Trihyperkähler reduction

Misha Verbitsky

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Cartan's formula and symplectomorphisms

We denote the Lie derivative along a vector field as $\operatorname{Lie}_x : \Lambda^i M \longrightarrow \Lambda^i M$, and contraction with a vector field by $i_x : \Lambda^i M \longrightarrow \Lambda^{i-1} M$.

Cartan's formula: $d \circ i_x + i_x \circ d = \operatorname{Lie}_x$.

REMARK: Let (M, ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms, and $\mathfrak{g}$ its Lie algebra. For any $g \in \mathfrak{g}$, denote by ρ_g the corresponding vector field. Then $\operatorname{Lie}_{\rho_g} \omega = 0$, giving $d(i_{\rho_g}(\omega)) = 0$. **We obtain that $i_{\rho_g}(\omega)$ is closed, for any $g \in \mathfrak{g}$.**

DEFINITION: A Hamiltonian of $g \in \mathfrak{g}$ is a function h on M such that $dh = i_{\rho_g}(\omega)$.

Moment maps

DEFINITION: (M,ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**.

DEFINITION: A G-invariant $c \in \mathfrak{g}^*$ is called **central**.

CLAIM: An equivariant moment map exists whenever $H^1(G, \mathfrak{g}^*) = 0$. In particular, when G is reductive and M is simply connected, an equivariant moment map exists.

Symplectic reduction and GIT

DEFINITION: (Weinstein-Marsden) (M,ω) be a symplectic manifold, G a compact Lie group acting on M by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of M, denoted by $M/\!\!/G$.

CLAIM: The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

THEOREM: Let (M,I,ω) be a Kähler manifold, $G_{\mathbb{C}}$ a complex reductive Lie group acting on M by holomorphic automorphisms, and G is compact form acting isometrically. **Then $M/\!\!/G$ is a Kähler orbifold.**

REMARK: In such a situation, $M/\!\!/G$ is called **the Kähler quotient**, or **GIT quotient**. The choice of a central element $c \in \mathfrak{g}^*$ is known as a choice of **stability data**.

REMARK: The points of $M/\!\!/G$ are in bijective correspondence with the orbits of $G_{\mathbb{C}}$ which intersect $\mu^{-1}(c)$. Such orbits are called **polystable**, and the intersection of a $G_{\mathbb{C}}$-orbit with $\mu^{-1}(c)$ is a G-orbit.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

REMARK:
The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is **holomorphic and symplectic** on (M, I).

Hyperkähler reduction

DEFINITION: Let G be a compact Lie group, ρ its action on a hyperkähler manifold M by hyperkähler isometries, and $\mathfrak{g}^*$ a dual space to its Lie algebra. **A hyperkähler moment map** is a G-equivariant smooth map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that $\langle \mu_i(v), g\rangle = \omega_i(v, d\rho(g))$, for every $v \in TM$, $g \in \mathfrak{g}$ and $i = 1, 2, 3$, where ω_i is one three Kähler forms associated with the hyperkähler structure.

DEFINITION: Let ξ_1, ξ_2, ξ_3 be three G-invariant vectors in $\mathfrak{g}^*$. The quotient manifold $M/\!/\!/G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of M.

THEOREM: (Hitchin, Karlhede, Lindström, Roček)
The quotient $M/\!/\!/G$ is hyperkaehler.

Holomorphic moment map

Let $\Omega := \omega_J + \sqrt{-1}\omega_K$. This is a holomorphic symplectic (2,0)-form on (M, I).

The proof of HKLR theorem. Step 1: Let μ_J, μ_K be the moment map associated with ω_J, ω_K, and $\mu_{\mathbb{C}} := \mu_J + \sqrt{-1}\,\mu_K$. Then $\langle d\mu_{\mathbb{C}}, g\rangle = i_{\rho_g}(\Omega)$ Therefore, $d\mu_{\mathbb{C}} \in \Lambda^{1,0}(M, I) \otimes \mathfrak{g}^*$.

Step 2: This implies that the map $\mu_{\mathbb{C}}$ is holomorphic. It is called **the holomorphic moment map.**

Step 3: By definition, $M /\!\!/\!\!/ G = \mu_{\mathbb{C}}^{-1}(c) /\!\!/ G$, where $c \in \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$ is a central element. **This is a Kähler manifold,** because it is a Kähler quotient of a Kähler manifold.

Step 4: We obtain 3 complex structures I, J, K on the hyperkähler quotient $M /\!\!/\!\!/ G$. **They are compatible in the usual way** (an easy exercise). ■

Quiver representations

DEFINITION: A **quiver** is an oriented graph. A **quiver representation** is a diagram of complex Hermitian vector spaces and arrows associated with a quiver:

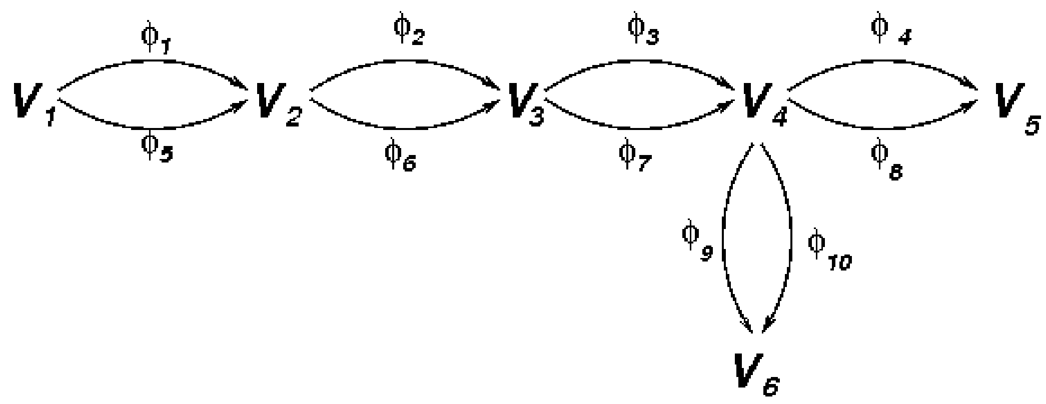
$$\begin{tikzcd} V_1 \arrow[r, "\phi_1"] & V_2 \arrow[r, "\phi_2"] & V_3 \arrow[r, "\phi_3"] & V_4 \arrow[r, "\phi_4"] \arrow[d, "\phi_9"'] & V_5 \\ & & & V_6 & \end{tikzcd}$$

Here, V_i are vector spaces, and φ_i linear maps.

REMARK: If one fixes the spaces V_i, the space of quiver representations is a Hermitian vector space.

Quiver varieties

Starting from a single graph, one can double it up, as follows, obtaining a Nakajima double quiver.



A Nakajima quiver for the Dynkin diagram D_5.

CLAIM: The space M of representations of a Nakajima's double quiver is a quaternionic vector space, and the group $G := U(V_1) \times U(V_2) \times ... \times U(V_n)$ acts on M preserving the quaternionic structure.

DEFINITION: A **Nakajima quiver variety** is a quotient $M /\!\!/\!\!/ G$.

Hyperkähler manifolds as quiver varieties

Many non-compact hyperkähler manifolds are obtained as quiver varieties.

EXAMPLE: A 4-dimensional ALE **(asymptotically locally Euclidean)** space obtained as a resolution of **a du Val singularity**, that is, a quotient $\mathbb{C}^2/G$, where $G \subset SU(2)$ is a finite group.

REMARK: Since finite subgroups of $SU(2)$ are classified by the Dynkin diagrams of type A,D,E, these ALE quotients are called **ALE spaces of A-D-E type**.

EXAMPLE: The moduli asymptotically flat Hermitian Yang-Mills connections on ALE spaces of A-D-E type.

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

THEOREM: (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **hyperkähler.**

This theorem is proved using quivers.

ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r, respectively. The **ADHM data** is maps

$$A, B \in \mathrm{End}(V), I \in \mathrm{Hom}(W, V), J \in \mathrm{Hom}(V, W).$$

We say that ADHM data is

stable,

if there is no subspace $S \subsetneq V$ such that $A(S), B(S) \subset S$ and $I(W) \subset S$;

costable,

if there is no nontrivial subspace $S \subset V$ such that $A(S), B(S) \subset S$ and $S \subset \ker J$;

regular,

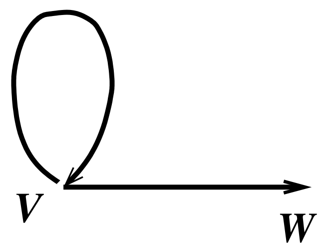
if it is both stable and costable.

The ADHM equation is $[A, B] + IJ = 0$.

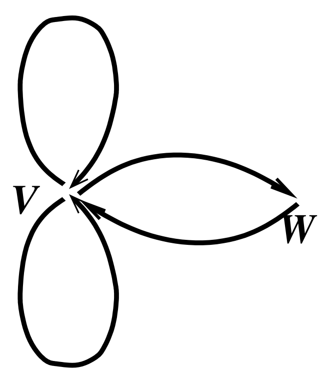
THEOREM: (Atiyah, Drinfeld, Hitchin, Manin) Framed rank r, charge c instantons on $\mathbb{C}P^2$ are in bijective correspondence with the set of equivalence classes of regular ADHM solutions. In other words, **the moduli of instantons on $\mathbb{C}P^2$ is identified with moduli of the corresponding quiver representation.**

ADHM spaces as quiver varieties

Consider the quiver



The ADHM data is the set Q of representations of the corresponding double quiver



The corresponding **holomorphic moment map is the ADHM equation** $A, B, I, J \longrightarrow [A, B] + IJ$ **with values in** $\operatorname{End}(V)$**.**

The set of equivalence classes of ADHM solutions is $Q /\!\!/\!\!/ U(V)$.

Twistor space

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\mathrm{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\mathrm{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x \mathrm{Tw}(M) \to T_x \mathrm{Tw}(M)$ satisfies $I_{\mathrm{Tw}}^2 = -\mathrm{Id}$. **It defines an almost complex structure on $\mathrm{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\mathrm{Tw}(M) = \mathrm{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\mathrm{Tw}(M)$ never admits a Kähler structure.

Rational curves on $\operatorname{Tw}(M)$.

REMARK: The twistor space **has many rational curves**.

DEFINITION: Denote by $\operatorname{Sec}(M)$ **the space of holomorphic sections** of the twistor fibration $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$.

DEFINITION: For each point $m \in M$, one has **a horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of π. The space of horizontal sections is denoted $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$

REMARK: The space of horizontal sections of π is identified with M. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$ is a smooth manifold of dimension $2\dim M$.**

DEFINITION: A twistor section $C \subset \operatorname{Tw}(M)$ is called **regular**, if $NC = \mathcal{O}(1)^{\dim M}$.

CLAIM: For any $I \neq J \in \mathbb{C}P^n$, consider the evaluation map $\operatorname{Sec}(M) \xrightarrow{E_{I,J}} (M,I) \times (M,J)$, $s \longrightarrow s(I) \times s(J)$. Then **$E_{I,J}$ is an isomorphism around the set $\operatorname{Sec}_0(M)$ of regular sections.**

Complexification of a hyperkähler manifold.

REMARK: Consider an anticomplex involution $\operatorname{Tw}(M) \xrightarrow{\iota} \operatorname{Tw}(M)$ mapping (m,t) to $(m,i(t))$, where $i: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is a central symmetry. Then $\operatorname{Sec}_{hor}(M) = M$ **is a component of the fixed set of** ι**.**

COROLLARY: $\operatorname{Sec}(M)$ **is a complexification of** M**.**

QUESTION: What are geometric structures on $\operatorname{Sec}(M)$?

Answer 1: For compact M, **a closure of** $\operatorname{Sec}(M)$ **is holomorphically convex** (Stein if $\dim M = 2$).

Answer 2: The space $\operatorname{Sec}_0(M)$ **admits a holomorphic, torsion-free connection** with holonomy $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

Mathematical instantons

DEFINITION: **A mathematical instanton** on $\mathbb{C}P^3$ is a stable bundle B with $c_1(B)=0$ and $H^1(B(-1))=0$. **A framed instanton** is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell=\mathbb{C}P^1\subset\mathbb{C}P^3$.

DEFINITION: **An instanton** on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B)=0$. **A framed instanton** is an instanton equipped with a trivialization $B|_x$ for some fixed point $x\in\mathbb{C}P^2$.

THEOREM: (Atiyah-Drinfeld-Hitchin-Manin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler.**

THEOREM: (Jardim–V.) The space $\mathbb{M}_{r,c}$ of framed mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections** $\operatorname{Sec}(\mathcal{M}_{r,c})$**.**

REMARK: This correspondence is not surprising if one realizes that $\operatorname{Tw}(\mathbb{H})=\mathbb{C}P^3\backslash\mathbb{C}P^1$.

The space of instantons on $\mathbb{C}P^3$

THEOREM: (Jardim–V.) **The space $\mathbb{M}_{r,c}$ is smooth.**

REMARK: To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that $\mathbb{M}_{r,c}$ is smooth, we develop **trihyperkähler reduction**, which is **a reduction defined on trisymplectic manifolds.**

We prove that $\mathbb{M}_{r,c}$ **is a trihyperkähler quotient** of a vector space by a reductive group action, hence smooth.

Trisymplectic manifolds

DEFINITION: Let $\mathbf{\Omega}$ be a 3-dimensional space of holomorphic symplectic 2-forms on a complex manifold. Suppose that

- **$\mathbf{\Omega}$ contains a non-degenerate 2-form**
- For each non-zero degenerate $\Omega \in \mathbf{\Omega}$, one has $\operatorname{rk}\Omega = \frac{1}{2}\dim V$.

Then $\mathbf{\Omega}$ is called **a trisymplectic structure on M**.

REMARK: The bundles $\ker\Omega$ are involutive, because Ω is closed.

THEOREM: (Jardim–V.) For any trisymplectic structure on M, M is equipped with a unique holomorphic, torsion-free connection, preserving the forms Ω_i. It is called **the Chern connection** of M.

REMARK: The Chern connection has holonomy in $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n}\otimes\mathbb{C}^2$.

Trisymplectic structure on $\operatorname{Sec}_0(M)$

EXAMPLE: Consider a hyperkähler manifold M. Let $I \in \mathbb{C}P^1$, and $ev_I : \operatorname{Sec}_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in \operatorname{Sec}_0(M)$ to $S(I)$. Denote by Ω_I the holomorphic symplectic form on (M, I). **Then** $ev_I^* \Omega_I$**,** $I \in \mathbb{C}P^1$ **generate a trisymplectic structure.**

COROLLARY: $\operatorname{Sec}_0(M)$ **is equipped with a holomorphic, torsion-free connection with holonomy in** $Sp(n, \mathbb{C})$**.**

Trihyperkähler reduction

DEFINITION: A trisymplectic moment map $\boldsymbol{\mu}_{\mathbb{C}}: M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*$ takes vectors $\Omega \in \boldsymbol{\Omega}, g \in \mathfrak{g} = \operatorname{Lie}(G)$ and maps them to a holomorphic function $f \in \mathcal{O}_M$, such that $df = \Omega \lrcorner g$, where $\Omega \lrcorner g$ denotes the contraction of Ω and the vector field g

DEFINITION: Let $(M, \boldsymbol{\Omega}, S_t)$ be a trisymplectic structure on a complex manifold M. Assume that M is equipped with an action of a compact Lie group G preserving $\boldsymbol{\Omega}$, and an equivariant trisymplectic moment map

$$\boldsymbol{\mu}_{\mathbb{C}}: M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*.$$

Let $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ be the corresponding **level set** of the moment map. Consider the action of the complex Lie group $G_{\mathbb{C}}$ on $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$. Assume that it is proper and free. Then the quotient $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$ is a smooth manifold called **the trisymplectic quotient** of $(M, \boldsymbol{\Omega}, S_t)$, denoted by $M /\!\!/\!\!/\!\!/ G$.

THEOREM: Suppose that the restriction of $\boldsymbol{\Omega}$ to $\mathfrak{g} \subset TM$ is non-degenerate. **Then $M /\!\!/\!\!/\!\!/ G$ is trisymplectic.**

Mathematical instantons and the twistor correspondence

REMARK: Using the monad description of mathematical instantons, **we prove that that the map $\operatorname{Sec}_0(\mathcal{M}_{r,c}) \longrightarrow \mathbb{M}_{r,c}$ to the space of mathematical instantons is an isomorphism** (Frenkel-Jardim, Jardim-V.).

REMARK: The smoothness of the space $\operatorname{Sec}_0(\mathcal{M}_{r,c}) = \mathbb{M}_{r,c}$ **follows from the trihyperkähler reduction procedure:**

THEOREM: Let M be flat hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler automorphisms. Suppose that the hyperkähler moment map exists, and the hyperkähler quotient $M/\!\!/\!\!/G$ is smooth. **Then there exists an open embedding**

$$\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G \xrightarrow{\Psi} \operatorname{Sec}_0(M/\!\!/\!\!/G),$$

which is compatible with the trisymplectic structures on $\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G$ and $\operatorname{Sec}_0(M/\!\!/\!\!/G)$.

THEOREM: If M is the space of quiver representations which gives $M/\!\!/\!\!/G = \mathcal{M}_{2,c}$, **$\Psi$ gives an isomorphism $\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G = \operatorname{Sec}_0(M/\!\!/\!\!/G)$.**

The 1-dimensional ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r, respectively. The **1-dimensional ADHM data** is maps

$$A_k, B_k \in \mathrm{End}(V), I_k \in \mathrm{Hom}(W,V), J_k \in \mathrm{Hom}(V,W), (k=0,1)$$

Choose homogeneous coordinates $[z_0 : z_1]$ on $\mathbb{C}P^1$ and define

$$\tilde{A} := A_0 \otimes z_0 + A_1 \otimes z_1 \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + B_1 \otimes z_1.$$

We say that 1-dimensional ADHM data is **globally regular** if $(\tilde{A}_p, \tilde{B}_p, \tilde{I}_p, \tilde{J}_p)$ is regular for every $p \in \mathbb{C}P^1$. The **1-dimensional ADHM equation** is $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p\tilde{J}_p = 0$, for all $p \in \mathbb{C}P^d$

THEOREM: (Marcos Jardim, Igor Frenkel) Let $C_1(r,c)$ denote the set of globally regular solutions of the 1-dimensional ADHM equation. Then **there exists a 1-1 correspondence between equivalence classes of globally regular solutions of the 1-dimensional ADHM equations and isomorphism classes of rank r instanton bundles** on $\mathbb{CP}^3$ framed at a fixed line ℓ, where $\dim W = \mathrm{rk}(E)$ and $\dim V = c_2(E)$.

1-dimensional ADHM construction and the trisymplectic moment map

THEOREM: (Jardim, V.) Consider the natural (flat) trisymplectic structure on $C_1(r,c)$, and let $\mu: C_1(r,c) \longrightarrow H^0(\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \operatorname{End}(V))$ be a map associating to $C \in C_1(r,c)$ and $p \in \mathbb{C}P^1$ the vector $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p \tilde{J}_p \in \mathcal{O}_{\mathbb{C}P^1}(2) \otimes \operatorname{End}(V))$. **Then μ is a trisymplectic moment map. This identifies the set of equivalence classes of solutions of the 1-dimensional ADHM equation with the trihyperkähler quotient $C_1(r,c) /\!\!/\!\!/\!\!/ U(V)$.**