Enumerative meaning of mirror maps for toric manifolds

Joint work with Kwokwai Chan,
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The University of Tokyo

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Target:

$$\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$$

for toric manifolds.

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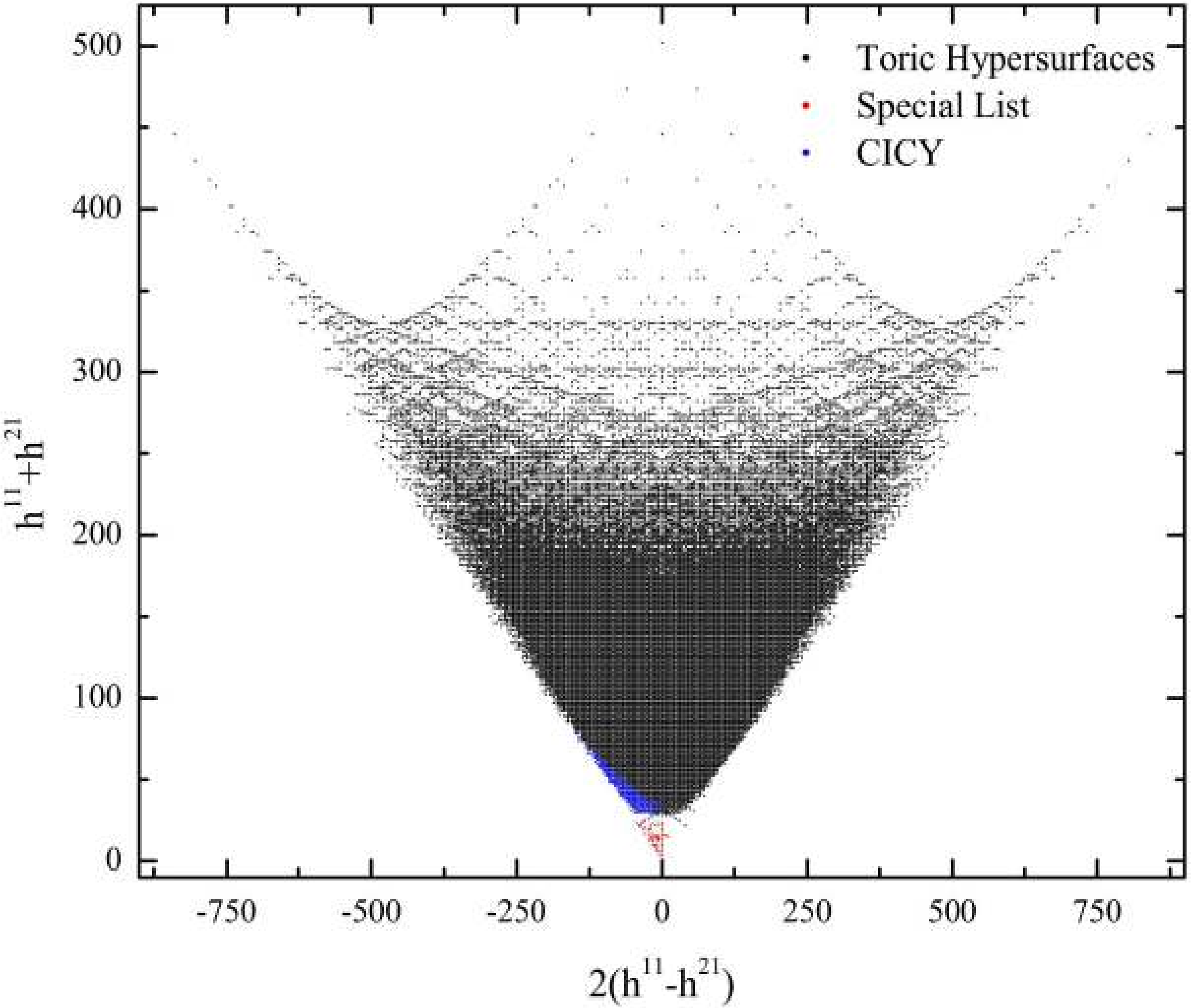
- Computation of open Gromov-Witten invariants
- Local models

What is mirror symmetry?



Ref:
arXiv:1110.1612

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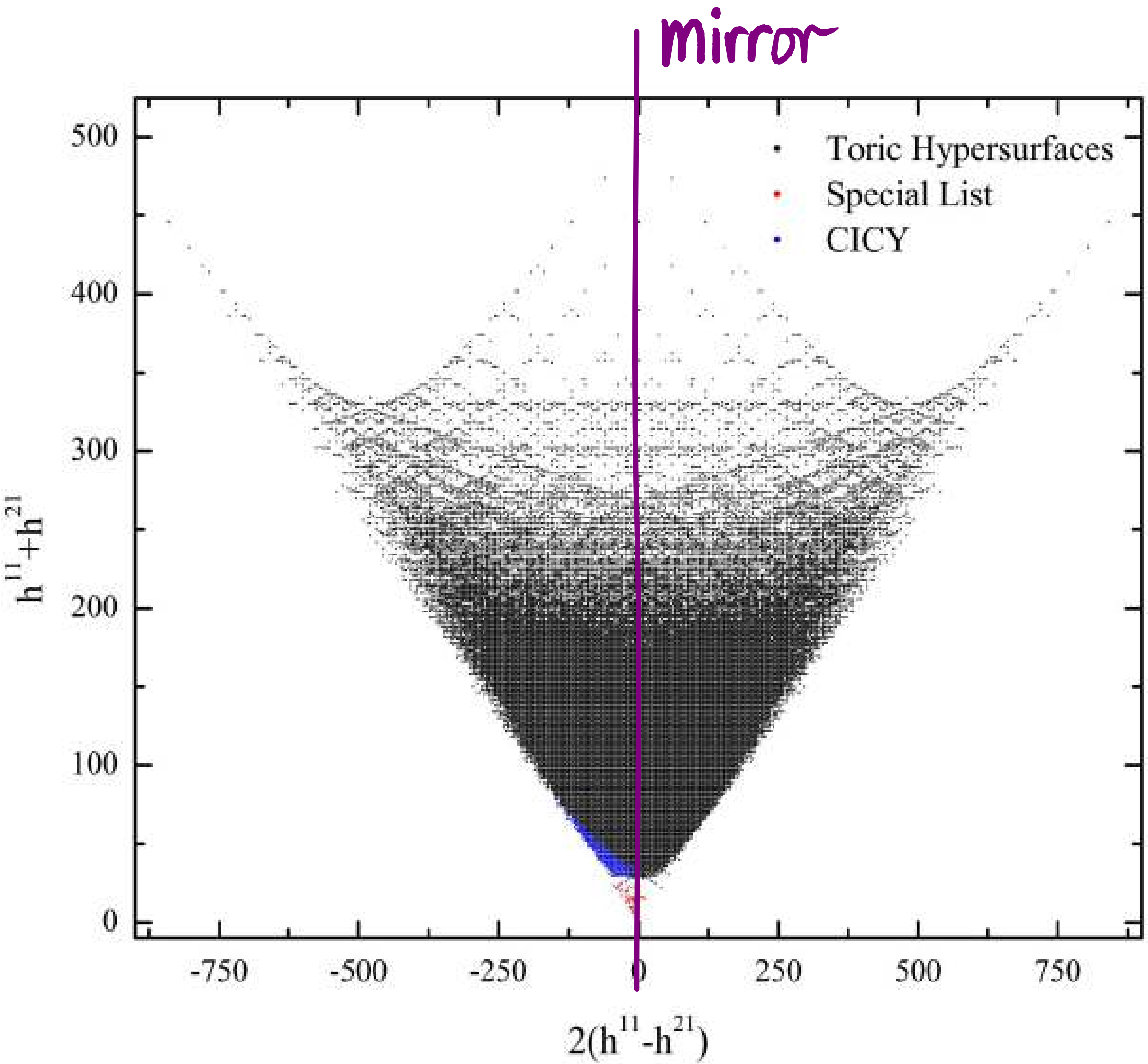

What is mirror symmetry?

X $\check{X}$

mirror pair

What is mirror symmetry?

$$X \qquad\qquad\qquad \check{X}$$

$$\mathcal{M}_X^{\text{Kähler}} \qquad\qquad\qquad \mathcal{M}_{\check{X}}^{\text{Complex}}$$

$$\Psi \qquad\qquad\qquad \Psi$$

$$[\omega] \qquad\qquad\qquad [\check{\Omega}]$$

What is mirror symmetry?

$$\begin{array}{ccc} X & & \check{X} \\ \mathcal{M}_X^{\text{Kähler}} & \xrightarrow[\mathcal{F}_{\text{mirror}}]{\qquad\qquad} & \mathcal{M}_{\check{X}}^{\text{Complex}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\omega] & & [\check{\Omega}] \end{array}$$

What is mirror symmetry?

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such that

$$d\mathcal{F}_{\text{mirror}} : T_{[\omega]}\mathcal{M}_X^{\text{Kähler}} \xrightarrow{\simeq} T_{[\check{\Omega}]}\mathcal{M}_{\check{X}}^{\text{Complex}}$$

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$$\mathcal{M}_X^{\text{Kähler}} \xrightarrow[\mathcal{F}_{\text{mirror}}]{} \mathcal{M}_{\check{X}}^{\text{Complex}}$$

$$[\omega] \in \mathcal{M}_X^{\text{Kähler}} \qquad\qquad [\check{\Omega}] \in \mathcal{M}_{\check{X}}^{\text{Complex}}$$

such that

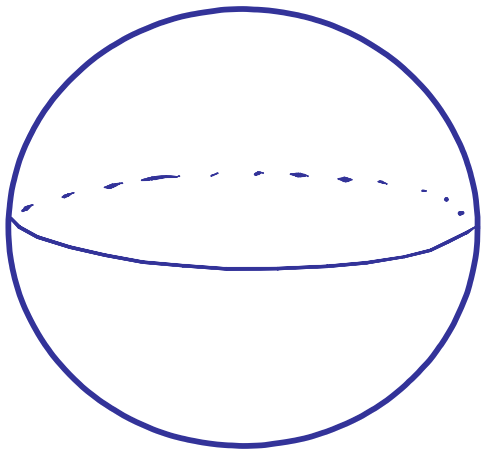
$$dF_{\text{mirror}} : T_{[\omega]}\mathcal{M}_X^{\text{Kähler}} \xrightarrow{\cong} T_{[\check{\Omega}]}\mathcal{M}_{\check{X}}^{\text{Complex}}$$

induces

$$\left(H^{\cdot}(X), \underset{[\omega]}{\cup}, (\ ,\ )_{\text{Poincaré}}\right) \xrightarrow{\cong} \left(H^{\cdot}(\check{X}), \underset{[\check{\Omega}]}{\cup}, (\ ,\ )_{\text{residue}}\right)$$

An example: a sphere

$X = S^2$.



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(complexified)

$$\mathcal{M}_X^{\text{Kähler}} \cong \mathbb{C}^\times$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\omega] & e^{-\int_X \omega} = & q \end{array}$$

$$\mathcal{M}_{\check{X}}^{\text{Complex}} = \mathbb{C}^\times$$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ \check{q} \end{array}$$

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$$\begin{array}{ccc} \mathcal{M}_X^{\text{Kähler}} & \xrightarrow[\check{q} = q]{\mathcal{F}_{\text{mirror}}} & \mathcal{M}_{\check{X}}^{\text{Complex}} \\ \Psi & & \Psi \\ \check{q} & & \check{q} \end{array}$$

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$d\mathcal{F}_{\text{mirror}}$ gives

$$\begin{array}{ccc} H^{\cdot}(X) & \longrightarrow & \mathrm{Jac}(W_{\check{q}}) = \dfrac{\mathbb{C}[z,z^{-1}]}{\langle z\frac{\partial}{\partial z}W_{\check{q}}\rangle} \cong \dfrac{\mathbb{C}[z^{\pm}]}{\langle z^2 - \check{q}\rangle} \\ \Psi & & \Psi \\ [x] = q\frac{\partial}{\partial q} & \longmapsto & \left[\check{q}\frac{\partial}{\partial \check{q}} W_{\check{q}}\right] = \left[\frac{\check{q}}{z}\right] \end{array}$$

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$$[X] \cup_q [X] = q \longleftrightarrow \left[\frac{\check{q}}{z}\right] \cdot \left[\frac{\check{q}}{z}\right] = \check{q}$$

Question: How does the mirror map come up from the intrinsic geometry of X?

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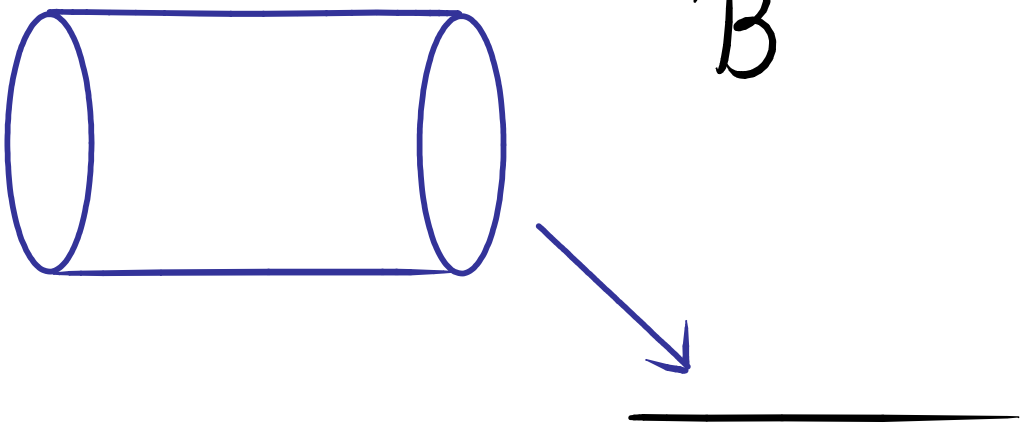
Answer by Strominger-Yau-Zaslow (SYZ):

T-duality.

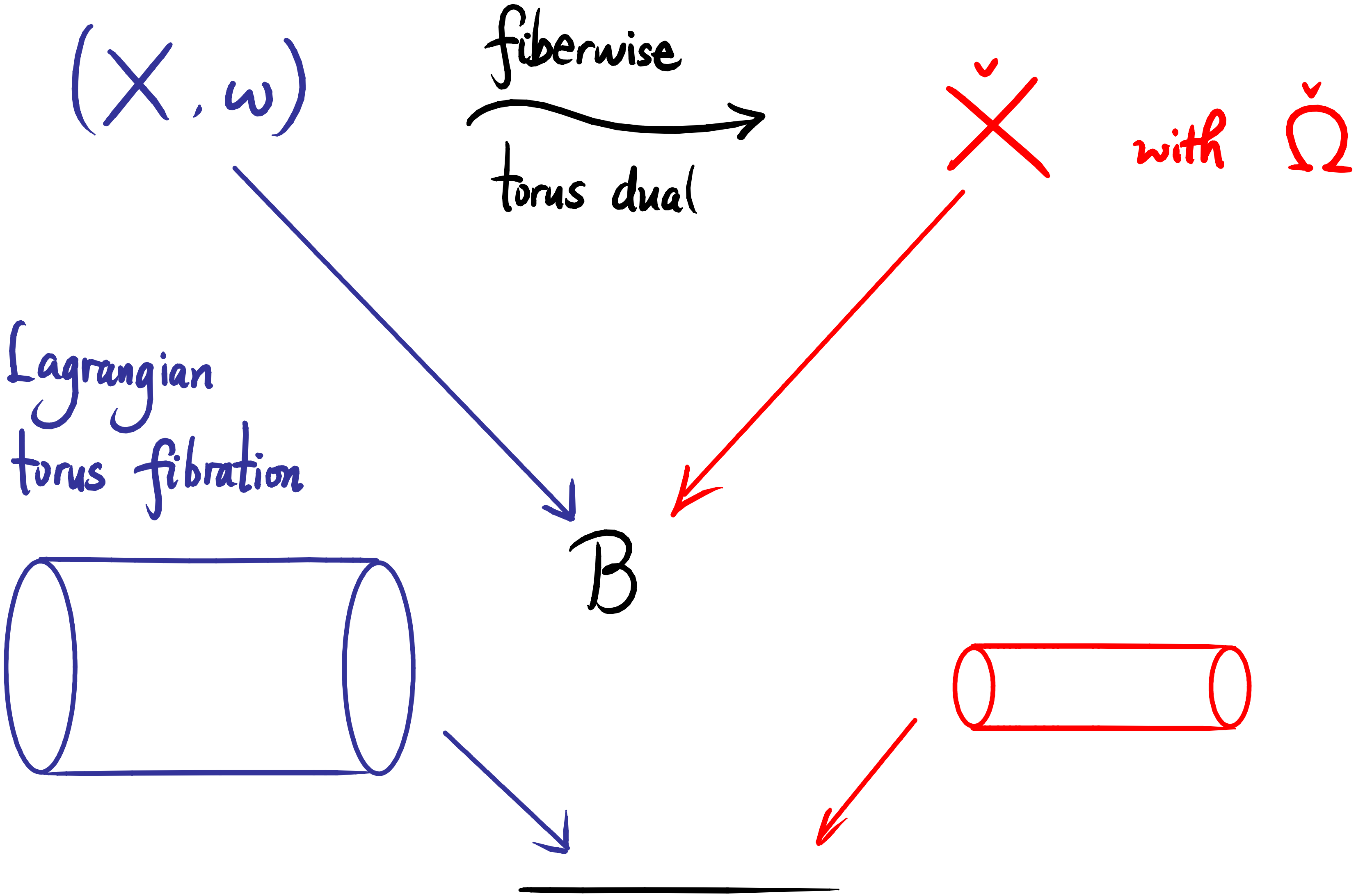
T-duality

$$(X, \omega) \longrightarrow B$$

Lagrangian torus fibration

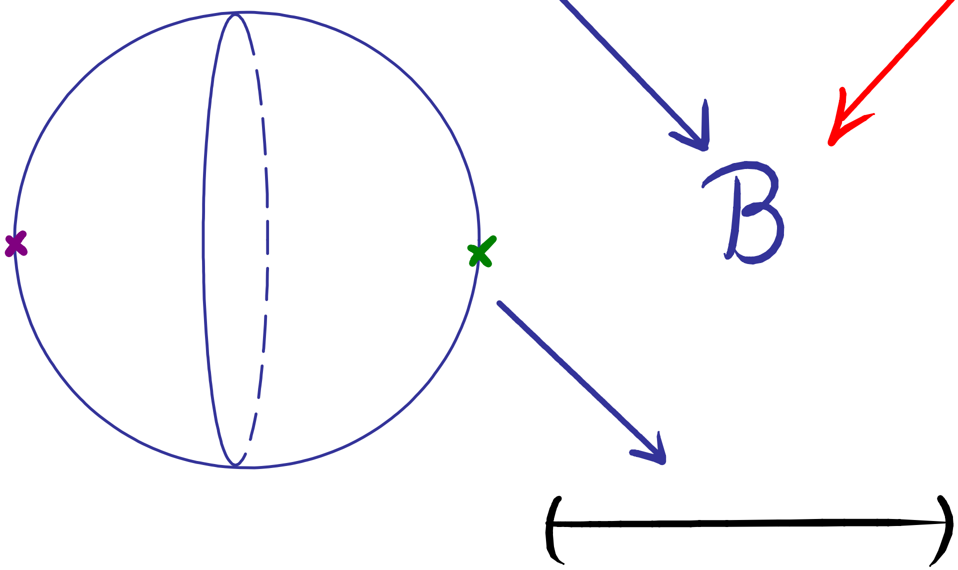


T-duality


(X, ω)
fiberwise
torus dual
$\check{X}$ with $\check{\Omega}$
Lagrangian
torus fibration
B

T-duality

$$\begin{tikzcd}
(X, \omega) \arrow[rr, squiggly, "\text{fiberwise}", "\text{torus dual}"'] \arrow[dr] & & \check{X} \arrow[dl] \\
& B &
\end{tikzcd}$$



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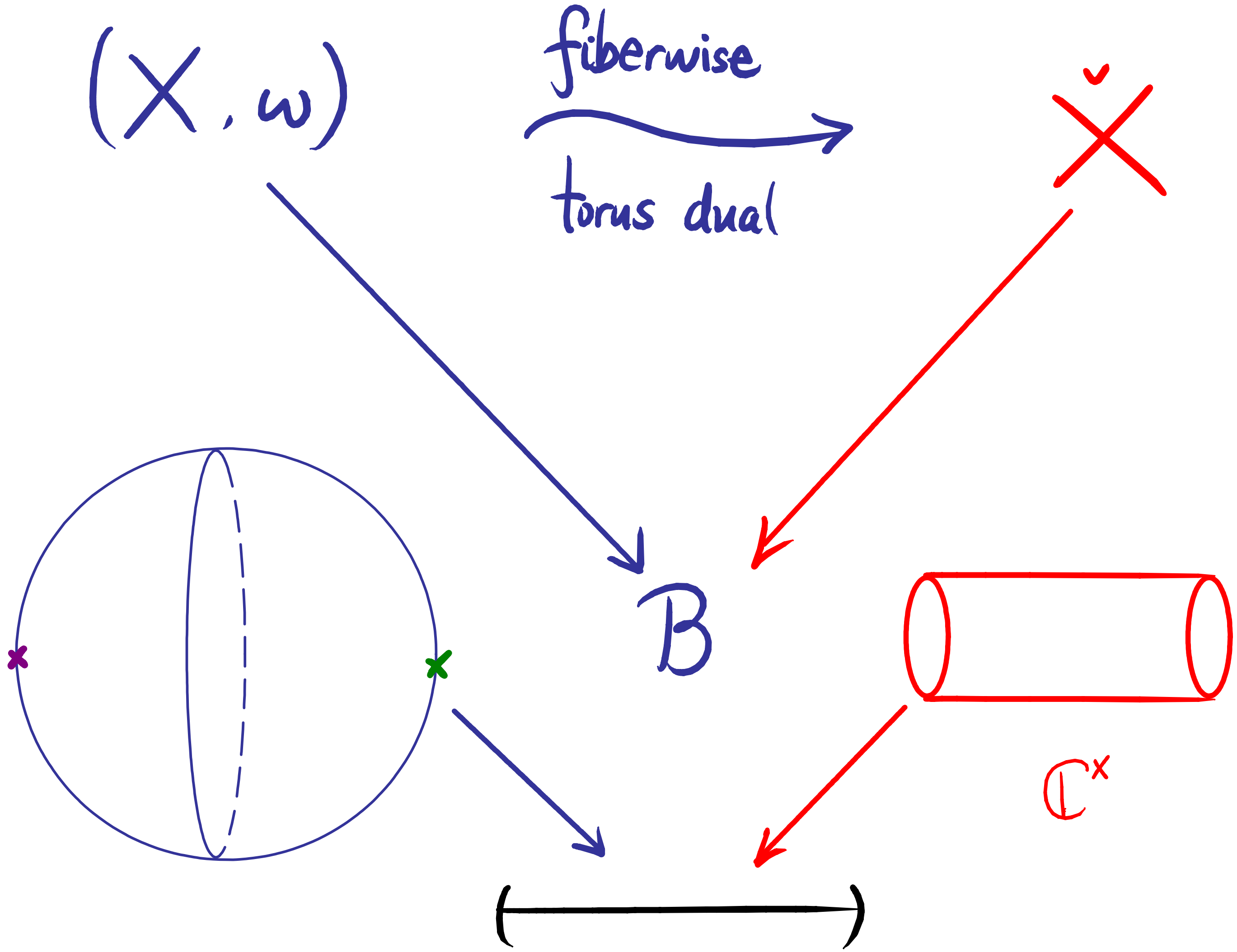

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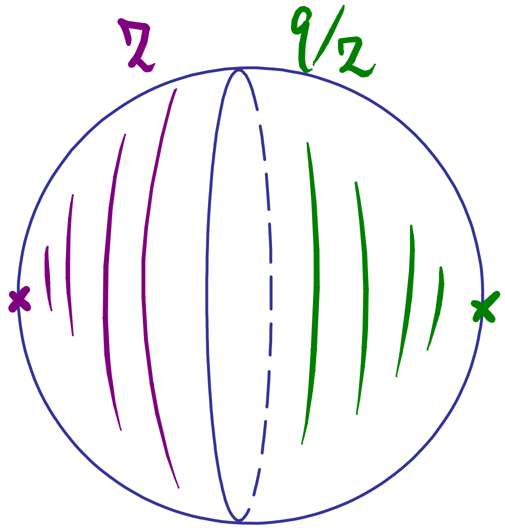

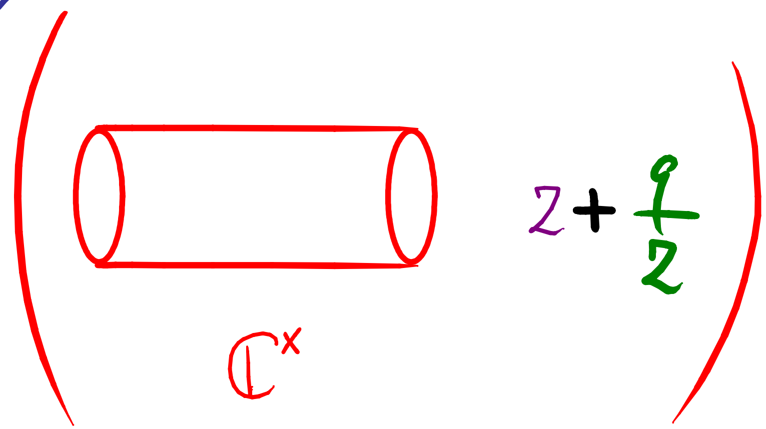

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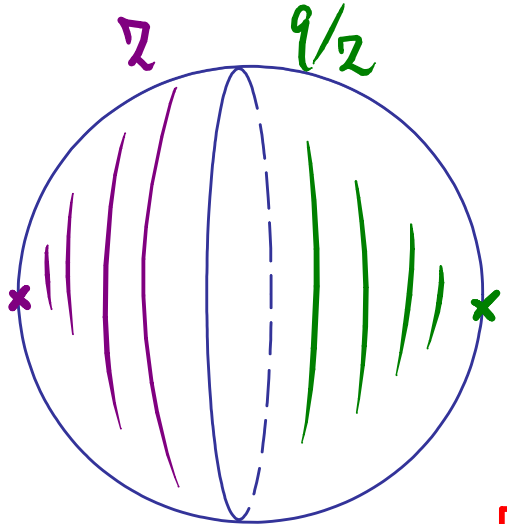

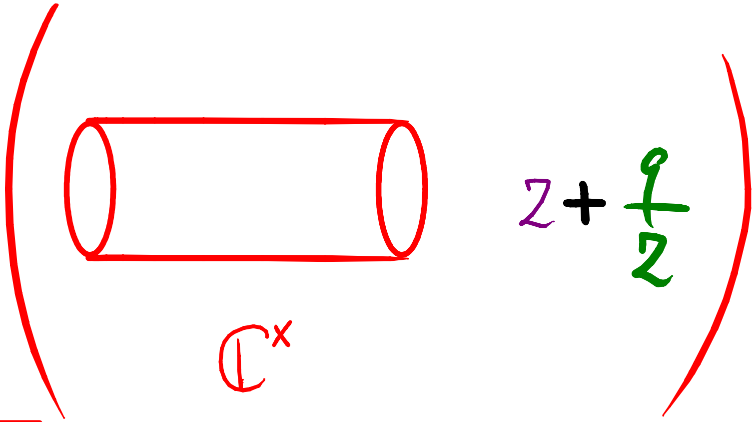

$$\boxed{\therefore\ \check{q} = q.}$$

$$
\begin{tikzcd}
\text{Open Gromov-Witten invariants} \arrow[d, leftrightarrow, "\text{SYZ}"] \\
\text{Mirror maps}
\end{tikzcd}
$$

Mirror map by SYZ:

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$$\mathscr{F}_{\mathrm{SYZ}}.$$

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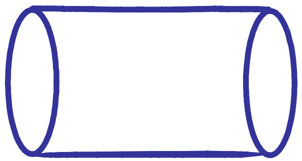
- Semi-flat cases by Leung-Yau-Zaslow 

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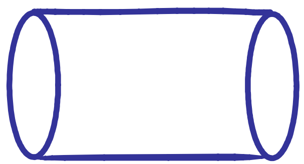
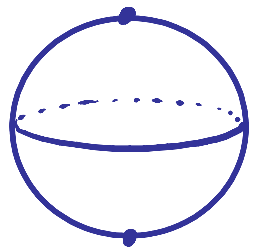
- Semi-flat cases by Leung-Yau-Zaslow 
- Compact toric manifolds by Chan-Leung, Cho-Oh and Fukaya-Oh-Ohta-Ono 

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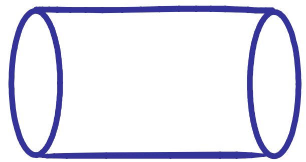

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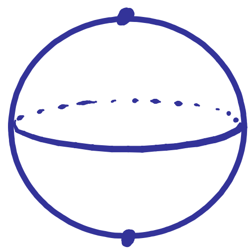
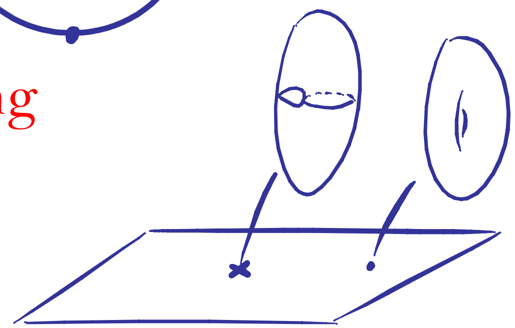


- Toric Calabi-Yau manifolds by Chan-Lau-Leung
  using the techniques of Auroux, Gross-Siebert and Fukaya-Oh-Ohta-Ono



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Theorem: (Chan-Lau-Leung-Tseng)

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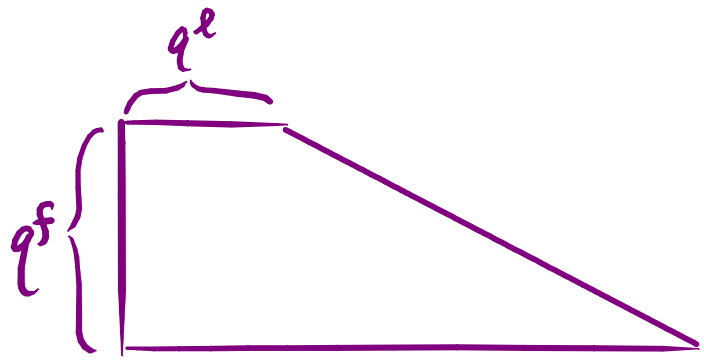
(2). X is a compact semi-Fano toric manifold, with the assumption that

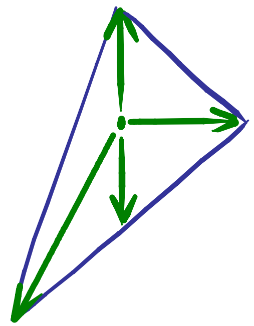
$\mathscr{F}_{\text{SYZ}}$ *converges.* satisfied when X is two-dimensional,

or $X = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$, Y compact toric Fano.

An example: $\mathbb{F}_2$.



fan:



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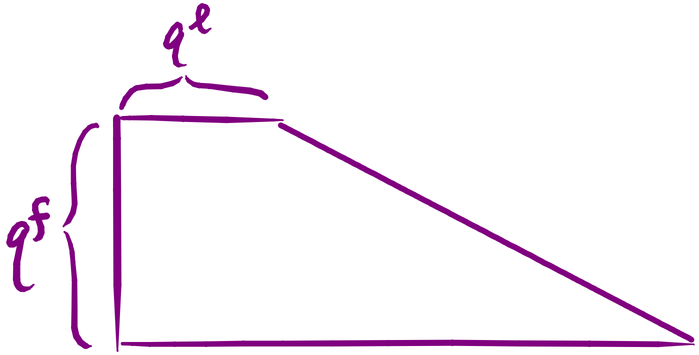

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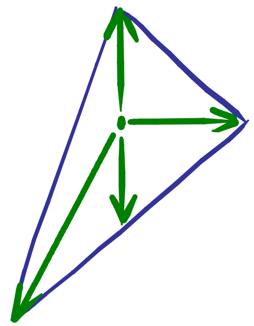

(Givental; Lian-Liu-Yau; Hori-Vafa)

$$W^{PDE}_{\check{q}^\ell,\check{q}^f} = z_1 + z_2 + \check{q}^f z_2^{-1} + \check{q}^{\ell+2f} z_1^{-1} z_2^{-2}.$$

where (mirror map)
$$\begin{cases} \check{q}^f = q^f(1+q^\ell); \\ \check{q}^\ell = q^\ell(1+q^\ell)^{-2}. \end{cases}$$

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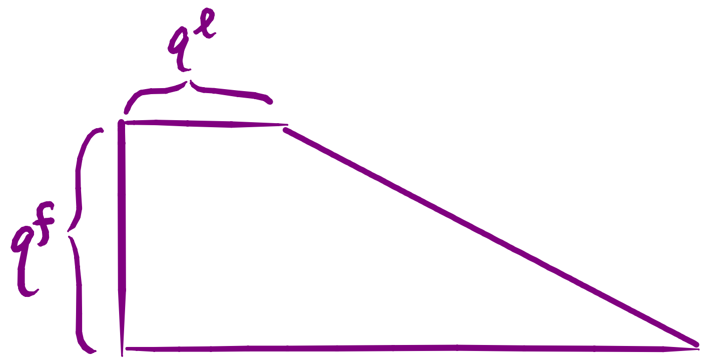

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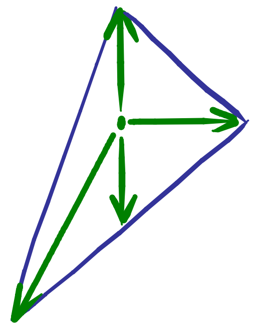

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where (mirror map) $\begin{cases} \check{q}^f = q^f(1+q^\ell); \\ \check{q}^\ell = q^\ell(1+q^\ell)^{-2}. \end{cases}$

$$\therefore W^{PDE}_{\check{q}^\ell, \check{q}^f} = z_1 + z_2 + q^f(1+q^\ell) z_2^{-1} + q^{\ell+2f} z_1^{-1} z_2^{-2}.$$

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$$\text{SYZ map}: W^{SYZ} = \sum_{\beta \in \pi_2(X,T)} n_\beta Z_\beta$$

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)

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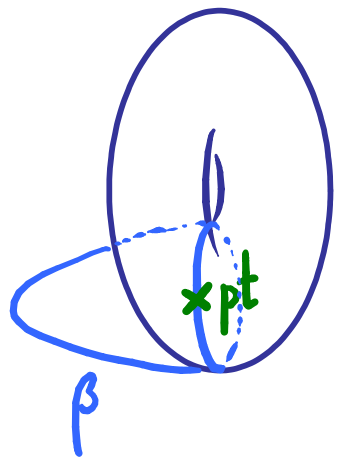
For $\beta \in \pi_2(X,T)$,

$$n_\beta \triangleq \int_{[\mathcal{M}_1(\beta)]^{\text{virt.}}} [pt].$$



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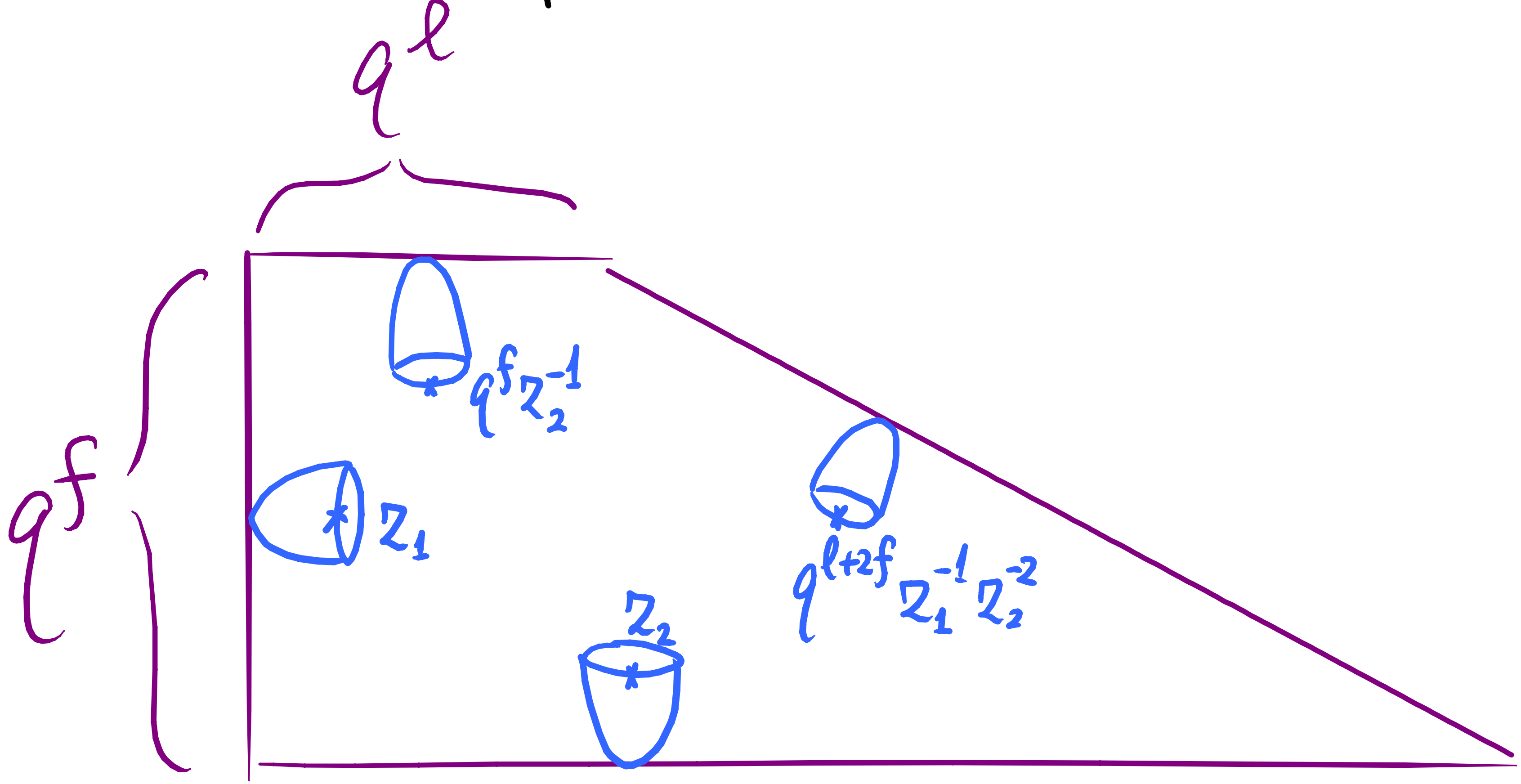

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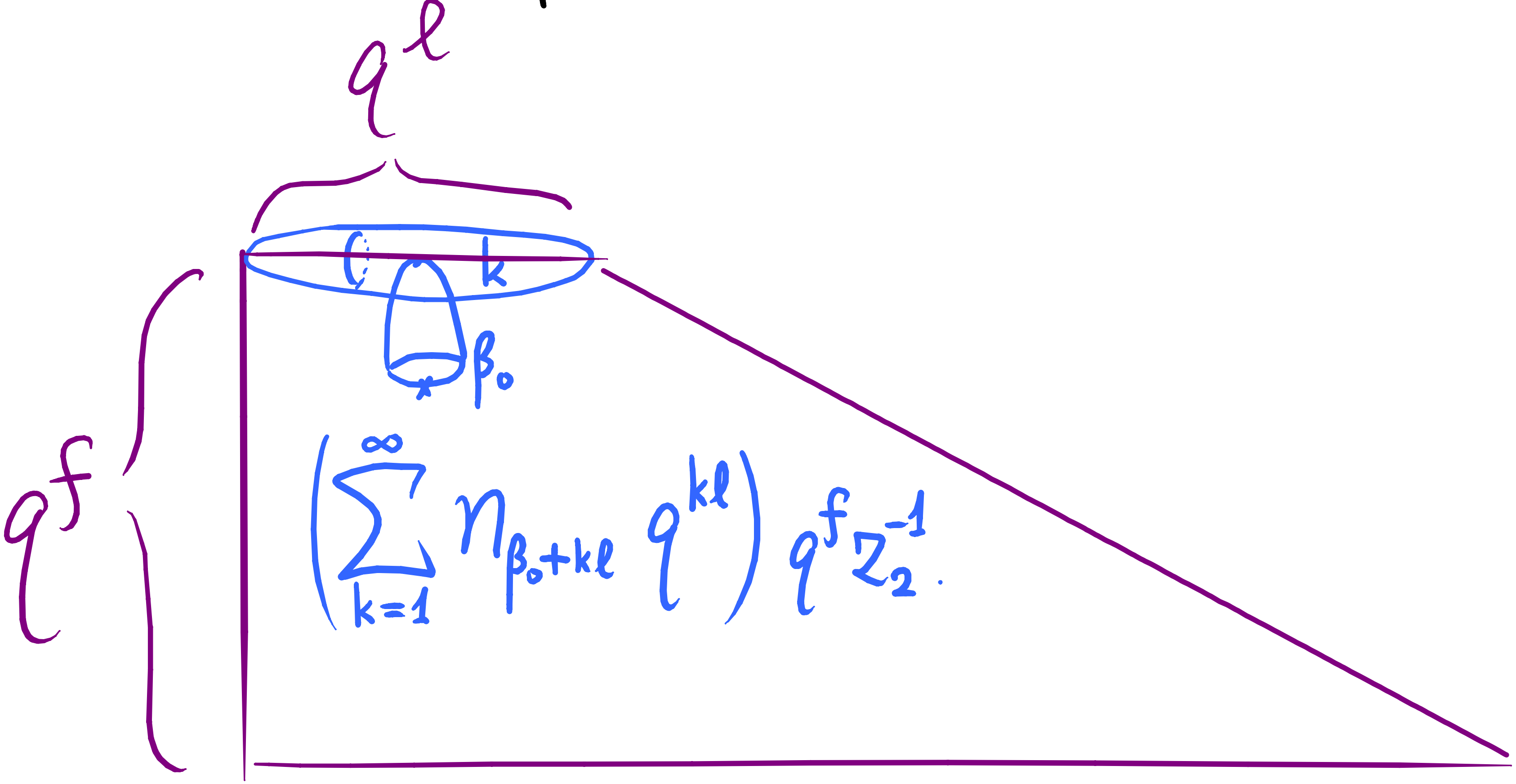

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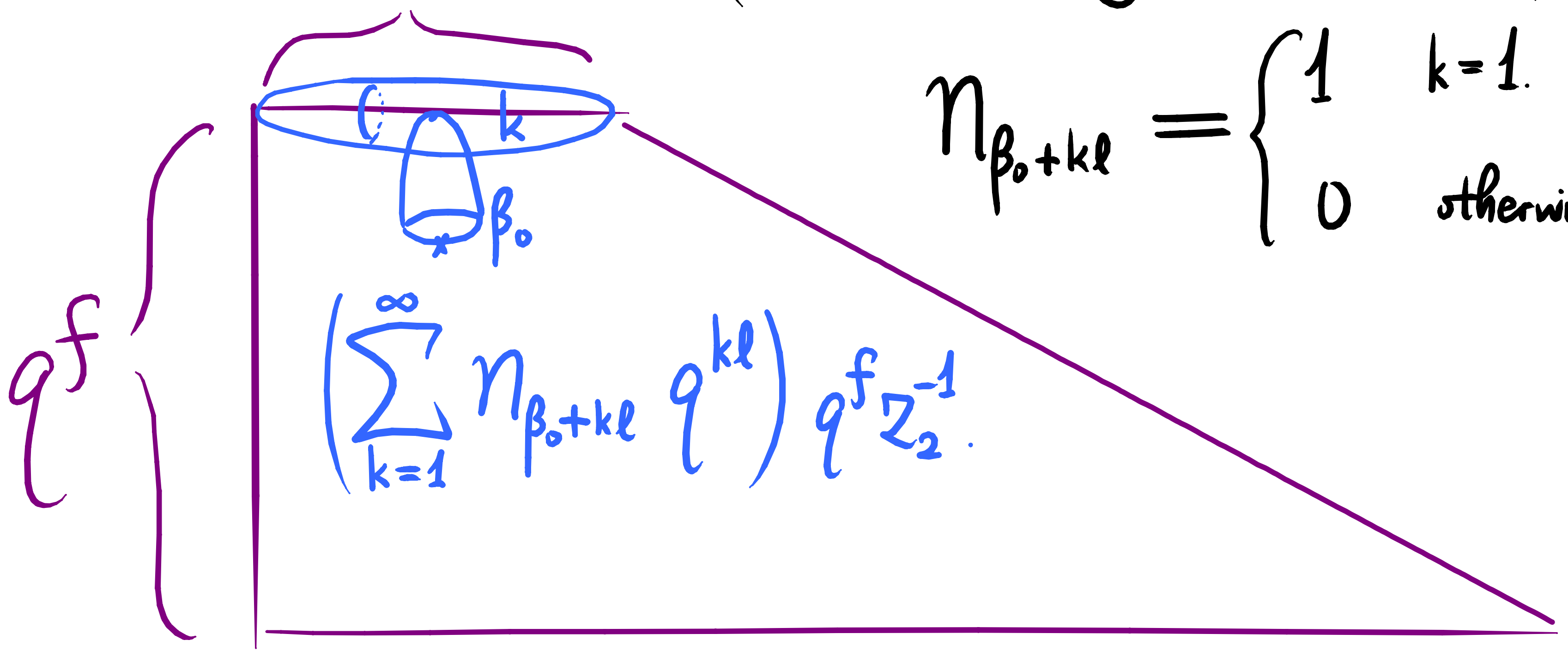

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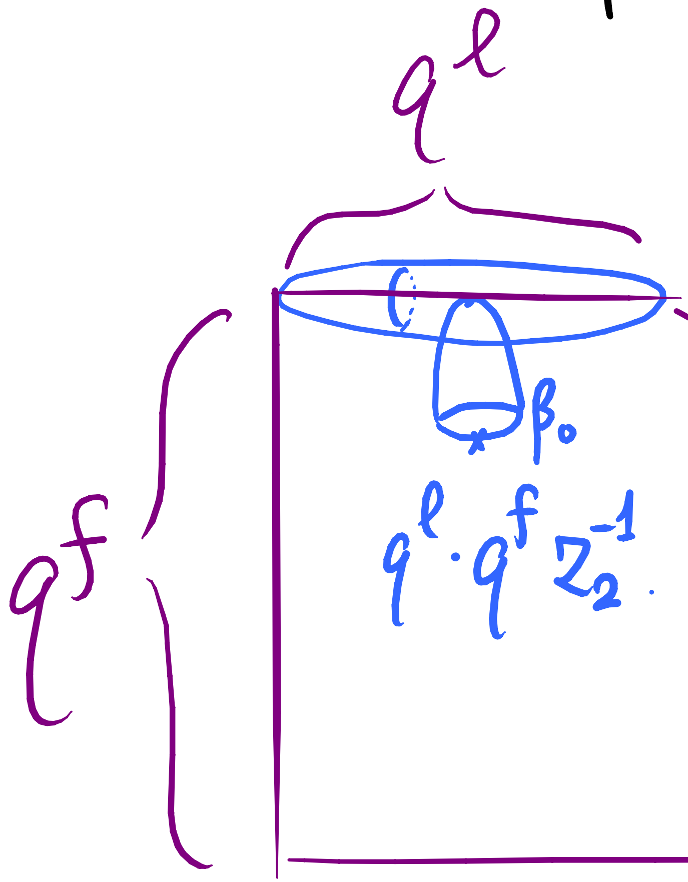

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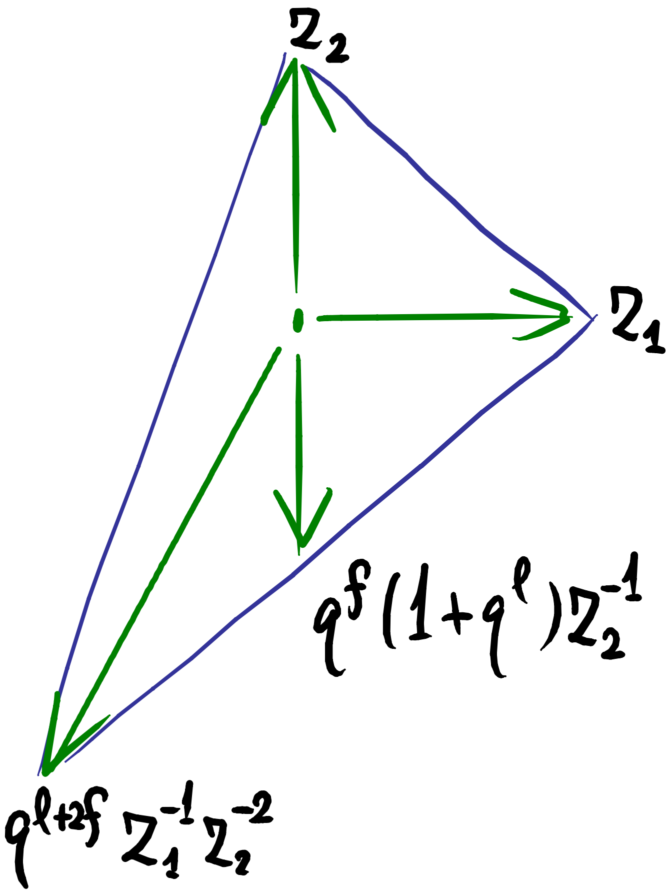

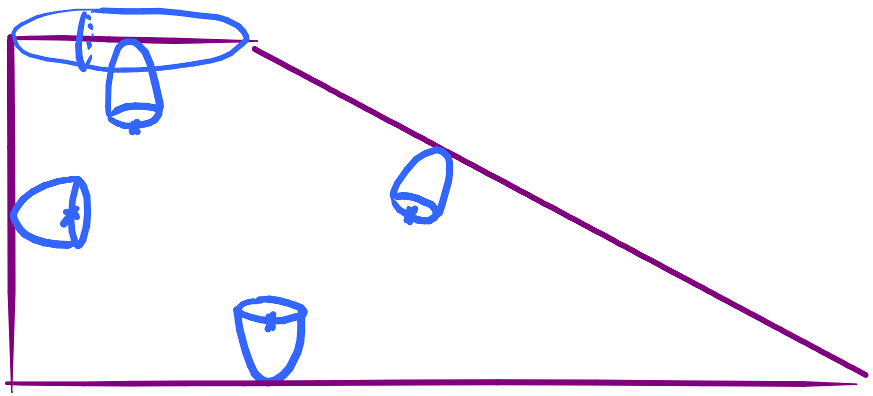

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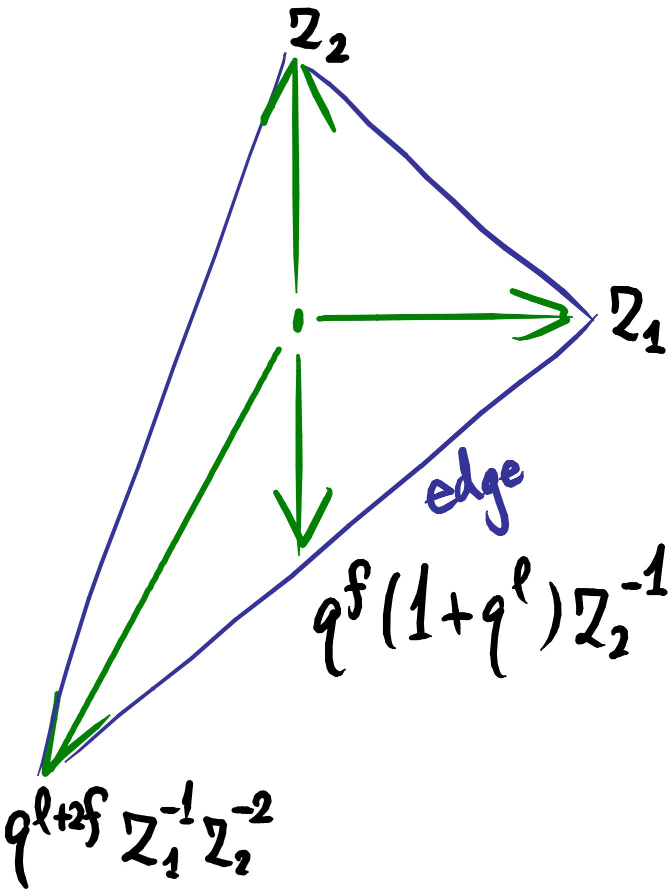

$$W\Big|_{edge} = z_1 + q^f(1+q^\ell)\, z_2^{-1} + q^{\ell+2f} z_1^{-1} z_2^{-2}$$

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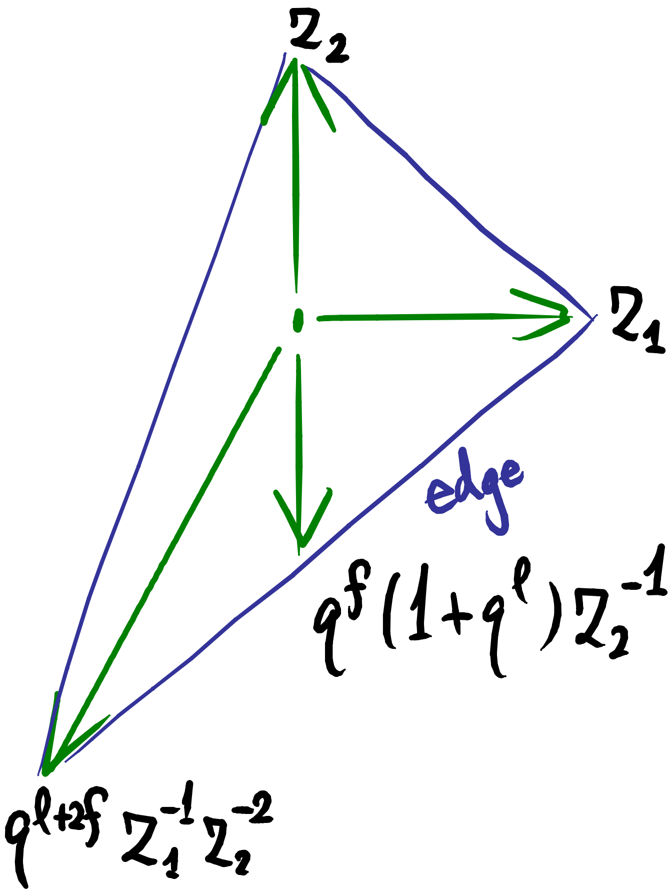

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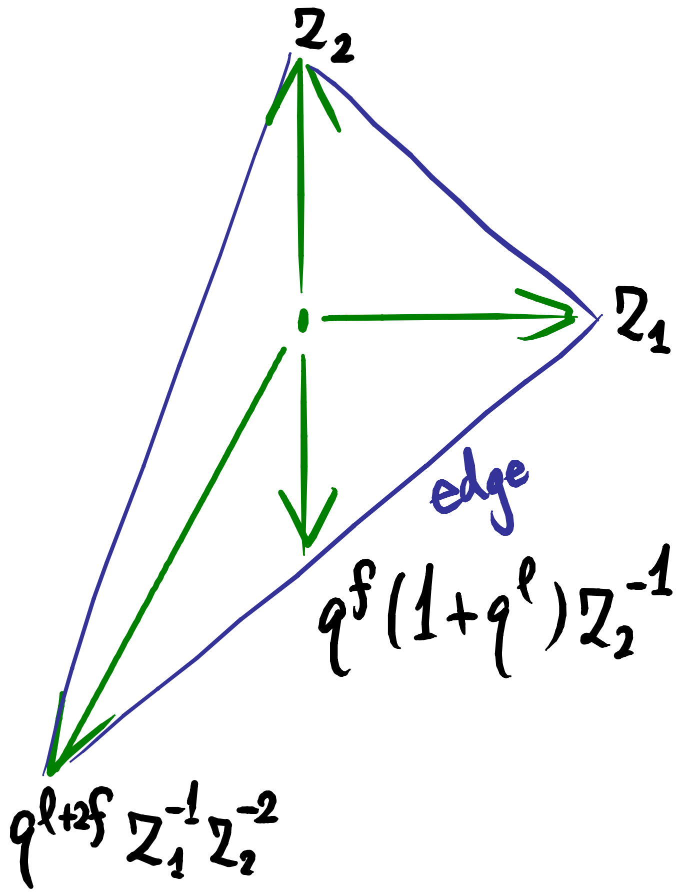

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$$= z_1\left(1 + q^f z_1^{-1} z_2^{-1}\right)^2$$

take $q^\ell = 1$

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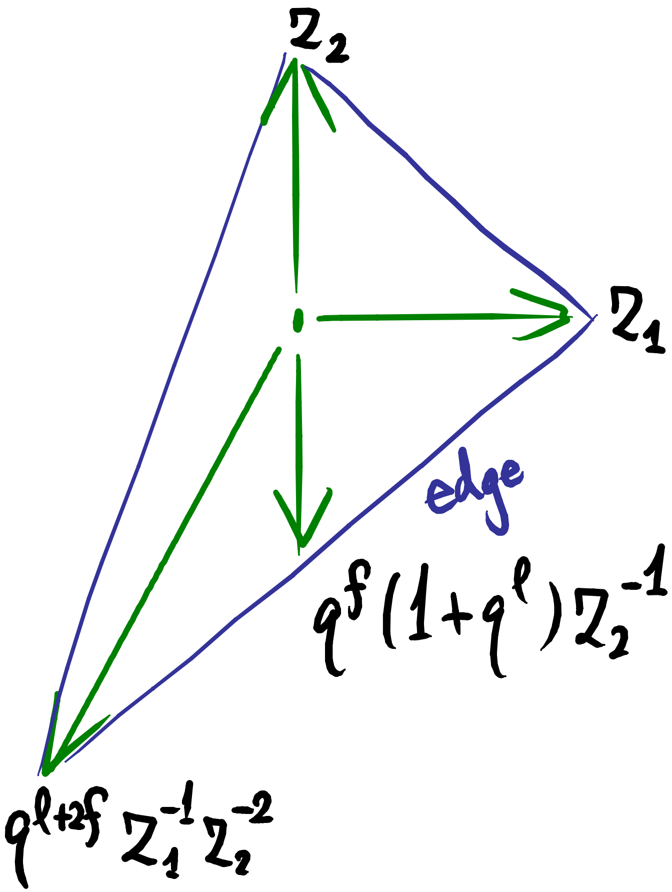
$$W^{SYZ} = z_1 + z_2 + q^f(1+q^\ell)\, z_2^{-1} + q^{\ell+2f} z_1^{-1} z_2^{-2} = W^{PDE}.$$



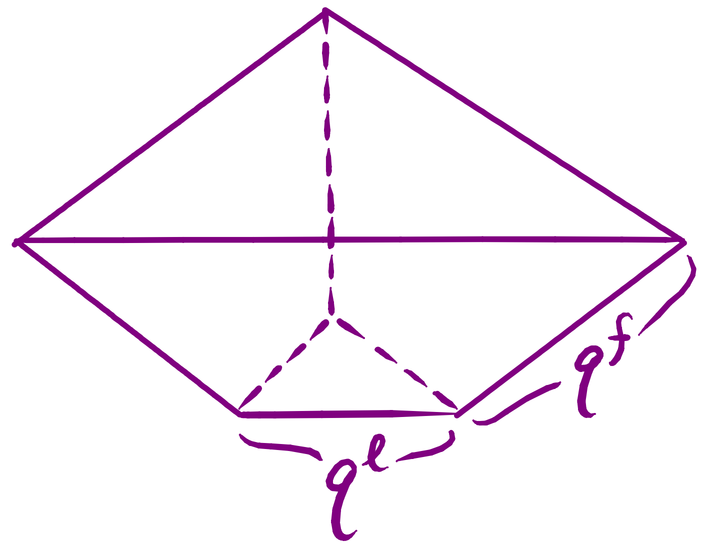
$$W\big|_{\text{edge}} = z_1 + q^f(1+q^\ell) z_2^{-1} + q^{\ell+2f} z_1^{-1} z_2^{-2}$$

$$= z_1 \left(1 + q^{f+\ell} z_1^{-1} z_2^{-1}\right)\left(1 + q^f z_1^{-1} z_2^{-1}\right)$$

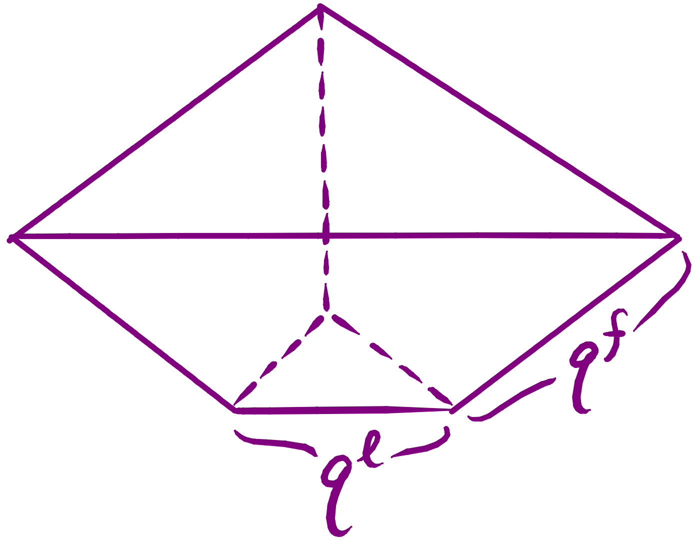
take $q^\ell = 1$ $\quad = z_1\left(1 + q^f z_1^{-1} z_2^{-1}\right)^2$

extremal Laurent polynomial in Corti's talk.

3D example: $\mathbb{P}(K_{\mathbb{P}^2} \oplus \mathcal{O})$.



3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.



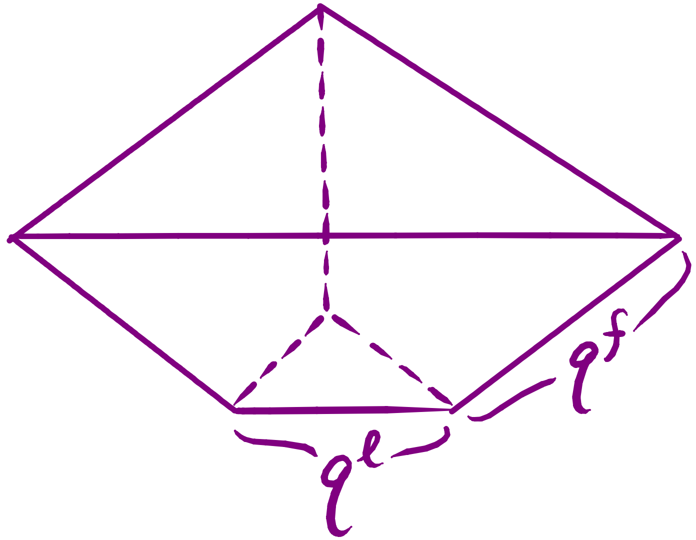
Mirror map:

(Givental; Lian-Liu-Yau; Hori-Vafa)

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

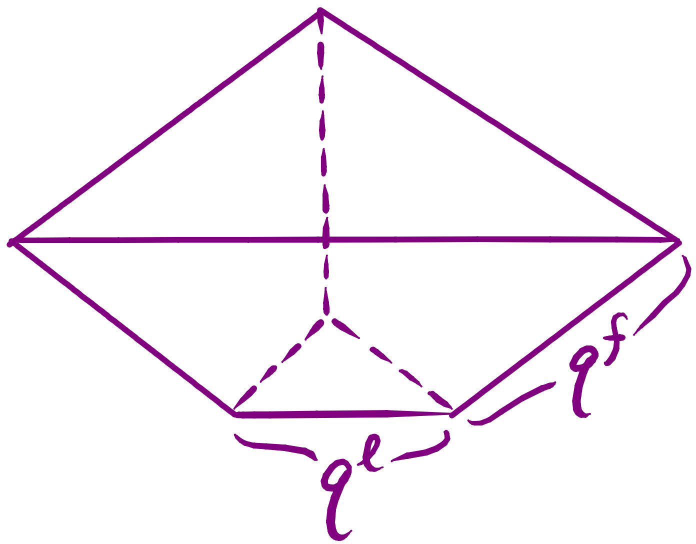
Mirror map :

(Givental ; Lian - Liu - Yau ; Hori - Vafa)



$$W^{PDE}_{\check{q}^\ell, \check{q}^f} = z_1 + z_2 + z_3 + \check{q}^f z_3^{-1} + \check{q}^\ell z_3^3 z_1^{-1} z_2^{-1}$$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.



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$$W^{PDE}_{\check{q}^\ell, \check{q}^f} = z_1 + z_2 + z_3 + \check{q}^f z_3^{-1} + \check{q}^\ell z_3^3 z_1^{-1} z_2^{-1}$$

where $\log q^\ell(\check{q})$, $\log q^f(\check{q})$ satisfies a Picard-Fuchs system.

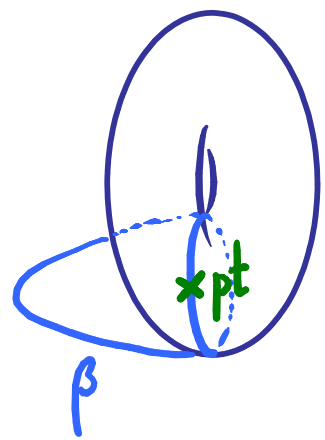
3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

$$\text{SYZ map}: W^{SYZ} = \sum_{\beta \in \pi_2(X,T)} n_\beta Z_\beta$$

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)

For $\beta \in \pi_2(X,T)$,

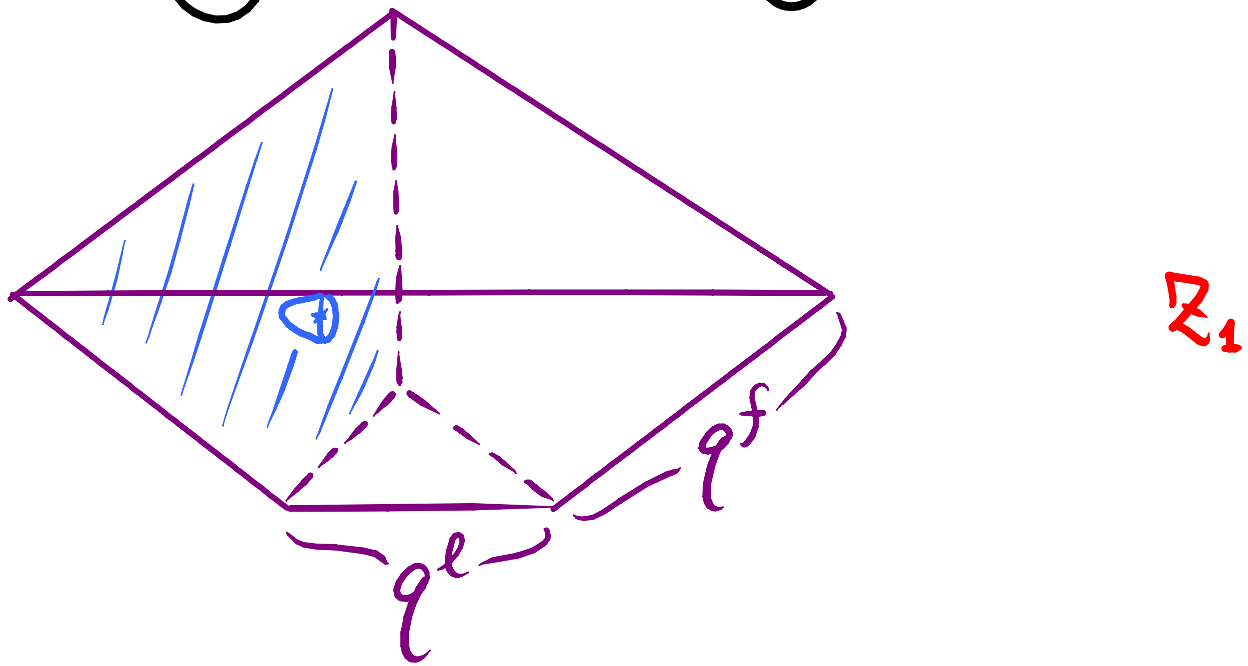
$$n_\beta \triangleq \int_{[\mathcal{M}_1(\beta)]^{\text{virt.}}} [pt].$$



3D example: $\mathbb{P}(K_{\mathbb{P}^2} \oplus \mathcal{O})$.

SYZ map:

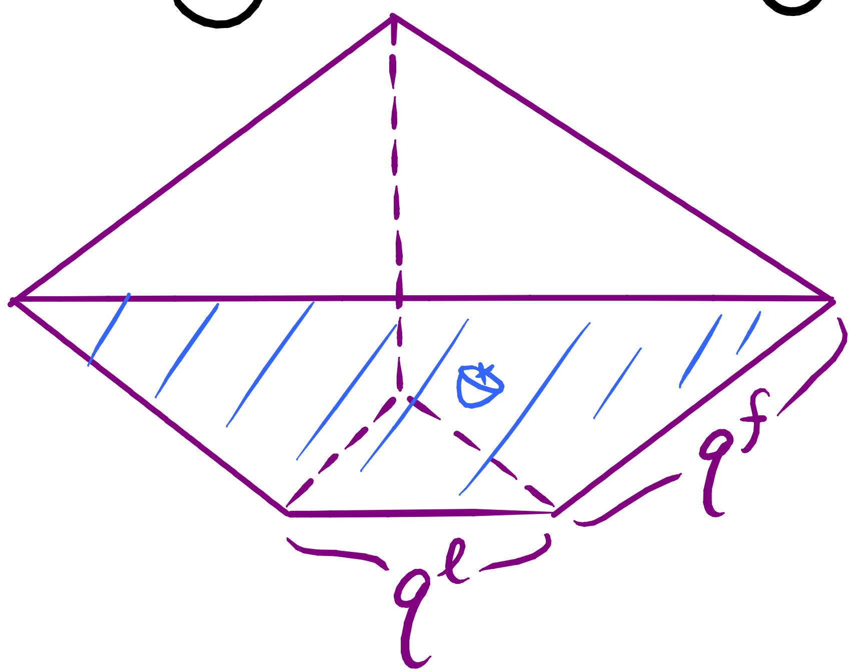
(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)



3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

SYZ map:

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)

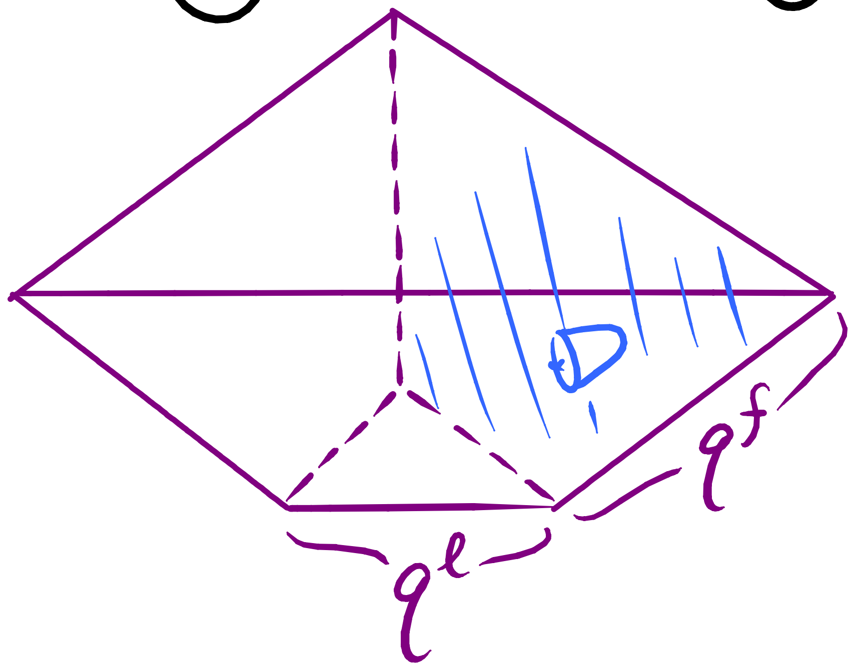


$\mathbb{Z}_2$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right).$

SYZ map:

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)

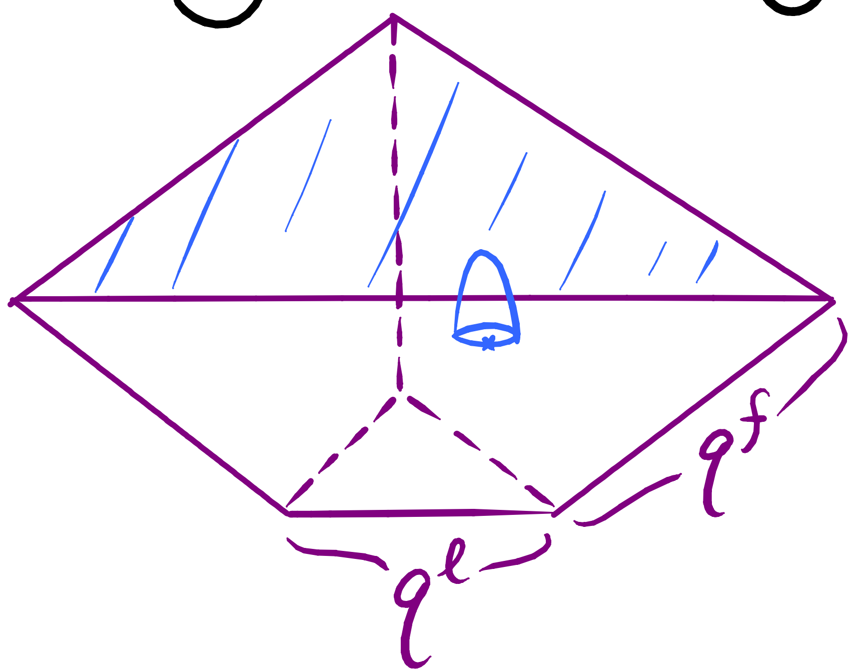


$q^\ell z_3^3 z_1^{-1} z_2^{-1}$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

SYZ map:

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)

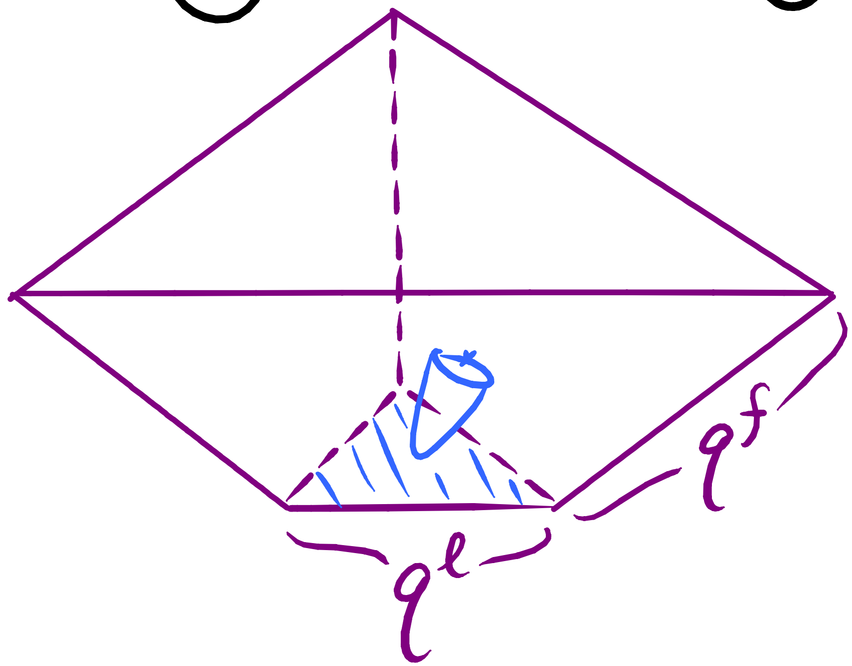


$q^f z_3^{-1}$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right).$

SYZ map:

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)

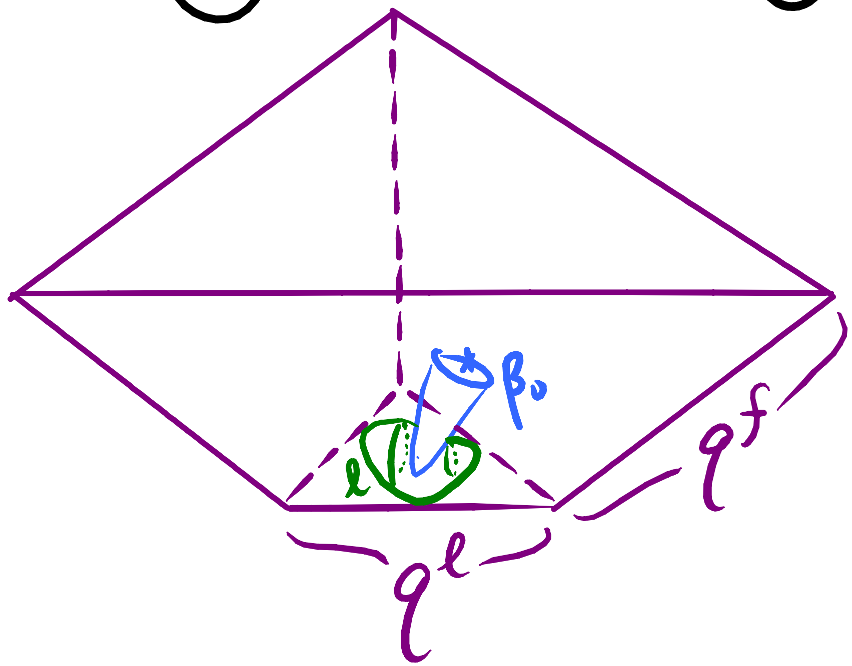


$\mathbb{Z}_3$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

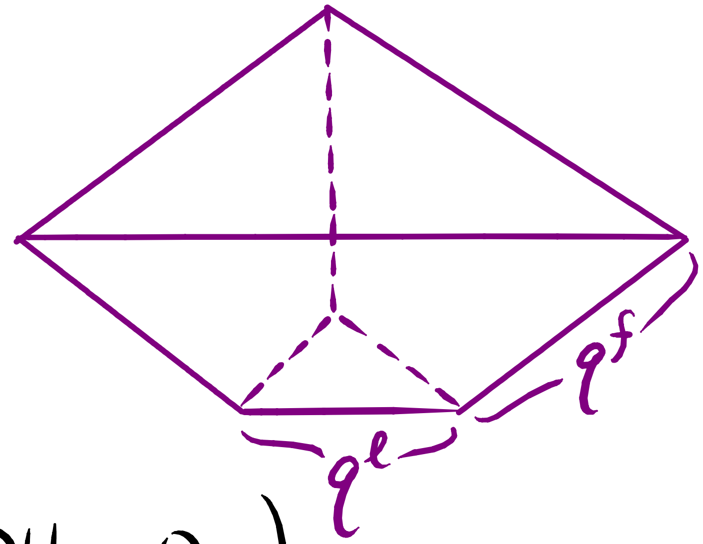
SYZ map:

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)



$$\left(1+\sum_{k=1}^{\infty} n_{\beta_0+k\ell}\, q^{k\ell}\right) Z_3$$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.



SYZ map:

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)

$$\sum n_\beta Z_\beta = Z_1 + Z_2 + \left(1 + \sum n_{\beta_0 + k\ell}\, q^{k\ell}\right) Z_3 + q^f Z_3^{-1} + q^\ell Z_3^3 Z_1^{-1} Z_2^{-1}.$$

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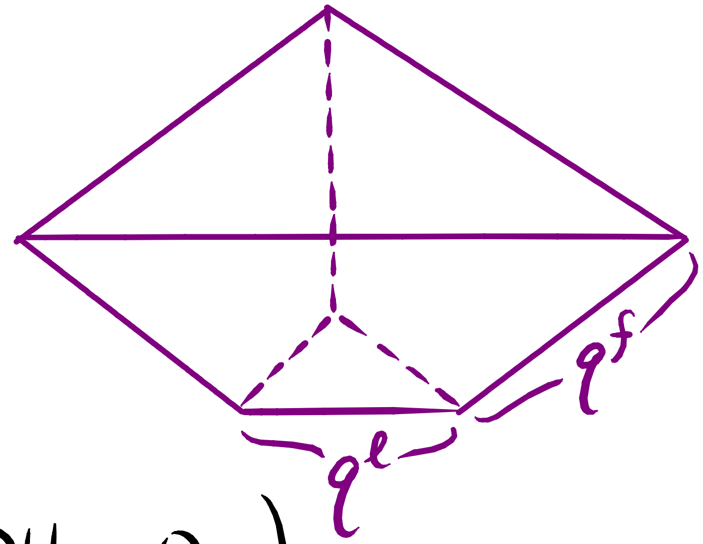

SYZ map:

(Chan-Leung; Cho-Oh; Fukaya-Oh-Ohta-Ono)

$$\sum n_\beta Z_\beta = Z_1 + Z_2 + \left(1 + \sum n_{\beta_0 + kl}\, q^{kl}\right) Z_3 + q^f Z_3^{-1} + q^l Z_3^3 Z_1^{-1} Z_2^{-1}.$$

By change of coordinates,

$$W^{SYZ} = Z_1 + Z_2 + Z_3 + q^f\left(1 + \sum n_{\beta_0 + kl}\, q^{kl}\right) Z_3^{-1} + \frac{q^l}{\left(1 + \sum n_{\beta_0 + kl}\, q^{kl}\right)^3} Z_3^3 Z_1^{-1} Z_2^{-1}.$$

Enumerative meaning of $\mathscr{F}_{\text{mirror}} = \mathscr{F}_{\text{SYZ}}$

$$W^{PDE}_{q^l, q^f} = z_1 + z_2 + z_3 + \check{q}^f z_3^{-1} + \check{q}^l z_3^3 z_1^{-1} z_2^{-1}$$

$$W^{SYZ} = z_1 + z_2 + z_3 + q^f\left(1 + \sum n_{\beta_0 + kl} q^{kl}\right) z_3^{-1} + \frac{q^l}{\left(1 + \sum n_{\beta_0 + kl} q^{kl}\right)^3} z_3^3 z_1^{-1} z_2^{-1}.$$

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$$W^{SYZ} = z_1 + z_2 + z_3 + q^f \left(1 + \sum n_{\beta_0 + kl} q^{kl}\right) z_3^{-1} + \frac{q^l}{\left(1 + \sum n_{\beta_0+kl} q^{kl}\right)^3} z_3^3 z_1^{-1} z_2^{-1}.$$

Equality $\Leftrightarrow$ $\check{q}^f = q^f \left(1 + \sum_{k=1}^{\infty} n_{\beta_0 + kl}\, q^{kl}\right).$

Enumerative meaning of $\mathscr{F}_{\text{mirror}} = \mathscr{F}_{\text{SYZ}}$

$$W^{PDE}_{q^l, q^f} = z_1 + z_2 + z_3 + \check{q}^f z_3^{-1} + \check{q}^l z_3^3 z_1^{-1} z_2^{-1}$$

$$W^{SYZ} = z_1 + z_2 + z_3 + q^f\left(1 + \sum \eta_{\beta_0+kl} q^{kl}\right) z_3^{-1} + \frac{q^l}{\left(1+\sum \eta_{\beta_0+kl} q^{kl}\right)^3} z_3^3 z_1^{-1} z_2^{-1}.$$

$$\text{Equality} \Leftrightarrow \underbrace{\check{q}^f}_{\text{Explicit in terms of } q} = q^f\left(1 + \sum_{k=1}^{\infty} \eta_{\beta_0+kl} q^{kl}\right).$$

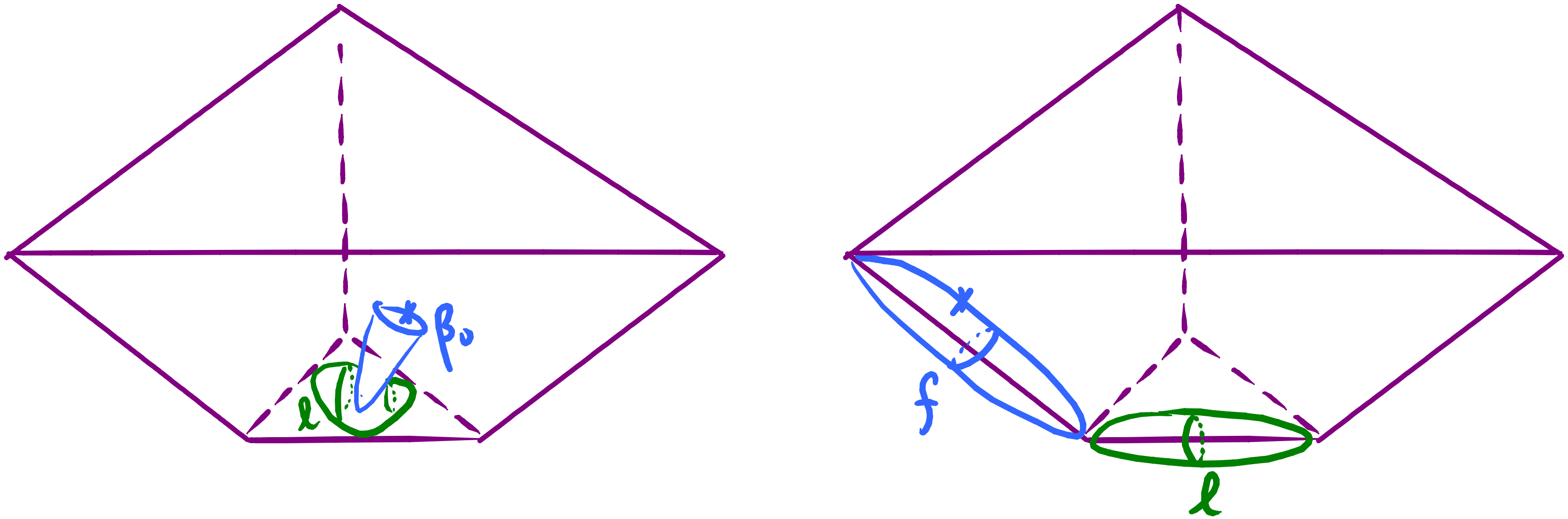
Get η_{β_0+kl}.

k	0	1	2	3	4	5	6
η_{β_0+kl}	1	-2	5	-32	286	-3038	35870

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by direct computation

Theorem: (Chan)

$$n_{\beta_0+kl} = \langle[\mathbf{pt}]\rangle_{0,1,f+kl}.$$

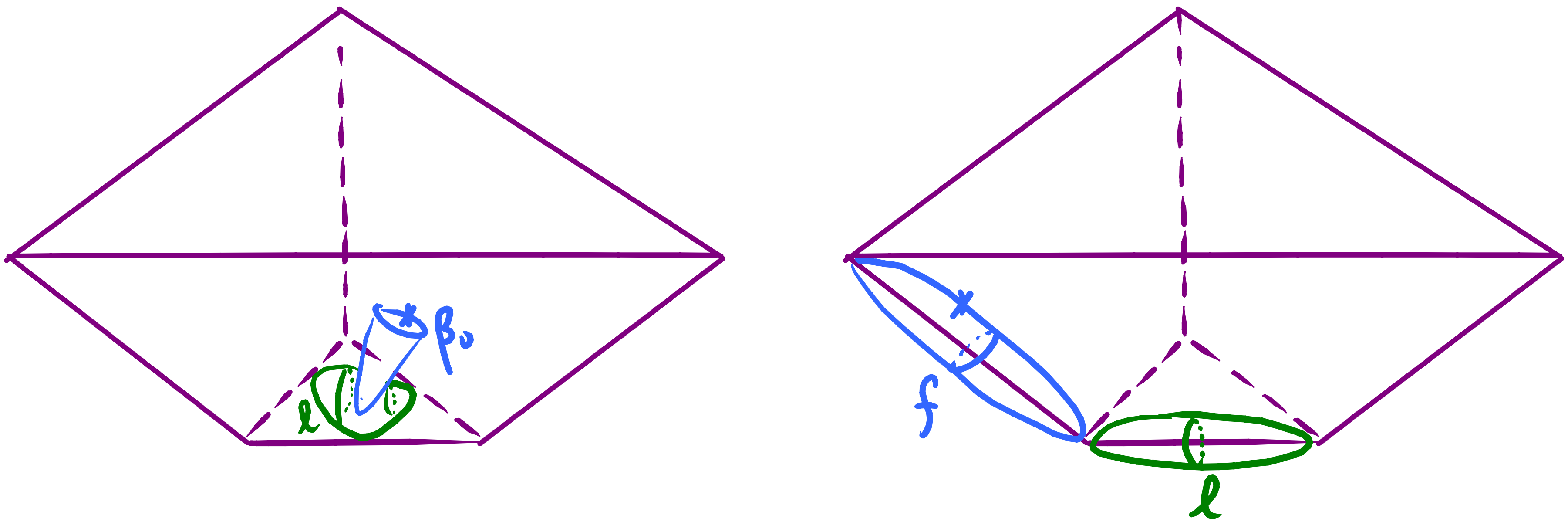


Proof of $\mathscr{F}_{\text{mirror}} = \mathscr{F}_{\text{SYZ}}$ by direct computation

appear in J function.

Theorem: (Chan)

$$n_{\beta_0+kl} = \langle [\mathbf{pt}] \rangle_{0,1,f+kl}.$$



Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ **by direct computation**

$$J(q) \sim \sum_{\substack{\alpha, \\ d \in H_2^{\mathrm{eff}}(X) \setminus \{0\}}} \frac{q^d}{z} \sum_{k \geq 1} \left(\langle \phi_\alpha \psi^{k-1} \rangle_{0,1,d} \frac{\phi^\alpha}{z^k} \right) \in H^{\cdot}(X, \mathbb{C})[[\tfrac{1}{z}]].$$

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H^0-part of $\frac{1}{z^2}$-coefficient

$$= q^f\left(1+\sum_{k=1}^{\infty}\langle pt\rangle_{0,1,f+kl}\, q^{kl}\right)$$

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H^0-part of $\frac{1}{z^2}$-coefficient

$$= q^f \left(1 + \sum_{k=1}^{\infty} \langle pt \rangle_{0,1,f+kl} \, q^{kl} \right)$$

$$= q^f \left(1 + \sum_{k=1}^{\infty} n_{\beta_0+kl} \, q^{kl} \right)$$

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ **by direct computation**

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$$= q^f\left(1+\sum_{k=1}^{\infty}\langle pt\rangle_{0,1,f+kl}\, q^{kl}\right)$$

$$= q^f\left(1+\sum_{k=1}^{\infty} n_{\beta_0+kl}\, q^{kl}\right) \underset{\text{Want}}{=} \check{q}^f$$

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by direct computation

Theorem: (Givental; Lian-Liu-Yau)

$$I(\check{q}(q)) = J(q).$$

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by direct computation

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$$\sum_{d\in H_2^{\mathrm{eff}}(X)\setminus\{0\}} \check{q}^d \prod_i \frac{\prod_{m=-\infty}^{0}(D_i + mz)}{\prod_{m=-\infty}^{D_i\cdot d}(D_i + mz)}$$

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$$\wr$$

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H^0-part of $\frac{1}{z^2}$-coefficient $= \check{q}^f$.

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H^0-part of $\frac{1}{z^2}$-coefficient $= \check{q}^f$.

$$\therefore\ q^f\left(1 + \sum_{k=1}^{\infty} n_{\beta_0 + kl}\, q^{kl}\right) = \check{q}^f.$$

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by mirror symmetry

- In general don't have

$$n_{\beta_0+kl} = \langle[\mathbf{pt}]\rangle_{0,1,f+kl}\,.$$

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$$n_{\beta_0+kl} = \langle[\mathbf{pt}]\rangle_{0,1,f+kl}\,.$$

Theorem: (Fukaya-Oh-Ohta-Ono)

$$\left(H^{\cdot}(X), \cup_q\right) \cong \left(\mathrm{Jac}\left(W_q^{SYZ}\right), \cdot\right)$$

$$\frac{\partial}{\partial q} \longmapsto \left[\frac{\partial}{\partial q} W_q^{SYZ}\right]$$

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by mirror symmetry

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$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \\ \dfrac{\partial}{\partial q} & \longmapsto & \left[\dfrac{\partial}{\partial q} W_q^{SYZ}\right] \end{array}$$

SYZ map has the property of mirror map!

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by mirror symmetry

Theorem: (Givental; Lian-Liu-Yau)

$$\left(H^{\cdot}(X), \cup_q\right) \cong \mathrm{Jac}\left(W_q^{PDE}\right)$$

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Want: such property characterizes the mirror map.

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ **by mirror symmetry**

Theorem: (Givental; Lian-Liu-Yau)

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$$\frac{\partial}{\partial q} \longmapsto \left[\frac{\partial}{\partial q} W_q^{PDE}\right]$$

Want: such property characterizes the mirror map.

$$\Downarrow$$

$$W^{SYZ} \equiv W^{PDE}.$$

Proof of $\mathscr{F}_{\text{mirror}} = \mathscr{F}_{\text{SYZ}}$ by mirror symmetry

$$\mathrm{Jac}(W_q^{\mathrm{PDE}}) \cong \mathrm{Jac}(W_q^{\mathrm{SYZ}}) \quad \text{as algebras.}$$

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by mirror symmetry

$$\mathrm{Jac}\left(W_q^{\mathrm{PDE}}\right) \cong \mathrm{Jac}\left(W_q^{\mathrm{SYZ}}\right) \quad \text{as algebras.}$$

$$\left[\frac{\partial}{\partial q} W_q^{\mathrm{PDE}}\right] \longleftrightarrow \left[\frac{\partial}{\partial q} W_q^{\mathrm{SYZ}}\right]$$

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by mirror symmetry

$$\mathrm{Jac}(W_q^{PDE}) \cong \mathrm{Jac}(W_q^{SYZ}) \quad \text{as algebras.}$$

$$\left[\frac{\partial}{\partial q} W_q^{PDE}\right] \longleftrightarrow \left[\frac{\partial}{\partial q} W_q^{SYZ}\right]$$

$\xRightarrow{\text{Semi-simple}}$ W_q^{PDE} and W_q^{SYZ} have the same set of critical values $\forall q$.

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by mirror symmetry

$$\mathrm{Jac}(W_q^{\mathrm{PDE}}) \cong \mathrm{Jac}(W_q^{\mathrm{SYZ}}) \quad \text{as algebras.}$$

$$\left[\frac{\partial}{\partial q} W_q^{\mathrm{PDE}}\right] \longleftrightarrow \left[\frac{\partial}{\partial q} W_q^{\mathrm{SYZ}}\right]$$

$\Longrightarrow$ W_q^{PDE} and W_q^{SYZ} have the same set of critical values.

Universal unfolding

$$\Longrightarrow \quad W_q^{\mathrm{PDE}} = W_q^{\mathrm{SYZ}}$$

Proof of $\mathscr{F}_{\mathrm{mirror}} = \mathscr{F}_{\mathrm{SYZ}}$ by mirror symmetry

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$$\left[\frac{\partial}{\partial q} W_q^{PDE}\right] \longleftrightarrow \left[\frac{\partial}{\partial q} W_q^{SYZ}\right]$$

$\Longrightarrow$ W_q^{PDE} and W_q^{SYZ} have the same set of critical values.

$$\boxed{\overset{\text{Universal unfolding}}{\Longrightarrow} \quad W_q^{PDE} = W_q^{SYZ}} \longleftarrow \text{Assume } W_q^{SYZ} \text{ converges.}$$

Conclusion

Theorem: (Chan-Lau-Leung-Tseng)

$$W^{PDE} = W^{SYZ}$$

for compact semi-Fano toric manifolds,

assuming W^{SYZ} converges.

Thanks