Instantons on $\mathbb{R}^3 \times S^1$ and Renormalons

Work w/ M. Ünsal, arXiv:1202.... (to appear)

Outline:

1. Motivation: Perturbation series and Borel resummation
2. QCD on $\mathbb{R}^4$: Instantons and renormalons
3. QCD on $\mathbb{R}^3 \times S^1$: Quick summary of results

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4. QCD(adj) on $\mathbb{R}^3 \times S^1$
5. Monopole-instantons on $\mathbb{R}^3 \times S^1$
6. Multi-instanton "molecules"
7. Further directions

- Brings together some 'old' topics:
  - Borel resummation & 't Hooft's renormalons
  - Polyakov picture of confinement in 3d
  - Bogomolny & Zinn-Justin on neutral topological molecules ("bounces")
- And some newer developments
  - M. Ünsal on gauge theories on $\mathbb{R}^3 \times S^1$
- Will try to review as necessary.

1. MOTIVATION / BOREL RESUMMATION

(When) is there a continuum lim'n of QFTs?

- ⇔ Can we make sense of perturbation theory/semi-class. expansion? [Eg $N \gg 2$ SYM]

- Problem: perturbation series in QFT,

$$F(g^2) \overset{?}{=} p_0 + p_1 g^2 + p_2 g^4 + \dots \equiv P$$

generally have 0-radius of convergence

- Instantons (saddle-point contributions to the path integral for small g^2) give non-pert. corrections

$$F(g^2) = P + e^{-8\pi^2/g^2}\underbrace{\left(i_0 + i_1 g^2 + \dots\right)}_{\equiv I}$$

[I = instanton
B = bounce = inst-inst]

$$+ e^{-2\cdot\left(\frac{8\pi^2}{g^2}\right)}\underbrace{\left(b_0 + b_1 g^2 + \dots\right)}_{\equiv B}$$

$$+ \dots$$

but does not improve convergence.

Borel resummation idea:

If P has convergent Borel transform

$$\mathcal{B}P(t) := \frac{p_0}{0!} + \frac{p_1}{1!}t^1 + \frac{p_2}{2!}t^2 + \dots$$

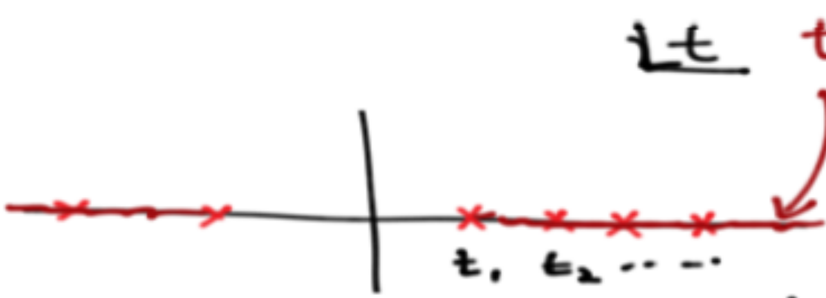
in neighborhood of $t=0$ (e.g., if $|p_n| < c\cdot n!$ as $n \to \infty$), then

$$\mathbb{P}(g^2) := \frac{1}{g^2}\int_0^\infty \mathcal{B}P(t)\, e^{-t/g^2}\, dt$$

formally gives back $P(g^2)$... But ambiguous if $\mathcal{B}P(t)$ has sings @ $t \in \mathbb{R}^+$ $\Longleftrightarrow$ pick contour (or give g^2 small Im part).

t-plane ("Borel plane")

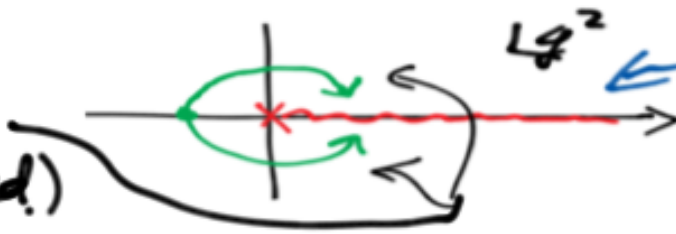
REV: Weinberg II, 20.7
't Hooft 77 Erice Sum.



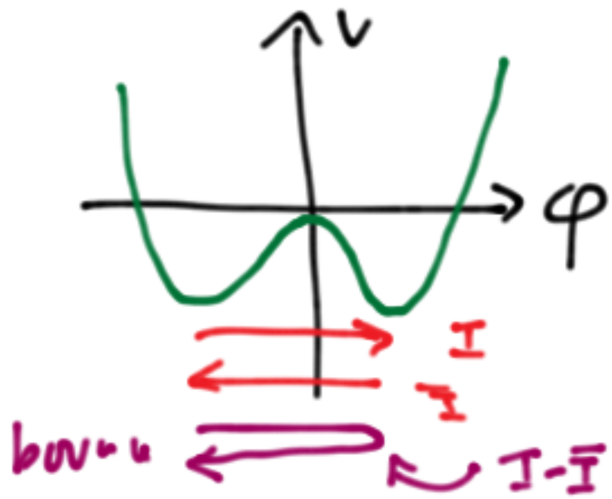
$\Longrightarrow \mathbb{P}(g^2) = \operatorname{Re}\mathbb{P} \pm i \operatorname{Im}\mathbb{P}$

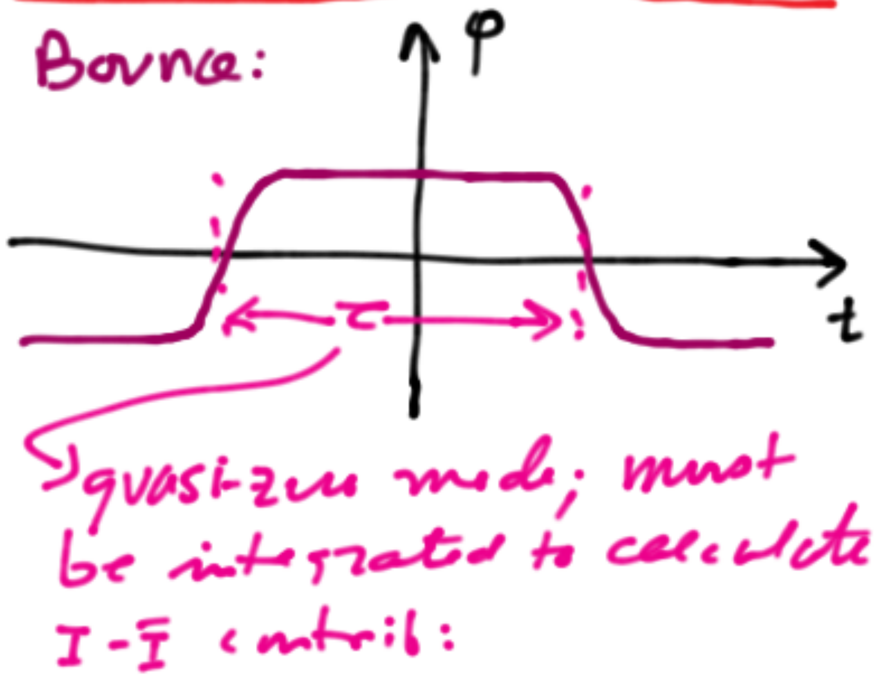
w/ $\operatorname{Im}\mathbb{P}(g^2) \simeq e^{-t_1/g^2} + \begin{pmatrix}\text{sub-leading from} \\ t_n > t_1 \quad \dots\end{pmatrix}$

analytic cont. is ambiguous.
(& often $\operatorname{Im}\mathbb{P}$ unphysical)

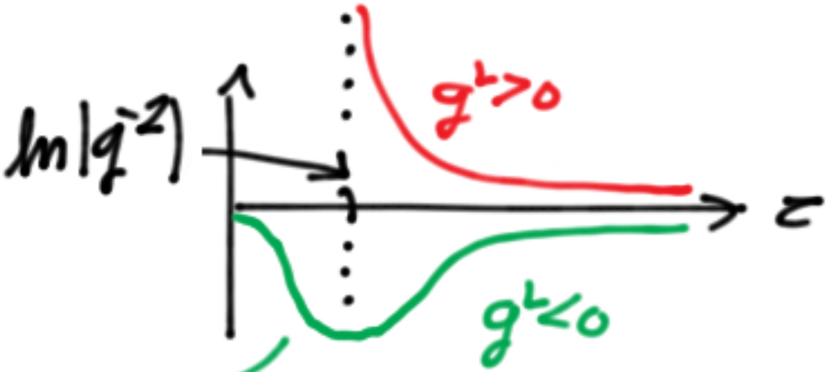


- Ambiguity in $\mathbb{P}(g^2)$ has same form as inst. contrib $\sim e^{-8\pi^2/g^2}$:
  - In late '70's Bogomolnyi, Zinn-Justin advocated/showed in double-well QM ... that inst-inst. ("bounce") contribs $\equiv \mathbb{B}(g^2)$ to vacuum quantities, evaluated @ $g^2<0$, & continued to $g^2>0$ in same way as $\mathbb{P}(g^2)$ gives unambiguous answer: Im part cancels in $\mathbb{P}+\mathbb{B}$:





$$\mathcal{Q}_{I\bar{I}} \sim \int_0^\infty d\tau\, \Big[\underbrace{\exp\!\big(g^{-2} e^{-\tau}\big)}_{I\text{-}\bar{I}\ \text{interaction}\ e^{-V}} \underbrace{-1}_{\text{non-interacting DIG.}}\Big]$$



Bogomolny PLB 91 (1980)
$\Rightarrow$ calc @ $g^2<0$ & continue ...

$$\mathcal{Q}(g^2_{<0}) = c + \ln(-g^2) \Rightarrow \mathcal{Q}(g^2_{>0}) = c + \ln g^2 \pm i\pi$$

$$\Rightarrow \mathbb{B} = \mathrm{Re} \pm i\pi e^{-2S_I}$$

From high orders in p.t. (Zinn-Justin text) $\Rightarrow$ NPB 192 ('81)

sing. in Borel plane @ $t = 2\cdot 8\pi^2 \Rightarrow$

$$\mathbb{P}(g^2) = \cdots \mp i\pi e^{-2S_I} \Rightarrow \boxed{\mathbb{P}+\mathbb{B} = \mathrm{Re},\ \text{unambig.}}$$

$\therefore$ Call "BZJ prescription".

Q: <u>Can this work for QFTs?</u>

e.g. QCD? Balitsky, <u>Yung</u> '86, '88.

A1: Not on $\mathbb{R}^4$ ... ('t Hooft)

A2: <u>Yes</u> on $\mathbb{R}^3 \times S^1$ (for some theories)

2. QCD on $\mathbb{R}^4$: Instantons and renormalons

Why doesn't this work for QCD on $\mathbb{R}^4$?

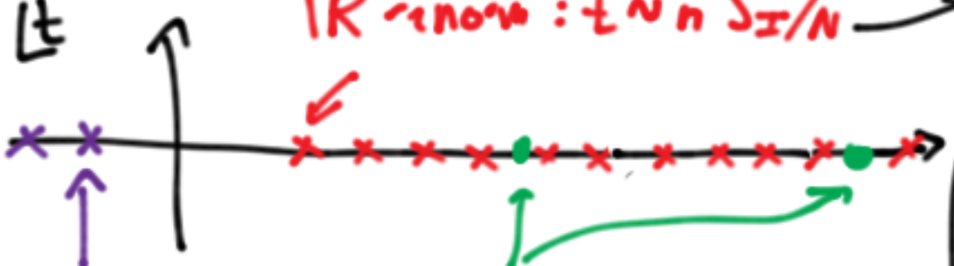
Inst-$\overline{\text{inst}}$ contribs can be calculated in the same way, give $\mathbb{B} \sim \pm i e^{-2S_I}$ ambig.

* always ∃ on general grounds Lipatov *

Borel transf $\mathbb{B}P(t)$ has sings @ $t_n = 2nS_I g^2$

$\Rightarrow \mathbb{P}(g^2) \sim \pm i e^{-t_n/g^2} \sim \pm i e^{-2nS_I}$. ✓ cancelled.

But $\mathbb{B}P(t)$ has other sings closer to origin of t-plane $t_r = r \cdot \frac{16\pi^2}{\beta_0}$ $r = 2, 3, \ldots$



ℂt

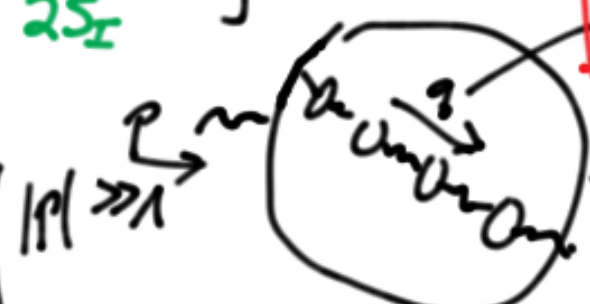
"IR renorm": $t \sim n S_I/N$ → $N = \text{rank}(G)$

$\Rightarrow$ leading (?) div. of p.t.

$\Downarrow$

"UV renorm": $t \sim -n S_I/N$

$I\bar{I}$: $t \sim n\, 2S_I$



p, $|p| \gg \Lambda$, q, $|q| \sim \Lambda$ "IR"

grows as $\left(\frac{n}{2}\right)!$ @ n^{th} order

Problem: no semi-classical saddlepoint gives corresp. contribution -- no prescription to make sense of p.t. in QCD.

Real problem: "power-law corrections" in QCD p.t. Beneke PR99

* <u>Lipatov argument</u> (from Weinberg v. II)

Say $F(g^2) = \int \mathcal{D}\Phi\, e^{+S_E[\Phi, g^2]}$ (eucl. PI)

w/ $S_E = \frac{1}{g^2}\hat{S}[\Phi]$.

$\Rightarrow$

$$F(g^2) = \sum_{n=0}^{\infty} p_n (g^2)^n \quad \text{w/}$$

$$p_n = \frac{1}{2\pi i}\int \mathcal{D}\phi \oint dg^2\, (g^2)^{-n-1} e^{+\frac{1}{g^2}\hat{S}}$$

$$= \frac{1}{2\pi i}\int \mathcal{D}\phi \oint dg^2 \exp\left\{+\frac{1}{g^2}\hat{S}[\phi] - (n+1)\ln g^2\right\}$$

$\downarrow$ saddle pnt: ($\phi = \phi_n$, $g = g_n$)

$$\underbrace{\frac{\delta \hat{S}}{\delta \phi}(\phi_n) = 0}_{} = \frac{\delta}{\delta g^2}\left(\frac{\hat{S}}{g^2} + (n+1)\ln g^2\right) = -\frac{S}{g_n^4} + \frac{n+1}{g_n^2}$$

$\Rightarrow$ instanton sol'n w/ $\underbrace{S_E = \frac{1}{g_n^2}\hat{S}_I = +(n+1)}$.

$\Rightarrow g_n^2 = \frac{+\hat{S}_I}{n+1}$

$$\Rightarrow p_n \approx (g_n^2)^{-n-1} e^{n+1}$$

$$\approx (n+1)^{n+1} e^{n+1} (+\hat{S}_I)^{-n-1}$$

$$\approx n!\, (+\hat{S}_I)^{-n}$$

$\Rightarrow$ pole in BP(t) @ $t = +\hat{S}_I$

$= +g^2 S_I$

3. QCD on $R^3 \times S^1$: Summary of results

- Studied extensively by M. Ünsal & collab.s so most of what follows is a summary of his work ...
- S^1: periodic B.C. for fermions (not thermal!)
- Small S^1, large class of theories, semi-class. dynamics $\Rightarrow G \to U(1)^n \Rightarrow$ 3d compact $U(1)$ w/ fermions.
- 3d $U(1) \xleftrightarrow{\text{EM dual}}$ periodic scalar σ "dual photon"

R^3 • Polyakov (PLB 59 (1975)): no ferm.s $\Rightarrow$ monopole-instantons generate $V(\sigma) \rightsquigarrow$ mass gap & confinement. ($\mathcal{M} \equiv$ monopole-inst.):

$$\mathcal{M} + \bar{\mathcal{M}} \rightsquigarrow V_m(\sigma) \sim \cos(\sigma)\, e^{-S_M} \Rightarrow m_\sigma .$$

R^3 • Affleck, Harvey, Witten (NPB 206 (1982)): w/ ferm.s mono-inst have ferm. 0-modes, $\Rightarrow$ do not generate $V(\sigma) \to [V(\sigma) \cdot \psi^{n_f}]$ ...

○ Ünsal (0709.3269): w/ fermions, $\exists\ \mathcal{M}_i \overline{\mathcal{M}}_j = B_{ij}$

$\mathbb{R}^3 \times S^1$ bound states "bions" w/ net magn. charge $\Rightarrow B + \bar{B} \rightsquigarrow V_B(\sigma) \sim \cos\sigma \cdot e^{-2S_M}$

generates $m_\sigma \Rightarrow$ mass gap, confinement.

○ Together w/ large-N vol. independence (Eguchi-Kawai, Kovtun, Ünsal, Yaffe), get int. qualitative & quantitative results, e.g. Poppitz-Ünsal 0905.0634 & 0906.5156; Ünsal-Yaffe 1006.2101

Will tell what B' is later.

○ <u>We</u> look at effect of $\mathcal{M}_i \overline{\mathcal{M}}_i =: \mathcal{B}_0$ $(i=j)$

magnetically <u>neutral</u> bions & higher top. mols

– quasi-0-mode $\int \Rightarrow$ needs to be continued to be converg.

– but analytic continuation <u>not</u> ambiguous.

– B-t same w/ $B'\bar{B}' \rightarrow$ but now gives ambig.

im part $\pm i\pi e^{-4S_M}$ & $S_M \cong S_I/N$.

– $\therefore$ B-ZJ $\Rightarrow$ predict pole in t-plane @

$$\boxed{t_r = r \cdot 4S_I g^2/N}$$

- These are at similar position as 4-d IR renormalon poles:

$$t_r^{4-d} = \underset{\substack{\uparrow \\ \mathbb{Z}}}{r} \cdot \frac{4S_I g^2}{\beta_0} \simeq r \cdot \frac{2S_I}{N} g^2$$

- Leads to following conjecture (questions):

(1) 3d $B\bar{B}$ pole is semiclassical realization of 4d IR renormalon pole.

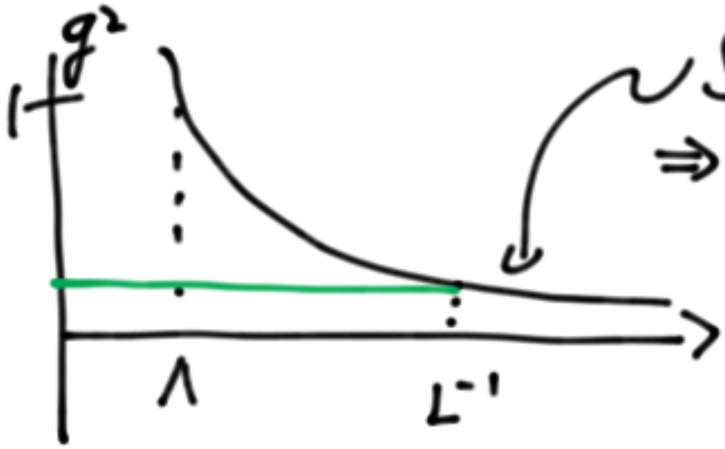
(1′) Same set of bubble diagrams in 4d which give IR renormalon pole, give $B\bar{B}$ pole in pert. thy around $U(1)^n$ vac.

(2) Abelianizing g.t.s on $\mathbb{R}^3 \times S^1$ have no closer poles in t-plane.

If true, then these theories may be defined through their semi-classical expansions.

4. QCD(adj) on $\mathbb{R}^3 \times S^1$

- Look at G w/ n_f adjoint fermions (e.g. SU(2) w/ n_f $\underline{3}$'s). } for simplicity &
- $\underline{\text{NB}}$ $n_f = 1 \equiv \underline{\mathcal{N}=1 \text{ SYM on } \mathbb{R}^3 \times S^1}$.
- Look at 3-d eff. act. @ scale $\mu \sim L^{-1}$ (L = size of S^1) with $L^{-1} \gg \Lambda$



$\int$-out KK modes $\Rightarrow$ 1-loop eff pot'l $V_0(\varphi)$ for $\underbrace{\varphi := \langle A_4 \rangle L}$

$\Rightarrow e^{i\varphi} \equiv$ gauge holonomy on S^1

- $\varphi \in CSA / \Gamma_r \rtimes W$ (adjoint Higgs breaking)
- $\dfrac{CSA}{\Gamma_r \rtimes W}$ = affine Weyl chamber:

$\langle \varphi \rangle \in$ interior $\Rightarrow G \to U(1)^r$

$\langle \varphi \rangle \in$ bdry $G \to SU(2) \times U(1)^{r-1} \ldots$

- We compute $V_0(\varphi) \Rightarrow SU(n), Sp(n)$
  pot'l min. $\in$ interior $\to U(1)^n$

[Other cases: show stays on bndry to orders pert. thry. Semi-classical non-pert. effects don't move it either ... Leaves a question of the strong-cp'g fate of QCD(adj) for $G = \{SO(N), E_n, F_4, G_2\}$.]

5. MONOPOLE-INSTANTONS on $\mathbb{R}^3 \times S^1$

- In $U(1)^N$ vac. ("$\langle\varphi\rangle \neq 0$"), $\exists$ microscopic (4-d) semiclassical BPS configs:
  1. Ad (BPST) instantons: $S_I = 8\pi^2/g^2$
  2. 4d monopole: $m = \frac{8\pi^2}{g^2}\langle A_4\rangle$, charge $\alpha_i^\vee$.

  rank(G) ~~simple roots~~ $(i = 1 \cdots N-1)$

- On S^1, length L, can wrap monopole world-line around S^1, giving 3d monopole-instantons w/ charge $\frac{1}{2\pi}\oint_{S_2} F = \alpha_i^\vee$, action $S_i = Lm = \frac{8\pi^2}{g^2}(\alpha_i^\vee, \varphi)$.

- K. Lee, P. Yi (9702107): $\exists$ another independent BPS monopole-instanton, 'twisting' monopole by large gauge-transf around the circle.

  Twisted monopole-instanton has charge $\alpha_0^\vee$ = lowest co-root, $S_0 = \frac{8\pi^2}{g^2}[1-(\alpha_0^\vee,\varphi)]$.

  [See Davies, Hollowood, Khoze (0006011) for review w/ $N=1$ SUSY.]

- At pert. vacua, $\langle\varphi\rangle$, find $[k_i-(\alpha_i^\vee,\varphi)] \sim \frac{1}{h^\vee} \sim \frac{1}{N}$

  $\Rightarrow \forall$ BPS monopole-inst have action

  $$S_M \sim S_I / N.$$

  & 4d inst. is unique mag'n-neutral sum of $h^\vee$ monopole-instantons.

- Thus, semi-classical expansion is organized in term of multi-monopole-inst configs.

  Calculate in (euclidean) dilute monopole plasma ...

6. Multi-instanton "molecules"

- <u>Note</u>: differs from "dilute instanton gas" b/c of $1/r$ forces.
- Also, $\mathcal{M}_i$ has $2n_f$ fermion 0-modes. So $\mathcal{M}$-plasma also has long range $\sim \ln r$ attraction from fermion exchange.
- Net. effective pot'l between $\mathcal{M}_i$ - $\bar{\mathcal{M}}_j$

$$V_{ij} \sim + \underbrace{\frac{(\alpha_i^\vee,\alpha_j^\vee)}{g^2}\,\frac{L}{r}}_{\text{(magn) Coul. pot'l}} + \underbrace{2n_f \ln r}_{\text{Fermion exchange}}$$

- $(\alpha_i,\alpha_j) \propto$ Cartan matrix: $(\alpha_i,\alpha_j) < 0 \begin{cases} i \neq j \ \& \\ i\text{-}j \text{ linked} \\ \text{in Dynk.} \end{cases}$

  & $(\alpha_i,\alpha_i) > 0$.

$\Rightarrow$ 

$i \neq j \rightarrow$ magnetic bion (size $r_0 \sim L/g^2$)

$i = j \rightarrow$ neutral bion

- $\int$ over 'r' quasi-0-mode: $\int_0^\infty r^2 dr\, e^{-V_{ij}(r)}$ same as Bogomolny's QM integrand w/ $r \to e^{\tau}$.

- The quasi-0-mode $\int$ using BZJ prescription:

$$\int d^3r\, e^{-V(r)} \sim \begin{cases} (g^2)^{2n_f-3} & \text{magn. bion} \\ (-g^2)^{2n_f-3} & \text{neutral bion} \end{cases}$$

↳ Since $n_f \in \mathbb{Z}$, <u>no branch pnt @ $g^2=0$ in neutral bion contrib.</u>

- In QM examples, find similar behavior: if constituents have fermionic 0-modes, then no branch pnt in analytic continuation of neutral amplitude.

- A check on the BZJ prescription: for $n_f=1$ it reproduces the $\mathcal{N}=1$ SYM result...

∘ Repeat analysis for B_{ij} - $\overline{B}_{ij}$ molecule (mag'n-bion — anti-mag-bion) (action $\sim 4S_0$)

∘ Now no ferm. 0-modes ⇒ only Coulomb attraction $V(r) \sim \frac{-1}{g^2}\frac{L}{r^2}$ ⇒

• Quasi-0-mode ∫:

$$I(g^2) = e^{-4S_0}\int d^3r\, e^{-V} = e^{-4S_0}\int_0^\infty dr \cdot r^2 e^{+\frac{1}{g^2 r}}$$ div's $r \to \infty$, $r \to 0$

∘ $r \to \infty$ div: overcounting of dilute-B-gas configs ... subtract off $r \to \infty$ divergence,

∘ $r \to 0$ div: use BZJ prescription: calc. @ $g^2 < 0$:

$$\Rightarrow I(g^2) \sim +\left(-\frac{1}{g^2}\right)^3 \ln(-g^2) \cdot e^{-4S_0}$$

∘ & analytically cont. back to $g^2 > 0$ ⇒

$$I(g^2) = -\left(\frac{1}{g^2}\right)^3 \left[\ln(g^2) \pm i\pi\right] e^{-4S_0}$$ ambig. im part ✓

- Then BZJ (Lipatov, 't Hooft) say: must $\exists$ pole @ $t_r = 4S_0 g^2$ $(\cdot r \in \mathbb{Z})$ ...

$S_0 \sim S_I/N \Rightarrow \boxed{t_r = \dfrac{4 \cdot 8\pi^2 \cdot r}{N}}$ $\leftarrow SU(N)$ ... but we can calc. precisely for general $\langle\varphi\rangle \in$ affine Weyl cell.

- This is then (1) a prediction of high-order behavior of p.t.
- Similarity: $t_r^{\mathbb{R}^4} \sim \dfrac{2 \cdot 8\pi^2 \cdot r}{N}$ & $t_r^{\mathbb{R}^3 \times S^1} \sim \dfrac{4 \cdot 8\pi^2 \cdot r}{N}$

leads to conjectures described before:

4-d $\mathbb{R}^4$ "renormalon" = $\mathbb{R}^3 \times S^1$ (m-bion)-($\overline{\text{m-bion}}$)

7. FURTHER DIRECTIONS

- High orders p-t in Weyl chamber QCD(adj) on $\mathbb{R}^3 \times S^1$
- Fate of SO-E-F-G QCD(adj)
- Other fermion reps ... (quivers, chiral)
- $\mathbb{R}^2 \times T^2$, $\mathbb{R} \times T^3$, T^4
- Anal. cont. of PI (Witten)