Asymptotic flatness in higher dimensional spacetimes

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Based on
1. Introduction/black holes in 4D
2. Higher dimensional black holes
3. Asymptotics
4. Applications
5. Summay
1. Introduction/4D BH
Why higher dimensions?

- String theory
- Gauge/gravity correspondence
- Black holes in LHC(?)
- A famous cartoon? - “4D” pocket, Doraemon -
BHs in 4D

- After gravitational collapse, black holes will be formed.

- Then spacetime will be stationary.

- The black hole is described by the Kerr solution.
The topology of stationary black holes is 2-sphere. [Hawking 1972]

It is shown that stationary black holes are axisymmetric (rigidity). [Hawking 1972]

Stationary and axisymmetric black hole is unique. The Kerr solution! [Carter 1973,…]

The Schwarzschild solution is unique static BH. [Israel 1967]
Then
The second Golden age?

A lot of remaining issues for higher dimensional BHs.

It is a chance to have a “gold”.
Focus on higher dimensional GR

- Vacuum Einstein equation
- Asymptotically flat
- Cosmic censorship conjecture [no naked singularities]
3. Higher dimensional black holes

~ brief review ~
Recent reviews

Eds. K.-I. Maeda, T. Shiromizu, T. Tanaka
*Higher dimensional black holes*,
Progress of Theoretical Physics Supplement No. 189, 2011

Ed. G. Horowitz,
*Black holes in higher dimensions*,
Cambridge Univ. Press 2012
Schwarzschild in higher dimensions

- spherical symmetric
- topology: (D-2)-sphere
- any dimensions

Tangherlini 1963
Kerr in higher dimensions

- Myers-Perry black holes
  - rotating
  - topology: (D-2)-sphere
  - any dimensions

Myers&Perry 1986
Black rings

Discovered by Emparan & Reall 2001

- $D=5$
- $S^1 \times S^2$
- rotation is important
Black object zoo

- **Black Saturn**  Elvang and Figueras, 2007
- **Di-Ring**    Iguchi and Mishima, 2007
- **Bi-Cycling Ring**  Izumi, 2008, Elvang and Rodriguez, 2008

and so on
Rigidity theorem

- Hawking 1972
  4D, stationary, rotating, asymptotically flat black holes are axisymmetric

- Hollands, Ishibashi & Wald 2006
  any dimensions, stationary, rotating, asymptotically flat black object spacetimes are axisymmetric.
Topology

- **Hawking 1972**
  
The topology of 4D, stationary, rotating, asymptotically flat black hole is 2-sphere

- **Galloway & Schoen 2005**
  
  In higher dimensions,\[ "\int_{S}^{(D-2)} RdS > 0" \]
  
  :Ricci scalar of BH cross section

  \[ D = 5: \quad S^3, \quad S^1 \times S^2, \text{ connected sum} \]
“Uniqueness” ≈ classification

- Gibbons, Ida & Shiromizu 2002
  Higher dimensional Schwarzschild spacetime is unique static vacuum BH.

- Morisawa & Ida 2004
  In D=5, the Myers-Perry solution is unique stationary vacuum BH if the topology is 3-sphere and 2 rotational symmetries are.

- ..., Hollands & Yazadjiev 2007
  In D=5, stationary BH with 2-rotational symmetries can be classified by mass, angular momentum and rod structure (~topology)
Stability

- Schwarzschild BH is stable [Ishibashi & Kodama 2003]

- Ultraspinning Myers-Perry (MP) solution (D>5) was conjecture to be unstable [Emparan & Myers 2003]

confirmed by a numerical study [Shibata & Yoshino 2009-2010]
settle down to BH with lower angular momentum.
Remaining issues

- **Stability**
  Although there are many efforts…

- **Complete classification**
  requirement of rod structure(≈topology,…) is too strong…

- **Construction of exact solution in D>5**
  need a new way to find (exact) solution…
  blackfold approach…? Emparan et al 2009
3. Asymptotics
Asymptotics

- Naïve concept

\[
g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{|t + x|^2}\right) \quad \text{null infinity}
\]

\[
g_{ij} = \delta_{ij} + O\left(1/|x|^{D-3}\right) \quad \text{spatial infinity}
\]

curries the information of dynamics
How to analyze

\[ 0 \times \infty = \text{finite} \]

\[ \Omega^2 \times \eta_{\mu \nu} = \tilde{g}_{\mu \nu} \]

infinity \iff \Omega = 0
Minkowski spacetime

embedded into the Einstein static universe

\[ \tilde{g}_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \]

\[ \Omega = 4(1+v^2)^{-1}(1+u^2)^{-1}, \quad v = t + r, u = t - r \]

\[ \tau = \tan^{-1} v + \tan^{-1} u, \quad \rho = \tan^{-1} v - \tan^{-1} u \]

\[ \tilde{g}_{\mu\nu} \, dx^\mu \, dx^\nu = -d\tau^2 + d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

Einstein static universe

\[ (-\pi < \tau + \rho < \pi, \quad -\pi < \tau - \rho < \pi, \quad 0 \leq \rho) \]
Conformal treatment in general

Near null infinity \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), where \( h_{\mu\nu} = O(1/r^{D/2-1}) \)

\[ \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega \sim 1/r \]

\( \Omega \) is used to be a coordinate in \( \tilde{g}_{\mu\nu} \)

\[ h_{\mu\nu} \sim \Omega^{D/2-1} \]

- The half-integer for odd dimensions.
- It is supposed that the conformally transformed spacetime is regular.
- Without solving the Einstein equation, one discusses the asymptotics in the physical spacetime.
- As a result, we encounter the bad behavior because of the half-integer for odd dimensions.
The conformal treatment does not work for odd dimensions...

Introduce the Bondi coordinate and then solve the Einstein equation near the null infinity.
Bondi coordinate

\[ ds^2 = -Ae^B du^2 - 2e^B dudr + r^2 h_{IJ} (dx^I + C^I du)(dx^J + C^J du) \]

\[ \sqrt{\det(h_{IJ})} = \omega_{D-2}, \]
\[ \omega_{D-2} : \text{volume of unit } (D-2)\text{-sphere} \]

Vacuum Einstein equation

\[ A = 1 + O(r^{-(D/2-1)}), B = O(r^{-(D-2)}), C^I = O(r^{-D/2}) \]
\[ h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)}) \]
Definition of asymptotic flatness

The spacetime is said to be asymptotically flat if the metric in the Bondi coordinate behaves like

\[
A = 1 + O(r^{-(D/2-1)}), \quad B = O(r^{-(D-2)}), \quad C^I = O(r^{-D/2})
\]

\[
h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)})
\]

Bondi coordinate

\[
ds^2 = -Ae^B du^2 - 2e^B dudr + r^2 h_{IJ} (dx^I + C^I du)(dx^J + C^J du)
\]

\[
\sqrt{\det(h_{IJ})} = \omega_{D-2}
\]
Bondi mass

\[ ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ} (dx^I + C^I du)(dx^J + C^J du) \]

\[ g_{uu} = -Ae^B = -1 - \sum_{k=0}^{k<D/2-2} \frac{A^{(k+1)}}{r^{D/2+k-1}} + \frac{m(u, x^I)}{r^{D-3}} + O(1/r^{D-5/2}) \]

\[ M_{Bondi}(u) = \frac{D-2}{16\pi} \int_{S^{D-2}} m d\omega \]

\( A^{(k+1)} \) does not contribute to the surface integral because of \( A^{(k+1)} \propto D^I D^J h_{IJ}^{(k+1)} \)
Bondi mass loss law

\[
\begin{align*}
 ds^2 &= -A e^B du^2 - 2 e^B dudr + r^2 h_{IJ} (dx^I + C^I du)(dx^J + C^J du) \\
 g_{uu} &= -A e^B = -1 - \sum_{k=0}^{k<D/2-2} \frac{A^{(k+1)}}{r^{D/2+k-1}} + \frac{m(u, x^I)}{r^{D-3}} + O(1/r^{D-5/2}) \\
 M_{\text{Bondi}}(u) &= \frac{D-2}{16\pi} \int_{S^{D-2}} m d\omega
\end{align*}
\]

The Einstein equation implies

\[
\partial_u M = -\frac{1}{2(D-2)} \partial_u h_{IJ}^{(1)} \partial_u h^{(1)IJ} + \text{total derivative}
\]

Flux of gravitational waves

\[
\frac{d}{du} M_{\text{Bondi}}(u) = -\frac{1}{32\pi} \int_{S^{D-2}} \partial_u h_{IJ}^{(1)} \partial_u h^{(1)IJ} \ d\omega \leq 0
\]
The Bondi mass loss law is given by:

$$\frac{d}{du} M_{\text{Bondi}}(u) = - \frac{1}{32\pi} \int_{S^{D-2}} \partial_u h_{IJ}^{(1)} \partial_u h^{(1)IJ} d\omega \leq 0$$

where $h_{IJ}$ are the Newman-Penrose coefficients.

The flux of gravitational waves is:

$$M_{\text{Bondi}}(-\infty) = M_{\text{initial}} = M_{\text{ADM}} \geq 0$$

and

$$M_{\text{Bondi}}(\infty) = M_{\text{final}} < M_{\text{initial}}$$

Gravitational wave curries the positive energy.
Asymptotic symmetry

Asymptotic symmetry is defined to be the transformation group which preserves the asymptotic structure at null infinity.

\[ ds^2 = -A e^B du^2 - 2e^B dudr + r^2 h_{ij} (dx^I + C^I du)(dx^J + C^J du) \]
\[ A = 1 + O(r^{-(D/2-1)}), B = O(r^{-D-2}), C^I = O(r^{-D/2}) \]
\[ h_{ij} = \omega_{ij} + O(r^{-(D/2-1)}) \]

Bondi coordinate condition

\[ \delta g_{rr} = 0, \quad \delta g_{rI} = 0, \quad g^{IJ} \delta g_{IJ} = 0 \]
\[ \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \]

\[ \xi^u = f(u, x^I), \]
\[ \xi^I = f^I(u, x^I) + \int dr \frac{e^B}{r^2} h^{ij} D_j f, \]
\[ \xi^r = -\frac{r}{D-2} (C^I D_I f + D_I f^I) \]

\( D_I : \text{covariant derivative w.r.t. } \omega_{IJ} \)
Asymptotic symmetry

$$ds^2 = -A e^B du^2 - 2 e^B dudr + r^2 h_{IJ} (dx^I + C^I du)(dx^J + C^J du)$$

$$A = 1 + O(r^{-(D/2-1)}), B = O(r^{-(D-2)}), C^I = O(r^{-D/2}), h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)})$$

The asymptotic behavior

$$\delta g_{uu} = O(1/r^{D/2-1}), \quad \delta g_{rl} = O(1/r^{D/2-2}), \quad \delta g_{ur} = O(1/r^{D-2}), \quad \delta g_{IJ} = O(1/r^{D/2-3})$$

$$\text{e.g.,} \quad \delta g_{IJ} = r^2 \left[ D_I f_J + D_J f_I - \frac{2D_K f^K}{D-2} \omega_{IJ} \right] - 2r \left[ D_I D_J f - \frac{D_K D^K f}{D-2} \omega_{IJ} \right] + O(1/r^{D/2-3})$$

$$\partial_u f^I = 0, \quad D_I f_J + D_J f_I = \frac{2D_K f^K}{D-2} \omega_{IJ}, \quad D_I f^I = (D-2) \partial_u f, \quad D_I D_J f = \frac{D_K D^K f}{D-2} \omega_{IJ}$$
Asymptotic symmetry is Poincare

The asymptotic symmetry is generated by $f$ and $f^I$

$$\partial_u f^I = 0, \quad D_I f_J + D_J f_I = \frac{2D_K f^K}{D-2} \omega_{IJ}, \quad D_I f^I = (D-2) \partial_u f, \quad D_I D_J f = \frac{D_K D^K f}{D-2} \omega_{IJ}$$

$f \Rightarrow \text{translation}$

$f^I \Rightarrow \text{Lorentz group}$
$D=4$ is special

\[ \delta_{IJ} = O(1/r^{D/2-3}) \quad \Rightarrow \quad \delta_{IJ} = O(r) \]

\[ \delta g_{IJ} = r^2 \left[ D_i f_j + D_j f_i - \frac{2D_K f^K}{D-2} \omega_{IJ} \right] - 2r \left[ D_I D_J f - \frac{D_K D^K f}{D-2} \omega_{IJ} \right] + O(1/r^{D/2-3}) \]

\[ O(r) \]

\[ \partial_u f^I = 0, \quad D_i f_j + D_j f_i = \frac{2D_K f^K}{D-2} \omega_{IJ}, \quad D_I f^I = (D-2) \partial_u f, \]

\[ \begin{cases} f \Rightarrow \text{translation} + \text{supertranslation} \\ f^I \Rightarrow \text{Lorentz group} \end{cases} \]

This gives us troublesome when one defines angular momentum at null infinity
4. Application
Stability of BHs

- The symmetry which exact BH solutions have are not enough to analyse the stability.

- Together with the general properties of dynamical BHs, we may be able to have some indications.
Stability of BHs

Figueras, Murata & Reall 2011, Hollands & Wald 2012

$J$ fixed

$M = M_{BH}(A) : \text{a solution}$

$M$ (mass)

$A$ (area)

Perturbed initial data

Stable

unstable

Area theorem

Bondi mass loss
Classification

Weyl tensor: \( C_{\mu \nu \alpha \beta} = \) trace free part of Riemann tensor \( R_{\mu \nu \alpha \beta} \)
contains information of spacetime (gravity)

Peeling property in 4D

\[
C_{\mu \nu \alpha \beta} = \frac{C^{(N)}_{\mu \nu \alpha \beta}}{r} + \frac{C^{(III)}_{\mu \nu \alpha \beta}}{r^2} + \frac{C^{(II,D)}_{\mu \nu \alpha \beta}}{r^3} + \frac{C^{(I)}_{\mu \nu \alpha \beta}}{r^4} + O(1/r^5)
\]

Algebraic classification of Weyl tensor (Petrov type) \( \Leftrightarrow \) asymptotic behavior

In Type D the equations will be significantly simplified and then solved completely. The Kerr solution, C-metric, ....

Using our result, a peeling properties has been discussed in higher dimensions [Godazgar & Reall 2012].

(i) \( D > 5 \)

\[
C_{\mu \nu \alpha \beta} = \frac{C^{(N)}_{\mu \nu \alpha \beta}}{r^{D/2-1}} + \frac{C^{(II)}_{\mu \nu \alpha \beta}}{r^{D/2}} + \frac{C^{(G)}_{\mu \nu \alpha \beta}}{r^{D/2+1}} + O(1/r^{D/2+2}) \text{[even D]} \text{ or } O(1/r^{D/2+3/2}) \text{[odd D]}
\]

(ii) \( D = 5 \)

\[
C_{\mu \nu \alpha \beta} = \frac{C^{(N)}_{\mu \nu \alpha \beta}}{r^{3/2}} + \frac{C^{(II)}_{\mu \nu \alpha \beta}}{r^{5/2}} + \frac{C^{(G)}_{\mu \nu \alpha \beta}}{r^{3}} + \frac{C^{(G)}_{\mu \nu \alpha \beta}}{r^{7/2}} + O(1/r^4)
\]
Classification

Similar to four dimensions, the classification of the Weyl tensor and the peeling property may give us a hint to classify and/or construct exact solutions in higher dimensions.
5. Summary and future issues
Summary

- For odd dimensions, we had to solve the Einstein equation to formulate the asymptotics at null infinity.

- We could show the Bondi mass loss law and the presence of the Poincare symmetry at null infinity.

- We also defined the momentum and angular momentum.

- There are a few efforts to discuss the stability and classification of higher dimensional BHs/spacetimes.
Remaining issues

Still…
- Stability…
- Classification…
- Construction…

New issues?
- Asymptotically anti-deSitter spacetimes
  ⇐ always unstable? Final fate?  [Dias, Horowitz, Santos, 2011]
- Asymptotics in spacetimes with compact space
  ⇐ final fate of black string?  [Latest numerical simulation: Lehner & Pretorius 2010]
YITP Molecule-type workshop on
Non-linear massive gravity
23rd July to 8th August

Core participants:
C. de Rham, S. Mukohyama, ..., T. Shiromizu (Chair), A. Tolley, ...
Appendix
Constraint and evolution eqs.
In Bondi coordinate, 

\[ ds^2 = -A e^B du^2 - 2e^B dudr + \gamma_{ij} (dx^I + C^I du)(dx^J + C^J du) \]

\[ u \leftrightarrow \text{time} \]

\[ R_{r\mu} = R_{\mu \nu} \gamma^{\mu \nu} = 0 \implies \text{constraint equations} \]

\[ R_{uu} = 0, R_{IJ} = 0 \implies \text{evolution equations} \]
Constraint equations

\[ ds^2 = -A e^B du^2 - 2e^B du dr + r^2 h_{ij} (dx^i + C^i du)(dx^j + C^j du) \]

\[ R_{rr} = 0 \Rightarrow B' = \frac{r}{4(D-2)} h_{ij}' h_{KL}' h^{IK} h^{JL} \]

\[ R_{\mu\nu} \gamma^{\mu\nu} = 0 \Rightarrow (D-2) \frac{(r^{D-3} A)'}{r^{D-2}} = -\omega^{(\omega)} \nabla I C^{I'} - \frac{2(D-2)\omega}{r} \nabla I C^I - \frac{r^2 e^{-B}}{2} h_{ij} C^{I'} C^{J'} \]

\[ -\frac{e^B}{2r^2} h^{II} \omega^{(\omega)} \nabla I B^{(\omega)} \nabla J B - \frac{e^B}{r^2} \omega^{(\omega)} \nabla I (h^{II} \omega^{(\omega)} \nabla J B) + \frac{e^B}{r^2} R \]

\[ R_{rl} = 0 \Rightarrow \frac{1}{r^{D-2}} (r^D e^{-B} h_{ij} C^J)' = -\omega^{(\omega)} \nabla I B' + \frac{D-2}{r} \omega^{(\omega)} \nabla I B + (h) \nabla h_{ij}' \]

Once \( h_{ij} \) are given on the initial \( u \)-const surface, \( A, B, C^I \) are determined
Constraint equations

$$ds^2 = -Ae^B du^2 - 2e^B dudr + r^2 h_{ij} (dx^i + C^i du)(dx^j + C^j du)$$

$$h_{ij} = \omega_{ij} + \sum_{k \geq 0} h^{(k+1)}_{ij} r^{-(D/2+k-1)} \quad (k \in \mathbb{Z} \text{ for even, } 2k \in \mathbb{Z} \text{ for odd dimensions})$$

$$\sqrt{\det(h_{ij})} = \omega_{D-2} \Rightarrow h^{(k+1)}_{ij} (k < D/2 - 1) \text{ is traceless}$$

$$B = B^{(1)} r^{-(D-2)} + O(1/r^{D-3/2}), \quad B^{(1)} = -\frac{1}{16} \omega^{IK} \omega^{JL} h^{(1)}_{IJ} h^{(1)}_{KL}$$

$$C^I = \sum_{k=0}^{k<D/2-1} \frac{C^{(k+1)I}}{r^{D/2+k}} + \frac{J^I(u,x^I)}{r^{D-1}} + O(1/r^{D-1/2}), \quad C^{(k+1)I} = \frac{2(D+2k-2)}{(D+2k)(D-2k-2)} \omega \nabla h^{(k+1)I}$$

$$A = 1 + \sum_{k=0}^{k<D/2-2} \frac{A^{(k+1)}}{r^{D/2+k-1}} - \frac{m(u,x^I)}{r^{D-3}} + O(1/r^{D-5/2}), \quad A^{(k+1)} = \frac{4(D+2k-4)}{(D+2k)(D-2k-2)(D-2k-4)} \omega \nabla I \omega \nabla h^{(k+1)I}$$
Evolution equations

\[ ds^2 = -A e^B du^2 - 2e^B dudr + r^2 h_{IJ} (dx^I + C^I du)(dx^J + C^J du) \]

\[ h_{IJ} = \omega_{IJ} + \sum_{k \geq 0} h_{IJ}^{(k+1)} r^{-(D/2+k-1)} \quad (k \in \mathbb{Z} \, \text{ for even}, \ 2k \in \mathbb{Z} \, \text{ for odd dimensions}) \]

\[ \sqrt{\det(h_{IJ})} = \omega_{D-2} \Rightarrow h_{IJ}^{(k+1)} (k < D/2 - 1) \text{ is traceless} \]

\[ R_{IJ} = 0 \Rightarrow (k+1) \dot{h}_{IJ}^{(k+2)} = -\frac{1}{2} (D-2k-4) A^{(k+1)} \omega_{IJ} + \frac{1}{8} [D^2 - 6D - 4(k^2 + k - 4)] h_{IJ}^{(k+1)} \]

\[ + \frac{1}{2} (-^{(\omega)} \nabla^2 h_{IJ}^{(k+1)} + 2^{(\omega)} \nabla_{(J}^{(\omega)} \nabla_{IJ)} h_{IJ}^{(k+1)}) - \frac{1}{2} (D-2k-4)^{(\omega)} \nabla_{(I} C_{J)_{(k+1)}^{(\omega)} - }^{(\omega)} \nabla^K C_{K_{(k+1)}^{(\omega)} \omega_{IJ}}} \]

\[ R_{\mu\nu} = 0 \Rightarrow \dot{m} = -\frac{1}{2(D-2)} \dot{h}_{IJ}^{(1)} \dot{h}_{IJ}^{(1)} + \frac{D-5}{D-2}^{(\omega)} \nabla^I C_{I \omega}^{(D/2-2)} + \frac{1}{D-2}^{(\omega)} \nabla^2 A^{(D/2-2)} \]
Asymptotic symmetry
Asymptotic symmetry

Asymptotic symmetry is defined to be the transformation group which preserves the asymptotic structure at null infinity

\[ ds^2 = -A e^B \, du^2 - 2e^B \, \, dudr + r^2 h_{IJ} (dx^I + C^I \, du)(dx^J + C^J \, du) \]
\[ A = 1 + O(r^{-(D/2-1)}) , \quad B = O(r^{-(D-2)}) , \quad C^I = O(r^{-D/2}) \]
\[ h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)}) \]

Bondi coordinate condition

\[ \delta g_{\mu \nu} = \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu} \]

\[ \delta g_{rr} = -2e^B \xi' = 0 \]
\[ \delta g_{rI} = -e^B(\gamma) \nabla_I \xi^u + \gamma_{IJ} C^J \xi^u + \gamma_{IJ} \xi^J' \]
\[ g^{IJ} \delta g_{IJ} = \xi^r (\log \gamma)' + \xi^u (\log \gamma) + 2^{(\gamma)} \nabla_I \xi^I + 2C^I (\gamma) \nabla_I \xi^u \]

\[ \xi^u = f(u, x^I), \]
\[ \xi^I = f^I (u, x^I) + \int dr \frac{e^B}{r^2} h^{IJ} D_J f, \]
\[ \xi^r = -\frac{r}{D-2} (C^I D_I f + D_I f^I) \]

\[ D_I : \text{covariant derivative w.r.t. } \omega_{IJ} \]
Asymptotic symmetry

\[ ds^2 = -Ae^B du^2 - 2e^B dudr + r^2 h_{ij} (dx^i + C^i du)(dx^j + C^j du) \]
\[ A = 1 + O(r^{-(D/2-1)}), B = O(r^{-(D-2)}), C^i = O(r^{-D/2}) \]
\[ h_{ij} = \omega_{ij} + O(r^{-(D/2-1)}) \]
\[ \sqrt{\det(h_{ij})} = \omega_{D-2} \]

\[ \delta g_{uu} = \frac{2r}{D-2} \partial_u^{(\omega)} \nabla^I f^I - \frac{2}{D-2} \partial_u^{(\omega)} \nabla^2 + (n-2)f + \frac{2C^{(l)}_I \partial_u f^I}{r^{D/2-2}} + O(1/r^{D/2-1}) \]

\[ \delta g_{ur} = \frac{1}{D-2} \left[ \partial_u^{(\omega)} \nabla^I f^I - (D-2) \partial_u f \right] - \sum_{k=0}^{k<D/2-2} \frac{D+2k-2}{(D-2)(D+2k)} h_{ij}^{(k+1)} \frac{\partial_u^{(\omega)} \nabla^I f^I}{r^{D/2+k}} + O(1/r^{D-2}) \]

\[ \delta g_{ui} = r^2 \partial_u f^I + \frac{r}{D-2} \partial_I \left[ \partial_u^{(\omega)} \nabla^I f^I - (D-2) \partial_u f \right] - \frac{1}{D-2} \partial_I \left[ \partial_u^{(\omega)} \nabla^2 + (D-2)f \right] + \frac{h_{ij}^{(1)} \partial_u f^I}{r^{D/2-3}} + O(1/r^{D/2-2}) \]

\[ \delta g_{IJ} = 2r^2 \left[ \partial_u^{(\omega)} \nabla_{(I} f_{J)} - \frac{\partial_u^{(\omega)} \nabla^K f^K}{D-2} \omega_{IJ} \right] - 2r \left[ \partial_u^{(\omega)} \nabla^I f^I - \frac{\partial_u^{(\omega)} \nabla^2 f}{D-2} \omega_{IJ} \right] + O(1/r^{D/2-3}) \]

\[ \partial_u f^I = 0, \quad \partial_u^{(\omega)} \nabla f^I + \partial_u^{(\omega)} \nabla f^I = \frac{2\partial_u^{(\omega)} \nabla^K f^K}{D-2} \omega_{IJ}, \quad \partial_u^{(\omega)} \nabla f^I = (D-2) \partial_u f, \quad \partial_u^{(\omega)} \nabla f^I = \frac{\partial_u^{(\omega)} \nabla^K f^K}{D-2} \omega_{IJ} \]
Asymptotic symmetry

\[ \partial_u f^I = 0, \quad (^{(\omega)} \nabla_I f_j + ^{(\omega)} \nabla_J f_I = \frac{2^{(\omega)} \nabla^K f^K}{D-2} \omega_{IJ}, \quad (^{(\omega)} \nabla_I f^I = (D-2) \partial_u f, \quad (^{(\omega)} \nabla_I (^{(\omega)} \nabla_J f = \frac{(^{(\omega)} \nabla^K (^{(\omega)} \nabla^K f}{D-2} \omega_{IJ} \]

\[ \partial_u f^I = 0, \quad F := ^{(\omega)} \nabla_I f^I = (D-2) \partial_u f \quad \Rightarrow \quad f = \frac{F(x^I)}{D-2} \alpha(x^I) \]

\[ (^{(\omega)} \nabla_I (^{(\omega)} \nabla_J f = \frac{(^{(\omega)} \nabla^K (^{(\omega)} \nabla^K f}{D-2} \omega_{IJ} \quad \Rightarrow \quad (^{(\omega)} \nabla_I (^{(\omega)} \nabla_J F = \frac{(^{(\omega)} \nabla^K (^{(\omega)} \nabla^K F}{D-2} \omega_{IJ} \]

\[ (^{(\omega)} \nabla_I (^{(\omega)} \nabla_J f = \frac{(^{(\omega)} \nabla^K (^{(\omega)} \nabla^K f}{D-2} \omega_{IJ} \quad \Rightarrow \quad (^{(\omega)} \nabla^2 + (D-2)) F = 0 \]

\[ (^{(\omega)} \nabla_I (^{(\omega)} \nabla_J f = \frac{(^{(\omega)} \nabla^K (^{(\omega)} \nabla^K f}{D-2} \omega_{IJ} \quad \Rightarrow \quad (^{(\omega)} \nabla_I (^{(\omega)} \nabla_J \alpha = \frac{(^{(\omega)} \nabla^K (^{(\omega)} \nabla^K \alpha}{D-2} \omega_{IJ} \]

\[ ^{(\text{trans})} f^I : \text{transverse part of } f, i.e., \quad (^{(\omega)} \nabla_I (^{\text{trans}} f^I = 0, \quad (^{(\omega)} \nabla_I (^{\text{trans}} f_j + (^{(\omega)} \nabla_J (^{\text{trans}} f_I = 0 \]

\[ l = 1 \text{ modes of the scalar harmonics on } S^{D-2} \]

\[ l = 0,1 \text{ modes of the scalar harmonics on } S^{D-2} \]

\[ \text{time/space translations} \]

\[ \text{Lorentz group} \]
Petrov type
null tetrad

\[ g_{\mu \nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu m^*_\nu + m^*_\mu m_\nu, \quad l_\mu n^\mu = -1, \quad m_\mu m^{*\mu} = 1, \quad l_\mu l^{\mu} = 0, \quad n_\mu n^{\mu} = 0 \]

Five complex functions

\[ \psi_0 = C_{\mu \nu \alpha \beta} l^\mu m^\nu l^\alpha m^\beta, \psi_1 = C_{\mu \nu \alpha \beta} l^\mu n^\nu l^\alpha m^\beta, \psi_2 = C_{\mu \nu \alpha \beta} l^\mu m^\nu n^\alpha m^{*\beta}, \]
\[ \psi_3 = C_{\mu \nu \alpha \beta} l^\mu n^\nu n^\alpha m^\beta, \psi_4 = C_{\mu \nu \alpha \beta} n^\mu m^\nu n^\alpha m^\beta \]

Transformation of null tetrad and principal null direction

\[ n^\mu \rightarrow n^\mu, \quad m^\mu \rightarrow m^\mu + an^\mu, \quad l^\mu \rightarrow l^\mu + am^{*\mu} + a^* m^\mu + a a^* n^\mu \]
\[ \psi_0 \rightarrow \psi_0 + 4a \psi_1 + 6a^2 \psi_2 + 4a^3 \psi_3 + a^4 \psi_4 \]

\[ \psi_0 \] can be made zero by satisfying the equation for \( a \)
\[ \psi_0 + 4a \psi_1 + 6a^2 \psi_2 + 4a^3 \psi_3 + a^4 \psi_4 = 0 \]

I (four distinct roots), II (two coincident), D (two distinct double), III (three coincident), N (four coincident)
Petrov classification

I (four distinct roots), II(two coincident ), D(two distinct double),
III(three coincident), N(four coincident)

\[
\begin{align*}
\text{I} & \quad \psi_0 = \psi_1 = 0 \\
\text{II} & \quad \psi_0 = \psi_1 = \psi_4 = 0 \\
\text{D} & \quad \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0 \\
\text{III} & \quad \psi_0 = \psi_1 = \psi_2 = \psi_4 = 0 \\
\text{N} & \quad \psi_0 = \psi_1 = \psi_2 = \psi_3 = 0
\end{align*}
\]