

# Quantum cohomology of flag varieties: Introduction

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# Flag varieties $X$

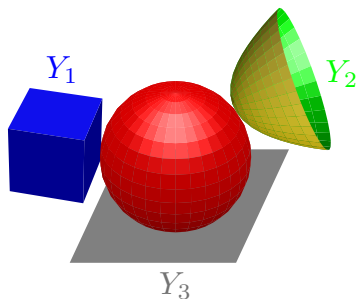
- Toy examples of flag varieties  $X$ :
  - ▶  $\mathbb{P}^1$ : lines in  $\mathbb{C}^2$ .
  - ▶  $\mathbb{P}^2$ : lines in  $\mathbb{C}^3$ .
  - ▶  $Fl_3 := \{V_1 \leq V_2 \leq \mathbb{C}^3 \mid \dim V_1 = 1, \dim V_2 = 2\}$ .

# Flag varieties $X$

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  - ▶  $Fl_3 := \{V_1 \leq V_2 \leq \mathbb{C}^3 \mid \dim V_1 = 1, \dim V_2 = 2\}$ .
- In general,  $X = G/P$ .
  - ▶  $G$ : a simply-connected complex simple Lie group.
  - ▶  $P$ : a parabolic subgroup of  $G$  ( $\iff G/P$  is a projective manifold).

# Why (SMALL) quantum cohomology??

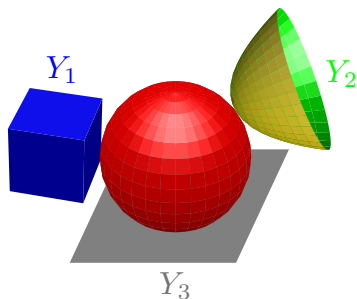
Let  $Y_1, Y_2, Y_3$  be subvarieties of a flag variety  $X$  in general position.



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■  $Vol(S^2) = 0 \rightsquigarrow \#Y_1 \cap Y_2 \cap Y_3 = ??$

# Motivation and Main results

$$\blacksquare \text{Vol}(\mathbb{S}^2) = 0 \quad \rightsquigarrow \quad \#Y_1 \cap Y_2 \cap Y_3 = ?? \quad \longleftarrow \quad H^*(X)$$

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## Theorem (Leung-Li, 2011)

*Under certain assumptions,*

$$\begin{aligned} & \#\{\mathbb{S}^2 \text{ of fixed volume that hits } Y_1, Y_2, Y_3\} \quad (\text{Gromov-Witten inv.}) \\ & = \#\widehat{Y}_1 \cap \widehat{Y}_2 \cap \widehat{Y}_3 \end{aligned}$$

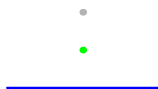


# Projective line $\mathbb{P}^1$ : point, line — basic subvarieties.

Caes I



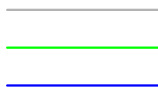
Caes II



Caes III



Caes IV

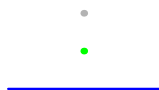


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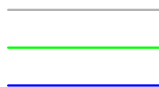
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$$e = P.D.(\mathbb{P}^1) \in H^0(\mathbb{P}^1, \mathbb{Z})$$

$$z = P.D.(pt) \in H^2(\mathbb{P}^1, \mathbb{Z})$$

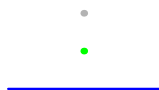
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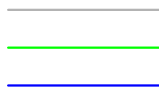
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$$\text{III} : e \cup z = z$$

$$\text{I} : z \cup z = 0 \cdot e$$

$$H^*(\mathbb{P}^1, \mathbb{Z}) \cong \frac{\mathbb{Z}[z]}{\langle z^2 \rangle}$$

Example of “quantum”:  $QH^*(\mathbb{P}^1)$ . (Note  $\mathbb{P}^1 \rightleftarrows G/B$ )

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represented by

$$z \star z = 1 \cdot q \cdot e$$

In fact,

$$QH^*(\mathbb{P}^1, \mathbb{Z}) \stackrel{\text{ring}}{=} \frac{\mathbb{Z}[z][q]}{\langle z^2 - q \rangle}.$$

(Note  $z \star z|_{q=0} = 0 = z \cup z$ .)

Example of “quantum”:  $QH^*(\mathbb{P}^2)$ . (Note  $\mathbb{P}^2 \rightsquigarrow G/P$ .)

$$\begin{aligned} H^*(\mathbb{P}^2, \mathbb{Z}) &= H^0(\mathbb{P}^2, \mathbb{Z}) \oplus H^2(\mathbb{P}^2, \mathbb{Z}) \oplus H^4(\mathbb{P}^2, \mathbb{Z}) \\ &= \mathbb{Z} \cdot e \oplus \mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y \\ \text{Here} \quad e &= P.D.(\mathbb{P}^2) \quad x = P.D.(\text{line}) \quad y = P.D.(\text{pt}) \end{aligned}$$





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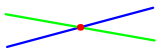
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•  $H^*(\mathbb{P}^2, \mathbb{Z}) \stackrel{\text{ring}}{=} \frac{\mathbb{Z}[x]}{\langle x^3 \rangle}$ .

▶  $x \cup x = y = 1 \cdot y$ ,

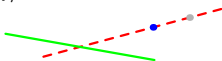
$x \cup y = 0 = 0 \cdot e$ .

▶



•  $QH^*(\mathbb{P}^2, \mathbb{Z}) \stackrel{\text{ring}}{=} \frac{\mathbb{Z}[x][q]}{\langle x^3 - q \rangle}$ .

$x \star y = 1 \cdot q \cdot e \rightsquigarrow$



(SMALL)  $QH^*(G/P) = (H^*(G/P) \otimes \mathbb{Q}[q_1, q_2, \dots, q_r], \star)$

- $$r = \text{rank of } H_2(G/P, \mathbb{Z}).$$
- $$\alpha \star \beta|_{\mathbf{q}=\mathbf{0}} = \alpha \cup \beta.$$
- “ $\star$ ” are defined by incorporating genus 0, three-pointed Gromov-Witten invariant of  $G/P$ .

$$QH^*(Fl_3) = H^*(Fl_3) \otimes \mathbb{Q}[q_1, q_2]$$

$$Fl_3 = \{V_1 \leq V_2 \leq \mathbb{C}^3\} \quad (\dim V_j = j)$$

↓

$$Gr(2,3) = \{V_2 \leq \mathbb{C}^3\}$$

$$QH^*(Fl_3) = H^*(Fl_3) \otimes \mathbb{Q}[q_1, q_2]$$

$$P/B: \mathbb{P}^1 = Gr(1, 2) = \{V_1 \leq \mathbb{C}^2 (= V_2)\}$$

$$\downarrow \iota$$

$$G/B: Fl_3 = \{V_1 \leq V_2 \leq \mathbb{C}^3\} \quad (\dim V_j = j)$$

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$$G/P: \mathbb{P}^2 \cong Gr(2, 3) = \{V_2 \leq \mathbb{C}^3\}$$

- Künneth formula  $\implies H^*(Fl_3) \stackrel{\text{vect. sp.}}{=} H^*(\mathbb{P}^1) \otimes H^*(\mathbb{P}^2)$ .

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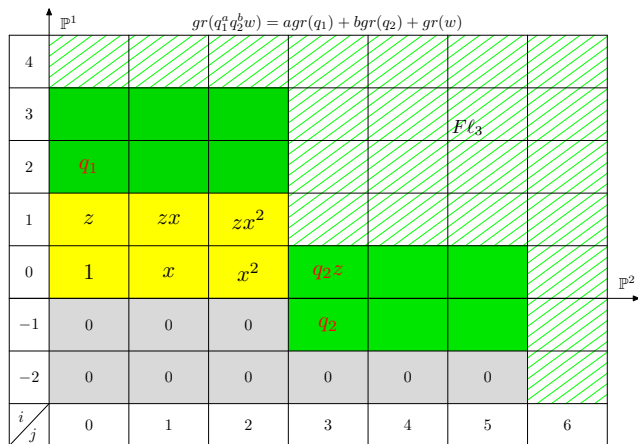
$$\rightsquigarrow \text{rank} H_2(Fl_3) = 2$$

$$\rightsquigarrow QH^*(Fl_3) = H^*(Fl_3) \otimes \mathbb{Q}[q_1, q_2].$$

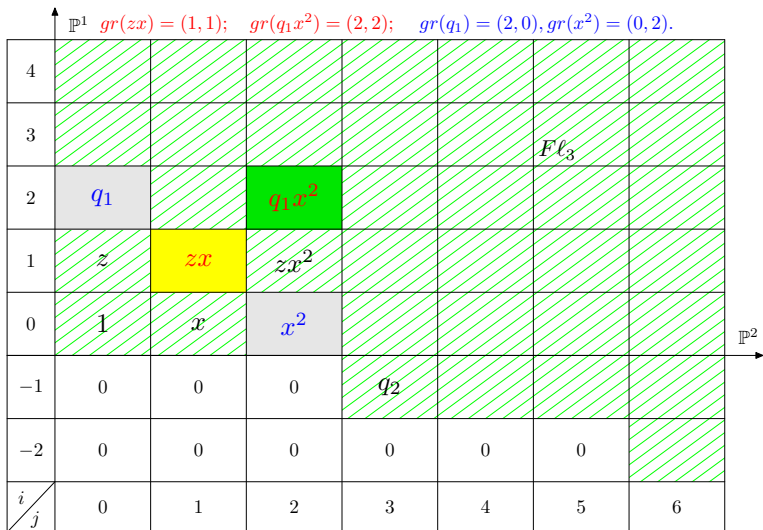
# Idea of the proof of Thm: NICE $\mathbb{Z}^2$ -grading $gr(\alpha) = (i, j)$ .

With respect to the fibration  $\mathbb{P}^1 \rightarrow Fl_3 \rightarrow \mathbb{P}^2$ :

- Leray spectral sequence  $\implies$  nice  $\mathbb{Z}^2$ -grading on  $H^*(Fl_3)$
- Leung-Li  $\implies$  nice  $\mathbb{Z}^2$ -grading on  $QH^*(Fl_3) = H^*(Fl_3) \otimes \mathbb{Q}[q_1, q_2]$ .

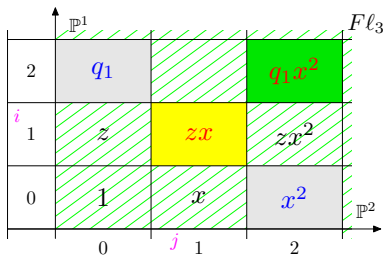


$$\mathbb{Z}^2\text{-grading} \rightsquigarrow \mathcal{F} \rightsquigarrow Gr^{\mathcal{F}}(QH^*(Fl_3)) = QH^*(\mathbb{P}^1) \otimes QH^*(\mathbb{P}^2)$$



# "Quantum = Classical"

In  $QH^*(F\ell_3)$ ,  $zx \star zx = Cq_1x^2 + \text{other terms (of smaller gradings)}$ .  
That is,  $Cq_1x^2 = (2,2)$ -grading part of  $((z \otimes x) \star_{total} (z \otimes x))$ .

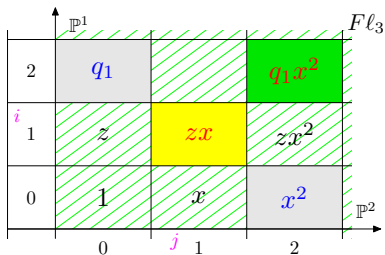




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Nice  $\mathbb{Z}^2$ -grading  $\rightsquigarrow \mathcal{F} \Rightarrow Gr^{\mathcal{F}}(QH^*(Fl_3)) \stackrel{gr. alg.}{=} QH^*(\mathbb{P}^1) \otimes QH^*(\mathbb{P}^2)$



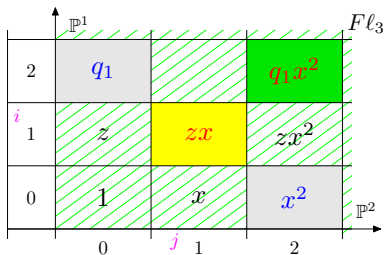
$$\begin{aligned}
 & Cq_1x^2 \\
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$$\therefore N_{zx, zx}^{x^2, q_1} = C = D = N_{x, x}^{x^2, 0}.$$

# Outline of Part II

- Definition of  $QH^*(G/P)$ .
- “Functorial relationships”: filtration on  $QH^*(G/B)$ .
- Applications and examples: classical aspects of Gromov-Witten invariants.

Thank you!!...