And Non-Holographic

Holographic C-theorems

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

with A. Sinha, H. Casini & M. Huerta
(1006.1263, 1011.5819, 1102.0440)

Overview:

1. Introductory remarks on c-theorem

2. Holographic c-theorem I: Einstein gravity

3. Holographic c-theorem II: Higher curvature gravity

4. a_d^*, Entanglement Entropy and Beyond

5. Concluding remarks

Zamolodchikov c-theorem (1986):

- renormalization-group (RG) flows can seen as one-parameter motion

$$\frac{d}{dt} \equiv -\beta^i(g)\,\frac{\partial}{\partial g^i}$$

in the space of (renormalized) coupling constants $\{g^i,\ i = 1, 2, 3, \cdots\}$ with beta-functions as "velocities"

- for unitary, renormalizable QFT's in two dimensions, there exists a positive-definite real function of the coupling constants $c(g)$:

  1. monotonically decreasing along flows: $\frac{d}{dt}c(g) \leq 0$
  2. "stationary" at fixed points $g^i = (g^*)^i$:

  $$\beta^i(g^*) = 0 \longleftrightarrow \frac{\partial}{\partial g^i}c(g) = 0$$

  3. at fixed points, it equals central charge of corresponding CFT

  $$c(g^*) = c$$

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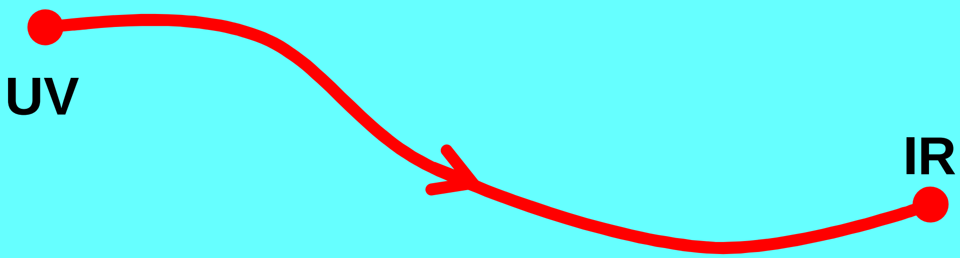
1. mon…

2. “stati…

3. at fix … nding CFT

Consequence for any RG flow:



$$c_{UV} > c_{IR}$$

C-theorems in higher dimensions??

d=2: $$\langle T_\mu{}^\mu \rangle = -\frac{c}{12} R$$

d=4: $$\langle T_\mu{}^\mu \rangle = \cancel{\frac{c}{16\pi^2}} I_4 - \frac{a}{16\pi^2} E_4 - \cancel{\frac{a'}{16\pi^2}} \nabla^2 R$$

$$I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$

- in 4 dimensions, have three central charges: $c,\ a,\ a'$
- do any of these obey a similar “c-theorem” under RG flows?

a'-theorem: a' is scheme dependent (not globally defined)

c -theorem: there are numerous counter-examples

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a -theorem: proposed by Cardy (1988)

- numerous nontrivial examples, eg, perturbative fixed points (Jack & Osborn), SUSY gauge theories (Anselmi et al; Intriligator & Wecht)

(Anselmi, Freedman, Grisaru & Johansen, hep-th/9708042)

SUSY example:

- SU(N_c) supersymmetric QCD with N_f flavors of massless quarks

  with $3/2 \leq N_f/N_c \leq 3$

- in UV, asymptotically free:

$$a_{UV} = \frac{1}{48}\left(9\,N_c^2 - 9 + 2\,N_f\,N_c\right)$$

$$c_{UV} = \frac{1}{24}\left(3\,N_c^2 - 3 + 2\,N_f\,N_c\right)$$

- in IR, flows to nontrivial conformal fixed point:

$$a_{IR} = \frac{3}{16}\left(2\,N_c^2 - 1 - 3\,\frac{N_c^4}{N_f^2}\right)$$

$$c_{IR} = \frac{1}{16}\left(7\,N_c^2 - 2 - 9\,\frac{N_c^4}{N_f^2}\right)$$

(Anselmi, Freedman, Grisaru & Johansen, hep-th/9708042)

SUSY example:

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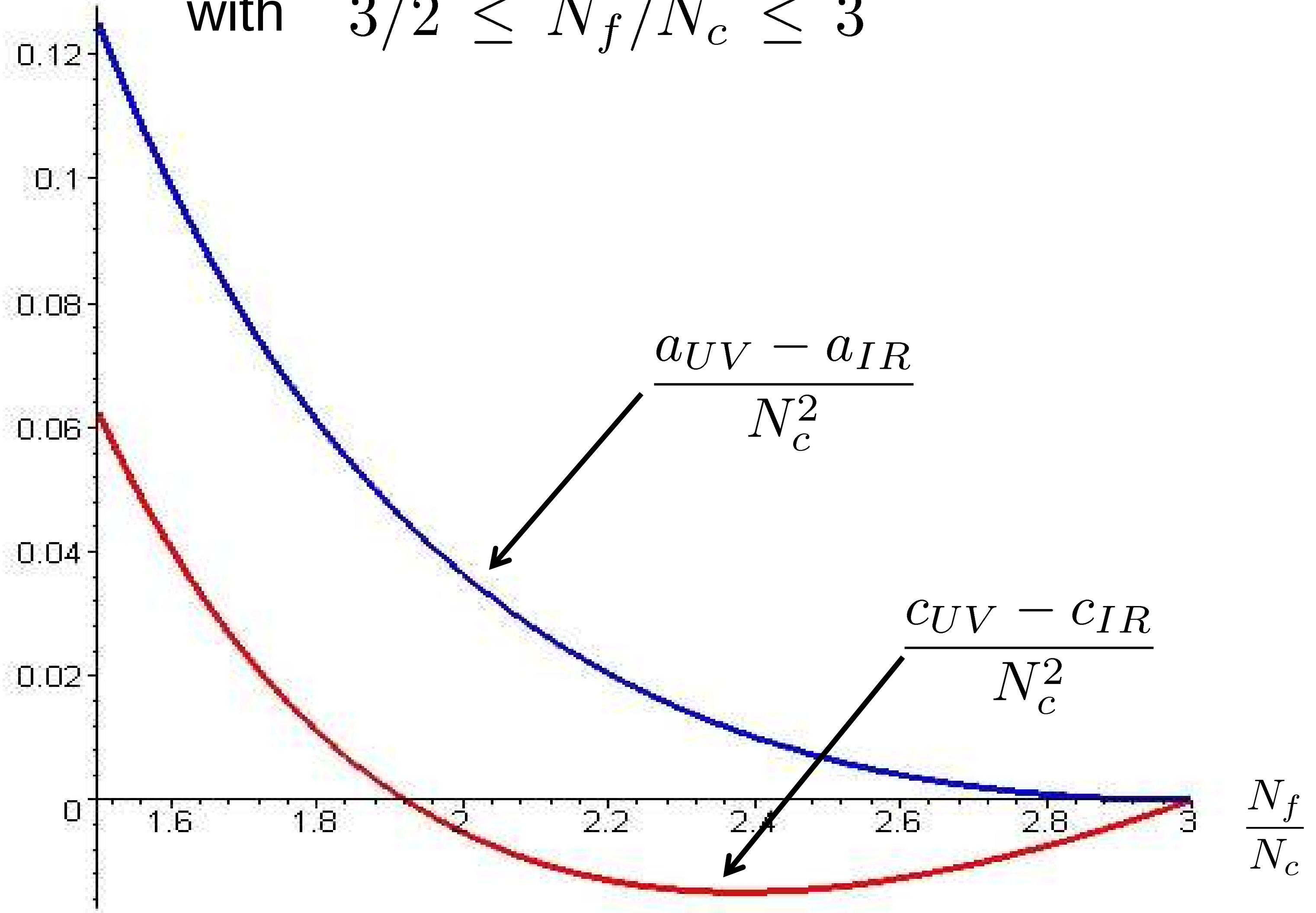

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? ~~• counterexample proposed: Shapere & Tachikawa, 0809.3238~~

(Gaiotto, Seiberg & Tachikawa)

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- holographic field theories with Einstein gravity dual
- holographic theories with higher curvature dual for any d
- F-theorem for d=3 (and general odd d)
- d=4 "proof" using dilaton effective action

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(Freedman, Gubser, Pilch & Warner, hep-th/9904017)
(Girardello, Petrini, Porrati and Zaffaroni, hep-th/9810126)

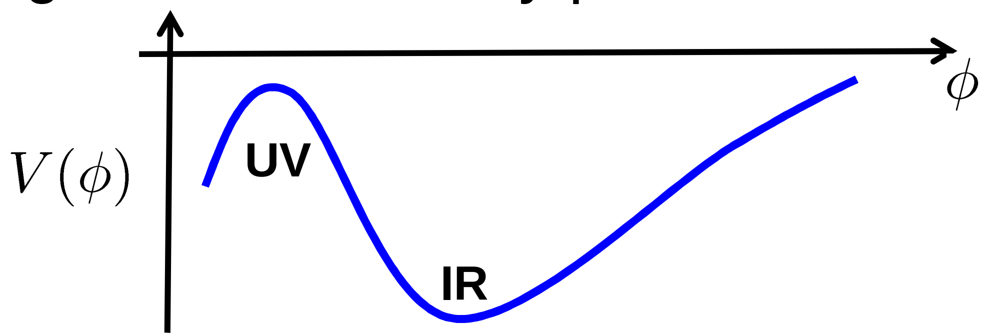
Holographic RG flows:

$$I = \frac{1}{2\ell_P^3}\int d^5x\,\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

- imagine potential has stationary points giving negative Λ

$$\longrightarrow \quad V(\phi_{i,cr}) = -\frac{12}{L^2}\alpha_i^2$$

- consider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2$
- at stationary points, AdS$_5$ vacuum: $A(r) = r/\tilde{L}$ with $\tilde{L} = L/\alpha_i$
- RG flows are solutions starting at one stationary point and ending at another



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Holographic RG flows:

- for general flow solutions, define: $a(r) \equiv \dfrac{\pi^2}{\ell_P^3 A'(r)^3}$

$$a'(r) = -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) = -\frac{\pi^2}{\ell_P^3 A'(r)^4} \left(T^t{}_t - T^r{}_r\right) \geq 0$$

Einstein equations **null energy condition**
$(T_{\mu\nu}\,\ell^\mu\,\ell^\nu \geq 0)$

- at stationary points, $a(r) \to a^* = \pi^2\,\tilde{L}^3/\ell_P^3$ and hence

$$\boxed{a_{UV}^* \geq a_{IR}^*}$$

- using holographic trace anomaly: $a^* = a$

(e.g., Henningson & Skenderis)

$\longrightarrow$ supports Cardy's conjecture

- for Einstein gravity, central charges equal $(a = c)$: $c_{UV} \geq c_{IR}$

(Freedman, Gubser, Pilch & Warner, hep-th/9904017)

Holographic RG flows:

$$I=\frac{1}{2\ell_P^{d-1}}\int d^{d+1}x\,\sqrt{-g}\left[R-\frac{1}{2}(\partial\phi)^2-V(\phi)\right]$$

- same story is readily extended to (d+1) dimensions

- defining: $a_d(r)\equiv\dfrac{\pi^{d/2}}{\Gamma\left(d/2\right)\left(\ell_P A'(r)\right)^{d-1}}$

$$a'(r)\quad=\quad-\frac{(d-1)\pi^{d/2}}{\Gamma\left(d/2\right)\ell_P^{d-1}A'(r)^d}A''(r)=-\frac{\pi^{d/2}}{\Gamma\left(d/2\right)\ell_P^{d-1}A'(r)^d}\left(T^t{}_t-T^r{}_r\right)\geq 0$$

Einstein equations **null energy condition**

- at stationary points, $a(r)\rightarrow a^*=\pi^{d/2}/\Gamma(d/2)\left(\tilde{L}/\ell_P\right)^{d-1}$ and so

$$\boxed{a^*_{UV}\geq a^*_{IR}}$$

- using holographic trace anomaly: $a^*\propto$ central charges

(for even d! what about odd d?) (e.g., Henningson & Skenderis)

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Improved Holographic RG Flows:

- add higher curvature interactions to bulk gravity action

  $\longrightarrow$ provides holographic field theories with, eg, $a \neq c$ so that we can clearly distinguish evidence of a-theorem

  (Nojiri & Odintsov; Blau, Narain & Gava)

- construct "toy models" with fixed set of higher curvature terms (where we can maintain control of calculations)

What about the swampland?

- constrain gravitational couplings with consistency tests (positive fluxes; causality; unitarity) and use best judgement

- seems an effective approach with, e.g., Gauss-Bonnet gravity

  (eg, Brigante, Liu, Myers, Shenker,Yaida, de Boer, Kulaxizi, Parnachev, Camanho, Edelstein, Buchel, Sinha, Paulos, Escobedo, Smolkin, Cremonini, Hofman, ...)

- ultimately one needs to fully develop string theory for interesting holographic backgrounds

(Myers & Robinsion, 1003.5357)

Quasi-Topological gravity:

$$I = \frac{1}{2\ell_P^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2} + R + L^2\frac{\lambda}{2}\chi_4 + L^4\frac{7\mu}{4}\mathcal{Z}_5\right]$$

with $\chi_4 = R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2$

$$\mathcal{Z}_5 = R_a{}^c{}_b{}^d R_d{}^e{}_c{}^f R_e{}^a{}_f{}^b + \frac{1}{56}\left(21R_{abcd}R^{abcd}R - 72R_{abcd}R^{abc}{}_e R^{de} + 120R_{abcd}R^{ac}R^{bd} + 144R_a{}^bR_b{}^cR_c{}^a - 132R_a{}^bR_b{}^aR + 15R^3\right)$$

- three dimensionless couplings, $L/\ell_P\,,\ \lambda\,,\ \mu\,,$ allow us to explore dual CFT's with most general three-point function $\langle T_{ab}\,T_{cd}\,T_{ef}\rangle$

"maintain control of calculations"

- analytic black hole solutions
- linearized eom in AdS$_5$ are second order (in fact, Einstein eq's!)
- can be extended to higher dimensions (D≥7)
- gravitational couplings constrained (Myers, Paulos & Sinha, 1004.2055)

Quasi-Topological gravity:

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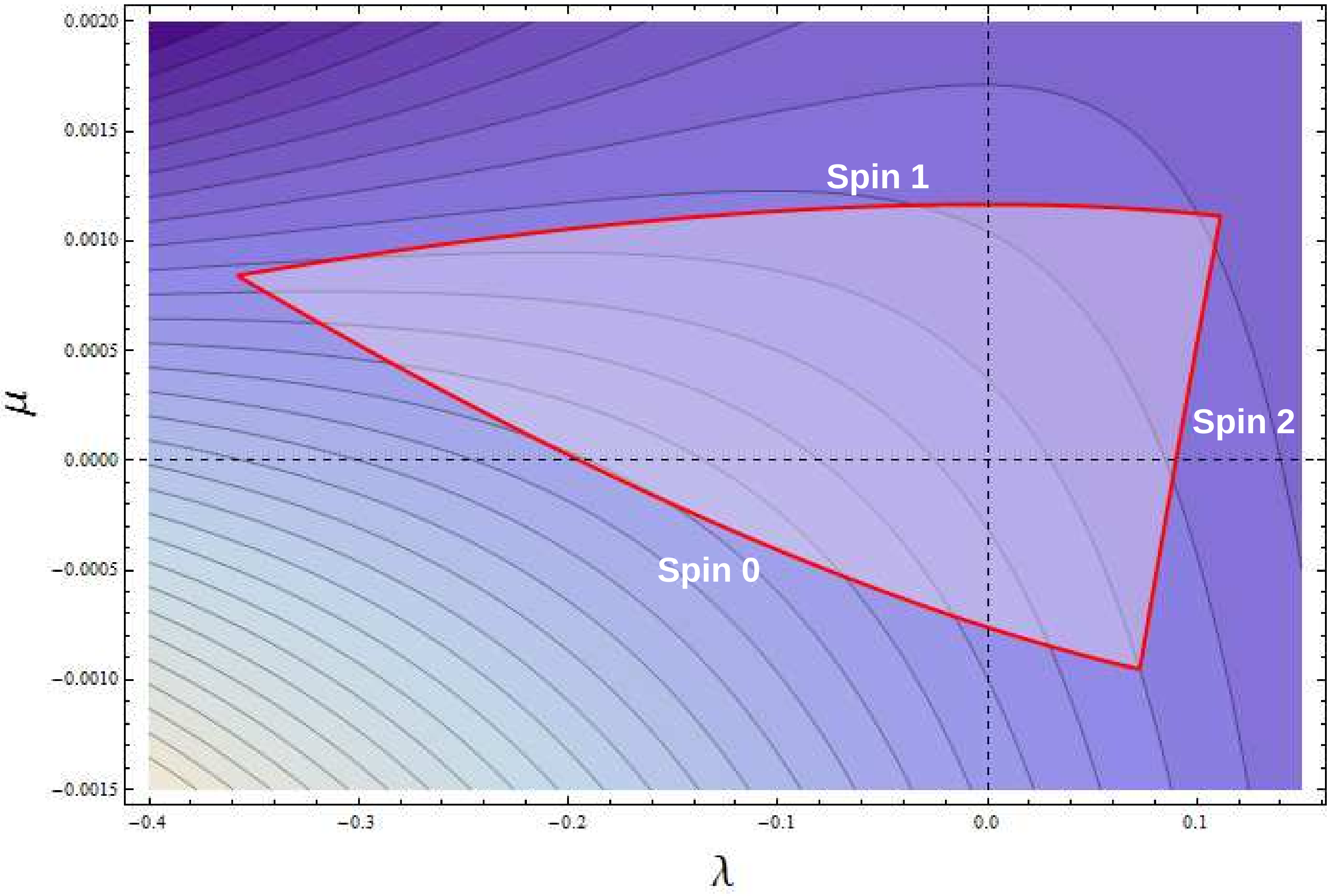

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- let's calculate!
- curvature in AdS$_5$ vacuum: $\dfrac{1}{\tilde{L}^2} = \dfrac{f_\infty}{L^2}$,

where $\quad \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

- holographic trace anomaly:

(Myers, Paulos & Sinha, 1004.2055)

$$a = \pi^2\frac{\tilde{L}^3}{\ell_P^3}\left(1 - 6\lambda f_\infty + 9\mu f_\infty^2\right)\,, \qquad c = \pi^2\frac{\tilde{L}^3}{\ell_P^3}\left(1 - 2\lambda f_\infty - 3\mu f_\infty^2\right)$$

RG flows in Quasi-Topological gravity:

- consider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2$

$\longrightarrow$ AdS_5 vacua: $A(r) = r/\tilde{L}$

- need to define “flow functions” which extend

$$a = \pi^2 \frac{\tilde{L}^3}{\ell_P^3} \left(1 - 6\lambda f_\infty + 9\mu f_\infty^2\right) \,,$$
$$c = \pi^2 \frac{\tilde{L}^3}{\ell_P^3} \left(1 - 2\lambda f_\infty - 3\mu f_\infty^2\right)$$

for general flows away from fixed points

RG flows in Quasi-Topological gravity:

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  $\longrightarrow$ AdS$_5$ vacua: $A(r) = r/\tilde{L}$

- natural to define "flow functions":

$$a(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3}\,\left(1 - 6\lambda L^2 A'(r)^2 + 9\mu L^4 A'(r)^4\right)$$

$$c(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3}\,\left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right)$$

  where at stationary points: $a(r) = a,\ c(r) = c$

- "simplest" r-dependent functions satisfying this condition

RG flows in Quasi-Topological gravity:

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where at stationary points: $a(r) = a, \; c(r) = c$

- in general flows:

$$\begin{aligned} a'(r) &= -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) \left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right) \\ &= -\frac{\pi^2}{\ell_P^3 A'(r)^4} \left(T^t{}_t - T^r{}_r\right) \geq 0 \end{aligned}$$

✓

assume null energy condition

gravitational equations of motion

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RG flows in Quasi-Topological gravity:

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- can try to be more creative in defining c(r) but we were unable to find a expression where flow is guaranteed to be monotonic

- our toy model seems to provide support for Cardy's "a-theorem" in four dimensions

Higher Dimensions: $D = d+1 \quad (\mu = 0 \text{ for } d=5)$

- straightforward to reverse engineer "a-theorem" flows
- eq's of motion:

$$T^t{}_t - T^r{}_r = (d-1)\,A''(r)\,\left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right)$$

- expression with natural flow:

$$a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2)\,(\ell_P A'(r))^{d-1}}\left(1 - \frac{2(d-1)}{d-3}\lambda L^2 A'(r)^2 - \frac{3(d-1)}{d-5}\mu L^4 A'(r)^4\right)$$

$$\Longrightarrow \quad a_d'(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)\ell_P^{d-1} A'(r)^d}\left(T^t{}_t - T^r{}_r\right) \geq 0$$

assume null energy condition

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- flow between stationary points (where $a_d^* \equiv a_d(r)|_{AdS}$)

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

What is a_d^* ??

$$a_d^* = \frac{\pi^{d/2}\tilde{L}^{d-1}}{\Gamma(d/2)\ell_P^{d-1}}\left(1 - \frac{2(d-1)}{d-3}\lambda f_\infty - \frac{3(d-1)}{d-5}\mu f_\infty^2\right)$$

where AdS curvature: $\frac{1}{\tilde{L}^2} = \frac{f_\infty}{L^2}\,, \quad \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

- a_d^* is NOT C_T , coefficient of leading singularity in

$$\langle\, T_{ab}(x)\, T_{cd}(0)\,\rangle = \frac{C_T}{x^{2d}}\,\mathcal{I}_{ab,cd}(x)$$

- a_d^* is NOT C_S , coefficient in entropy density: $s = C_S\, T^{d-1}$

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- trace anomaly for CFT's with even d: (Deser & Schwimmer)

$$\langle\, T_\mu{}^\mu\,\rangle = \sum B_i(\text{Weyl invariant})_i - 2(-)^{d/2} A\,(\text{Euler density})_d$$

- verify that we have precisely reproduced central charge

$$\boxed{a_d^* = A}$$

(Henningson & Skenderis; Nojiri & Odintsov; Blau, Narain & Gava; Imbimbo, Schwimmer, Theisen & Yankielowicz)

$\longrightarrow$ agrees with Cardy's proposal

What is a_d^* ??

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What is a_d^* for odd d?? **(One moment!)**

How robust is Holographic C-theorem?:

- quasi-topological gravity obeys c-theorem in very nontrivial way
- generalize to start with arbitrary curvature-cubed action

$$I = \frac{1}{2\ell_P^{d-1}} \int \mathrm{d}^{d+1}x \sqrt{-g} \left[\frac{d(d-1)}{L^2}\alpha^2 + R + L^2 \widetilde{\mathcal{X}} + L^4 \widetilde{\mathcal{Z}} \right]$$

where
$$\begin{aligned}
\widetilde{\mathcal{X}} &= b_1 R_{abcd}R^{abcd} + b_2 R_{ab}R^{ab} + b_3 R^2 \ , \\
\widetilde{\mathcal{Z}} &= c_1\, R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b + c_2\, R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} + c_3\, R_{abcd}R^{abc}{}_e R^{de} \\
&\quad + c_4\, R_{abcd}R^{abcd}R + c_5\, R_{abcd}R^{ac}R^{bd} + c_6\, R_a{}^b R_b{}^c R_c{}^a \\
&\quad + c_7\, R_a{}^b R_b{}^a R + \ c_8\, R^3 \, .
\end{aligned}$$

- AdS vacua: $\dfrac{1}{\tilde{L}^2} = \dfrac{f_\infty}{L^2}$, where $\quad \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

$$\begin{aligned}
\lambda &= \frac{d-3}{d-1}\,(2\, b_1 + d\, b_2 + d(d+1)\, b_3) \\
\mu &= -\frac{d-5}{d-1}\,((d-1)\, c_1 + 4\, c_2 + 2d\, c_3 + 2d(d+1)\, c_4 \\
&\qquad + d^2\, c_5 + d^2\, c_6 + d^2(d+1)\, c_7 + d^2(d+1)^2 c_8)
\end{aligned}$$

More Improved Holographic RG Flows:

- is it reasonable to expect any theory to obey a c-theorem? **NO**
- how do we constrain theory to be physically reasonable?
- recall one of the nice properties of quasi-top. gravity was that linearized graviton equations in AdS were 2nd order
- greatly facilitates calculations but deeper physical significance
- analogy with higher derivative scalar field eq. (in flat space)

$$\left(\nabla^2+\frac{a}{M^2}(\nabla^2)^2\right)\phi=0 \longrightarrow \frac{1}{q^2(1-a\,q^2/M^2)}=\frac{1}{q^2}\underset{\substack{\uparrow\\ \text{ghost}}}{-}\frac{1}{q^2-M^2/a}$$

- graviton ghosts will be generic with 4th order equations

$\longrightarrow$ couple to additional non-unitary tensor operator in dual CFT

More Improved Holographic RG Flows:

$$I = \frac{1}{2\ell_P^{d-1}} \int \mathrm{d}^{d+1}x \sqrt{-g} \left[\frac{d(d-1)}{L^2}\alpha^2 + R + L^2 \widetilde{\mathcal{X}} + L^4 \widetilde{\mathcal{Z}} \right]$$

where
$$\begin{aligned}
\widetilde{\mathcal{X}} &= b_1 R_{abcd}R^{abcd} + b_2 R_{ab}R^{ab} + b_3 R^2 \ , \\
\widetilde{\mathcal{Z}} &= c_1\, R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} + c_2\, R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} + c_3\, R_{abcd}R^{abc}{}_e R^{de} \\
&\quad + c_4\, R_{abcd}R^{abcd} R + c_5\, R_{abcd}R^{ac}R^{bd} + c_6\, R_a{}^b R_b{}^c R_c{}^a \\
&\quad + c_7\, R_a{}^b R_b{}^a R + c_8\, R^3 \, .
\end{aligned}$$

- demand linearized graviton equations are 2nd order in RG flow

i.e., around background geometry: $ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}^2) + dr^2$

More Improved Holographic RG Flows:

$$I=\frac{1}{2\ell_P^{d-1}}\int \mathrm{d}^{d+1}x\,\sqrt{-g}\,\left[\frac{d(d-1)}{L^2}\alpha^2+R+L^2\widetilde{\mathcal{X}}+L^4\widetilde{\mathcal{Z}}\right]$$

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\widetilde{\mathcal{X}} &= b_1 R_{abcd}R^{abcd}+b_2 R_{ab}R^{ab}+b_3 R^2\ , \\
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&\quad +c_7\, R_a{}^bR_b{}^aR+\ c_8\, R^3\,.
\end{aligned}$$

- demand linearized graviton equations are 2nd order in RG flow

$$b_2=-4b_1\ ,\qquad b_3=b_1$$

R^2 interaction is χ_4

$$c_8 = \frac{1}{d(d+1)}\Big(-\frac{d+5}{2(d+1)}c_1+\frac{2(d+9)}{d+1}c_2+\frac{d+8}{3}c_3$$

5 constraints → 3 free parameters

$$c_7 = \frac{1}{d(d+1)}\Big(3c_1-24c_2-4(d+1)c_3-4d(d+1)c_4-(2d-1)c_5-3d\,c_6\Big)$$

. . . .

More Improved Holographic RG Flows:

- as before, try to reverse engineer "c-theorem" flows
- with above constraints, flow eq's of motion yield:

$$T^t{}_t - T^r{}_r = (d-1)\, A''(r)\, \left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right)$$

- expression with natural flow:

$$a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2)\,(\ell_P A'(r))^{d-1}} \left(1 - \frac{2(d-1)}{d-3}\lambda L^2 A'(r)^2 - \frac{3(d-1)}{d-5}\mu L^4 A'(r)^4\right)$$

$$\Longrightarrow \quad a_d'(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)\ell_P^{d-1} A'(r)^d}\left(T^t{}_t - T^r{}_r\right) \geq 0$$

- flow between stationary points (where $a_d^* \equiv a_d(r)|_{AdS}$)

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

- also extends to Lovelock and R^n theories (Liu, Sabra & Zhao)
- for even d, find same match: $a_d^* = A \longrightarrow$ Cardy's proposal

What about odd d?

Overview:

1. Introductory remarks on c-theorem
2. Holographic c-theorem I: Einstein gravity
3. Holographic c-theorem II: Higher curvature gravity
4. a_d^*, Entanglement Entropy and Beyond
5. Concluding remarks

General result for any CFT

(Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose S^{d-2} with radius R

→ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$

→ by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and $T=1/2\pi R$

$$\boxed{S_{EE} = S_{thermal}}$$

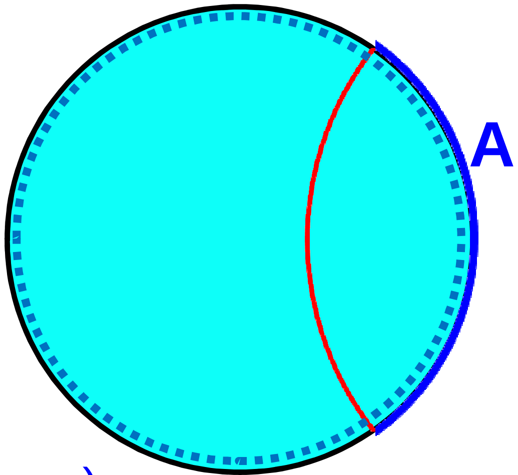
AdS/CFT correspondence:

- thermal bath in CFT = black hole in AdS

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

- only need to find appropriate black hole

→ topological BH with hyperbolic horizon which intersects A on AdS boundary



(Aminneborg et al; Emparan; Mann; . . .)

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

- desired “black hole” is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2\, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2}\, d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \longrightarrow T = \frac{1}{2\pi R}$$

- apply Wald’s formula (for any gravity theory) for horizon entropy:

$$S = \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right)\, \frac{a_d^*}{R^{d-1}}\, V\left(H^{d-1}\right)$$

$$ds^2 = R^2\left[\frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2}\right]$$

universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \log\left(2R/\delta\right) + \cdots \quad \text{for even d}$$

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots \quad \text{for odd d}$$

- discussion extends to case with background: $R^{1,d-1} \rightarrow R \times S^{d-1}$

Conjecture:

- entanglement entropy of ground state of CFT across sphere S^{d-2} of radius R has universal contribution:

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^* \log(2R/\delta) & \text{for even } d \\ (-)^{\frac{d-1}{2}}\, 2\pi\, a_d^* & \text{for odd } d \end{cases}$$

(any gravitational action)

- in RG flows between fixed points (any CFT in even d with $a_d^* = A$)

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

("unitary" models)

→ behaviour discovered for holographic model but conjecture that result applies generally (outside of holography)

→ gives framework to consider c-theorem for **odd** or even *d*

and Beyond:

- Susskind & Witten: density of degrees of freedom in N=4 SYM connected to area of holographic screen at large R in AdS_5

$$\frac{V_3}{\delta^3} \times N_c^2 \sim \frac{A(R)}{\ell_P^3}$$

cut-off scale defined by regulator radius: $\frac{1}{\delta} = \frac{R}{L^2}$

- given higher curvature bulk action, natural extension is to evaluate Wald entropy on holographic screen at large R

$$S = -2\pi \oint d^{d-1}x\sqrt{h}\ \hat{\varepsilon}^{ab}\ \hat{\varepsilon}_{cd}\ \frac{\partial \mathcal{L}_{bulk}}{\partial R^{ab}{}_{cd}}$$

- straightforward evaluate “entropy” for count of density of dof

$$\boxed{S = \frac{2}{\pi}\, a_d^*\, \frac{V_{d-1}}{\delta^{d-1}}}$$

for any covariant action: $\mathcal{L}_{bulk} = \mathcal{L}_{bulk}\left(g^{ab}, R^{ab}{}_{cd}, \nabla_e R^{ab}{}_{cd}, \cdots\right)$

F-theorem:

(Jafferis, Klebanov, Pufu & Safdi)

- examine partition function for broad classes of 3-dimensional quantum field theories (SUSY and non-SUSY) on three-sphere
- in all examples, F= – log Z >0 and decreases along RG flows
- **coincides with our conjectured c-theorem!** (Casini, Huerta & RCM)
- consider S_{EE} of d-dimensional CFT for sphere S^{d-2} of radius R
- conformal mapping: causal domain $\mathcal{D} \to$ (static patch of) dS_d

curvature ~ 1/R and thermal state: $\rho = \exp[-2\pi R\, H_\tau]/Z$

$$\longrightarrow \quad S_{EE} = S_{thermal} = \beta\langle H_\tau\rangle + \log Z$$

F-theorem:

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  curvature ~ 1/R and thermal state: $\rho = \exp[-2\pi R\, H_\tau]/Z$

$$\longrightarrow \quad S_{EE} = S_{thermal} = \cancel{\beta\langle H_\tau\rangle} + \log Z$$

- stress-energy fixed by trace anomaly – vanishes for odd d!
- upon passing to Euclidean time with period $2\pi R$:

$$S_{EE} = \log Z|_{S^d} \quad \text{for any odd d}$$

F-theorem:

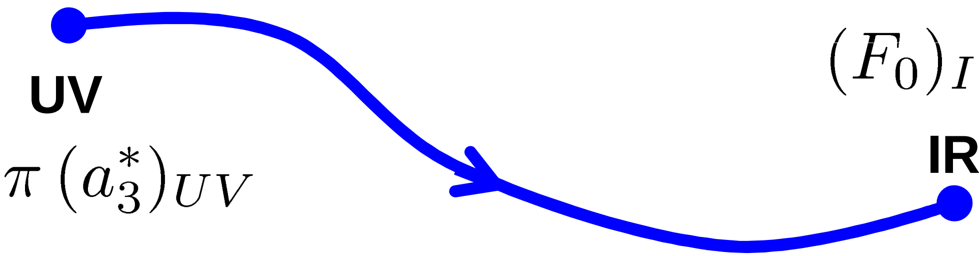
- must focus on renormalized or universal contributions, eg,

$$F_0 = -\log Z|_{finite} = -S_{univ} = 2\pi\, a_3^* \,.$$

- generalizes to general odd d:

$$(-)^{\frac{d-1}{2}} \log Z|_{finite} = (-)^{\frac{d-1}{2}} S_{univ} = 2\pi\, a_d^* \,.$$

- equivalence shown only for fixed points but good enough:



$$(F_0)_{IR} = 2\pi\, (a_3^*)_{IR}$$

$$(F_0)_{UV} = 2\pi\, (a_3^*)_{UV}$$

- evidence for F-theorem (SUSY, perturbed CFT’s & O(N) models) supports present conjecture and our holographic analysis provides additional support for F-theorem

(Komargodski & Schwimmer ; see also: Schwimmer & Theisen)

RG Flows and Dilaton Effective Action

- think of RG flow as “spontaneously broken conformal symmetry”
- couple theory to “dilaton” (conformal compensator) and organize effective action in terms of $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$

  diff X Weyl invariant: $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \quad \tau \to \tau + \sigma$

- introduce UV kinetic term: $\frac{f^2}{6}\int d^4x\sqrt{\hat{g}}\hat{R} \;=\; f^2\int d^4x e^{-2\tau}(\partial\tau)^2$
- follow effective action to IR fixed point, eg,

$$S_{anomaly} = -\delta a\int d^4x\sqrt{-g}\Big(\tau E_4 + 4(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\,\tau + 2(\partial\tau)^4\Big)$$

$\delta a = a_{UV} - a_{IR}$: ensures UV & IR anomalies match

- with $g \to \eta$,this term is only contribution to 4pt scattering:

$$S_{anomaly} = 2\,\delta a\int d^4x\,(\partial\tau)^4$$

- causality or analyticity arguments demand: $\delta a > 0$ (Adams et al)

(Komargodski & Schwimmer ; see also: Schwimmer & Theisen)

RG Flows and Dilaton Effective Action

- causality or analyticity arguments demand:

$$\delta a = a_{UV} - a_{IR} > 0$$

- proof of a-theorem??:

$\longrightarrow$ assumes RG flow from UV fixed pt to IR fixed pt

(Fortin, Grinstein & Stergiou)

- recent work suggests there exist d=4 QFT's which are scale invariant but **not** conformally invariant
- further suggests RG flows exhibit limit cycle behaviour

$\longrightarrow$ full structure of d=4 RG flows still to explore

Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories
- toy theories with higher-R interactions extend class of CFT's

$\longrightarrow$ maintain calculational control with LL or quasi-top. gravity

- consistency (e.g., causality & unitarity) constrains couplings
- provide interesting insights into RG flows
- naturally support Cardy's version of "A-theorem" with d even
- suggests extension of c-theorem to d odd
- a_d^* seems to play a privileged role in holography
- further implications for holographic dualities??
- can entanglement entropy lead to proof of a-theorem?

(Casini & Huerta)

Lots to explore!