Quantum cohomology of flag varieties: Introduction

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November 16, 2011

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Flag varieties X

- Toy examples of flag varieties X:
  - $\mathbb{P}^1$: lines in $\mathbb{C}^2$.
  - $\mathbb{P}^2$: lines in $\mathbb{C}^3$.
  - $F\ell_3 := \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3 \mid \dim V_1 = 1, \dim V_2 = 2\}$.

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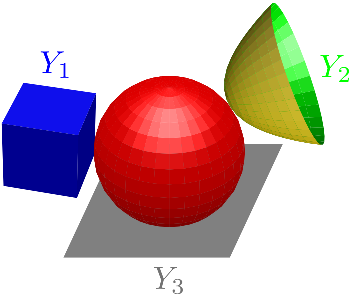
- In general, $X = G/P$.
  - G: a simply-connected complex simple Lie group.
  - P: a parabolic subgroup of G $\left(\Longleftrightarrow G/P \text{ is a projective manifold}\right)$.

Why (SMALL) quantum cohomology??

Let Y_1, Y_2, Y_3 be subvarieties of a flag variety X in general position.



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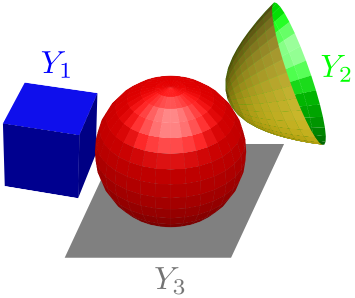

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Motivation and Main results

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- General cases $\quad \Longleftarrow \quad QH^*(X)$

Motivation and Main results

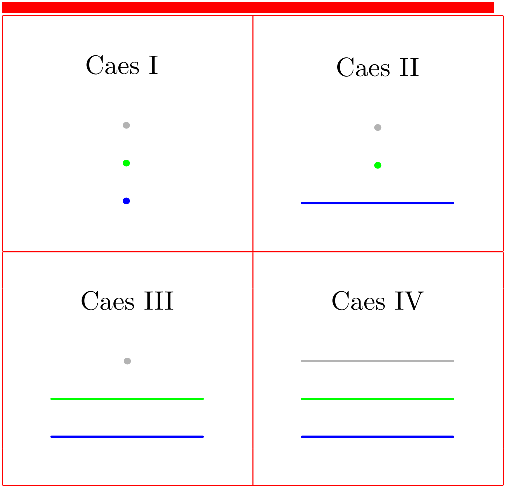
- $Vol(\mathbb{S}^2) = 0 \quad \rightsquigarrow \quad \sharp Y_1 \cap Y_2 \cap Y_3 = ?? \quad \Longleftarrow \quad H^*(X)$
- General cases $\quad \Longleftarrow \quad QH^*(X)$

Theorem (Leung-Li, 2011)

Under certain assumptions,

$$\sharp\{\mathbb{S}^2 \textit{ of fixed volume } \textit{ that hits } Y_1, Y_2, Y_3\} \quad (\textit{Gromov-Witten inv.})$$
$$= \sharp \widehat{Y_1} \cap \widehat{Y_2} \cap \widehat{Y_3}$$

Projective line $\mathbb{P}^1$: point, line — basic subvarieties.



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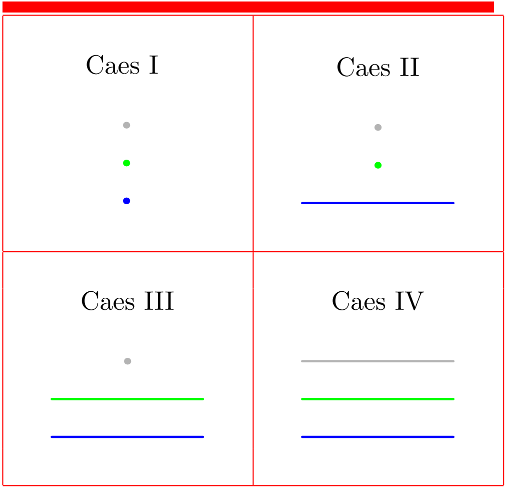

$$e = P.D.(\mathbb{P}^1) \in H^0(\mathbb{P}^1, \mathbb{Z})$$

$$z = P.D.(\mathrm{pt}) \in H^2(\mathbb{P}^1, \mathbb{Z})$$

$$H^*(\mathbb{P}^1, \mathbb{Z}) = H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1)$$

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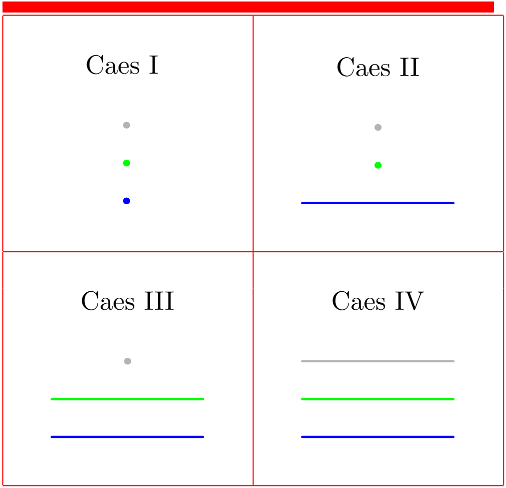

$$\begin{aligned} e &= P.D.(\mathbb{P}^1) \in H^0(\mathbb{P}^1, \mathbb{Z}) \\ z &= P.D.(\mathrm{pt}) \in H^2(\mathbb{P}^1, \mathbb{Z}) \\ H^*(\mathbb{P}^1, \mathbb{Z}) &= H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1) \end{aligned}$$

$$\mathrm{III} : e \cup z = z$$

$$\mathrm{I} : z \cup z = 0 \cdot e$$

$$H^*(\mathbb{P}^1, \mathbb{Z}) \cong \frac{\mathbb{Z}[z]}{\langle z^2 \rangle}$$

Example of "quantum": $QH^*(\mathbb{P}^1)$. (Note $\mathbb{P}^1 \leftrightsquigarrow G/B$)

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In fact,

$$QH^*(\mathbb{P}^1, \mathbb{Z}) \overset{\text{ring}}{=} \frac{\mathbb{Z}[z][[q]]}{\langle z^2 - q \rangle}.$$

(Note $z \star z|_{q=0} = 0 = z \cup z$.)

Example of "quantum": $QH^*(\mathbb{P}^2)$. (Note $\mathbb{P}^2 \leftrightsquigarrow G/P$.)

$$\begin{array}{ccccccc}
H^*(\mathbb{P}^2,\mathbb{Z}) & = & H^0(\mathbb{P}^2,\mathbb{Z}) & \oplus & H^2(\mathbb{P}^2,\mathbb{Z}) & \oplus & H^4(\mathbb{P}^2,\mathbb{Z}) \\
 & = & \mathbb{Z}\cdot e & \oplus & \mathbb{Z}\cdot x & \oplus & \mathbb{Z}\cdot y \\
\text{Here} & & e = P.D.(\mathbb{P}^2) & & x = P.D.(\text{line}) & & y = P.D.(\text{pt})
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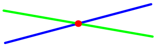
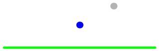
- $H^*(\mathbb{P}^2,\mathbb{Z}) \overset{\text{ring}}{=} \dfrac{\mathbb{Z}[x]}{\langle x^3\rangle}$.
  - $x \cup x = y = 1 \cdot y$, $\quad x \cup y = 0 = 0 \cdot e$.
  -





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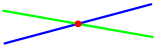

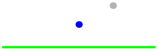

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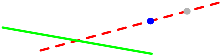
$x \star y = 1 \cdot q \cdot e \quad \leftrightsquigarrow$



(SMALL) $QH^*(G/P) = (H^*(G/P) \otimes \mathbb{Q}[q_1, q_2, \cdots, q_r], \star)$

- $$r = \text{rank of } H_2(G/P, \mathbb{Z}).$$

- $$\alpha \star \beta\big|_{\mathbf{q}=\mathbf{0}} = \alpha \cup \beta.$$

- "$\star$" are defined by incorporating genus 0, three-pointed Gromov-Witten invariant of G/P.

$QH^*(F\ell_3) = H^*(F\ell_3) \otimes \mathbb{Q}[q_1, q_2]$

$$\begin{array}{rc} F\ell_3 = & \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3\} \\ & \downarrow \\ Gr(2,3) = & \{V_2 \leqslant \mathbb{C}^3\} \end{array} \qquad \big(\dim V_j = j\big)$$

$QH^*(F\ell_3) = H^*(F\ell_3) \otimes \mathbb{Q}[q_1, q_2]$

$$
\begin{array}{lll}
P/B: & \mathbb{P}^1 = Gr(1,2) = \{V_1 \leqslant \mathbb{C}^2 (= V_2)\} & \\
 & \qquad\qquad\downarrow \iota & \\
G/B: & F\ell_3 = \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3\} & \big(\dim V_j = j\big) \\
 & \qquad\qquad\Big\downarrow \pi & \\
G/P: & \mathbb{P}^2 \cong Gr(2,3) = \{V_2 \leqslant \mathbb{C}^3\} &
\end{array}
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- Künneth formula $\Longrightarrow$ $H^*(F\ell_3) \overset{\text{vect. sp.}}{=} H^*(\mathbb{P}^1) \otimes H^*(\mathbb{P}^2)$.

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$\rightsquigarrow \quad \mathrm{rank} H_2(F\ell_3) = 2$

$\rightsquigarrow QH^*(F\ell_3)) = H^*(F\ell_3) \otimes \mathbb{Q}[q_1, q_2]$.

Idea of the proof of Thm: NICE $\mathbb{Z}^2$-grading $gr(\alpha) = (i, j)$.

With respect to the fibration $\mathbb{P}^1 \to F\ell_3 \longrightarrow \mathbb{P}^2$:

- Leray spectral sequence $\Longrightarrow$ nice $\mathbb{Z}^2$-grading on $H^*(F\ell_3)$
- Leung-Li $\Longrightarrow$ nice $\mathbb{Z}^2$-grading on $QH^*(F\ell_3)=H^*(F\ell_3) \otimes \mathbb{Q}[q_1, q_2]$.

$\mathbb{P}^1$ $\qquad gr(q_1^a q_2^b w) = a gr(q_1) + b gr(q_2) + gr(w)$

4							
3						$F\ell_3$	
2	q_1						
1	z	zx	zx^2				
0	1	x	x^2	$q_2 z$			
-1	0	0	0	q_2			
-2	0	0	0	0	0	0	
i / j	0	1	2	3	4	5	6

$\mathbb{P}^2$

$\mathbb{Z}^2$-grading $\rightsquigarrow \mathcal{F} \rightsquigarrow Gr^{\mathcal{F}}(QH^*(F\ell_3)) = QH^*(\mathbb{P}^1) \otimes QH^*(\mathbb{P}^2)$

$\mathbb{P}^1$ $gr(zx) = (1,1)$; $gr(q_1x^2) = (2,2)$; $gr(q_1) = (2,0), gr(x^2) = (0,2)$.

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3						$F\ell_3$	
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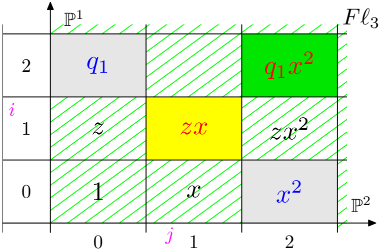
1	z	zx	zx^2				
0	1	x	x^2				
-1	0	0	0	q_2			
-2	0	0	0	0	0	0	
$i \backslash j$	0	1	2	3	4	5	6

$\mathbb{P}^2$

"Quantum = Classical"

In $QH^*(F\ell_3)$, $zx \star zx = Cq_1x^2 +$ other terms (of smaller gradings).
That is, $Cq_1x^2 =$ $(2,2)$-grading part of $\big((z \otimes x) \star_{total} (z \otimes x)\big)$.



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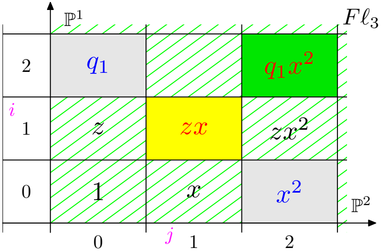
Nice $\mathbb{Z}^2$-grading $\rightsquigarrow \mathcal{F} \Rightarrow Gr^{\mathcal{F}}(QH^*(F\ell_3)) \overset{gr.alg.}{=} QH^*(\mathbb{P}^1) \otimes QH^*(\mathbb{P}^2)$



$$
\begin{aligned}
& Cq_1x^2 \\
=& (2,2)\text{-part of } \big((z \star_{fiber} z) \otimes (x \star_{base} x)\big) \\
=& q_1 \otimes \big((0,2)\text{-part of } x \star_{base} x\big) \\
=& q_1 \otimes (Dx^2)
\end{aligned}
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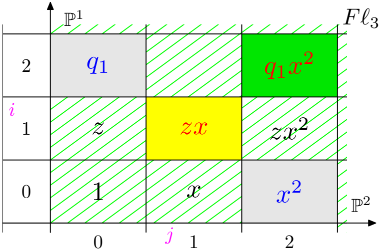

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\end{aligned}
$$

$$\therefore \quad N_{zx,zx}^{x^2,q_1} = C = D = N_{x,x}^{x^2,0}.$$

Outline of Part II

- Definition of $QH^*(G/P)$.
- "Functorial relationships": filtration on $QH^*(G/B)$.
- Applications and examples: classical aspects of Gromov-Witten invariants.

Thank you!!...