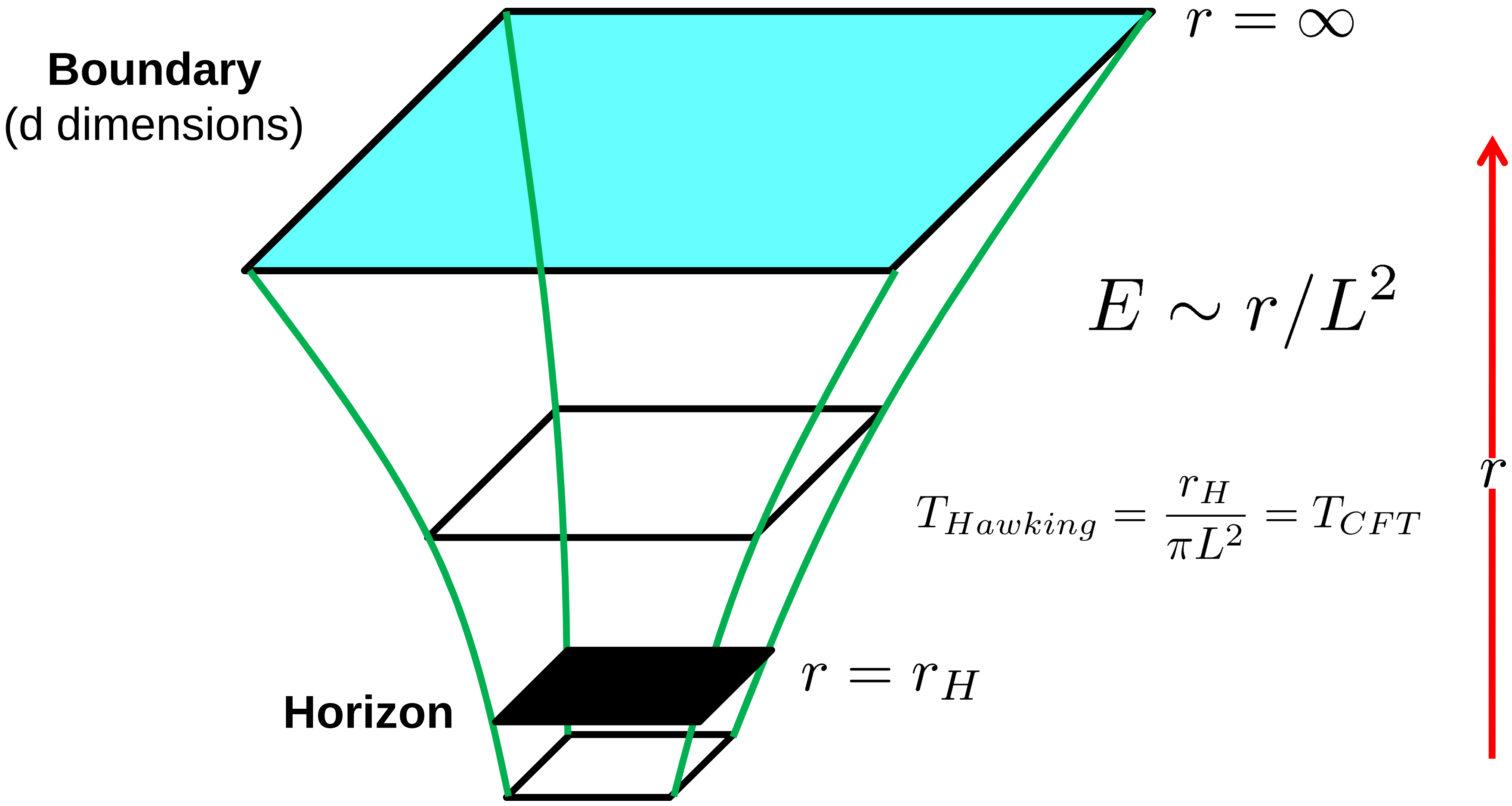
maximally symmetric geometry with negative curvature

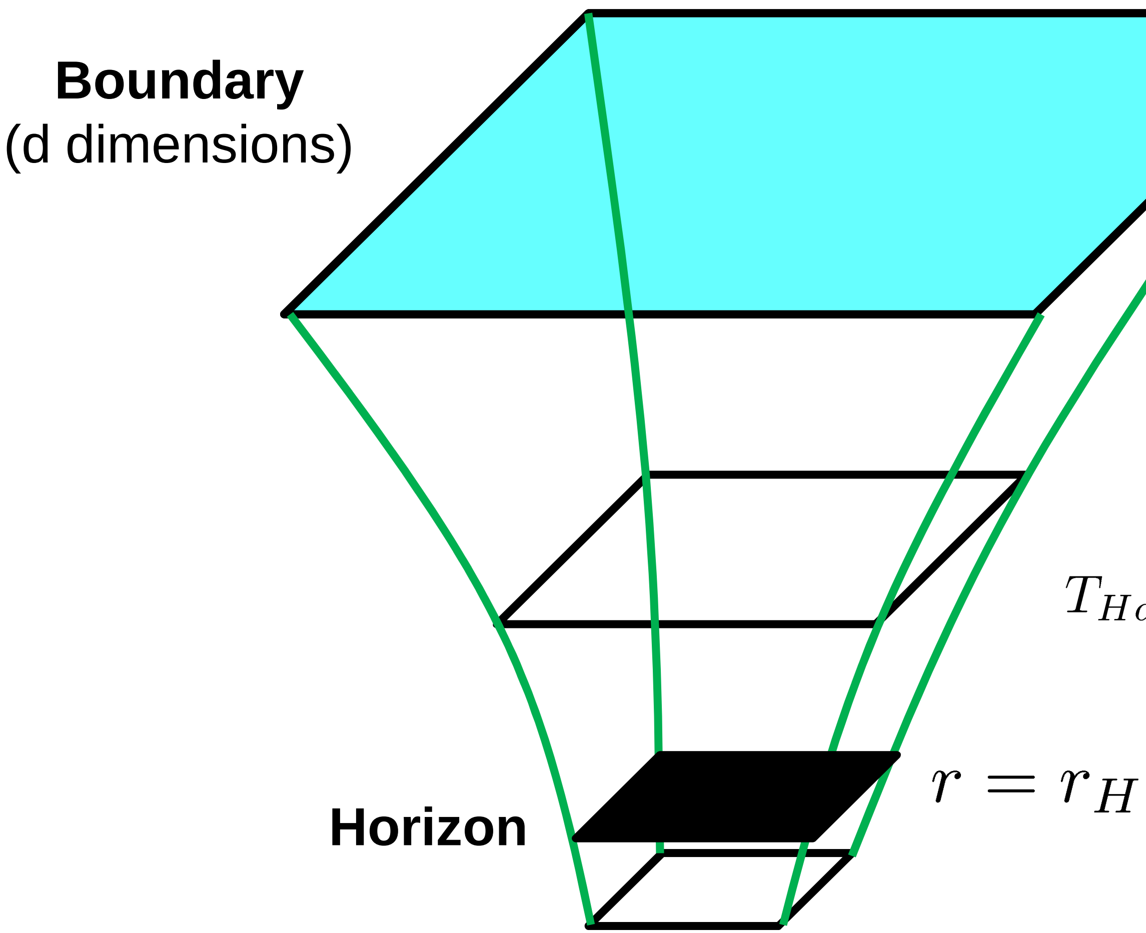


AdS Black Hole = CFT Thermal State

anti-de Sitter space:

$$R \sim -\frac{1}{L^2}$$

maximally symmetric geometry with negative curvature



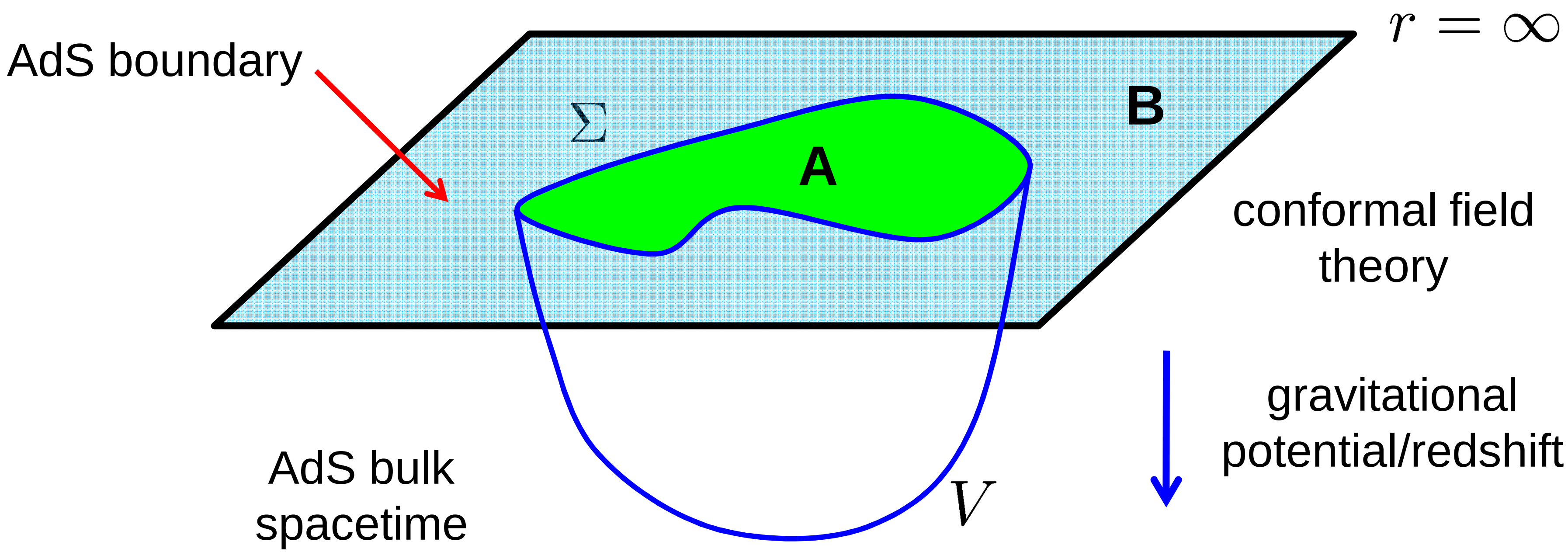
$$E \sim r/L^2$$

$$T_{Hawking} = \frac{r_H}{\pi L^2} = T_{CFT}$$

Horizon Entropy = Entropy of Thermal State $\quad S = \frac{A_{\rm horizon}}{4G_N}$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



$$S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N}$$

looks like BH entropy!

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N}$$

conjecture

Extensive consistency tests:

1) leading contribution yields "area law" $S = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

. . . . *(more tests)* . . . .

6) for general even d, connection to central charges of CFT

(Hung, RCM & Smolkin, arXiv:1101.5813)

7) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RCM, arXiv:1102.044)

(see also: RCM & Sinha, arXiv:1011.5819)

6) for general even d, connection to central charges of CFT

(Hung, RCM & Smolkin, arXiv:1101.5813)

- trace anomaly in CFT (with even d) defines central charges

d=4: $$\langle T_\mu{}^\mu \rangle = \frac{\mathbf{c}}{16\pi^2} I_4 - \frac{\mathbf{a}}{16\pi^2} E_4$$

$$I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$

- universal/logarithmic contribution to entanglement entropy determined by central charges using trace anomaly, eg,

$$S_{uni} = \log(R/\delta) \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[\mathbf{c} \left(C^{ijkl} \, \tilde{g}^{\perp}_{ik} \, \tilde{g}^{\perp}_{jl} - K^{i\,b}_{a} K^{ia}_{b} + \frac{1}{2} K^{i\,a}_{a} K^{i\,b}_{b} \right) - \mathbf{a} \, \mathcal{R} \right]$$

(Solodukhin)

- R&T proposal for holographic EE reproduce precisely this result

- extends to certain higher curvature theories (eg, GB gravity)

$$S = \min_{\partial V = \Sigma} \frac{2\pi}{\ell_p^3} \int_V d^3x \sqrt{h} \left[1 + \lambda L^2 \mathcal{R} \right]$$

7) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RCM, arXiv:1102.044)

(see also: RCM & Sinha, arXiv:1011.5819)

- holographic translation for standard calculation of EE is difficult
- new calculation for special case:

CFT in d-dim. flat space and choose $\Sigma = S^{d-2}$ with radius R

→ by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and T=1/2πR

$$S_{EE} = S_{thermal}$$

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- holographic translation for standard calculation of EE is difficult
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⟶ by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and T=1/2πR

$$S_{EE} = S_{thermal} = S_{horizon}$$

⟶ thermal bath in CFT = black hole in AdS

- can calculate holographic EE for any bulk gravity theory

universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \log(2R/\delta) + \cdots \quad \text{for even d}$$

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots \quad \text{for odd d}$$

Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories

- holographic entanglement entropy is part of an interesting dialogue has opened between string theorists and physicists in a variety of fields (eg, condensed matter, nuclear physics, . . .)

- potential to learn new lessons about general properties of entanglement entropy that have application beyond the context of AdS/CFT correspondence

- potential to learn new lessons about general properties of quantum gravity or string theory

Lots to explore!

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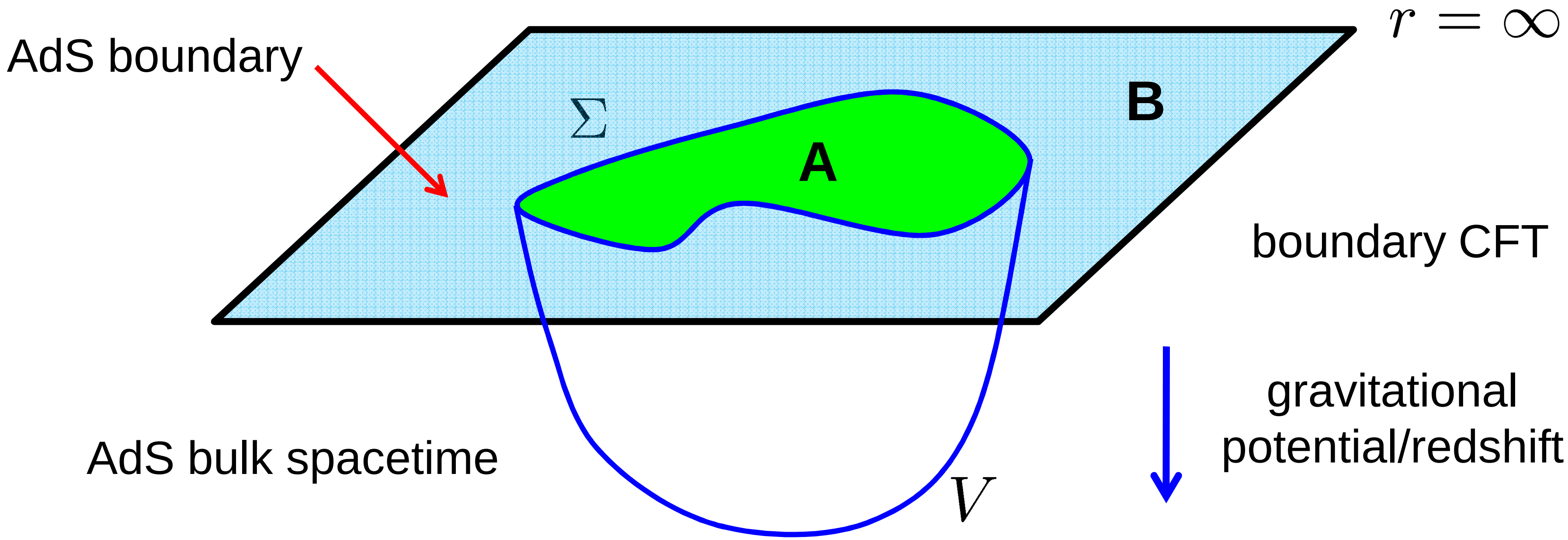
Holographic Entanglement Entropy:



$$S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N}$$

looks like BH entropy!

(Ryu & Takayanagi `06)

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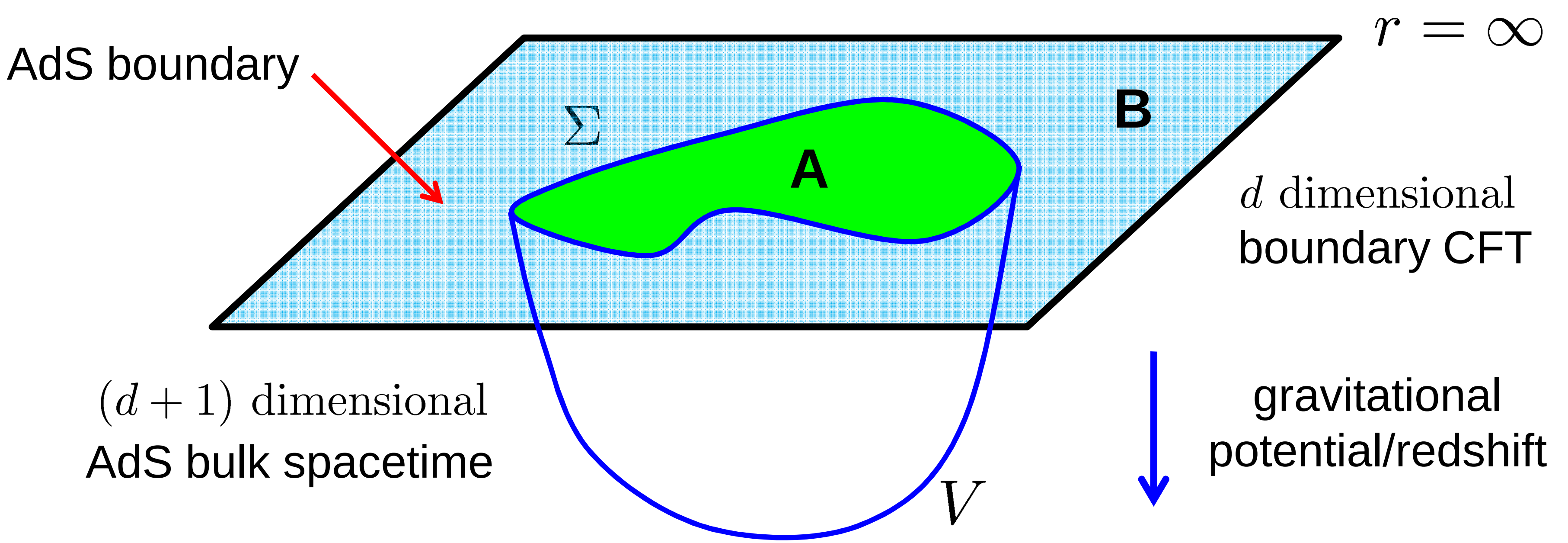


$(d-1)$ dimensional

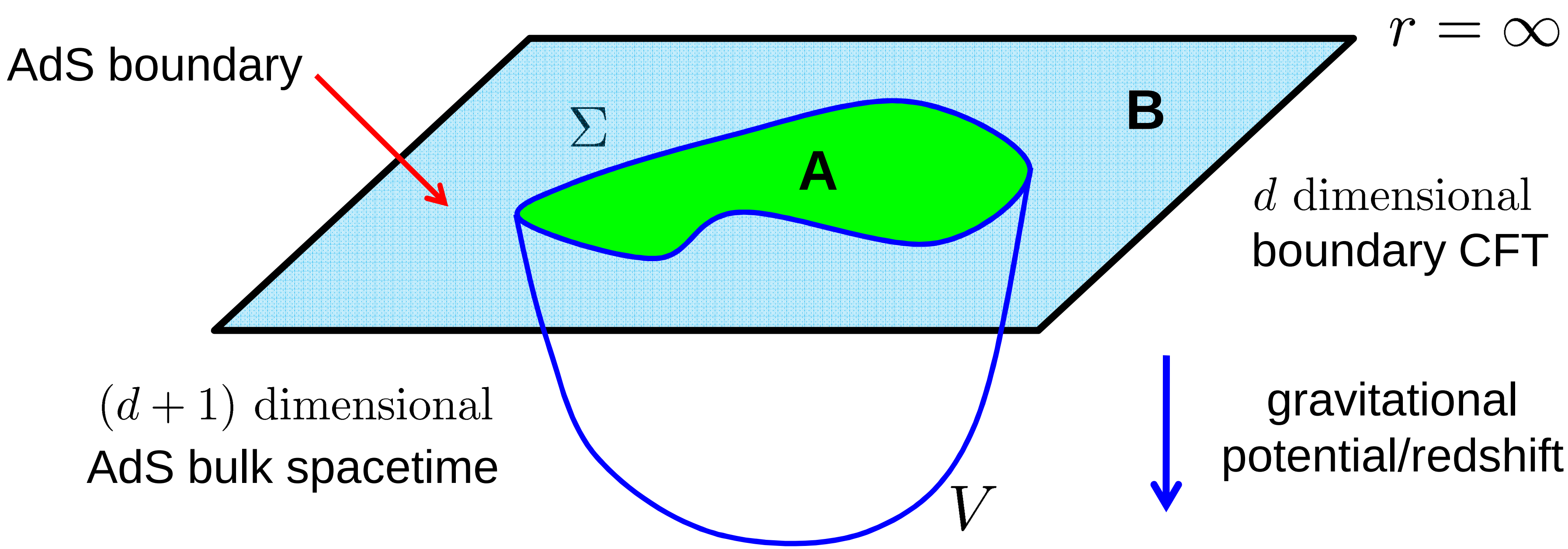
$$S(A) = \underset{V \sim A}{\text{ext}} \frac{A_V}{4G_N}$$

technicalities

looks like
BH entropy!

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

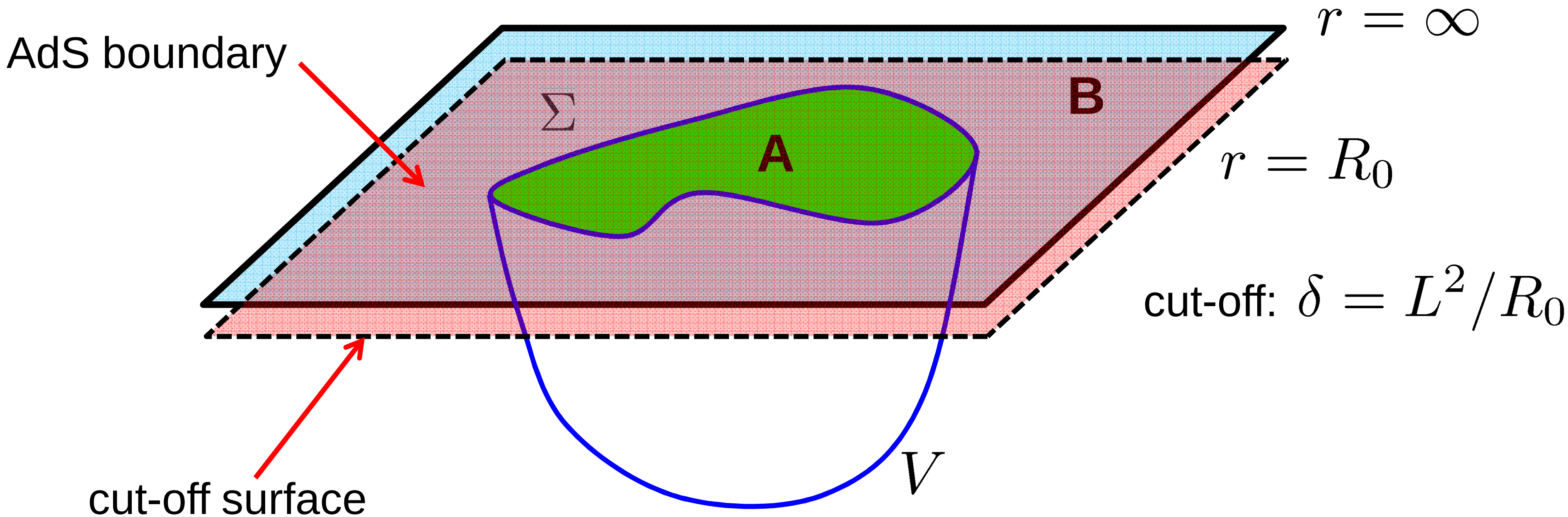


$$S(A) = \underset{V\sim A}{\text{ext}} \frac{A_V}{4G_N} = \infty!!$$

- "UV divergence" because area integral extends to $r=\infty$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



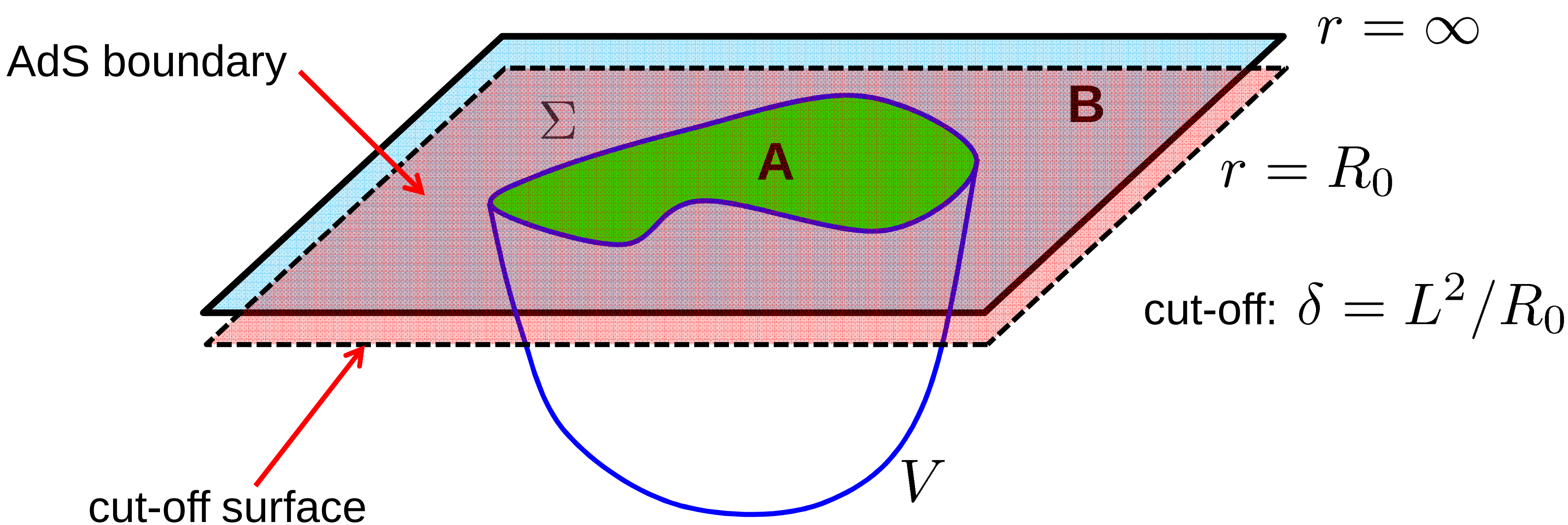
$$S(A) = \underset{V\sim A}{\text{ext}} \frac{A_V}{4G_N} \simeq \frac{L^{d-1}}{G_N}\frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$$

- "UV divergence" because area integral extends to $r=\infty$
- finite result by stopping radial integral at large radius: $r=R_0$

  ⟶ short-distance cut-off in boundary theory: $\delta = L^2/R_0$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



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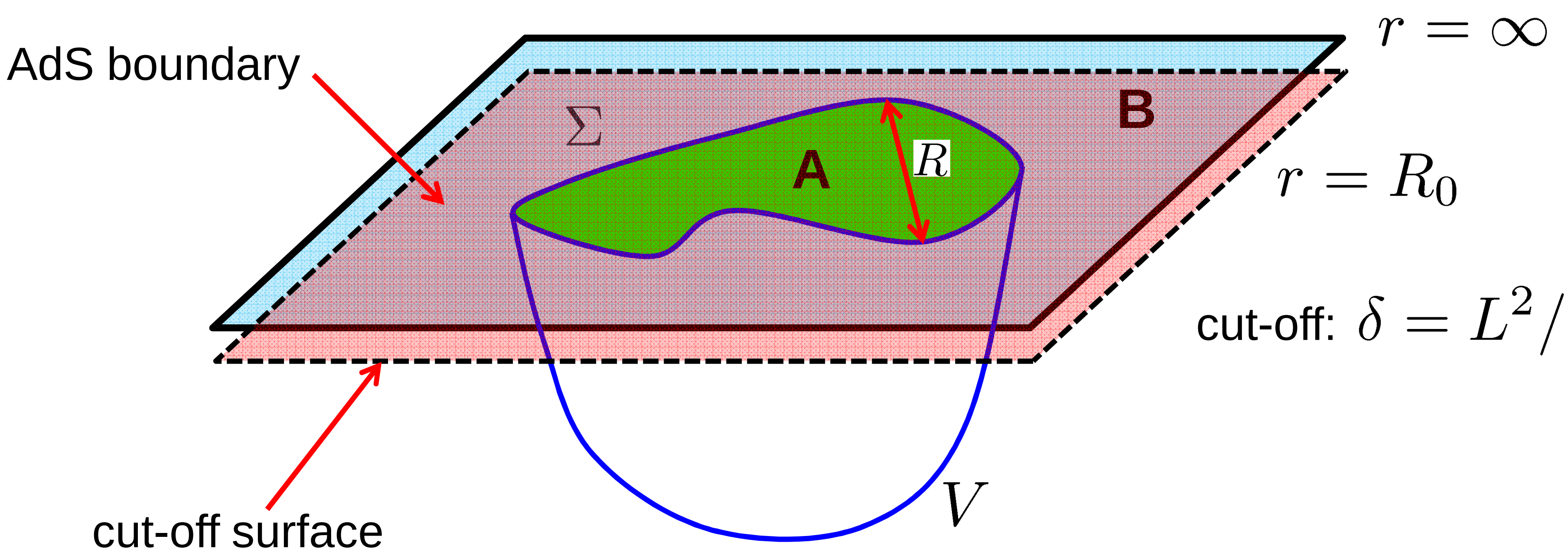
central charge
(counts dof) $(L/\ell_{Planck})^{d-1}$

"Area Law"

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



general expression (as desired):

$$S(A) \simeq c_0(R/\delta)^{d-2} + c_2(R/\delta)^{d-4} + \cdots$$

$$\begin{cases} +c_{d-2}\log(R/\delta) + \cdots & \text{(d even)} \\ +\ c_{d-2}\ + \cdots & \text{(d odd)} \end{cases}$$

universal contributions

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{V\sim A}{\text{ext}} \frac{A_V}{4G_N}$$

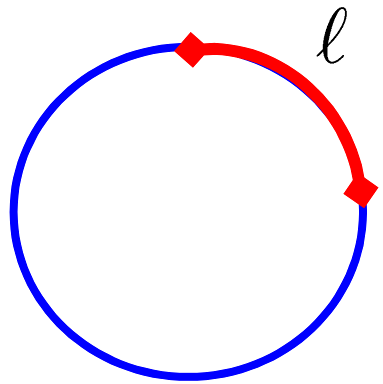
conjecture

Extensive consistency tests:

1) leading contribution yields "area law" $S \simeq \frac{L^{d-1}}{G_N}\frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

2) recover known results of Calabrese & Cardy for d=2 CFT

$$S = \frac{c}{3}\log\left(\frac{C}{\pi\,\delta}\sin\frac{\pi\ell}{C}\right)$$



$C = \text{circumference}$

(also result for thermal ensemble)

(Ryu & Takayanagi `06)

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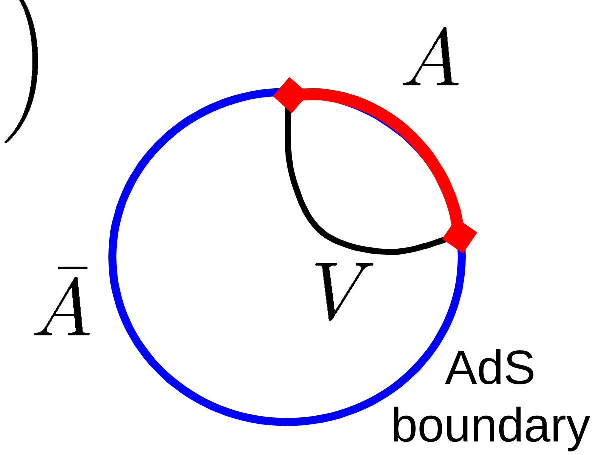


3) $S(A) = S(\bar{A})$ in a pure state

→ A and $\bar{A}$ both yield same bulk surface V

(not pure state → horizon in bulk; $S(A) \neq S(\bar{A})$ for thermal state)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

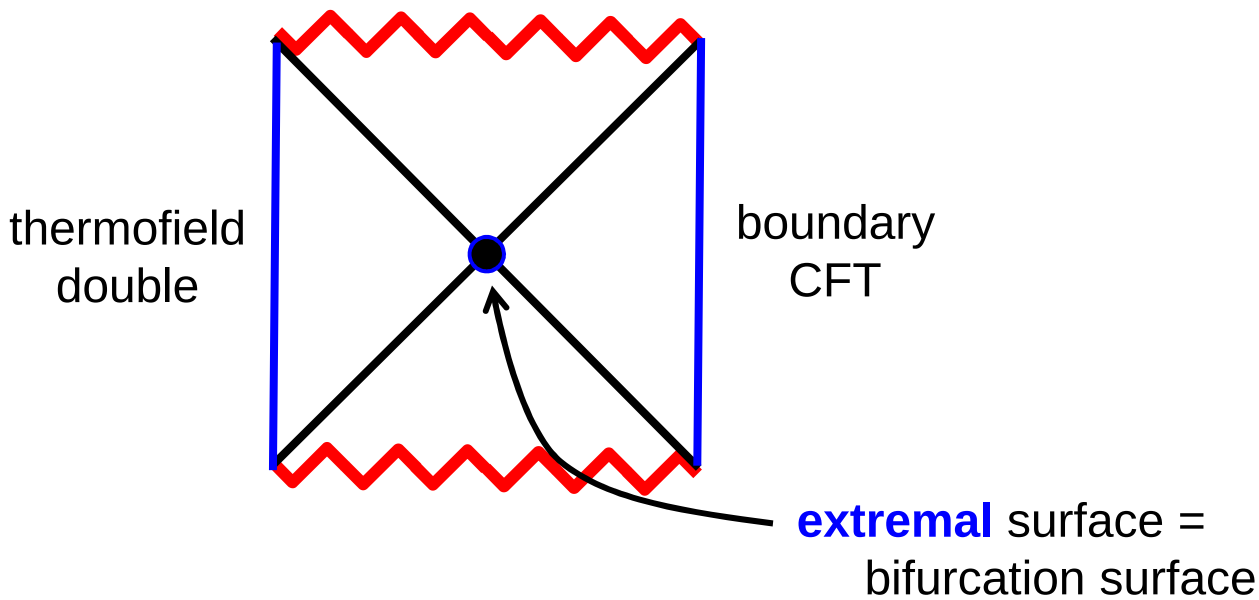
$$S(A) = \underset{V \sim A}{\text{ext}} \frac{A_V}{4G_N}$$

conjecture

Extensive consistency tests:

4) Entropy of eternal black hole =
entanglement entropy of boundary CFT & thermofield double

(Headrick)



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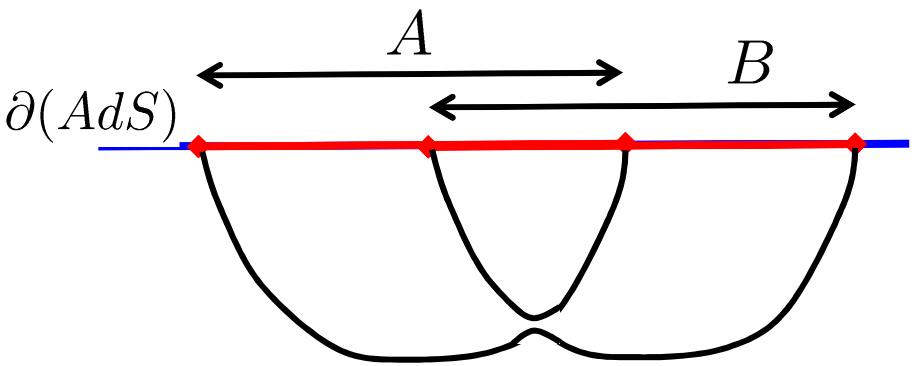
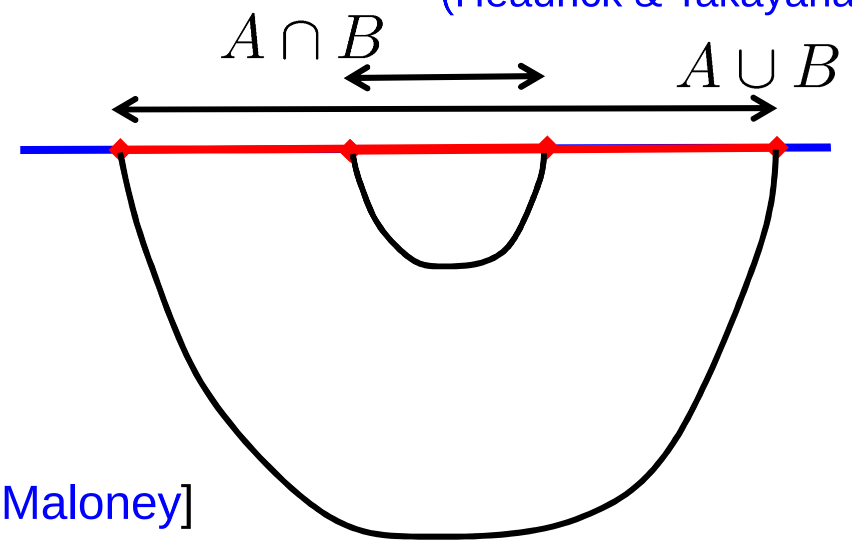
5) sub-additivity: $S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$

(Headrick & Takayanagi)





["all" other inequalities: Hayden, Headrick & Maloney]

(Ryu & Takayanagi `06)

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(see also: RCM & Sinha, arXiv:1011.5819)

Central charges and trace anomaly:

d=2: $$\langle T_\mu{}^\mu \rangle = -\frac{\mathbf{c}}{12} R$$

d=4: $$\langle T_\mu{}^\mu \rangle = \frac{\mathbf{c}}{16\pi^2} I_4 - \frac{\mathbf{a}}{16\pi^2} E_4$$

$I_4 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and $E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$

- in higher (even) dimensions, number of central charges grows

$$\langle T_\mu{}^\mu \rangle = \sum \mathbf{B}_i (\text{Weyl invariants})_i - 2(-)^{d/2} \mathbf{A} (\text{Euler density})_d$$

(Deser & Schwimmer)

- universal contribution to entanglement entropy determined using trace anomaly (for even d)

(Holzhey, Larsen & Wilczek;Calabrese & Cardy;Takayanagi & Ryu;Schwimmer & Theisen)

$$S_{univ} = \log(R/\delta)\; 2\pi \int_\Sigma d^{d-2}x \sqrt{h}\, \frac{\partial \langle T_\lambda{}^\lambda \rangle}{\partial R^{\mu\nu}{}_{\rho\sigma}}\, \hat{\varepsilon}^{\mu\nu}\, \hat{\varepsilon}_{\rho\sigma}$$

(RCM & Sinha)

- partial result! needs rotational symmetry on entangling surface Σ

Central charges and trace anomaly:

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- in higher (even) dimensions, numbers of central charges grows
- universal contribution to entanglement entropy determined using trace anomaly (for even d)

d=2:
$$S = \frac{\mathbf{c}}{3} \log\left(\frac{C}{\pi\,\delta} \sin\frac{\pi \ell}{C}\right)$$

(Holzhey, Larsen & Wilczek; Calabrese & Cardy)

d=4:
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corrections for general (smooth) Σ

(Solodukhin)

Central charges and trace anomaly:

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- in higher dimensions, numbers of central charges grows
- universal contribution to entanglement entropy determined using trace anomaly (for even d)
- central charges identified in AdS/CFT using holographic trace anomaly: (Henningson &Skenderis)

  e.g., for (boundary) d=4: $\boxed{a = c} = \pi^2 L^3/\ell_P^3$
- for general d, central charges $\propto (L/\ell_P)^{d-1}$
- for Einstein gravity, all central charges equal for any d
- distinguishing central charges requires higher curvature gravity

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy:

- consider more general gravity theory in AdS:

$$I = \int d^{d+1}x \sqrt{-g}\, \mathcal{L}(g^{ab}, R^{ab}{}_{cd}, \nabla_e R^{ab}{}_{cd}, \dots, matter)$$

- how do we evaluate holographic entanglement entropy?

⟶ take direction from tests of R&T prescription

(Ryu & Takayanagi `06)

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conjecture

Extensive consistency tests:

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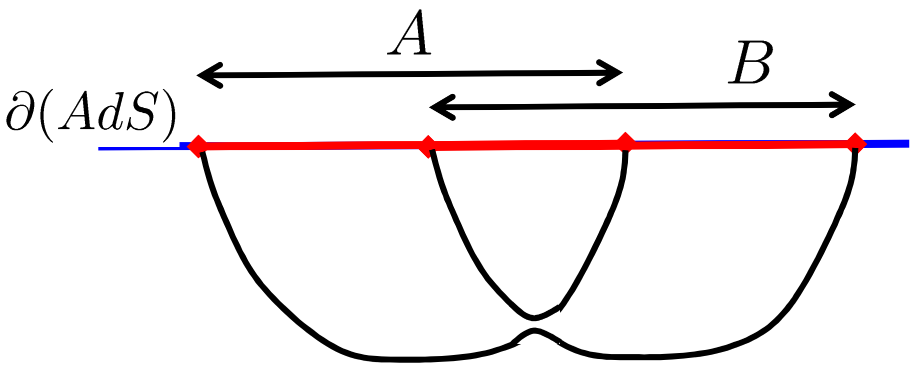

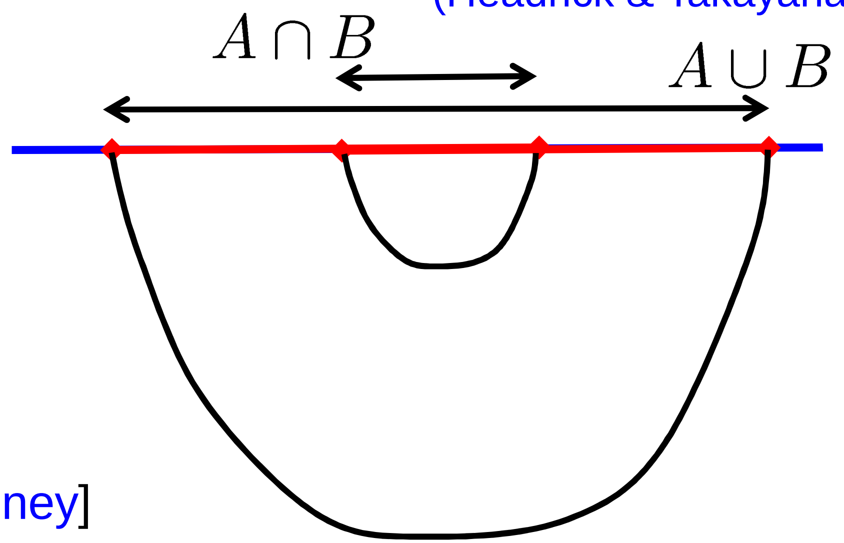

[other inequalities: Hayden, Headrick & Maloney]

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy:

- consider more general gravity theory in AdS:

$$I = \int d^{d+1}x\sqrt{-g}\,\mathcal{L}(g^{ab}, R^{ab}{}_{cd}, \nabla_e R^{ab}{}_{cd}, \dots, matter)$$

- natural conjecture: extremize Wald's entropy formula

$$S = -2\pi \int d^{d-1}x\sqrt{h}\,\frac{\partial \mathcal{L}}{\partial R^{\mu\nu}{}_{\rho\sigma}}\,\hat{\varepsilon}^{\,\mu\nu}\,\hat{\varepsilon}_{\rho\sigma}$$

- focus on universal term for d=4: (Solodukhin)

$$S_{uni} = \log(R/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\,\left[\mathbf{c}\left(C^{ijkl}\,\tilde{g}^{\perp}_{ik}\,\tilde{g}^{\perp}_{jl} - K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

- holographic calculation following above conjecture yields

$$S_{uni} = \log(R/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\,\left[\mathbf{a}\left(C^{ijkl}\,\tilde{g}^{\perp}_{ik}\,\tilde{g}^{\perp}_{jl} - K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

$\longrightarrow$ conjecture wrong 

(Hung, Myers & Smolkin)

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- triumph of R&T prescription in Einstein gravity!! (**c** = **a**)
- for general gravity action, conjecture is wrong
- there is nothing wrong with Wald's formula!!

⟶ to proceed further, focus on special gravity actions

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy:

- consider special case of Gauss-Bonnet gravity:

$$I = \frac{1}{2\ell_p^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2} + R + L^2\frac{\lambda}{2}\,\underbrace{\left(R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2\right)}_{\text{4d Euler density}}\right]$$

4d Euler density

- higher curvature but eom are still **second order!!** (Lovelock)
- studied in detail for stringy gravity in 1980's

(Zwiebach; Boulware & Deser; Wheeler; Myers & Simon; . . . .)

- interest recently in AdS/CFT studies – a toy model with $c \neq a$

(eg, Brigante, Liu, Myers, Shenker,Yaida, de Boer, Kulaxizi, Parnachev, Camanho, Edelstein, Buchel, Sinha, Paulos, Escobedo, Smolkin, Cremonini, Hofman, ...)

- black hole entropy: (Jacobson & Myers)

$$S_{JM} = \frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\,\left[1 + \lambda\,L^2\,\mathcal{R}\right]$$

- not precisely same as Wald entropy – agree when K^i_{ab} vanish

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy: (deBoer, Kulaxizi & Parnachev)

- consider special case of Gauss-Bonnet gravity:

$$I = \frac{1}{2\ell_p^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2} + R + L^2\frac{\lambda}{2}\,\underbrace{\left(R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2\right)}_{\text{4d Euler density}}\right]$$

- second conjecture: extremize JM entropy formula

$$S_{JM} = \frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\,\left[1 + \lambda\,L^2\,\mathcal{R}\right]$$

- again consider universal term for d=4: (Solodukhin)

$$S_{univ} = \log(\ell/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\,\left[\mathbf{c}\left(C^{ijkl}\,\tilde{g}^{\perp}_{ik}\,\tilde{g}^{\perp}_{jl} - K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

- holographic calculation following above conjecture yields

$$S_{univ} = \log(\ell/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\,\left[\mathbf{c}\left(C^{ijkl}\,\tilde{g}^{\perp}_{ik}\,\tilde{g}^{\perp}_{jl} - K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

$\longrightarrow$ passes nontrivial test

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy: (deBoer, Kulaxizi & Parnachev)

- consider special case of Gauss-Bonnet gravity:

$$I = \frac{1}{2\ell_p^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2}+R+L^2\frac{\lambda}{2}\,\underbrace{\left(R^{abcd}R_{abcd}-4R_{ab}R^{ab}+R^2\right)}_{\text{4d Euler density}}\right]$$

- second conjecture: extremize JM entropy formula

$$S_{JM}=\frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\,\left[1+\lambda\,L^2\,\mathcal{R}\right]$$

✓ reproduces universal term for any smooth surface in d=4

✓ partial results for d=6 (geometries with rotational symmetry; found new curvature corrections when $K^i_{ab}=0$)

✓ extends to general Lovelock theories for d≥6

? still no general result for completely general gravity action ?

⟶ with sufficient symmetry, Wald entropy seems correct

? curious instability to adding handles for $\lambda > 0$? (Ogawa & Takayanagi)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) \;=\; \underset{V\sim A}{\mathrm{ext}}\;\frac{A_V}{4G_N}$$

conjecture

Extensive consistency tests:

4) Entropy of eternal black hole =
entanglement entropy of boundary CFT & thermofield double

(Headrick)

5) strong subadditivity: $S(A\cup B)+S(A\cap B)\leq S(A)+S(B)$

(Headrick & Takayanagi)

6) for general even d, connection to central charges of CFT

(Hung, RCM & Smolkin, arXiv:1101.5813)

7) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RCM, arXiv:1102.044)

(see also: RCM & Sinha, arXiv:1011.5819)

Calculating Entanglement Entropy:

$$S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$$

- "standard" approach relies on replica trick and calculating Renyi entropy first and taking n $\longrightarrow$ 1 limit

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

- replica trick involves path integral of QFT in singular n-fold cover of background spacetime

- problematic in holographic framework

  $\longrightarrow$ produce singularity in dual gravity description

  (resolved by quantum gravity/string theory?)

(Fursaev; Headrick)

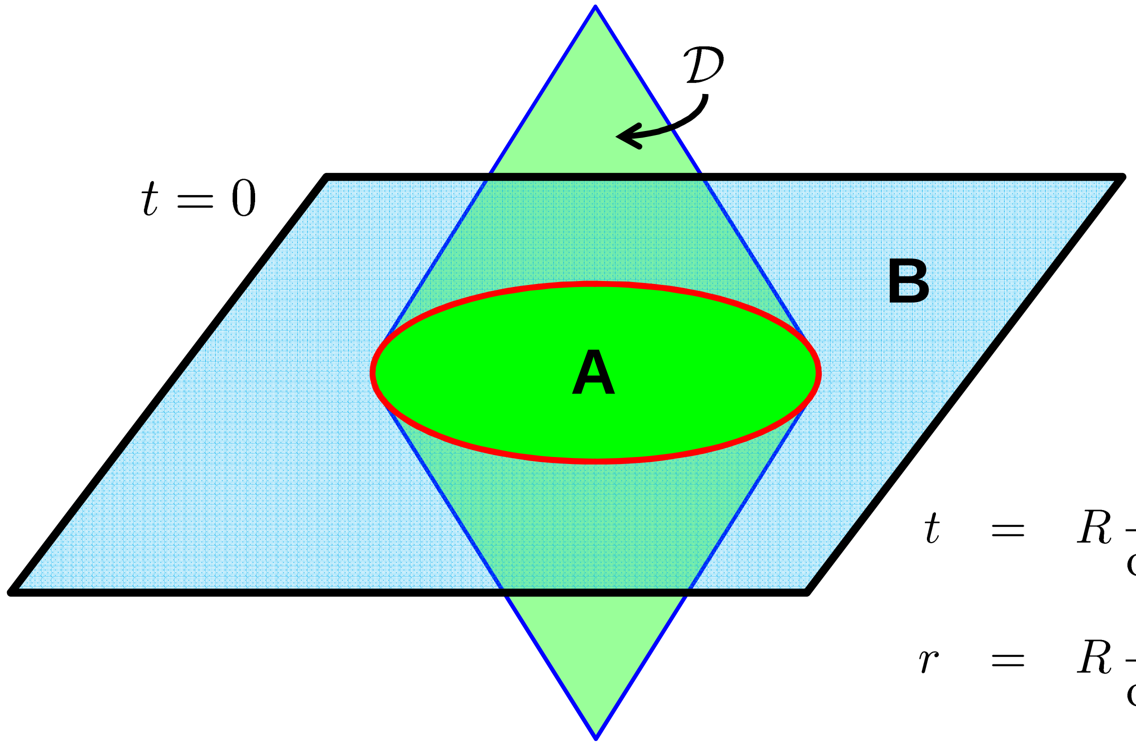
- need another calculation with simpler holographic translation

Calculating Entanglement Entropy:

(Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose $\Sigma = S^{d-2}$ with radius R

  → entanglement entropy: $S_{EE} = -Tr\left[\rho_A \, \log \rho_A\right]$



$$t = R\,\frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)}$$

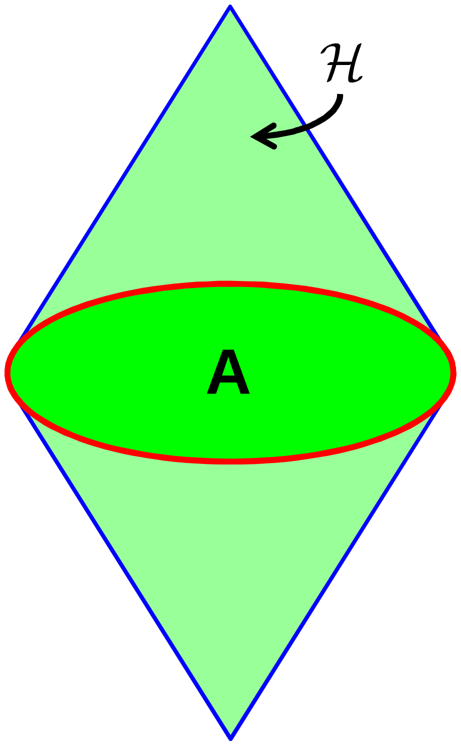
$$r = R\,\frac{\sinh u}{\cosh u + \cosh(\tau/R)}$$

- density matrix ρ_A describes physics in entire causal domain $\mathcal{D}$
- conformal mapping: $\mathcal{D} \rightarrow \mathcal{H} = R \times H^{d-1}$

General result for any CFT (Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose S^{d-2} with radius R

→ entanglement entropy: $S_{EE} = -Tr\,[\rho_A\,\log\rho_A]$



$$t = R\,\frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)}$$
$$r = R\,\frac{\sinh u}{\cosh u + \cosh(\tau/R)}$$

- conformal mapping: $\mathcal{D} \to \mathcal{H} = R \times H^{d-1}$

curvature scale: 1/R temperature: T=1/2πR !!

- for CFT: $\rho_{thermal} = U\,\rho_A\,U^{-1}$ → $S_{EE} = S_{thermal}$

General result for any CFT (Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose S^{d-2} with radius R

→ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$

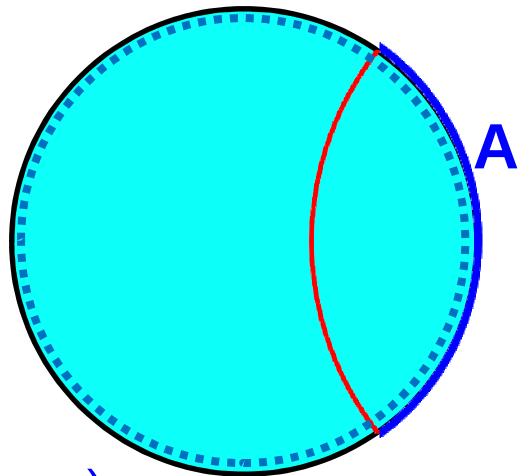
→ by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and $T=1/2\pi R$

$$S_{EE} = S_{thermal}$$

AdS/CFT correspondence:

- thermal bath in CFT = black hole in AdS

$$S_{EE} = S_{thermal} = S_{horizon}$$



- only need to find appropriate black hole

→ topological BH with hyperbolic horizon which intersects A on AdS boundary

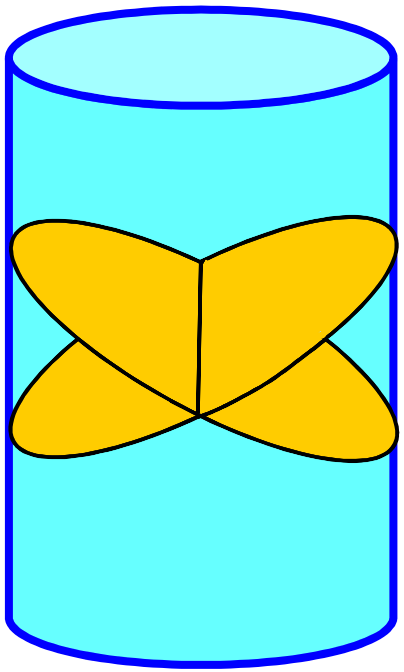
(Aminneborg et al; Emparan; Mann; . . .)

$$S_{EE} = S_{thermal} = S_{horizon}$$

- desired "black hole" is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right) d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \quad \longrightarrow \quad T = \frac{1}{2\pi R}$$

- "Rindler coordinates" of AdS space



$$S_{EE} = S_{thermal} = S_{horizon}$$

- desired “black hole” is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2\, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2}\, d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \quad \longrightarrow \quad T = \frac{1}{2\pi R}$$

- apply Wald’s formula (for any gravity theory) for horizon entropy:

$$\begin{aligned} S &= -2\pi \int d^{d-1}x \sqrt{h}\, \frac{\partial \mathcal{L}}{\partial R^{\mu\nu}{}_{\rho\sigma}}\, \hat{\varepsilon}^{\,\mu\nu}\, \hat{\varepsilon}_{\rho\sigma} \\ &= \frac{2\pi}{\pi^{d/2}} \Gamma\left(d/2\right)\, \frac{a_d^*}{R^{d-1}}\, V\left(H^{d-1}\right) \end{aligned}$$

(RCM & Sinha)

where a_d^* = central charge for “A-type trace anomaly”

for even d

= entanglement entropy defines effective central charge

for odd d

$$S_{EE} = S_{thermal} = S_{horizon}$$

- desired “black hole” is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2\, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2}\, d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \quad \longrightarrow \quad T = \frac{1}{2\pi R}$$

- apply Wald’s formula (for any gravity theory) for horizon entropy:

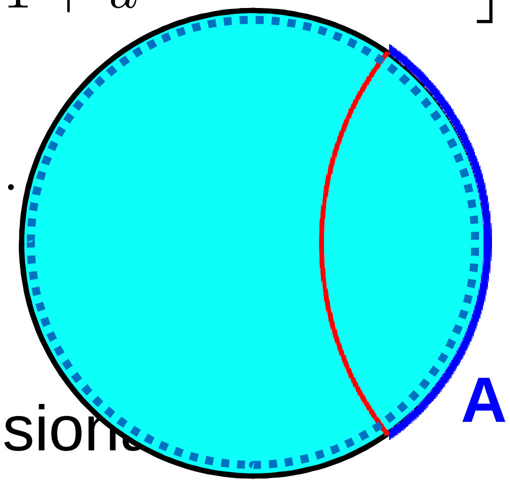
$$S = \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right)\, \frac{a_d^*}{R^{d-1}}\, V\left(H^{d-1}\right)$$

$$ds^2 = R^2\left[\frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2}\right]$$

intersection with standard regulator surface: $z_{min} = \delta$

$$S = a_d^*\, \frac{4\pi^{\frac{d-3}{2}}}{(d-2)\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{R}{\delta}\right)^{d-2} + \cdots$$

“area law” for d-dimension



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$$S = \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right)\,\frac{a_d^*}{R^{d-1}}\,V\left(H^{d-1}\right)$$

$$ds^2 = R^2\left[\frac{du^2}{1+u^2} + u^2\,d\Omega_2^{d-2}\right]$$

universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\,a_d^*\ \log\left(2R/\delta\right) + \cdots \quad \text{for even d}$$

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\,a_d^* + \cdots \quad \text{for odd d}$$

- discussion extends to case with background: $R^{1,d-1} \rightarrow R \times S^{d-1}$

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- turn to Renyi entropy (close cousin of entanglement entropy)

$$S_n = \frac{1}{1-n} \log Tr \left[\rho_A^n\right] \qquad S_{EE} = \lim_{n\to 1} S_n$$

- universal contribution (for even d)

$$S_n = \cdots + constant \times \log\left(R/\delta\right) + \cdots$$

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- turn to Renyi entropy (close cousin of entanglement entropy)

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad S_{EE} = \lim_{n\to 1} S_n$$

- universal contribution (for even d)

d=2: $$S_n = \cdots + \frac{c}{6}\left(1+\frac{1}{n}\right) \log\left(R/\delta\right) + \cdots$$

(Calabrese & Cardy)

- (almost) no calculations for d > 2

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

• turn to Renyi entropy (close cousin of entanglement entropy)

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

• "standard" calculation involves singular n-fold cover of spacetime

⟶ problematic for translation to dual AdS gravity

• our previous derivation lead to thermal density matrix

$$\rho_A = U^{-1}\frac{e^{-H/T_0}}{Tr\left[e^{-H/T_0}\right]}U \qquad \text{with} \qquad T_0 = \frac{1}{2\pi R}$$

⇒ $$Tr\left[\rho_A^n\right] = \frac{Tr\left[e^{-nH/T_0}\right]}{Tr\left[e^{-H/T_0}\right]^n}$$ ⟵ partition function at new temperature, $T = T_0/n$

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

• turn to Renyi entropy (close cousin of entanglement entropy)

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

• "standard" calculation involves singular n-fold cover of spacetime
⟶ problematic for translation to dual AdS gravity

• with bit more work, find convenient formula:

$$S_n = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S(T)dT \quad \text{where} \quad T_0 = \frac{1}{2\pi R}$$

↑ Renyi entropy for spherical Σ

↑ thermal entropy on hyperbolic space H^{d-1}

• in holographic framework, need to know topological black hole solutions for arbitrary temperature

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- Renyi entropy of CFT for spherical entangling surface:

$$S_n = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S(T)dT \quad \text{where} \quad T_0 = \frac{1}{2\pi R}$$

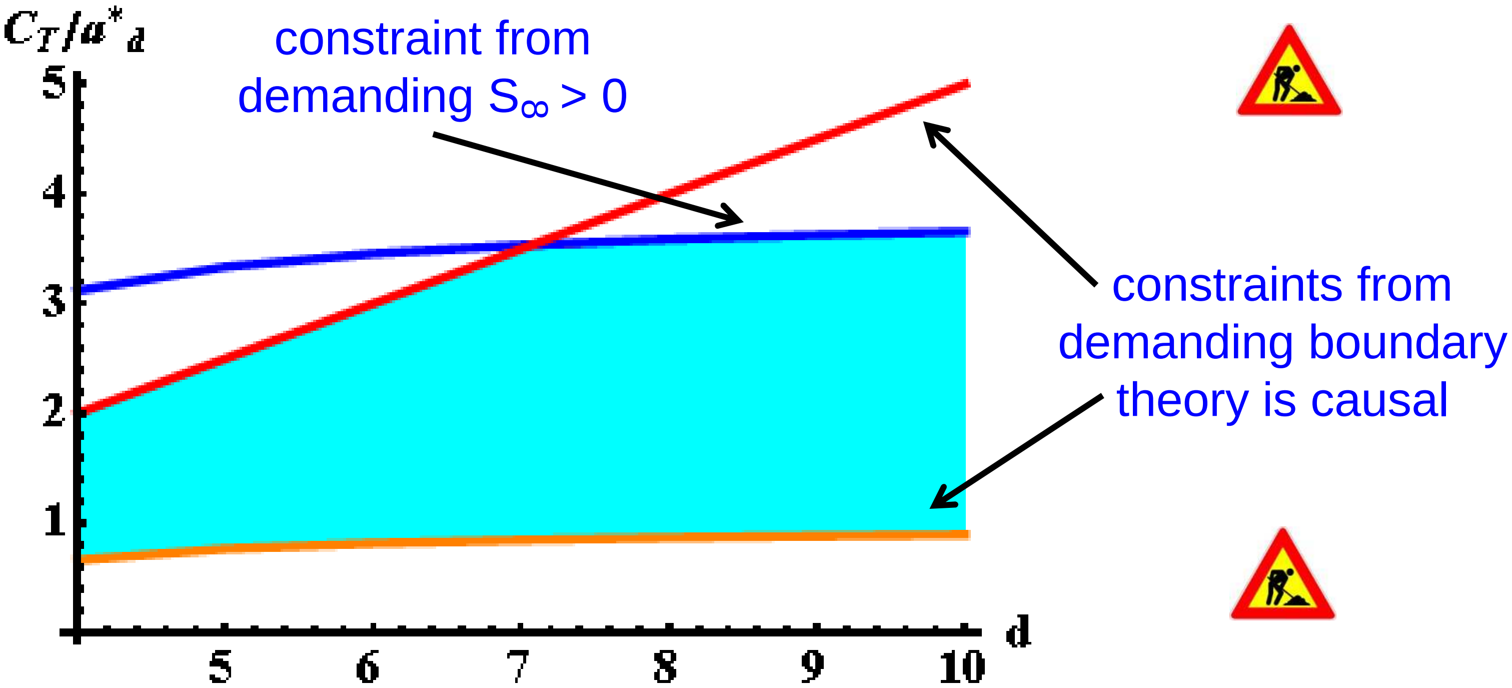
- need to know topological black holes for arbitrary temperature
- focus on gravity theories where we can calculate: Einstein, Gauss-Bonnet, Lovelock, quasi-topological, …..
- for example, with GB gravity and (boundary) d=4:

$$S_n = \frac{n}{n-1}\frac{V(H^3)}{4\pi}\frac{3c-a}{3a-c}(1-x^2)\Big[(5a-c)x^2-(13a-5c) + 4a\frac{2ax^2-(a-c)}{(3a-c)x^2-(a-c)}\Big]$$

$$\text{where} \quad 0 = x^3 - \frac{3a-c}{5a-c}\left(\frac{x^2}{n}+x\right) + \frac{1}{n}\frac{a-c}{5a-c}$$

- no elegant result as was found for d=2 CFT, ie, S_n depends on both central charges and dependence on n does not factor out
- further work (with quasi-topological gravity) shows the universal coefficient depends on more data from the boundary CFT than central charges appearing in the trace anomaly (eg, t_4)
- preliminary work indicates positivity of Renyi entropies may constrain gravitational couplings in higher curvature models

GB gravity:



Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories

- holographic entanglement entropy is part of an interesting dialogue has opened between string theorists and physicists in a variety of fields (eg, condensed matter, nuclear physics, . . .)

- potential to learn lessons about issues in boundary theory
  - eg, readily calculate Renyi entropies for wide class of theories in higher dimensions

- potential to learn lessons about issues in bulk gravity theory
  - eg, holographic entanglement entropy may give new insight into quantum gravity or emergent spacetime

(eg, van Raamsdonk)

Lots to explore!