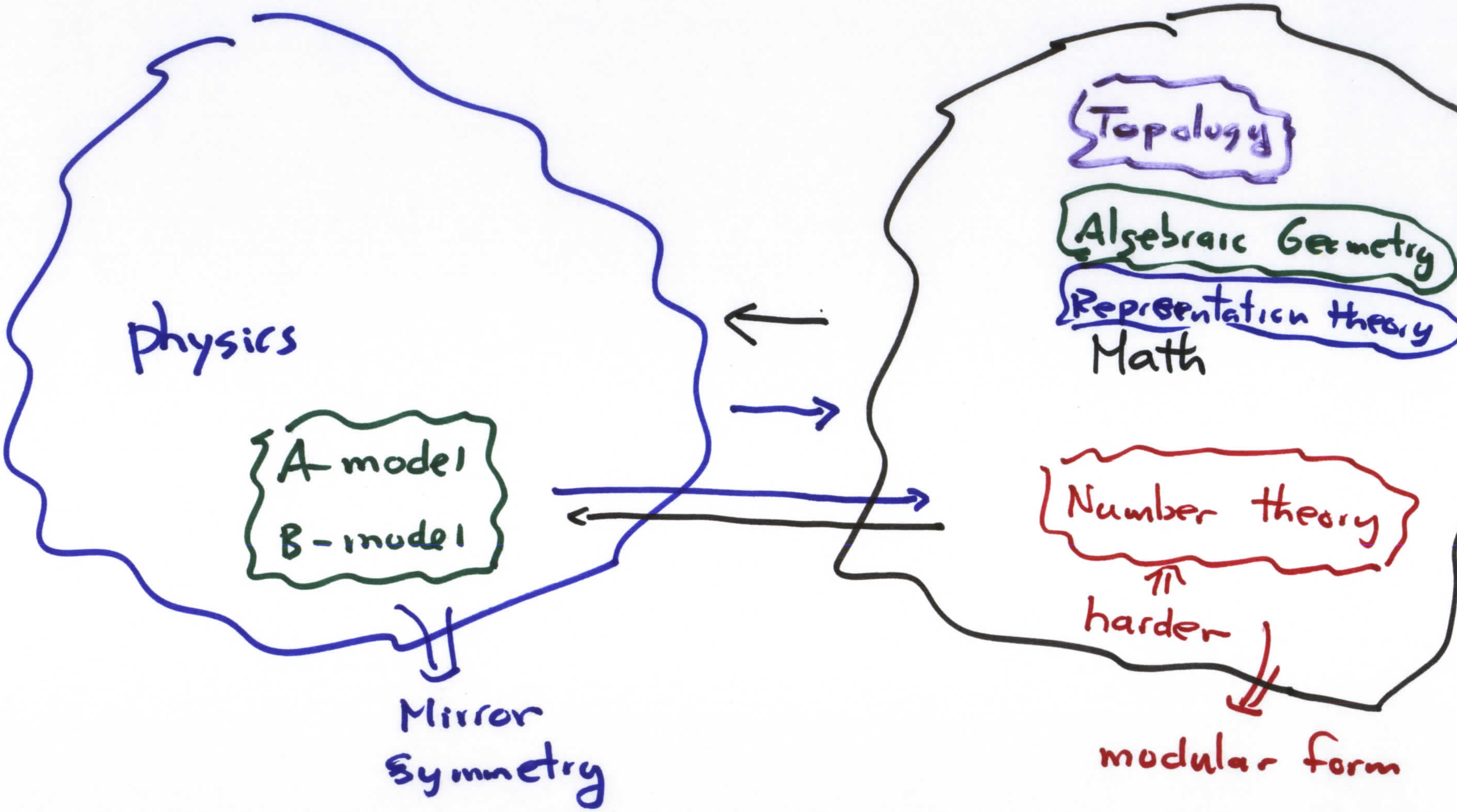
Mirror Symmetry & Modular form

YONGBIN RUAN

University of Michigan


Physics
A-model
B-model
Mirror Symmetry
Topology
Algebraic Geometry
Representation theory
Math
Number theory
harder
modular form

Modular form:

"There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and ...... modular forms"

Eichler quote (Math overflow)

One of THE most beautiful objects in Math!

(appear in Wiles' proof of Fermat Last Theorem)

Modular form (from Wikipedia)

$f : \mathbb{H} \longrightarrow \mathbb{C}$ - complex analytic funct satisfying

upper half plane following transformation property

- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

  weight

  "$a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$"

- f has Fourier expansion

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z} = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i z}$$

Rmk: one can restrict $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL_2(\mathbb{Z})$. Then, we say f is a modular form of Γ.

Examples: (as a counting function)

- Eisenstein Series

$$E_{2k}(\tau) = \sum_{(n,m)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}}$$

$(k>2)$

~ a weighted count of integral points

- Weierstrass Δ - funct

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_n a_n q^n \qquad q = e^{2\pi i \tau}$$

a_n: a count of partition

- Many More examples in number theory

Why should you be interested in modular form

Modular forms form a ring!

- f_1, f_2 - weight k $\Rightarrow$ $f_1 + f_2$ - weight k
- f_1 - weight k_1, f_2 - weight k_2 $\Rightarrow$ $f_1 \cdot f_2$ - weight $k_1 + k_2$

Space of modular forms $\mathcal{M}(\Gamma)$ - ring

Key Theorem: $M(\Gamma)$ is often finitely generated!

Example: $M(SL_2(\mathbb{Z}))$ is generated by E_4, E_6

Application: want to compute $f \in M(\Gamma)$, write
f - poly of generators $\Leftarrow$ determine coeff

Relation to Mirror Symmetry:

A quick review

Type II A string theory

A-model

interchange $\Longleftrightarrow$

Type II B string theory

B-model

$\Downarrow$ Fourier expansion

$\Downarrow$ modular transformation

modular form

B-model: X - Calabi-Yau n-fold

$$H^{n,0}(X) \to L$$ - vacuum line bundle.

$$\downarrow$$

fundamental object: $\mathcal{M}_B^X$ = moduli space of complex strs on X

$X = E$ - elliptic curves

upper half plan

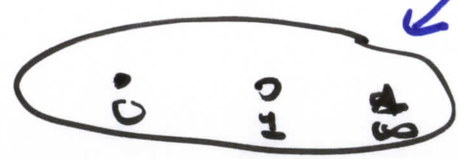
$\mathbb{H}/SL_2(\mathbb{Z})$

modular group

$$L \downarrow$$

$$\mathcal{M}_B^E = \{ y^2 = x(x-1)(x-\sigma) \} =$$

parameter



$= \mathbb{P}^1 - \{0, 1, \infty\}$

$s \in \Gamma(L^k)$ $\Longleftrightarrow$ f transform as modular form of weight k

expansion at $\sigma = \infty$

Fourier expansion (at $\tau = i\infty$)

Other interesting examples

- $\mathcal{M}_B = \{ x_1^3 + x_2^3 + x_3^3 - \sigma x_1 x_2 x_3 = 0 \} = \mathbb{H} / \Gamma_3$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} a = d = 1 \\ b = c = 0 \end{matrix} \bmod (N) \right\}$$

- $\mathcal{M}_B = \{ x^2 + x y^2 + y z^3 + \sigma x y z = 0 \} = \mathbb{H} / \Gamma(4)$

- $\mathcal{M}_B = \{ x^2 + x y^3 + z^3 + \sigma x y z = 0 \} = \mathbb{H} / \Gamma(6) \leftarrow ?$

Question: Find explicit examples for other modular groups ?

Dimension one : Number theorist know a lot about modular forms $\Longrightarrow$ help physics

dim 2 = K3-surfaces: (potential ground to join physics
math)

$$\mathcal{M}_B = \left(O(2,k) \Big/ \underset{\Uparrow}{SO(2) \times O(k)} \right) \Big/ \Gamma \qquad 1 \le k \le 19$$

hermitian symmetry domain of type IV

$s \in \Gamma(L^k) \rightsquigarrow$ automorphic form

- A very rich subject in number theory
- Not much is known (difficult)
- physics is very useful (orbifold Gromov-Witten Theory)

dim $\geq$ 3 ?

- Less interesting in number theory
- Same strategy can be applied to physics

The object of interest: (Gromov-Witten Theory)

Big idea:

B-model GW-Theory: $\mathcal{F}_{B,g} \in \overset{\text{modular transformation}}{\overset{\Uparrow}{\Gamma}}(L^{2g-2})$

$\Big\|$ Mirror symmetry $\Rightarrow$ modular form on $\mathcal{M}_B$

A-model GW-Theory: $\mathcal{F}_{A,g} = \sum_{n \geq 0} N_{g,n} \, q^n$

$\Downarrow$

Fourier expansion

NoT too fast!

holomorphic anomaly $\Longrightarrow$ $\mathcal{F}_{B,g}$ is not holomorphic

$\Downarrow$

quasi-modular form

Bershadsky – Cecotti – Ooguri – Vafa holomorphic anomaly equation

- $\exists$ a (conjectural) B-model $\mathcal{F}_g^B(\tau,\bar\tau)$ which is modular but not holomorphic

- $\mathcal{F}_g^B(\tau,\bar\tau)$ satisfies holomorphic anomaly equation

$$\partial_{\bar\tau}\mathcal{F}_g^B = \Gamma_{h<g}(\mathcal{F}_h^B)$$

<u>Results from physics</u>: BCOV, Klemm's group.

(1) solution to Holomorphic anomaly equation is equivalent to expansion $\mathcal{F}_g^B(\tau,\bar\tau) = \mathcal{F}_g^B(\tau) + \sum_{i=1}^{k} \mathcal{F}_{g,i}^B(\tau)(\operatorname{Im}\tau)^{-i}$

quasi-modular form

(2) Explicit formula of $\mathcal{F}_{g,i}^B(\tau)$ using ~~feyman~~ diagram expansion.

Quasi - Modular form:

- $f: \mathbb{H} \to \mathbb{C}$ holomorphic and can be completed to

anti-holomorphic generator

$$\hat{f}(\tau, \bar{\tau}) = f + \sum_{i=1}^{k} f_i(\tau)\, (\operatorname{im}\tau)^{-i}$$

modular

holomorphic (mysterious)

almost holomorphic modular form

- f has Fourier expansion

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z} = \sum_{n \geq 0} a_n q^n \qquad q = e^{2\pi i z}$$

Rmk: ① Quasi-Modular forms are studied a lot in number theory too, further generalization (Mock modular form)

② directly relevant to physics.

A geometric Application:

- Gromov-Witten Inv: (restrict to Calabi-Yau mfd)

$$N_{g,d} = \# \left\{ f : \right.$$  $$\left. \xrightarrow{\text{holomorphic}} X \right\}$$

genus g   degree $d = f_*[\Sigma]$

generating funct

$$\mathcal{F}_g = \sum_{d \geq 0} N_{g,d}\, q^d \ldots$$

counting funct similar to modular form except **MUCH** harder to compute

one of most important problems in geometry and physics

- One may hope

  (i) Technique from modular form can help to compute Gromov-Witten Inv

  (ii) Gromov-Witten Inv provide new examples of (quasi)-modular form

Recap:

Want
- construct $\mathcal{F}_{B,g} \in \Gamma(L^{2g-2})$ satisfying holomorphic anomaly equation
- Prove mirror symmetry $\mathcal{F}_{A,g} = \lim_{\tau \to \omega} \mathcal{F}_{B,g}$

$\Downarrow$

$\mathcal{F}_{A,g}$ – quasi-modular form

or $\mathcal{F}_{B,g}$ – almost holomorphic form

Results

$\nearrow$ modular group $\Gamma = SL_2(\mathbb{Z})$

- Theorem for elliptic curve (Okounkov-Pandharipande, Costello-Li)
- Trivial for K3-surface: "$\mathcal{F}_{A,g} = 0$"

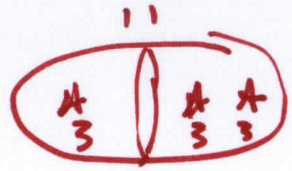
First generalization: orbifold Calabi-Yau X/G

Examples

- dim = 1:

$E/\mathbb{Z}_2$ =

$E/\mathbb{Z}_3$ =



$E/\mathbb{Z}_4$ =

$E/\mathbb{Z}_6$ =

- dim = 2: $K3/G$ – thousands of examples

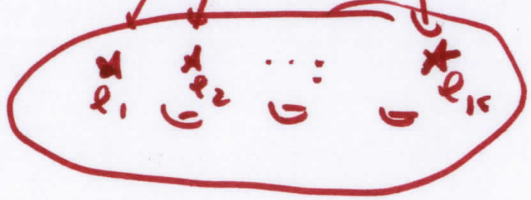
- dim = 3: Even more. mirror quintic $X_5/\mathbb{Z}_5^3$ ...

- Orbifolds appear naturally in both math and physics

- Provide much bigger playfield

Orbifold Gromov-Witten Theory:

orbifold singularities

$N_{g,\beta}^{\ell_1\dots\ell_k} = \Big\{$ 

$\xrightarrow[f]{\text{holomorphic}} X/G \Big\}$

$f_*[C] = \beta$

$$F^{A}_{g,\ell_1\dots\ell_k} = \sum_{\beta\geq 0} N^{\ell_1\dots\ell_k}_{g,\beta}\, q^{\beta}$$ — counting funct of orbifold holomorphic map

$H^*_{orb}(X/G)$

Remark:

- $F_{A,g,\ell_1\dots\ell_k}$ are much more complicated due to extra indices $\ell_1\dots\ell_k$
- $F_{A,g,\ell_1\dots\ell_k}$ posses extra-structures to make them much easier to compute

Mirror of orbifold: Landau - Ginzburg model (LG)

Berglund - Hubsch - Krawitz Mirror

$$W = \sum_{j=1}^{N} c_j \prod_{i=1}^{N} z_i^{n_{ij}} \Leftarrow (\# \text{ of monomials} \overset{\text{invertible}}{\Updownarrow} = \# \text{ of variables})$$

transpose

$$\text{dual} = W^T = \sum_{j=1}^{N} c_j \prod_{i=1}^{N} z_i^{n_{ji}}$$

$$X_W = \{ W = 0 \}$$

Conj: $X_W / \mathrm{Aut}(W) \xleftrightarrow{\text{Mirror}}$ "Landau - Ginzburg model" of W^T

Examples:

- $E/\mathbb{Z}_3 \longleftrightarrow W^T = x_1^3 + x_2^3 + x_3^3 \Leftarrow \mathbb{P}(3)$
- $E/\mathbb{Z}_4 \longleftrightarrow W^T = x^2 + xy^2 + yz^3 \Leftarrow \mathbb{P}(4)$
- $E/\mathbb{Z}_6 \longleftrightarrow W^T = x^2 + xy^3 + z^3 \Leftarrow \mathbb{P}(6)$

LG - B - model:

use $W = x^3 + y^3 + z^3$ to illustrate the idea

CY - B - model: $\mathcal{M}_B = \{ x^3 + y^3 + z^3 + \underset{\uparrow \text{parameter}}{\sigma} xyz = 0 \}$.

LG - B - model: $LG_B = \{ x^3 + y^3 + z^3 + \sigma xyz + s_0 + s_1 x + s_2 y + s_3 z + s_4 xy + s_5 xz + s_6 yz \}$

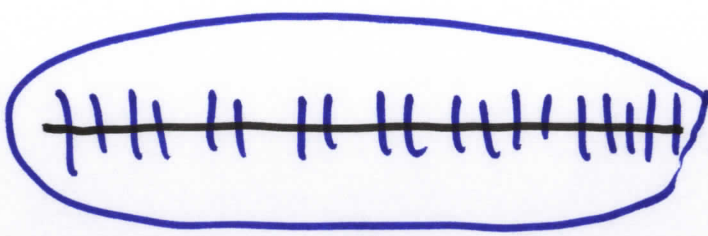
$\Uparrow$

8 - dim space

$\shortparallel$

$\dim H^*_{orb}(E/\mathbb{Z}_3)$

$\sigma \in \mathbb{P}^1 - \{0, 1, \infty\}$, $|s_i| \le \varepsilon$

LG_B:  $\mathcal{M}_B$

Vacuum line bundle

$$\begin{tikzcd} L \arrow[d] & L \arrow[l, "\text{extend}"'] \arrow[d] \\ \mathcal{M}_B \subset & LG_B \end{tikzcd}$$

$\underline{\mathcal{F}_{B,g}}$?

CY-case: $\mathcal{F}^{CY}_{B,g}$ (BCOV, Costello-Li)

LG-case: $\mathcal{F}^{LG}_{B,g}$ (Hertling, Coates-Iritani, Iritani-Milanov-Ruan-Shen)

Advantage of LG-model : Semi-simplicity

- $x^3+y^3+z^3+\sigma xyz$ has a single, but degenerate singularity at $(x,y,z)=(0,0,0)$
- For $s_i \neq 0$, $x^3+y^3+z^2+\sigma xyz+s_0+s_1x+\cdots$ split $(0,0,0)$ into 8-different but non-degenerate (Morse) singularities

much easier to compute $\Leftarrow$

Mathematical Results:

I: (Krawitz-Shen-Milanov-Ruan)

$\mathcal{F}^{A}_{E/G,\ell_1\cdots\ell_k}$ are quasi-modular forms of $\Gamma(3), \Gamma(4), \Gamma(6)$)

$E/\mathbb{Z}_3, E/\mathbb{Z}_4, E/\mathbb{Z}_6$

II: (Iritani-Milanov-Ruan-Shen)

For orbifold quintic $X_5/\mathbb{Z}_5^4$, $\mathcal{F}_{A,\ell_1\cdots\ell_k}$ are "quasi-modular forms"

$K3/G$: (i) GW-theory is highly nontrivial

(ii) modularity is expected in this case

Second generalization: cycle-valued modular form

Generating / counting - funct of cycles

$$F(q) = \sum_{n \geq 0} a_n q^n$$

a_n - cycles in some algebraic varieties

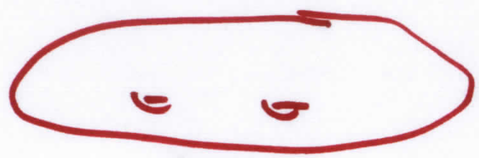
Modularity of $F(q)$:
(i) Much harder
(ii) Much deeper

Question: Can we see cycle-valued modular form in mirror symmetry ?

Answer: Yes !

Gromov-Witten Theory is, by definition, cycle valued

X – Calabi-Yau n-fold

$\{$  $\xrightarrow{\text{holomorphic}} X \} = \overline{\mathcal{M}}_{g,\beta}(X)$

genus = g

moduli space of stable maps

form a space of "virtual dim" $(3-n)(g-1)$

$$[\overline{\mathcal{M}}_{g,\beta}(X)]^{vir} \in H_{(3-n)(g-1)}(\overline{\mathcal{M}}_g, \mathbb{Q})$$

moduli space of Riemann surface

- $n=3$, $\deg([\overline{\mathcal{M}}_{g,\beta}(X)]^{vir}) = 0$
- $n=1$, $\deg(\quad) = 2g-2$

cycle-valued generating/counting funct

- $\mathcal{F}_{A,g} = \sum_{\beta \geq 0} [\overline{\mathcal{M}}_{g,\beta}(X)]^{vir} q^{\beta} \in H_*(\overline{\mathcal{M}}_g, \mathbb{Q})$

- B-model cycle valued $\mathcal{F}_{B,g}$?
  - Yes at $LG_{B,g}$ at dim = 1 (Milanov-Ruan-Shen)
  - OK for $LG_{B,g}$ at dim > 1
  - $\mathcal{F}_{B,g}$?

Cycle valued Mirror symmetry ?

- Calabi-Yau case : Much harder than numerical cases
- Landau-Ginzburg case: Semi-simplicity ⊕ Teleman Result

Theorem : (Milanov-Ruan-Shen)

Cycle-valued Gromov-Witten funct of $E/\mathbb{Z}_3$, $E/\mathbb{Z}_4$, $E/\mathbb{Z}_6$ are cycle-valued quasi-modular forms of $\Gamma(3)$, $\Gamma(4)$, $\Gamma(6)$

- Not known for elliptic curve !

THANK YOU!