Overview:

1. Introductory remarks on c-theorem
2. Holographic c-theorem I: Einstein gravity
3. Holographic c-theorem II: Higher curvature gravity
4. $a_d^*$, Entanglement Entropy and Beyond
5. Concluding remarks
Zamolodchikov c-theorem (1986):

- renormalization-group (RG) flows can seen as one-parameter motion
  \[
  \frac{d}{dt} = -\beta^i(g) \frac{\partial}{\partial g^i}
  \]
  in the space of (renormalized) coupling constants \( \{g^i, \, i = 1, 2, 3, \ldots\} \) with beta-functions as “velocities”

- for unitary, renormalizable QFT’s in two dimensions, there exists a positive-definite real function of the coupling constants \( c(g) \):
  1. monotonically decreasing along flows: \( \frac{d}{dt} c(g) \leq 0 \)
  2. “stationary” at fixed points \( g^i = (g^*)^i \):
     \[
     \beta^i(g^*) = 0 \iff \frac{\partial}{\partial g^i} c(g) = 0
     \]
  3. at fixed points, it equals central charge of corresponding CFT
     \[
     c(g^*) = c
     \]
Zamolodchikov c-theorem (1986):

• renormalization-group (RG) flows can be seen as one-parameter motion

\[ \frac{d}{dt} \equiv -\beta^i(g) \frac{\partial}{\partial g^i} \]

in the space of (renormalized) coupling constants \(\{g^i, \ i = 1, 2, 3, \cdots\}\) with beta-functions as “velocities”

• for unitary, renormalizable QFT’s in two dimensions, there exists a positive-definite real function of the coupling constants \(c(g)\):

  1. monotonically decreasing along flows:
  2. “stationary” at fixed points:
  3. at fixed points, it equals the central charge of the corresponding CFT

Consequence for any RG flow:

\( C_{UV} > C_{IR} \)
C-theorems in higher dimensions??

d=2: \[ \langle T_\mu^\mu \rangle = -\frac{c}{12} R \]
d=4: \[ \langle T_\mu^\mu \rangle = \frac{1}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2} \nabla^2 R \]

\[ I_4 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \]

• in 4 dimensions, have three central charges: \(c, a, a'\)
• do any of these obey a similar “c-theorem” under RG flows?

\[ a' \text{-theorem: } a' \text{ is scheme dependent (not globally defined)} \]

\[ c \text{-theorem: } \text{there are numerous counter-examples} \]
C-theorems in higher dimensions??

\[
\begin{align*}
d=2: & \quad \langle T^\mu_\mu \rangle = -\frac{c}{12} R \\
d=4: & \quad \langle T^\mu_\mu \rangle = \frac{1}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2} \nabla^2 R \\
I_4 &= C^{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2
\end{align*}
\]

- in 4 dimensions, have three central charges: \(c, a, a'\)
- do any of these obey a similar “c-theorem” under RG flows?

\(a\)-theorem: proposed by Cardy (1988)

- numerous nontrivial examples, eg, perturbative fixed points (Jack & Osborn), SUSY gauge theories (Anselmi et al; Intriligator & Wecht)
SUSY example:

- SU($N_c$) supersymmetric QCD with $N_f$ flavors of massless quarks

  with \[ \frac{3}{2} \leq \frac{N_f}{N_c} \leq 3 \]

- in UV, asymptotically free:

  \[
  a_{UV} = \frac{1}{48} \left( 9 N_c^2 - 9 + 2 N_f N_c \right)
  \]

  \[
  c_{UV} = \frac{1}{24} \left( 3 N_c^2 - 3 + 2 N_f N_c \right)
  \]

- in IR, flows to nontrivial conformal fixed point:

  \[
  a_{IR} = \frac{3}{16} \left( 2 N_c^2 - 1 - 3 \frac{N_c^4}{N_f^2} \right)
  \]

  \[
  c_{IR} = \frac{1}{16} \left( 7 N_c^2 - 2 - 9 \frac{N_c^4}{N_f^2} \right)
  \]
**SUSY example:**

- SU($N_c$) supersymmetric QCD with $N_f$ flavors of massless quarks with \( \frac{3}{2} \leq \frac{N_f}{N_c} \leq 3 \)

\[
\frac{a_{UV} - a_{IR}}{N_c^2} \quad \frac{c_{UV} - c_{IR}}{N_c^2}
\]

\[
\frac{N_f}{N_c}
\]
C-theorems in higher dimensions??

\[ \langle T_{\mu}^{\mu} \rangle = -\frac{c}{12} R \]

\[ \langle T_{\mu}^{\mu} \rangle = \frac{S}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2} \nabla^2 R \]

\[ I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \text{ and } E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \]

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- holographic field theories with Einstein gravity dual
C-theorems in higher dimensions??

\[ \langle T^\mu_\mu \rangle = -\frac{c}{12} R \]

for \( d=2 \):

\[ \langle T^\mu_\mu \rangle = \frac{1}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2} \nabla^\mu R \]

for \( d=4 \):

\[ I_4 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \]

• in 4 dimensions, have three central charges: \( c, a, a' \)
• do any of these obey a similar “c-theorem” under RG flows?

\textbf{a-theorem:} proposed by Cardy (1988)

✓ • numerous nontrivial examples, eg, perturbative fixed points
  (Jack & Osborn), SUSY gauge theories (Anselmi et al; Intriligator & Wecht)

✓ • holographic field theories with Einstein gravity dual

✓ • counterexample proposed: Shapiro & Tachikawa, 0809.3238
  (Gaiotto, Seiberg & Tachikawa)
C-theorems in higher dimensions??

\[ \langle T_{\mu}^{\mu} \rangle = -\frac{c}{12} R \]

\[ \langle T_{\mu}^{\mu} \rangle = \frac{1}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2} \nabla^2 R \]

\[ I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \]

- in 4 dimensions, have three central charges: \( c, a, a' \)
- do any of these obey a similar “c-theorem” under RG flows?

\[ a \text{-theorem: proposed by Cardy (1988)} \]
- numerous nontrivial examples, eg, perturbative fixed points (Jack & Osborn), SUSY gauge theories (Anselmi et al; Intriligator & Wecht)
- holographic field theories with Einstein gravity dual
- holographic theories with higher curvature dual for any \( d \)
- F-theorem for \( d=3 \) (and general odd \( d \))
- \( d=4 \) “proof” using dilaton effective action
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Holographic RG flows:

\[ I = \frac{1}{2\ell_P^3} \int d^5 x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right] \]

• imagine potential has stationary points giving negative \( \Lambda \)

\[ V(\phi_{i,cr}) = -\frac{12}{L^2} \alpha_i^2 \]

• consider metric: \( ds^2 = e^{2A(r)} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2 \)

• at stationary points, AdS\(_5\) vacuum: \( A(r) = r/\tilde{L} \) with \( \tilde{L} = L/\alpha_i \)

• RG flows are solutions starting at one stationary point and ending at another
Holographic RG flows:

- for general flow solutions, define:
  \[ a(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3} \]
  \[ a'(r) = -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) = -\frac{\pi^2}{\ell_P^3 A'(r)^4} (T^t_t - T^r_r) \geq 0 \]
  Einstein equations \[ \leftrightarrow \] null energy condition \( (T_{\mu\nu} \ell^\mu \ell^\nu \geq 0) \)

- at stationary points, \( a(r) \rightarrow a^* = \pi^2 \frac{\tilde{L}^3}{\ell_P^3} \) and hence
  \[ a_{UV} \geq a_{IR} \]

- using holographic trace anomaly: \( a^* = a \)
  (e.g., Henningson & Skenderis)

  \[ \rightarrow \text{supports Cardy’s conjecture} \]

- for Einstein gravity, central charges equal \( a = c \): 
  \( c_{UV} \geq c_{IR} \)
Holographic RG flows:

\[
I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right]
\]

- same story is readily extended to (d+1) dimensions

- defining: \( a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2) (\ell_P A'(r))^{d-1}} \)

\[
a'(r) = -\frac{(d-1)\pi^{d/2}}{\Gamma(d/2) \ell_P^{d-1} A'(r)^d} A''(r) = -\frac{\pi^{d/2}}{\Gamma(d/2) \ell_P^{d-1} A'(r)^d} (T^t_t - T^r_r) \geq 0
\]

Einstein equations

null energy condition

- at stationary points, \( a(r) \to a^* = \frac{\pi^{d/2}}{\Gamma(d/2) (\tilde{L}/\ell_P)^{d-1}} \) and so \( a^*_{UV} \geq a^*_{IR} \)

- using holographic trace anomaly: \( a^* \propto \) central charges

(for even d! what about odd d?)

(e.g., Henningson & Skenderis)
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**Improved Holographic RG Flows:**

- add higher curvature interactions to bulk gravity action

  provides holographic field theories with, eg, $a \neq c$

  so that we can clearly distinguish evidence of a-theorem

  (Nojiri & Odintsov; Blau, Narain & Gava)

- construct “toy models” with fixed set of higher curvature terms

  (where we can maintain control of calculations)

**What about the swampland?**

- constrain gravitational couplings with consistency tests

  (positive fluxes; causality; unitarity) and use best judgement

- seems an effective approach with, e.g., Gauss-Bonnet gravity

  (eg, Brigante, Liu, Myers, Shenker, Yaida, de Boer, Kulaxizi, Parnachev, Camanho, Edelstein, Buchel, Sinha, Paulos, Escobedo, Smolkin, Cremonini, Hofman, . . . . )

- ultimately one needs to fully develop string theory for interesting holographic backgrounds
Quasi-Topological gravity: (Myers & Robinson, 1003.5357)

\[ I = \frac{1}{2\ell_P^3} \int d^5x \sqrt{-g} \left[ \frac{12}{L^2} + R + L^2 \frac{\lambda}{2} \chi_4 + L^4 \frac{7\mu}{4} \mathcal{Z}_5 \right] \]

with \( \chi_4 = R^{abcd} R_{abcd} - 4R_{ab}R^{ab} + R^2 \)

\[ \mathcal{Z}_5 = R^{\, c\, d\, e\, f\, b}_{\, a\, b\, c\, f\, d} + \frac{1}{56} \left( 21R_{abcd}R^{abcd}R - 72R_{abcd}R^{abc}eR^{de} \right. \]

\[ + 120R_{abcd}R^{ac}R^{bd} + 144R_{a}^{\, b\, c}R_{b}^{\, c\, a}R_{c}^{\, a} - 132R_{a}^{\, b\, a}R_{b}^{\, a\, a}R + 15R^3 \) \]

- three dimensionless couplings, \( L/\ell_P, \lambda, \mu \), allow us to explore dual CFT’s with most general three-point function \( \langle T_{ab} T_{cd} T_{ef} \rangle \)

“maintain control of calculations”

- analytic black hole solutions
- linearized eom in AdS\(_5\) are second order (in fact, Einstein eq’s!)
- can be extended to higher dimensions (D≥7)
- gravitational couplings constrained (Myers, Paulos & Sinha, 1004.2055)
Quasi-Topological gravity:  
(Myers, Paulos & Sinha, 1004.2055)
Quasi-Topological gravity:  

\[ I = \frac{1}{2\ell_P^3} \int d^5x \sqrt{-g} \left[ \frac{12}{L^2} \alpha^2 + R + \frac{\lambda}{2} \chi_4 + L^4 \frac{\mu}{4} Z_5 \right] \]

with \( \chi_4 = R^{abcd} R_{abcd} - 4 R_{ab} R^{ab} + R^2 \)

\[ Z_5 = R^{c \, d} R_{d \, c} R^{a \, f} R_{e \, f} + \frac{1}{56} \left( 21 R_{abcd} R^{abcd} R - 72 R_{ab} R^{ab} R_{c} + 144 R_a R^b R_c R + 132 R_a R_b R_c R + 15 R^3 \right) \]

\( \text{let's calculate!} \)

\( \text{curvature in AdS}_5 \) vacuum:

\[ \frac{1}{L^2} = \frac{f_\infty L^2}{L^2} , \]

where \( \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0 \)

\( \text{holographic trace anomaly:} \) 

\[ a = \pi^2 \frac{\tilde{L}^3}{\ell_P^3} \left( 1 - 6\lambda f_\infty + 9\mu f_\infty^2 \right) , \quad c = \pi^2 \frac{\tilde{L}^3}{\ell_P^3} \left( 1 - 2\lambda f_\infty - 3\mu f_\infty^2 \right) \]
RG flows in Quasi-Topological gravity:

• consider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2$

  $\rightarrow$ AdS$_5$ vacua: $A(r) = r/\tilde{L}$

• need to define “flow functions” which extend

  $a = \pi^2 \frac{\tilde{L}^3}{\ell_P^3} \left( 1 - 6\lambda f_\infty + 9\mu f_\infty^2 \right)$,

  $c = \pi^2 \frac{\tilde{L}^3}{\ell_P^3} \left( 1 - 2\lambda f_\infty - 3\mu f_\infty^2 \right)$

for general flows away from fixed points
RG flows in Quasi-Topological gravity:

• consider metric: \( ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2 \)

\[ \text{AdS}_5 \text{ vacua: } A(r) = r / \tilde{L} \]

• natural to define “flow functions”:

\[
\begin{align*}
 a(r) &\equiv \frac{\pi^2}{\ell_P^3 A'(r)^3} \left( 1 - 6\lambda L^2 A'(r)^2 + 9\mu L^4 A'(r)^4 \right) \\
c(r) &\equiv \frac{\pi^2}{\ell_P^3 A'(r)^3} \left( 1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4 \right)
\end{align*}
\]

where at stationary points: \( a(r) = a, \ c(r) = c \)

• “simplest” \( r \)-dependent functions satisfying this condition
RG flows in Quasi-Topological gravity:

\[ a(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3} \left( 1 - 6\lambda L^2 A'(r)^2 + 9\mu L^4 A'(r)^4 \right) \]

\[ c(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3} \left( 1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4 \right) \]

where at stationary points: \( a(r) = a, \ c(r) = c \)

• in general flows:

\[ a'(r) = -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) \left( 1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4 \right) \]

\[ = -\frac{\pi^2}{\ell_P^3 A'(r)^4} \left( T^t_t - T^r_r \right) \geq 0 \]

assume null energy condition

gravitational equations of motion
RG flows in Quasi-Topological gravity:

\[ a(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3} \left( 1 - 6\lambda L^2 A'(r)^2 + 9\mu L^4 A'(r)^4 \right) \]

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\[ = -\frac{\pi^2}{\ell_P^3 A'(r)^4} \left( T_{tt} - T_{rr} \right) \geq 0 \]

\[ c'(r) = -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) \left( 1 - \frac{2}{3} \lambda L^2 A'(r)^2 - \mu L^4 A'(r)^4 \right) \]

\[ = -\frac{\pi^2}{\ell_P^3 A'(r)^4} \frac{1 - \frac{2}{3} \lambda L^2 A'(r)^2 - \mu L^4 A'(r)^4}{1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4} \left( T_{tt} - T_{rr} \right) \]
RG flows in Quasi-Topological gravity:

\[ a(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3} \left( 1 - 6\lambda L^2 A'(r)^2 + 9\mu L^4 A'(r)^4 \right) \]

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\[ = -\frac{\pi^2}{\ell_P^3 A'(r)^4} \left( T^t_{\ t} - T^r_{\ r} \right) \geq 0 \]

- can try to be more creative in defining \( c(r) \) but we were unable to find a expression where flow is guaranteed to be monotonic

- our toy model seems to provide support for Cardy’s “a-theorem” in four dimensions
**Higher Dimensions:** \( D = d + 1 \) (\( \mu = 0 \) for \( d = 5 \))

- straightforward to reverse engineer “a-theorem” flows

- eq’s of motion:

\[
T^t_t - T^r_r = (d - 1) A''(r) \left( 1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4 \right)
\]

- expression with natural flow:

\[
a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2) (\ell_P A'(r))^{d-1}} \left( 1 - \frac{2(d-1)}{d-3} \lambda L^2 A'(r)^2 - \frac{3(d-1)}{d-5} \mu L^4 A'(r)^4 \right)
\]

\[
a'_d(r) = -\frac{\pi^{d/2}}{\Gamma(d/2) \ell_P^{d-1} A'(r)^d} \left( T^t_t - T^r_r \right) \geq 0
\]

assume null energy condition
Higher Dimensions: \( D = d + 1 \) (\( \mu = 0 \) for \( d = 5 \))

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\]

\[
a'_d(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)\ell_P^{d-1} A'(r)^d} \left( T^t_{\ t} - T^r_{\ r} \right) \geq 0
\]

- flow between stationary points (where \( a_d^* \equiv a_d(r)|_{AdS} \))

\[
(a_d^*)_{UV} \geq (a_d^*)_{IR}
\]
What is $a_d^*$? 

$$a_d^* = \frac{\pi^{d/2} \tilde{L}^{d-1}}{\Gamma(d/2) \ell_P^{d-1}} \left( 1 - \frac{2(d-1)}{d-3} \lambda f_\infty - \frac{3(d-1)}{d-5} \mu f_\infty^2 \right)$$

where AdS curvature: $\frac{1}{\tilde{L}^2} = \frac{f_\infty}{L^2}$, $\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

- $a_d^*$ is NOT $C_T$, coefficient of leading singularity in

$$\langle T_{ab}(x) T_{cd}(0) \rangle = \frac{C_T}{x^{2d}} \mathcal{I}_{ab,cd}(x)$$

- $a_d^*$ is NOT $C_S$, coefficient in entropy density: $s = C_S T^{d-1}$
What is $a_d^*$??

$$a_d^* = \frac{\pi^{d/2} \tilde{L}^{d-1}}{\Gamma(d/2) \ell_P^{d-1}} \left( 1 - \frac{2(d-1)}{d-3} \lambda f_\infty - \frac{3(d-1)}{d-5} \mu f_\infty^2 \right)$$

where AdS curvature: \[ \frac{1}{\tilde{L}^2} = \frac{f_\infty}{L^2}, \quad \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0 \]

• trace anomaly for CFT’s with even $d$: \[(\text{Deser & Schwimmer})\]

\[ \langle T^{\mu}_{\nu} \rangle = \sum B_i (\text{Weyl invariant})_i - 2(-)^{d/2} A (\text{Euler density})_d \]

• verify that we have precisely reproduced central charge \[ a_d^* = A \]

(\text{Henningson & Skenderis; Nojiri & Odintsov; Blau, Narain & Gava; Imbimbo, Schwimmer, Theisen & Yankielowicz})

→ agrees with Cardy’s proposal
What is $a_d^*$ ??

$$a_d^* = \frac{\pi^{d/2} \tilde{L}^{d-1}}{\Gamma(d/2) \ell_P^{d-1}} \left( 1 - \frac{2(d-1)}{d-3} \lambda f_\infty - \frac{3(d-1)}{d-5} \mu f_\infty^2 \right)$$

where AdS curvature: $\frac{1}{\tilde{L}^2} = \frac{f_\infty}{L^2}, \quad \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

• trace anomaly for CFT’s with even $d$: (Deser & Schwimmer)

$$\langle T_\mu^\mu \rangle = \sum B_i (\text{Weyl invariant})_i - 2(-)^{d/2} A (\text{Euler density})_d$$

• verify that we have precisely reproduced central charge

$$a_d^* = A$$

(Henningson & Skenderis; Nojiri & Odintsov; Blau, Narain & Gava; Imbimbo, Schwimmer, Theisen & Yankielowicz)

What is $a_d^*$ for odd $d$?? (One moment!)
How robust is Holographic C-theorem?:

- quasi-topological gravity obeys c-theorem in very nontrivial way
- generalize to start with arbitrary curvature-cubed action

\[ I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g} \left[ \frac{d(d-1)}{L^2} \alpha^2 + R + L^2 \tilde{\chi} + L^4 \tilde{Z} \right] \]

where

\[ \tilde{\chi} = b_1 R_{abcd} R^{abcd} + b_2 R_{ab} R^{ab} + b_3 R^2, \]
\[ \tilde{Z} = c_1 R^{c}_{a\ b} R^{d}_{c\ d} R_{e\ f} R^{a \ b} + c_2 R_{ab} R^{cd} R^{ef}_{a\ f} R^{ab} + c_3 R_{abcd} R^{abc}_{e} R^{de} \]
\[ + c_4 R_{ab} R^{abcd} R + c_5 R_{ab} R^{ac} R^{bd} + c_6 R^{b}_{a} R^{c}_{b} R^{a}_{c} \]
\[ + c_7 R^{b}_{a} R^{a}_{b} R + c_8 R^3. \]

- AdS vacua: \( \frac{1}{L^2} = \frac{f_\infty}{L^2} \), where \( \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0 \)

\[ \lambda = \frac{d-3}{d-1} (2b_1 + db_2 + d(d+1)b_3) \]
\[ \mu = -\frac{d-5}{d-1} ((d-1)c_1 + 4c_2 + 2dc_3 + 2d(d+1)c_4 \]
\[ + d^2c_5 + d^2c_6 + d^2(d+1)c_7 + d^2(d+1)^2c_8) \]
More Improved Holographic RG Flows:

• is it reasonable to expect any theory to obey a c-theorem? **NO**

• how do we constrain theory to be physically reasonable?

• recall one of the nice properties of quasi-top. gravity was that linearized graviton equations in AdS were 2\textsuperscript{nd} order

• greatly facilitates calculations but deeper physical significance

• analogy with higher derivative scalar field eq. (in flat space)

\[
\left( \nabla^2 + \frac{a}{M^2} (\nabla^2)^2 \right) \phi = 0 \quad \rightarrow \quad \frac{1}{q^2 (1 - a q^2 / M^2)} = \frac{1}{q^2} - \frac{1}{q^2 - M^2 / a}
\]

• graviton ghosts will be generic with 4\textsuperscript{th} order equations

\[\rightarrow\] couple to additional non-unitary tensor operator in dual CFT
More Improved Holographic RG Flows:

\[
I = \frac{1}{2 \ell_P^{d-1}} \int d^{d+1} x \sqrt{-g} \left[ \frac{d(d-1)}{L^2} \alpha^2 + R + L^2 \tilde{\chi} + L^4 \tilde{Z} \right]
\]

where

\[
\begin{align*}
\tilde{\chi} &= b_1 R_{abcd} R^{abcd} + b_2 R_{ab} R^{ab} + b_3 R^2, \\
\tilde{Z} &= c_1 R^{c}{}_{d}{}^{f} R^{e}{}_{f} R^{a}{}_{b} + c_2 R_{ab} R^{cd} R^{ef} R^{ab} + c_3 R_{abcd} R^{abc} R^{de} \\
&\quad + c_4 R_{abcd} R^{abcd} R + c_5 R_{abcd} R^{ac} R^{bd} + c_6 R^{b} R^{c} R^{a} + c_7 R^{b} R^{a} R + c_8 R^3.
\end{align*}
\]

• demand linearized graviton equations are 2\textsuperscript{nd} order in RG flow

\[i.e., \text{around background geometry: } ds^2 = e^{2A(r)} (-dt^2 + dx^2) + dr^2\]
More Improved Holographic RG Flows:

\[ I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g} \left[ \frac{d(d-1)}{L^2} \alpha^2 + R + L^2 \tilde{\chi} + L^4 \tilde{Z} \right] \]

where

\[
\tilde{\chi} = b_1 R_{abcd} R^{abcd} + b_2 R_{ab} R^{ab} + b_3 R^2 ,
\]

\[
\tilde{Z} = c_1 R_{a}^{\ c} R_{d}^{\ e} f R_{ef}^{\ ab} + c_2 R_{ab}^{\ cd} R_{ef}^{\ cd} + c_3 R_{abcd} R^{abc} e R^{de} + c_4 R_{abcd} R^{abcd} + c_5 R_{abcd} R^{ac} R^{bd} + c_6 R_{a}^{\ b} R_{b}^{\ c} R_{c}^{\ a} + c_7 R_{a}^{\ b} R_{b}^{\ a} R + c_8 R^3 .
\]

• demand linearized graviton equations are 2\textsuperscript{nd} order in RG flow

\[ b_2 = -4b_1 , \quad b_3 = b_1 \]

\[ R^2 \text{ interaction is } \chi_4 \]

\[ c_8 = \frac{1}{d(d+1)} \left( - \frac{d + 5}{2(d+1)} c_1 + \frac{2(d+9)}{d+1} c_2 + \frac{d+8}{3} c_3 \right) \]

\[ 5 \text{ constraints } \rightarrow 3 \text{ free parameters} \]

\[ c_7 = \frac{1}{d(d+1)} \left( 3c_1 - 24c_2 - 4(d+1)c_3 - 4d(d+1)c_4 - (2d-1)c_5 - 3d c_6 \right) \]
More Improved Holographic RG Flows:

- as before, try to reverse engineer “c-theorem” flows
- with above constraints, flow eq’s of motion yield:

\[ T_t^t - T_r^r = (d - 1) A''(r) \left( 1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4 \right) \]

- expression with natural flow:

\[
\begin{align*}
    a_d(r) &\equiv \frac{\pi^{d/2}}{\Gamma(d/2) (\ell_P A'(r))^{d-1}} \left( 1 - \frac{2(d-1)}{d-3} \lambda L^2 A'(r)^2 - \frac{3(d-1)}{d-5} \mu L^4 A'(r)^4 \right) \\
    a'_d(r) &\equiv -\frac{\pi^{d/2}}{\Gamma(d/2) \ell_P^{d-1} A'(r)^d} (T_t^t - T_r^r) \geq 0
\end{align*}
\]

- flow between stationary points (where \( a_d^* \equiv a_d(r)\big|_{\text{AdS}} \))

\[
(a_d^*)_{UV} \geq (a_d^*)_{IR}
\]

- also extends to Lovelock and R^n theories (Liu, Sabra & Zhao)

- for even \( d \), find same match: \( a_d^* = A \) \( \Rightarrow \) Cardy’s proposal

What about odd \( d \)?
Overview:

1. Introductory remarks on c-theorem
2. Holographic c-theorem I: Einstein gravity
3. Holographic c-theorem II: Higher curvature gravity
4. $a_d^*$, Entanglement Entropy and Beyond
5. Concluding remarks
General result for any CFT (Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose $S^{d-2}$ with radius $R$
- entanglement entropy: $S_{EE} = -Tr \left[ \rho_A \log \rho_A \right]$
- by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $R \sim 1/R^2$ and $T=1/2\pi R$

$$S_{EE} = S_{\text{thermal}}$$

AdS/CFT correspondence:

- thermal bath in CFT = black hole in AdS

$$S_{EE} = S_{\text{thermal}} = S_{\text{horizon}}$$

- only need to find appropriate black hole

  topological BH with hyperbolic horizon which intersects $A$ on AdS boundary

  (Aminneborg et al; Emparan; Mann; . . . )
\[ S_{EE} = S_{thermal} = S_{horizon} \]

- desired “black hole” is a hyperbolic foliation of empty AdS space

\[
 ds^2 = \frac{L^2 \, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2} \, d\tau^2 + \rho^2 \, d\Sigma_{d-1}^2 \quad \rightarrow \quad T = \frac{1}{2\pi R} 
\]

- apply Wald’s formula (for any gravity theory) for horizon entropy:

\[
 S = \frac{2\pi}{\pi^{d/2}} \Gamma (d/2) \frac{a_d^*}{R^{d-1}} \, V \left( H^{d-1} \right) 
\]

universal contributions:

\[
 S = \cdots + (-)^{\frac{d}{2} - 1} 4 \, a_d^* \log \left( 2R/\delta \right) + \cdots \quad \text{for even } d \\
 + \cdots + (-)^{\frac{d-1}{2}} 2\pi \, a_d^* + \cdots \quad \text{for odd } d 
\]

- discussion extends to case with background: \( R^{1,d-1} \rightarrow R \times S^{d-1} \)
Conjecture:

• entanglement entropy of ground state of CFT across sphere $S^{d-2}$ of radius $R$ has universal contribution:

$$S_{univ} = \begin{cases} 
(-)^{d-1} \frac{d}{2} 4 a_d^* \log(2R/\delta) & \text{for even } d \\
(-)^{\frac{d-1}{2}} 2\pi a_d^* & \text{for odd } d
\end{cases}$$

(any gravitational action)

• in RG flows between fixed points (any CFT in even $d$ with $a_d^* = A$)

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

(“unitary” models)

→ behaviour discovered for holographic model but conjecture that result applies generally (outside of holography)

→ gives framework to consider c-theorem for odd or even $d$
and Beyond:

- **Susskind & Witten**: density of degrees of freedom in N=4 SYM connected to area of holographic screen at large $R$ in $\text{AdS}_5$

\[
\frac{V_3}{\delta^3} \times N_c^2 \sim \frac{A(R)}{\ell_P^3}
\]

cut-off scale defined by regulator radius: \( \frac{1}{\delta} = \frac{R}{L^2} \)

- given higher curvature bulk action, natural extension is to evaluate Wald entropy on holographic screen at large $R$

\[
S = -2\pi \int d^{d-1}x \sqrt{h} \hat{\varepsilon}^{ab} \hat{\varepsilon}_{cd} \frac{\partial L_{\text{bulk}}}{\partial R^{ab}_{\cdots cd}}
\]

- straightforward evaluate “entropy” for count of density of dof

\[
S = \frac{2}{\pi} a^* \frac{V_{d-1}}{\delta^{d-1}}
\]

for any covariant action: \( L_{\text{bulk}} = L_{\text{bulk}} (g^{ab}, R^{ab}_{cd}, \nabla_e R^{ab}_{cd}, \cdots) \)
F-theorem:  
(Jafferis, Klebanov, Pufu & Safdi)

- examine partition function for broad classes of 3-dimensional quantum field theories (SUSY and non-SUSY) on three-sphere
- in all examples, \( F = -\log Z > 0 \) and decreases along RG flows
- **coincides with our conjectured c-theorem!**  
  (Casini, Huerta & RCM)

- consider \( S_{EE} \) of d-dimensional CFT for sphere \( S^{d-2} \) of radius \( R \)
- conformal mapping: causal domain \( \mathcal{D} \rightarrow (\text{static patch of}) \ dS_d \)

\[
\text{curvature} \sim 1/R \quad \text{and thermal state:} \quad \rho = \exp[-2\pi R H_\tau]/Z
\]

\[
\rightarrow S_{EE} = S_{\text{thermal}} = \beta \langle H_\tau \rangle + \log Z
\]
**F-theorem:** (Jafferis, Klebanov, Pufu & Safdi)

- examine partition function for broad classes of 3-dimensional quantum field theories (SUSY and non-SUSY) on three-sphere.
- in all examples, $F = -\log Z > 0$ and decreases along RG flows.

**coincides with our conjectured c-theorem!** (Casini, Huerta & RCM)

- consider $S_{EE}$ of d-dimensional CFT for sphere $S^{d-2}$ of radius $R$.
- conformal mapping: causal domain $\mathcal{D} \rightarrow$ (static patch of) $dS_d$.

  curvature $\sim 1/R$ and thermal state: $\rho = \exp[-2\pi R H_\tau]/Z$.

  $S_{EE} = S_{\text{thermal}} = \rho \langle H_\tau \rangle + \log Z$.

- stress-energy fixed by trace anomaly — vanishes for odd $d$.
- upon passing to Euclidean time with period $2\pi R$:

  $S_{EE} = \log Z|_{S^d}$ for any odd $d$. 

F-theorem:

- must focus on renormalized or universal contributions, eg,
  \[ F_0 = - \log Z\big|_{\text{finite}} = -S_{\text{univ}} = 2\pi a_3^*. \]

- generalizes to general odd d:
  \[ (-)^{d-1 \over 2} \log Z\big|_{\text{finite}} = (-)^{d-1 \over 2} S_{\text{univ}} = 2\pi a_d^*. \]

- equivalence shown **only for fixed points** but good enough:

\[ (F_0)_{\text{IR}} = 2\pi (a_3^*)_{\text{IR}} \]

\[ (F_0)_{\text{UV}} = 2\pi (a_3^*)_{\text{UV}} \]

- evidence for F-theorem (SUSY, perturbed CFT’s & O(N) models) supports present conjecture and our holographic analysis provides additional support for F-theorem
RG Flows and Dilaton Effective Action

• think of RG flow as “spontaneously broken conformal symmetry”

• couple theory to “dilaton” (conformal compensator) and organize effective action in terms of \( \hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu} \)

  \[ \text{diff X Weyl invariant: } \quad g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu} \quad \tau \rightarrow \tau + \sigma \]

• introduce UV kinetic term:
  \[ \frac{f^2}{6} \int d^4 x \sqrt{g} \hat{R} = \mathcal{O}(\frac{f^2}{6}) \int d^4 x e^{-2\tau}(\partial\tau)^2 \]

• follow effective action to IR fixed point, eg,
  \[ S_{\text{anomaly}} = -\delta a \int d^4 x \sqrt{-g} \left( \tau E_A + 4(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \Box \tau + 2(\partial\tau)^4 \right) \]

  \[ \delta a = a_{\text{UV}} - a_{\text{IR}} \quad \text{: ensures UV & IR anomalies match} \]

• with \( g \rightarrow \eta \), this term is only contribution to 4pt scattering:
  \[ S_{\text{anomaly}} = 2 \delta a \int d^4 x (\partial\tau)^4 \]

• causality or analyticity arguments demand: \( \delta a > 0 \)  

(Adams et al)
RG Flows and Dilaton Effective Action

• causality or analyticity arguments demand:
  \[ \delta a = a_{UV} - a_{IR} > 0 \]

• proof of a-theorem??:
  \[ \longrightarrow \text{ assumes RG flow from UV fixed pt to IR fixed pt} \]
  (Fortin, Grinstein & Stergiou)

• recent work suggests there exist d=4 QFT’s which are scale invariant but not conformally invariant

• further suggests RG flows exhibit limit cycle behaviour
  \[ \longrightarrow \text{ full structure of d=4 RG flows still to explore} \]
Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories.
- **toy theories** with higher-R interactions extend class of CFT’s maintain calculational control with LL or quasi-top. gravity
- consistency (e.g., causality & unitarity) constrains couplings
- provide interesting insights into RG flows
- naturally support Cardy’s version of “A-theorem” with d even
- suggests extension of c-theorem to d odd
- $a^*_d$ seems to play a privileged role in holography
- further implications for holographic dualities??
- can entanglement entropy lead to proof of a-theorem? 

Lots to explore! (Casini & Huerta)