# On The Strong Coupling Behavior of Wilson loops in N=2 Superconformal Gauge Theories

Takao Suyama (SNU)

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1,000,000 USD
99,000,000 JPY
1,300,000,000 KRW

Attractive !!

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**Too difficult ! Let's compromise !**

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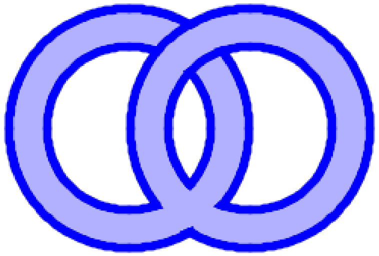

creation/annihilation
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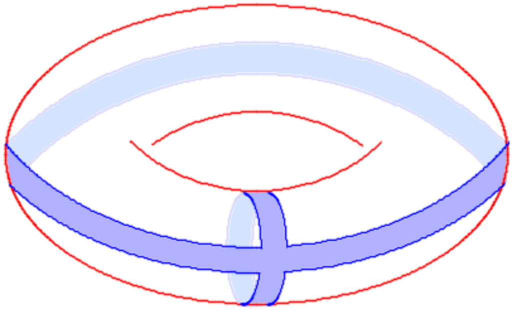

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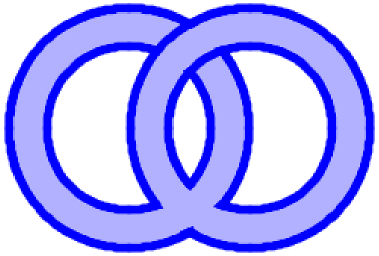

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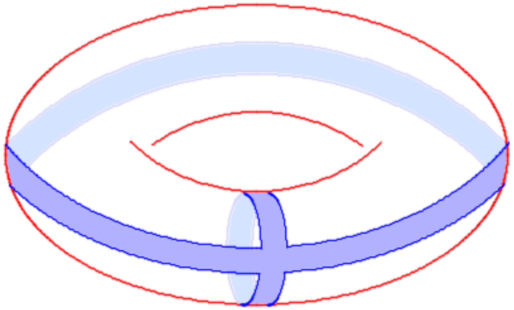

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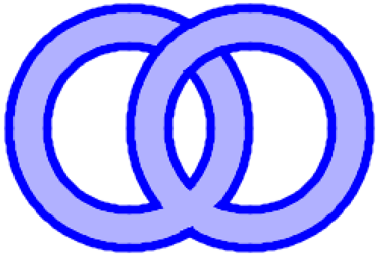

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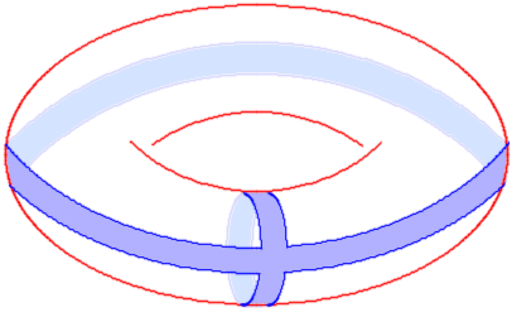

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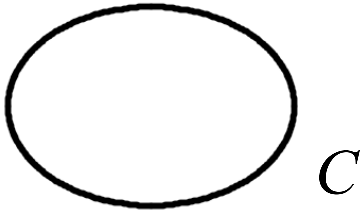

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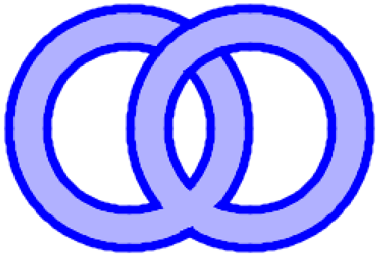

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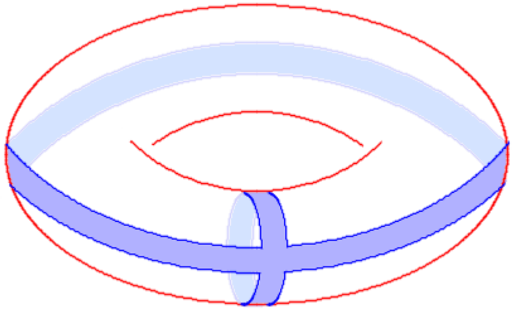

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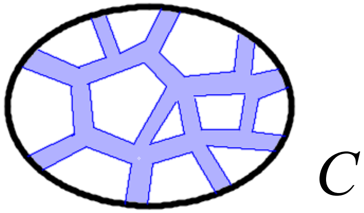

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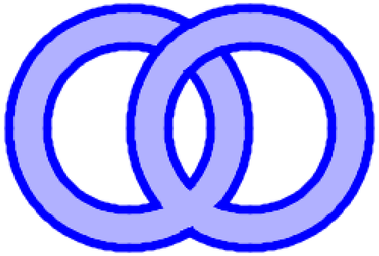

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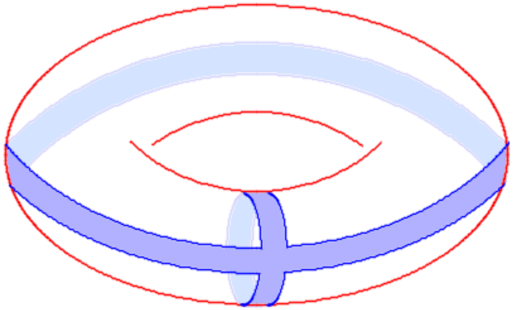

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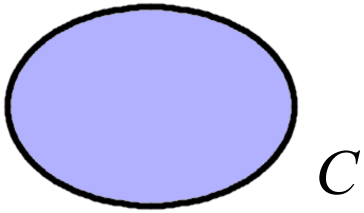

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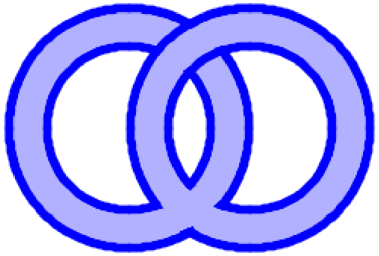

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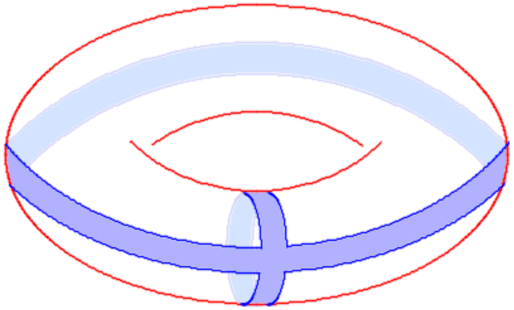

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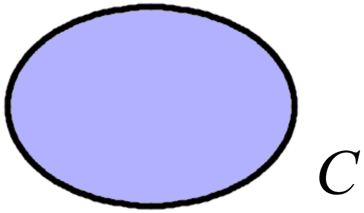

⟸ **Which string theory?**

**Tractable?**

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**We are going to study gravity by gauge theory, and vice versa!**

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$\langle W[C] \rangle$ can be evaluated by a finite-dimensional integral.

Recall the saddle-point approximation of a path-integral:

$$\int D\phi\, O(\phi)\, e^{-S[\phi]} \sim \sum_{\phi_c} O(\phi_c) \underline{\frac{\Delta_F(\phi_c)}{\Delta_B(\phi_c)}}\, e^{-S[\phi_c]}$$

1-loop : quantum fluctuations

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N x N Hermitian

N=4 SYM : $\langle W[C]\rangle \propto \int dX\, \mathrm{tr}\left(e^{2\pi X}\right) e^{-(4\pi^2/\lambda)\, \mathrm{tr}\, X^2}$

$$\xrightarrow{N\to\infty} \underline{\frac{2}{\sqrt{2\lambda}} I_1(\sqrt{2\lambda})} \sim e^{\sqrt{2\lambda}}, \quad (\lambda\to\infty)$$

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e.g. in the 1st theory, Wilson loop does not grow exponentially.

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Amounts to 100 USD ???

To be continued...

# Motivation

AdS$_4$ / CFT$_3$ correspondence [ABJM]

ABJM theory i.e. ⇔ M-theory on

N=6 Chern-Simons-Matter theory $AdS_4 \times S^7 / \mathbf{Z}_k$

with $U(N)_k \times U(N)_{-k}$

Wilson loops: $W[C], \overline{W}[C]$ Two distinct operators.

BPS Wilson loop: [Rey, TS, Yamaguchi]

$$W[C] = \frac{1}{N} \mathrm{Tr} P \exp\left[ i \int_C (A + M_I{}^J Y^I Y_J^\dagger) \right]$$

with a specific $M_I{}^J$. (also for $\overline{W}[C]$ )

Wilson loop / IIA string correspondence is expected to hold. [Maldacena] [Rey-Yee]

Problems:

1. At most 1/6 BPS Wilson loop ⟺ 1/2 BPS string

⟹ Wilson loop for a smeared string ??? [Drukker-Plefka-Young]

2. Two Wilson loops ⟺ single kind of string

A proposal for 2: [RSY]

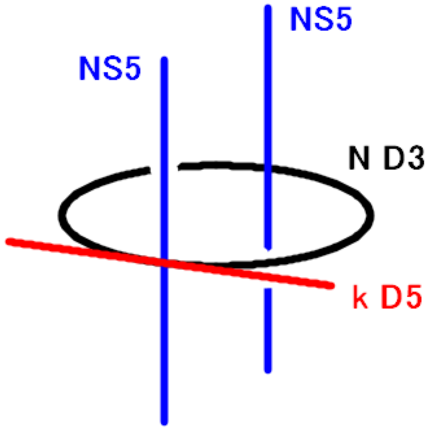
A linear combination $\mathrm{W}_N[C] = \left(W_N[C]+\overline{W}_N[C]\right)/2$ should correspond to IIA string.

( under time-reversal, $T\,\mathrm{W}_N[C]T^{-1}=\mathrm{W}_{\overline{N}}[C]$ )

On the other hand, $\langle W_N[C]-\overline{W}_N[C]\rangle = 0$ by symmetry.

⟹ This plays no role in the planar limit.

# Bulk counterparts of Wilson loops are still unclear...



Recall the brane construction of ABJM theory.

In the UV, it is T-dual to $A_1$-singularity with branes.

⇨ w.v. theory is of the quiver type (3-dim.)

gauge group $U(N)\times U(N)$ ⇨ multiple Wilson loops

However, even in 4-dim. it seems

Wilson loops in quiver theory $\overset{???}{\Longleftrightarrow}$ Some objects in $AdS_5$

is not well-understood.

## Aim

Determine the strong coupling behavior of Wilson loops in N=2 superconformal gauge theories in 4-dim.

($N_f = 2N_c$ & $A_1$ quiver theory in detail)

⇨ Provide information for gravity duals, if exist.

## How to do

Wilson loop in N=4 SYM is determined by localization. [Pestun]

⇨ Extend to N=2 theories.

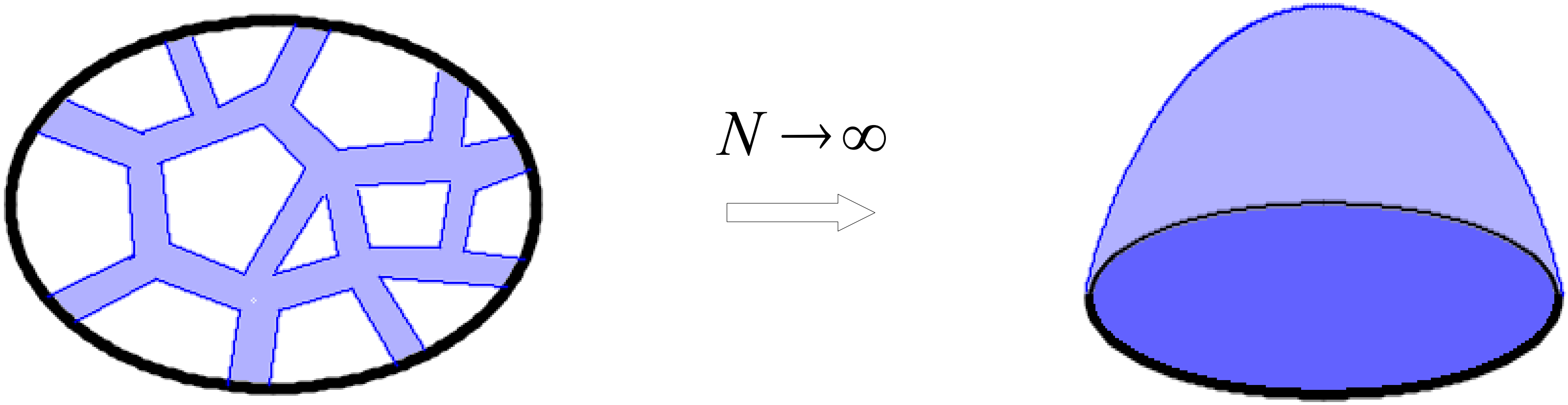
- Reduction to matrix model
- 1-loop determinant is non-trivial, non-Gaussian

## Results

$N_f = 2N_c$ : non-exponential

quiver : $\langle W_1[C] \rangle, \langle W_2[C] \rangle \sim e^{\sqrt{2\lambda}}$

Note : We ignore non-perturbative corrections.



$N \to \infty$

Dense Feynman diagrams are assumed to be dominant in evaluating Wilson loops in the planar limit.

# Localization for N=4

[Pestun]

Consider N=4 SYM on $S^4$ with radius $r$.

$$S = \int d^4 \underset{\underset{x^1_, x^2_, x^3_, x^4}{\uparrow}}{x} \sqrt{h}\ \mathrm{tr}\left[ -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{r^2} A_{\underset{\underset{0,5,6\sim 9}{\uparrow}}{S}} A^S - \frac{i}{2} \bar{\Psi} \Gamma^M D_{\underset{\underset{0\sim 9}{\nearrow}}{M}} \Psi + \frac{1}{2} \underset{\underset{\text{auxiliary fields}}{\nwarrow}}{K_I} K^{\overset{1\sim 7}{\searrow}\, I} \right]$$

$$F_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N] \quad \text{etc.}$$

This action is invariant under a fermionic transf.

$$\begin{cases} QA_M = -i\bar{\xi}\Gamma_M \Psi \\ Q\Psi = \frac{1}{2} F_{MN}\Gamma^{MN}\xi - 2\Gamma^S \tilde{\xi} A_S + \nu_I K^I \\ QK^I = i\bar{\nu}^I \Gamma^M D_M \Psi \end{cases}$$

where $\xi, \tilde{\xi}, \nu^I$ are suitable bosonic spinors.

For 1/2 BPS Wilson loop, $\underline{Q \cdot W[C] = 0.}$ $\quad C$ : circle on the equator

Deform the action as follows:

$$S_t = S + t\int QV \quad \text{where} \quad V \sim \overline{\Psi}\,Q\,\Psi\,, \quad \boxed{\int Q^2 V = 0.}$$

⇨ $\underline{\langle W[C]\rangle_t := Z_t^{-1}\int e^{-S_t}\,W[C] \text{ is independent of } t.}$

One can show that

$$QV = \sum (\text{bosonic})^2 \quad + \quad (\text{fermionic terms})$$

⇨ In the large *t* limit, the path-integral is localized to

$$\Omega := \left\{ \left. QV\right|_{bosonic} = 0 \right\}$$

$$\Longleftrightarrow \quad \left\{ \begin{array}{l} A_{0,}\, K^{I} \propto \Phi : \text{Hermitian matrix} \\ \text{others} = 0. \end{array} \right.$$

Action ⇨ $S = \dfrac{4\pi^2}{g^2} \operatorname{tr} \Phi^2$

1-loop det. ⇨ trivial

} Gaussian matrix model

$$\langle W[C] \rangle_t = \lim_{t \to +\infty} \langle W[C] \rangle_t = \left\langle \frac{1}{N} \operatorname{tr} e^{2\pi\Phi} \right\rangle_{Gauss}$$

In the large $N$ limit,

$$\left\langle N^{-1} \operatorname{tr} e^{2\pi\Phi} \right\rangle = \frac{2}{\sqrt{2\lambda}} I_1(\sqrt{2\lambda}) \sim \sqrt{\frac{2}{\pi}} (2\lambda)^{-3/4} e^{\sqrt{2\lambda}}$$

where $\lambda = g^2 N$.

[Erickson-Semenoff-Zarembo]

⇨ Generalize this to N=2 theories.

# Localization for N=2

N=2 gauge theory on $S^4$ with radius $r$ :

$$
\begin{aligned}
S = \int d^4x\sqrt{h}\Big[ & -\frac{1}{4}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\,\mathrm{tr}\,\bar{\lambda}\,\Gamma^{\mu}D_{\mu}\lambda - D_{\mu}q_{\alpha}D^{\overset{\substack{0\sim5\\ \swarrow}}{\mu}}q^{\alpha} - i\bar{\psi}\,\Gamma^{\mu}D_{\mu}\psi \\
& + g\,\bar{\lambda}^{A}\gamma^{\alpha}q_{\alpha}T_{A}\psi + g\,\bar{\psi}\,\gamma_{\alpha}T_{A}q^{\alpha}\lambda^{A} - g^{2}(q_{\alpha}T^{A}q^{\beta})^{2} + \frac{1}{2}g^{2}(q_{\underset{\substack{\nearrow\\ 1,\,2}}{\alpha}}T_{A}q^{\alpha})^{2} \\
& - \frac{1}{r^{2}}\,\mathrm{tr}\,A_{\underset{\substack{\nwarrow\\ 0,\,5}}{a}}A^{a} - \frac{2}{r^{2}}q_{\alpha}q^{\alpha} + \frac{1}{2}K^{\dot{m}}K_{\underset{\substack{\nwarrow\\ 1,\,2,\,3}}{\dot{m}}} + K_{\alpha}K^{\alpha}\Big].
\end{aligned}
$$

$T^A$ : generators of Lie($G$) in rep. $R$

This action is invariant under

$$
\left\{
\begin{aligned}
QA_{\mu} &= -i\,\bar{\xi}\,\Gamma_{\mu}\lambda \\
Q\lambda^{A} &= (1/2)F_{\mu\nu}{}^{A}\Gamma^{\mu\nu}\xi + igq_{\alpha}T^{A}q^{\beta}\gamma^{\alpha}{}_{\beta}\xi - 2\Gamma^{a}\tilde{\xi}A_{a}{}^{A} + K^{\dot{m}A}\nu_{\dot{m}} \\
QK^{\dot{m}A} &= -\bar{\nu}^{\dot{m}}\left[-i\,\Gamma^{\mu}D_{\mu}\lambda^{A} + g\,\gamma^{\alpha}q_{\alpha}T^{A}\psi - g\,\gamma_{\alpha}\psi^{*}T^{A}q^{\alpha}\right]
\end{aligned}
\right.
\qquad \text{etc.}
$$

There is a 1/2 BPS Wilson loop s.t. $Q\cdot W[C] = 0.$

Reduction to matrix model turns out to be straightforward.

However, 1-loop determinant is non-trivial ( non-polynomial! )

⇨ Tractable problem ???

Recall matrix model calculation. In the planar limit,

$$W = \int dx\, \rho(x)\, e^{2\pi x} \qquad \text{where} \qquad \rho(x) \geqslant 0, \; \int dx\, \rho(x) = 1.$$

Let $a := \max\{\mathrm{supp}(\rho)\}$. ⇨ $W \leqslant e^{2\pi a}$, $a = a(\lambda)$

Assume $\lim_{\lambda\to+\infty} a(\lambda) = +\infty$. (otherwise, no classical AdS dual)

$$\Rightarrow \quad W \sim \beta\, \Gamma(\alpha+1)(2\pi a)^{-\alpha-1} e^{2\pi a} \qquad (\lambda \to +\infty)$$

where $a\,\rho(x) \sim \beta(1-x/a)^{\alpha}$ $\quad (x \to a-0)$.

- 1-loop det. for large eigenvalues of the matrix is relevant.

Quadratic terms for $q^\alpha$ are

$$-q_\alpha \underline{\Delta^\alpha{}_\beta} q^\beta + \underline{\frac{1}{r^2}\Phi^A \Phi^B q_\alpha T_A T_B q^\beta}$$

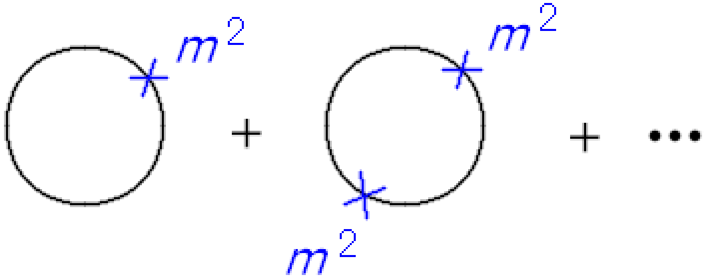
Laplacian on $S^4$
+ some terms

$$\begin{cases} \dfrac{2N}{r^2}\displaystyle\sum_{i=1}^{N}(\phi_i)^2 q_{i\alpha} q_i{}^\alpha & (N_f = 2N_c) \\ \dfrac{2}{r^2}\displaystyle\sum_{i\neq j}(\phi_i{}^{(1)} - \phi_j{}^{(2)})^2 q_{ij\alpha} q_{ji}{}^\alpha & (A_1 \text{ quiver}) \end{cases}$$

⇨ Necessary to obtain the large-mass behavior of the 1-loop det. for each component.

Note: 1-loop effective action $\frac{1}{2}\mathrm{Tr}\log\left(-\Delta + \frac{m^2}{r^2}\right)$

diagrammatic expansion ⟺ small-mass expansion



⇨ A resummation of the diagrams is required.

# Zeta-function regularization

Suppose that the eigenfunctions $\psi_k$ of $\Delta$ are known:

$$-\Delta \psi_k = \lambda_k \psi_k .$$

Then, formally

$$\mathrm{Det}\left[-\Delta + \frac{m^2}{r^2}\right] = \prod_k \left(\lambda_k + \frac{m^2}{r^2}\right).$$

Define a zeta-function

$$\zeta(s, m) := r^{-2s} \sum_k \left(\lambda_k + \frac{m^2}{r^2}\right)^{-s} .$$

well-defined
for a right-half
*s*-plane.

⇨ $$\log \mathrm{Det}\left[-\Delta + \frac{m^2}{r^2}\right] := -\partial_s \zeta(s, m)\Big|_{s=0} + \mathrm{const.}$$

# Heat-kernel expansion

The zeta-function $\zeta(s,m)$ can be written as

$$\zeta(s,m) = \frac{r^{-2s}}{\Gamma(s)}\int_0^\infty dt\ t^{s-1} e^{-m^2 t/r^2} K(t)$$

where

$$K(t) = \mathrm{Tr}(e^{t\Delta}) \sim \sum_{i=0}^{\infty} t^{i-2} a_{2i}(\Delta) \quad : \underline{\text{heat-kernel expansion}}$$

For large $m$,

$$\zeta(s,m) \sim \sum_{i=0}^{\infty} a_{2i}(\Delta) r^{2i-4} \frac{\Gamma(s+i-2)}{\Gamma(s)} m^{-2s-2i+4}.$$

⇨ 1-loop effective action:

$$F = \underline{\left(\frac{1}{2} m^4 \log m^2 - \frac{3}{4} m^4\right) a_0(\Delta) r^{-4}} - \left(m^2 \log m^2 - m^2\right) a_2(\Delta) r^{-2} + O(\log m^2).$$

↑

canceled by SUSY

Quadratic terms for $\psi$ are

$$i\bar{\psi}\not{D}\psi \;:=\; i\bar{\psi}\Gamma^m\nabla_m\psi-\frac{i}{r}\bar{\psi}\Gamma^0\Phi^A T_A\psi+\frac{i}{2}(\bar{\xi}\Gamma_{\mu\nu}\tilde{\xi})\bar{\psi}\Gamma^0\Gamma^{\mu\nu}\psi\,.$$

$$\Rightarrow \quad -\mathrm{Tr}\log(i\not{D}) \;=\; -\frac{1}{2}\log\left(-\underline{\Delta_F}+\frac{m^2}{r^2}\right), \qquad m^2=\phi_i^{\,2} \ \text{ or } \ (\phi_i-\phi_j)^2$$

$$\uparrow$$

including linear term in $m$.
(each order of $1/m$ is a finite sum)

Adding bosonic and fermionic contributions,

$$F_h \;=\; \begin{cases} 2N\sum_i F(\phi_i) & (N_f = 2N_c) \\ 2\sum_{i\neq j} F(\phi_i^{(1)}-\phi_j^{(2)}) & (A_1 \ \text{quiver}) \end{cases}$$

where

$$F(x) \;\sim\; \frac{3}{4}x^2\log x^2 \qquad (x\to\infty)$$

In N=4 SYM, 1-loop determinant is trivial. [Pestun]

⇨ $$F_v = -F_h(\text{adj.}) = -\sum_{i\neq j} F(\phi_i - \phi_j)$$

for U(N) vector multiplet.

In $A_1$ quiver theory,

$$F_v = -\sum_{i\neq j} F(\phi_i^{(1)} - \phi_j^{(1)}) - \sum_{i\neq j} F(\phi_i^{(2)} - \phi_j^{(2)})$$

# Saddle-point equations

$N_f = 2N_c$: We have to solve

$$\frac{8\pi^2}{\lambda}\phi_k + 2F'(\phi_k) - \frac{2}{N}\sum_{i\neq k} F'(\phi_k - \phi_i) = \frac{2}{N}\sum_{i\neq k}\frac{1}{\phi_k - \phi_i}.$$

Rescaling $\phi_k \to \lambda^\gamma \phi_k$ , ( $\gamma$ is determined s.t. rescaled distribution is finite)

$$\Rightarrow \quad 8\pi^2\phi_k + \underbrace{2\lambda^{1-\gamma}\left[F'(\lambda^\gamma\phi_k) - \frac{1}{N}\sum_{i\neq k}F'(\lambda^\gamma(\phi_k - \phi_i))\right]}_{\sim\ \lambda} = \frac{2}{N}\lambda^{1-2\gamma}\sum_{i\neq k}\frac{1}{\phi_k - \phi_i}.$$

⇒ Leading order equation for large $\lambda$ is

$$\frac{1}{\phi_k} = \frac{1}{N}\sum_{i\neq k}\frac{1}{\phi_k - \phi_i} \quad \Rightarrow \quad \rho(x) = \delta(x).$$

contradiction!

For this case, $\boxed{\langle W[C]\rangle = o(e^{c\lambda^\gamma}) \quad \forall c, \gamma > 0.}$

## $A_1$ quiver :

There are two saddle-point equations.

$$\frac{8\pi^2}{\lambda_1}\phi_k^{(1)}+\frac{2}{N}\sum_{i=1}^{N}F'(\phi_k^{(1)}-\phi_i^{(2)})-\frac{2}{N}\sum_{i\neq k}F'(\phi_k^{(1)}-\phi_i^{(1)}) = \frac{2}{N}\sum_{i\neq k}\frac{1}{\phi_k^{(1)}-\phi_i^{(1)}}$$

$$\frac{8\pi^2}{\lambda_2}\phi_k^{(2)}+\frac{2}{N}\sum_{i=1}^{N}F'(\phi_k^{(2)}-\phi_i^{(1)})-\frac{2}{N}\sum_{i\neq k}F'(\phi_k^{(2)}-\phi_i^{(2)}) = \frac{2}{N}\sum_{i\neq k}\frac{1}{\phi_k^{(2)}-\phi_i^{(2)}}$$

Define $\rho(x) := (\rho_1(x)+\rho_2(x))/2, \qquad \int dx\, \rho(x)=1$

$$\delta\rho(x) := (\rho_1(x)-\rho_2(x))/2, \qquad \int dx\, \delta\rho(x)=0$$

$$\Rightarrow \begin{cases} \underline{\dfrac{4\pi^2}{\lambda}\phi = \fint d\phi' \dfrac{\rho(\phi')}{\phi-\phi'}} \\ \dfrac{4\pi^2 b}{\lambda}\phi - 8\fint d\phi'\delta\rho(\phi')F'(\phi-\phi') = 4\fint d\phi' \dfrac{\delta\rho(\phi')}{\phi-\phi'} \end{cases}$$

where $\displaystyle \frac{1}{\lambda} := \frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right), \qquad \frac{b}{\lambda} := \frac{1}{2}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right).$

Note: Obviously, $\delta\rho(x)=b\cdot\delta\rho_0(x).$

We obtain

$$\boxed{\langle W_{1,2}[C]\rangle = \frac{2}{\sqrt{2\lambda}} I_1(\sqrt{2\lambda}) \pm b\int dx\, \delta\rho_0(x) e^{2\pi x}.}$$

One can easily show that

- $\mathrm{supp}(\delta\rho_0) \subset \mathrm{supp}(\rho)$
- $\rho(x) \sim (1-x/a)^{\alpha}, \delta\rho_0(x) \sim (1-x/a)^{\alpha'} \quad \Longrightarrow \quad \alpha \leq \alpha'$
- The leading order equation:

$$\int d\phi' \frac{\delta\rho_0(\phi')}{\phi-\phi'} = 0 \quad \Longrightarrow \quad \delta\rho_0(x) = o(\rho(x)).$$

⇨ 2nd term is negligible compared with 1st.

i.e. $\underline{\langle W_{1,2}[C]\rangle \sim e^{\sqrt{2\lambda}}.}$

# Summary

- Localization is applied to N=2 superconformal gauge theories.
- Wilson loops are evaluated using non-Gaussian matrix models.
- $$\begin{cases} N_f = 2N_c : & \langle W[C] \rangle = o(e^{c\lambda^{\gamma}}) \quad \forall\, c, \gamma > 0. \\ A_1 \text{ quiver} : & \langle W_{1,2}[C] \rangle \sim e^{\sqrt{2\lambda}} \end{cases} \qquad (\lambda \to \infty)$$

## Open issues

- Gravity duals of N=2 theories ( in detail )

$$\begin{cases} N_f = 2N_c : & \text{No classical AdS dual.} \\ A_1 \text{ quiver} : & \text{Each Wilson loop may have a string dual.} \end{cases}$$

# Observation

For $A_1$ quiver case,

$$\langle W_{1,2}[C]\rangle = \frac{2}{\sqrt{2\lambda}} I_1(\sqrt{2\lambda}) \pm \frac{i}{\pi}\delta W(\lambda)\underline{\sum_{n\neq0}\frac{(-1)^n}{n}e^{n\pi i b}}.$$

contribution from B-field??

$$S_{ws} = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial X^\mu \partial X^\nu\left[G_{\mu\nu}(X) + iB_{\mu\nu}(X)\right]$$