Gravitational Lensing and Topology

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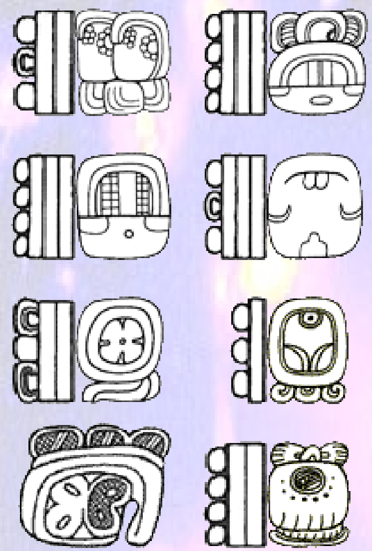
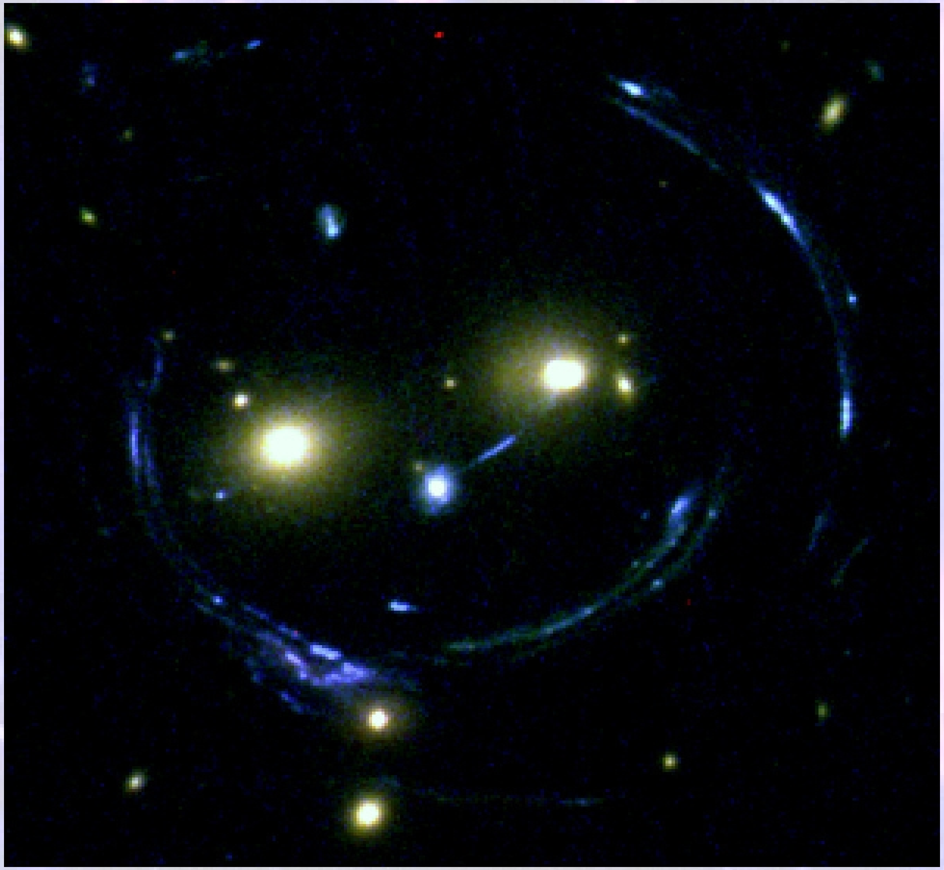
Colloquium
12-12-12

Another curious date coming up...



Today's Maya date: 12.19.19.17.11, 8 Chuwen, G9, 14 Mak
9 days till 13.0.0.0.0, 4 Ajaw, G9, 3 K'ank'in
(using GMT correlation constant, FAMSI)

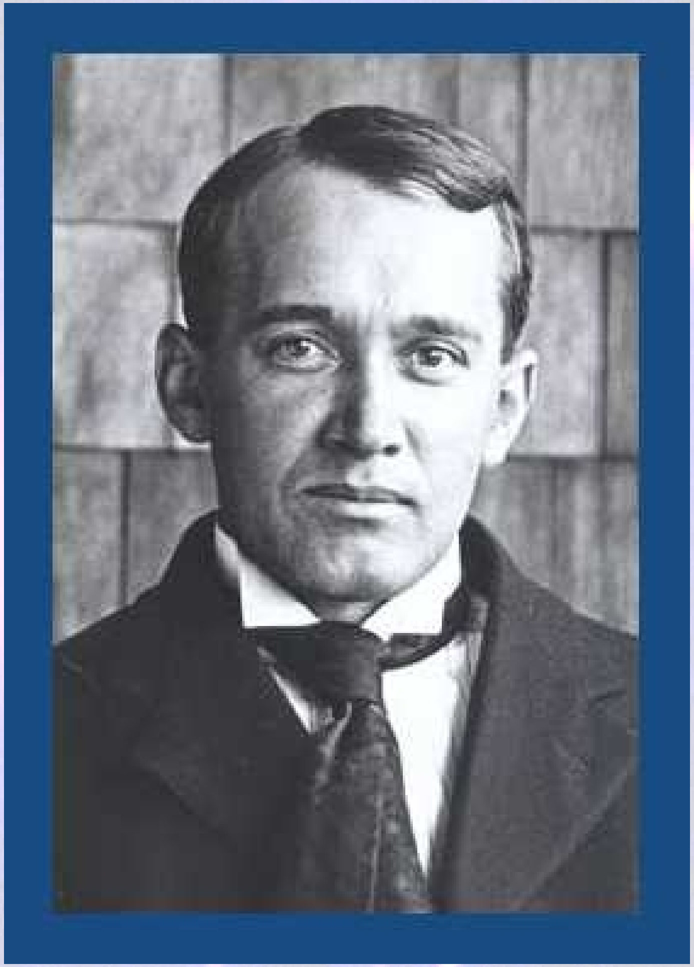
Gravitational lensing: the Cheshire Cat



CSWA 2. Two lensing galaxies: z=0.43; left source galaxy: z=0.97; right source galaxy: z>1.4. [Left image: V. Belokurov et al. 2008. Right image: Tim Burton's “Alice in Wonderland” © Disney 2010]

Pioneers of astrophysics

Vesto Melvin Slipher (1875-1969), discovered redshifts in galactic spectra, at Lowell Observatory in 1912.[Picture: Lowell Obs.]



Hertzsprung-Russell diagram (1910): stellar luminosity versus surface temperature from spectra, to study stellar evolution.

Pre-astrophysics: astronomy and geometry

Classical astronomy: astrometry
> Positions of stars
> Magnitudes of stars

e.g. at University of Cambridge:
Lowndean Chair of Astronomy
and Geometry, endowed 1749

First holder of the Lowndean Chair:
Roger Long (1680-1770),
built the first planetarium
[Picture: Wikipedia]

Gravitational lensing in a broad sense: heir of "classical astronomy"

At least in principle, fundamental to observational astronomy

Applications in three interconnected fields:
> Theoretical physics: testing fundamental theory of gravity
> Astronomy: dark matter distribution, extrasolar planets
> Mathematics: singularity theory, topology

Three approaches to gravitational lensing theory:
> Geometry of null geodesics in Lorentzian manifolds
> Geometry of spatial light rays: optical geometry,
  also called Fermat geometry, optical reference geometry
> Framework used in astronomy: impulse approximation

Outline: “strong” gravitational lensing and topology

Introduction to history, basic theory

Gravitational lensing in astronomy: impulse approximation
> Image counting and topology, odd number theorem
> Bounds on image numbers
> Magnification invariants, Lefschetz fixed point theory

Optical geometry in general relativity:
> Schwarzschild, and singular isothermal sphere
> Image counting and the Gauss-Bonnet theorem
> Kerr-Randers optical geometry

Historical overview

> Einstein's first estimate of gravitational light deflection, June 1911
> First calculations of gravitational lensing (unpublished), April 1912: Nova Geminorum
> Eddington's 1919 solar eclipse expedition corroborating the general relativity value of light deflection
> Einstein's 1936 paper on microlensing, Zwicky's 1937 paper on strong lensing by galaxies
> First extragalactic lens system, a double quasar, discovered by Walsh, Carswell and Weymann in 1979

[Cf. Sauer (2008)]

The "true pioneer": František Link

[Valls-Gabaud (2012): 1206.1165]

> Czech astronomer, 1906-84, discussing the first detailed microlensing calculations, March 1936 (*Comptes Rendus)* and 1937, including: magnification and light curve, finite source size and arclets (!)

> Compare conclusions of
Link: *"It is extremely interesting to look systematically in all the domains of stellar astronomy for favourable instances where such events can take place"*
Einstein, December 1936 (*Science): "Of course, there is no hope of observing this phenomenon directly"*

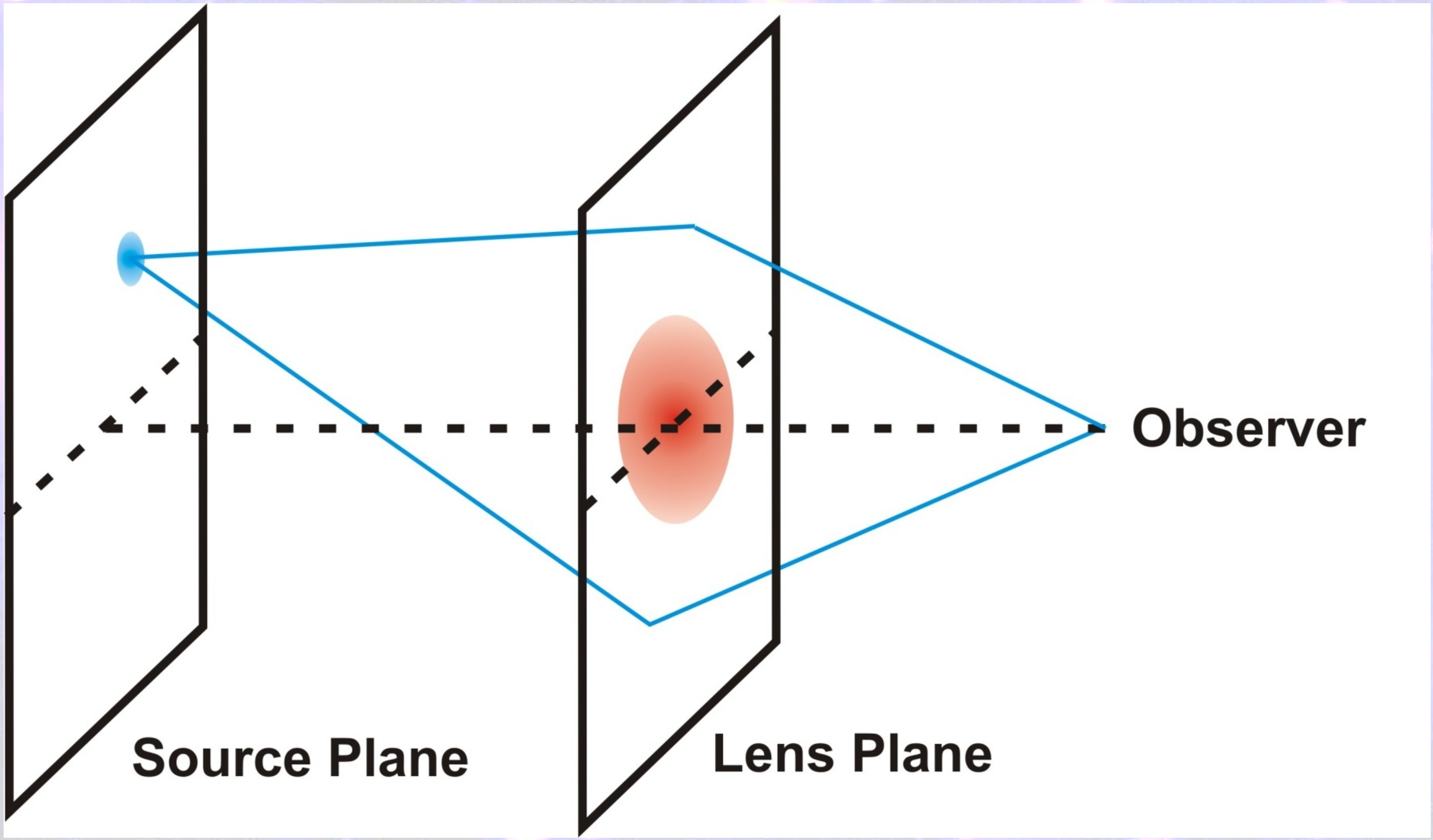
Gravitational lensing theory

Quasi-Newtonian impulse approximation: a useful framework for lensing problems in astronomy, in the limit of

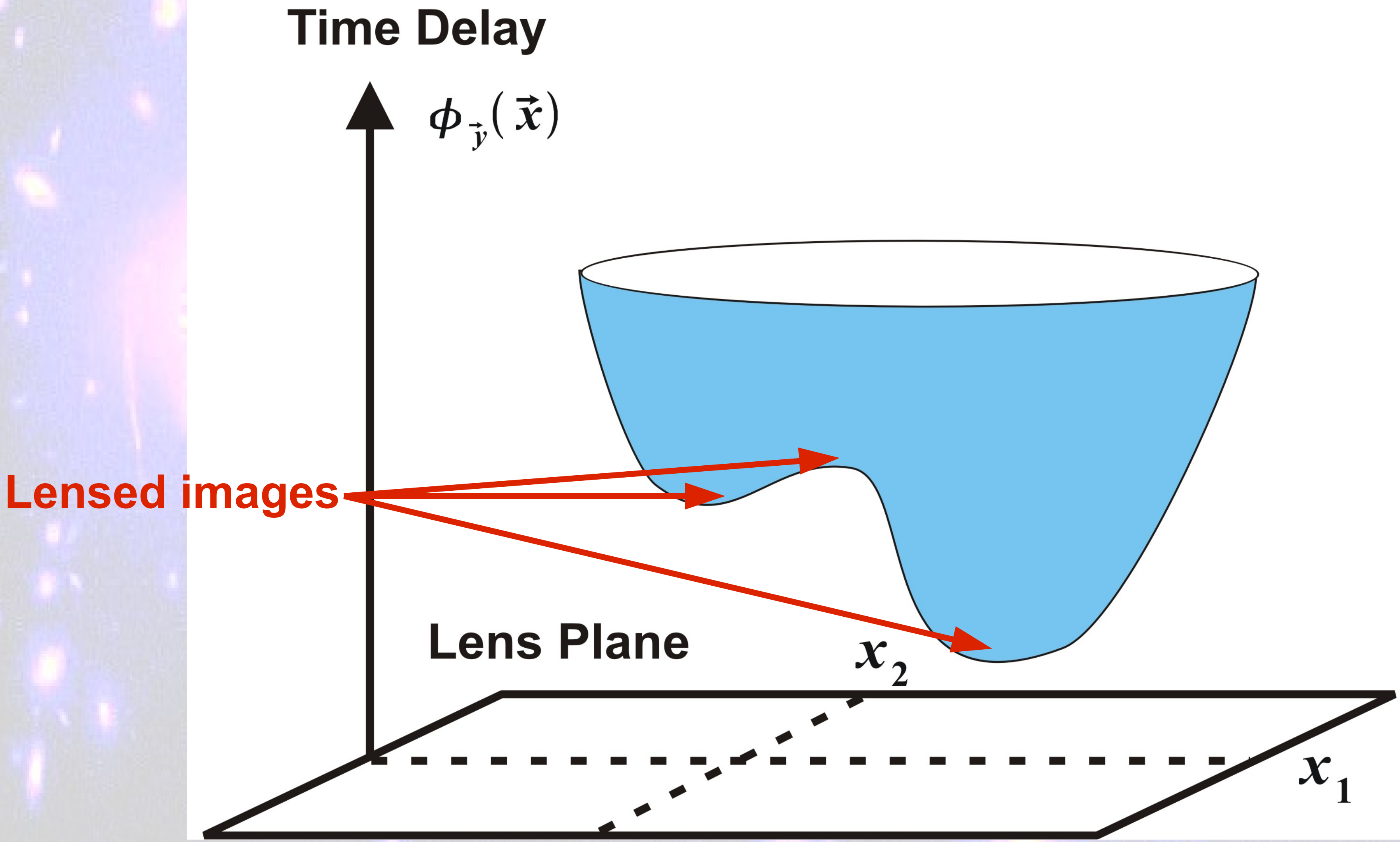
> Geometrical optics
> Small deflection angles
> Euclidean space(s), can be extended to cosmology
> Thin lenses, compared to the length of the light rays

Consider the parallel lens plane $L=\mathbb{R}^2$ and source plane $S=\mathbb{R}^2$ containing deflecting masses and light sources, respectively, at $\vec{x}\in L\,,\vec{y}\in S$.

Gravitational lensing theory



Basic theory: the Fermat surface



The lensing map

Geometrical and gravitational time delay combined yields the Fermat potential $\Phi_{\vec{y}}(\vec{x}): L \times S \rightarrow \mathbb{R}$ given by

$$\Phi_{\vec{y}}(\vec{x}) = \frac{1}{2}|\vec{x}-\vec{y}|^2 - \Psi(\vec{x})$$

Then the Fermat's principle $\nabla \Phi_{\vec{y}}(\vec{x}) = \vec{0}$ implies the lens equation

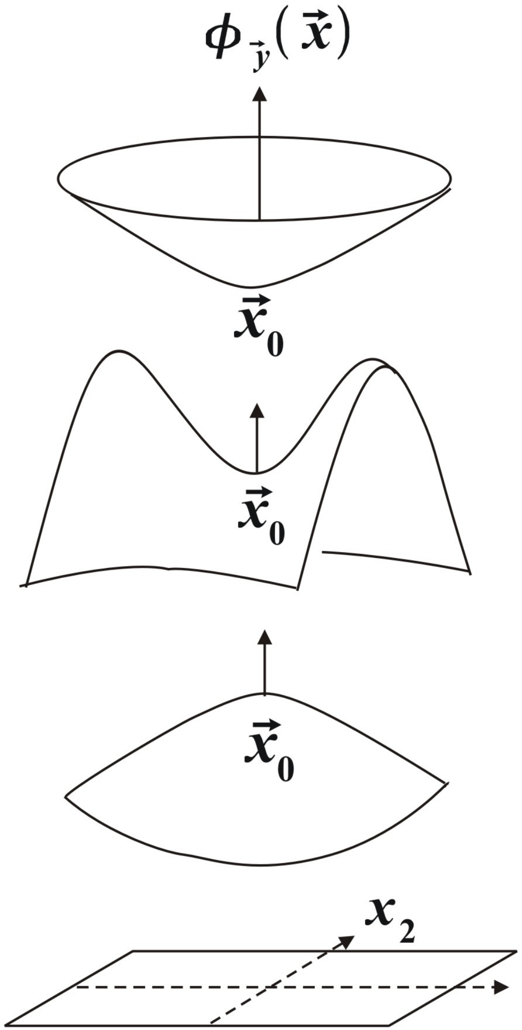
$$\vec{y} = \vec{x} - \nabla \Psi(\vec{x})$$

mapping (physical) images at $\vec{x}$ surjectively to the source.

The lensing map is $\eta: L \rightarrow S, \vec{x} \rightarrow \vec{y}$.

Image properties

Critical Points of the Fermat Surface



Minimum:

$$\kappa_1 > 0, \kappa_2 > 0$$

$$\phi_{\vec{y}}(\vec{x}) = \phi_{\vec{y}}(\vec{x}_0) + (x_1 - x_{0,1})^2 + (x_2 - x_{0,2})^2$$

$$\lambda = 0$$

Saddle:

$$\kappa_1 > 0, \kappa_2 < 0$$

$$\phi_{\vec{y}}(\vec{x}) = \phi_{\vec{y}}(\vec{x}_0) + (x_1 - x_{0,1})^2 - (x_2 - x_{0,2})^2$$

$$\lambda = 1$$

Maximum:

$$\kappa_1 < 0, \kappa_2 < 0$$

$$\phi_{\vec{y}}(\vec{x}) = \phi_{\vec{y}}(\vec{x}_0) - (x_1 - x_{0,1})^2 - (x_2 - x_{0,2})^2$$

$$\lambda = 2$$

Lens Plane

x_1

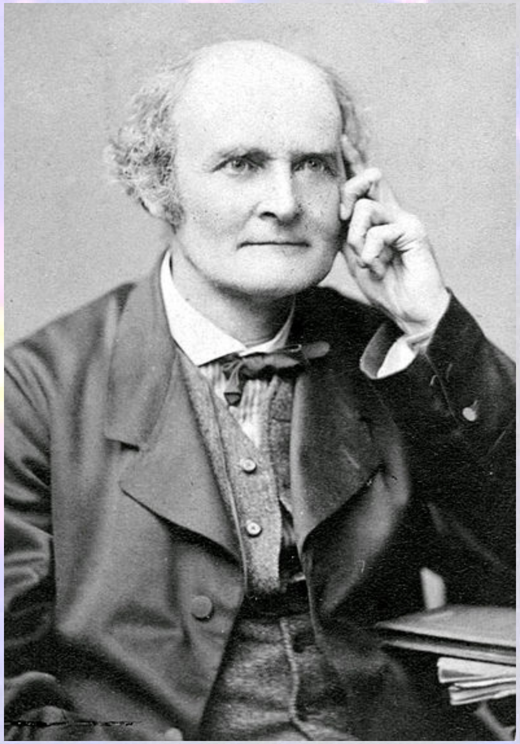
Question: how many lensed images can occur?

Theorem:
For an isolated, non-singular gravitational lens, the number of lensed images of a given light source is *odd*.

This fact is a *topological* property:
It remains true for any continuous deformation of the lens or source position.

Proof:
By Fermat's Principle, images are local maxima/ saddles/ minima of the time delay surface, so need to count critical points of the Fermat surface.

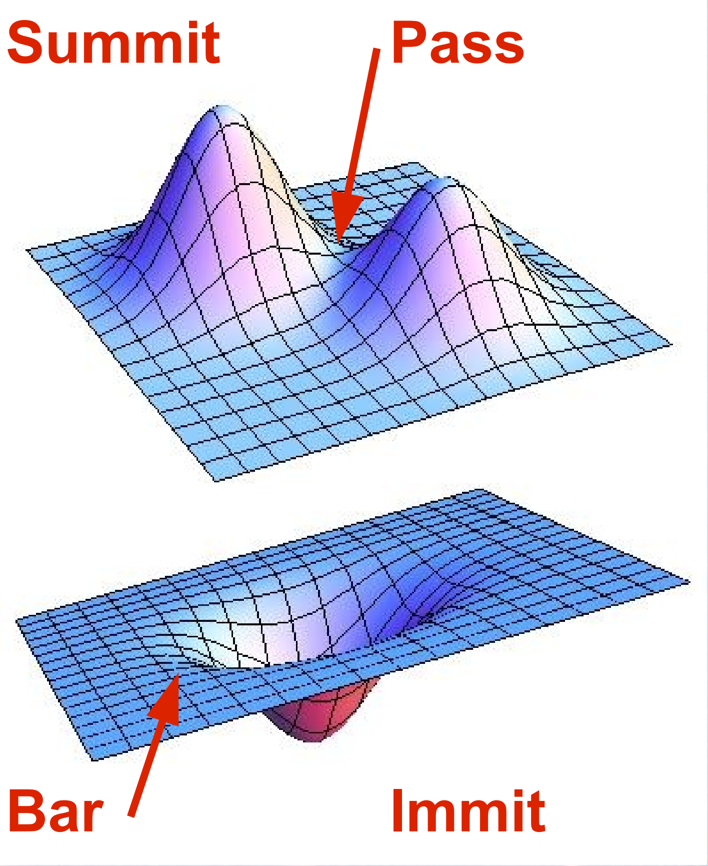
A precursor of Morse Theory: the idea of Cayley (1859) and Maxwell (1870)



Arthur Cayley
(1821-95)

James Clerk Maxwell
(1831-79)

“On Hills and Dales” (1870) and gravitational lensing



Summits (maxima) and passes:

$$n_{summits} = n_{passes} + 1$$

Immits (minima) and bars:

$$n_{immits} = n_{bars}$$

Saddles: $n_{saddles} = n_{passes} + n_{bars}$

Hence, the total number (e.g., of lensed images) is odd:

$$n_{total} = n_{summits} + n_{immits} + n_{saddles} = 2n_{saddles} + 1$$

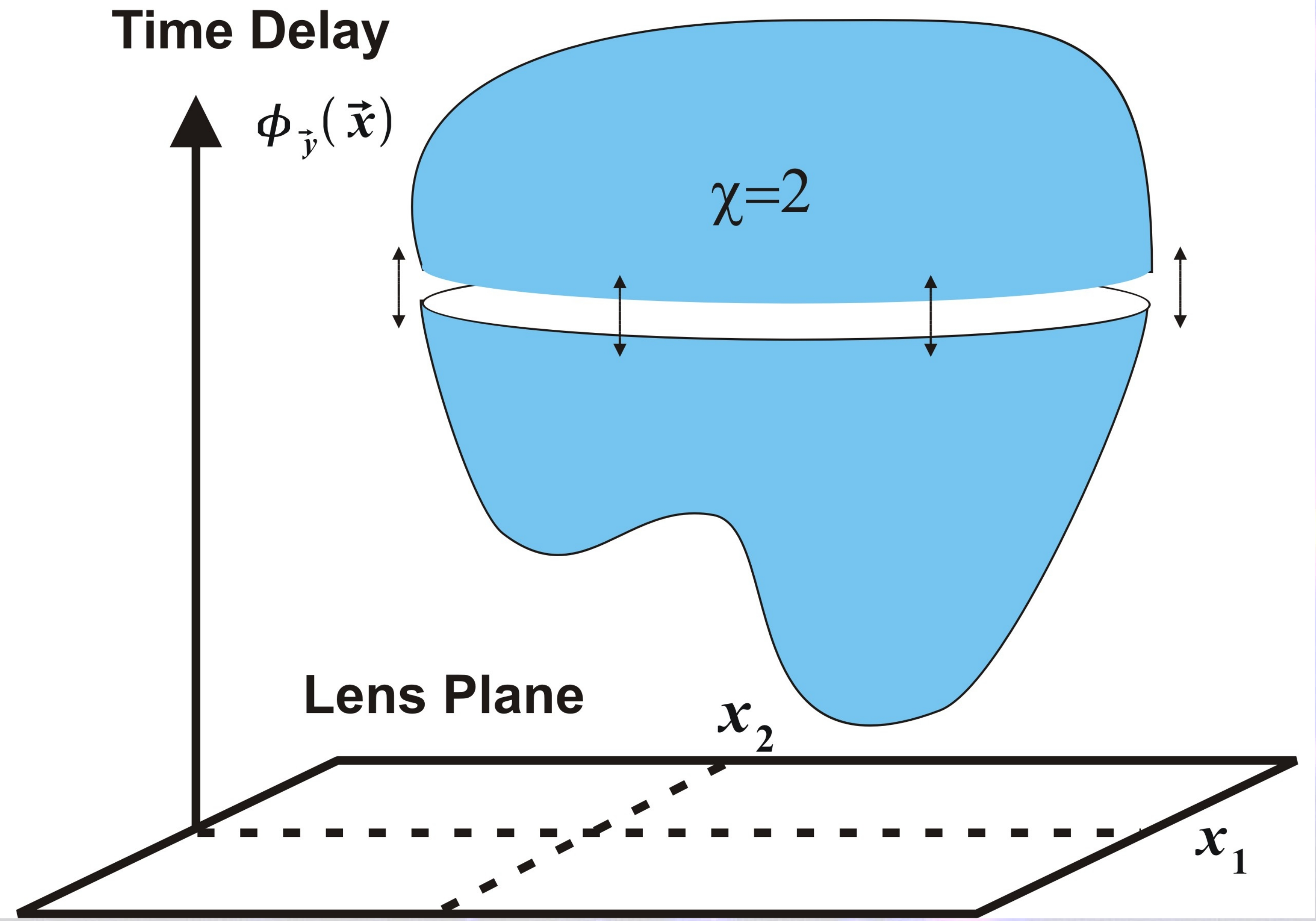
More formally,

Images are non-degenerate critical points of the Morse function $\Phi_{\vec{y}}(\vec{x})$, with Morse index λ .

Theorem [Morse, 1925]. Let M be a smooth closed real manifold with dimension d and Euler characteristic $\chi(M)$, and n_λ non-degenerate critical points with Morse index λ . Then

$$\sum_{\lambda=0}^{d} (-1)^\lambda n_\lambda = \chi(M)$$

Closing the Fermat surface



The odd number theorem...

Hence from Morse theory:

$$n_{min} - n_{sad} + n_{max} + 1 = \chi = 2$$

Total number of images:

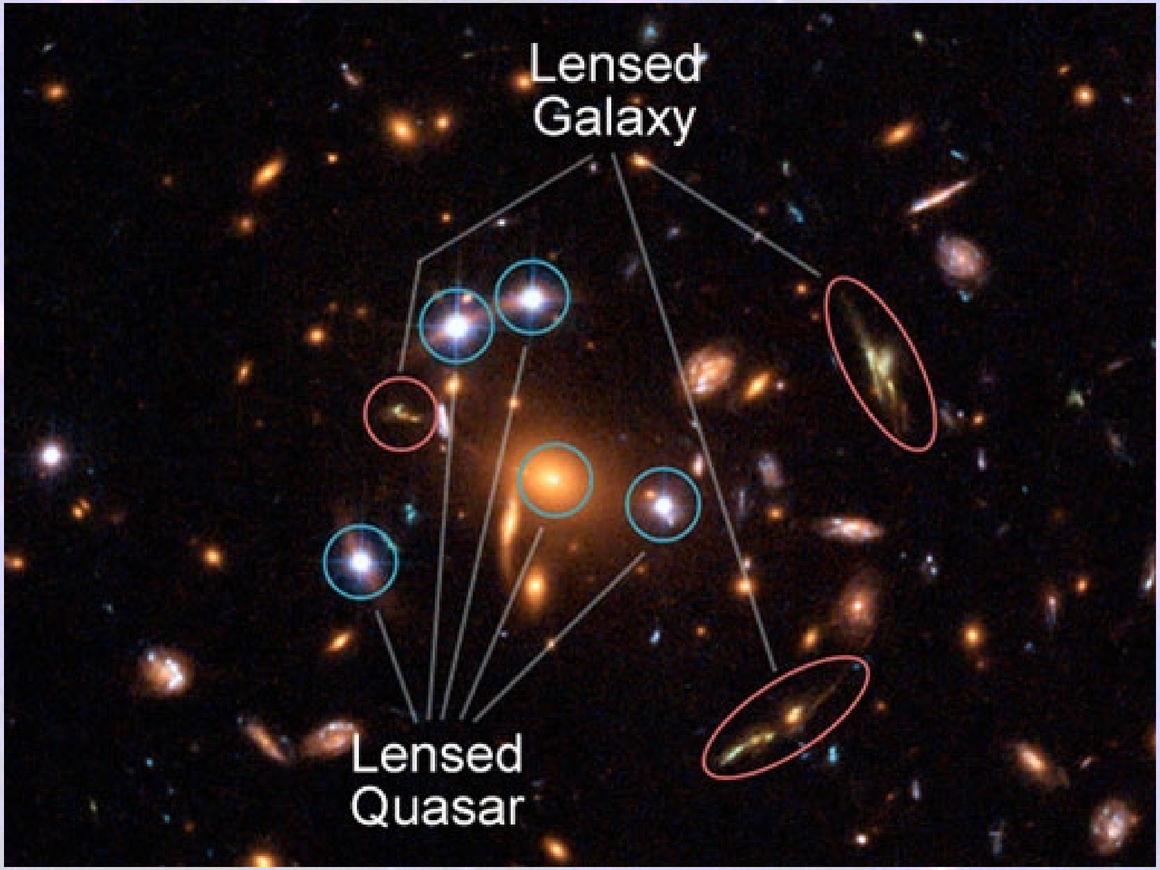
$$n_{min} + n_{sad} + n_{max} = n_{tot}$$

Therefore, the odd number theorem follows:

$$n_{tot} = 2\, n_{sad} + 1$$

[Cf. Burke (1981), Petters (1995). Spacetime version: McKenzie (1985)]

...seems to work!



SDSS J1004+4112. Cluster: z=0.68; quasar: z=1.73; galaxy: z=3.33.

[Image: ESA, NASA, K. Sharon, E. Ofek, also Kavli IPMU's M. Oguri!]

Bounds on image numbers: a simple case

Consider $N>1$ coplanar point lenses with masses proportional to $m_i>0, 1\leq i\leq N$.

Question: what is the maximum number N_{max} of lensed images (of any type) that can be produced (by suitably arranging the lenses the plane)?

Well-known cases: $N=2: N_{max}=5$

$$N=3: N_{max}=10$$

Conjecture (Rhie 2001): $N_{max}=5(N-1)$

Sharpness (Rhie 2003): $N-1$ equal masses in regular polygon plus tiny mass at centre.

Bounds on image numbers: complexification

Complexify lens plane coordinates $z = x_1 + ix_2$, and source plane coordinates $w = y_1 + iy_2$.

Then the lens equation becomes $w = z - \sum_{i=1}^{N} \frac{m_i}{\bar{z} - \bar{z}_i}$.

Theorem [Khavinson and Neumann, 2005]: Let $r(z), deg\, r = N > 1$ be a rational harmonic function, then

$$number(z : r(z) - \bar{z} = 0) \leq 5(N-1)$$

proving Rhie's conjecture, sharpness provided by Rhie's result - a case of astronomy informing mathematics!

Bounds on image numbers

Expository article:

Khavinson and Neumann:
Not. Amer. Math. Soc. **55**
(2008), 666

Research paper:

Khavinson and Neumann:
Proc. Amer. Math. Soc. **134**
(2005), 1077

From the Fundamental Theorem of Algebra to Astrophysics: A "Harmonious" Path

Dmitry Khavinson and Genevra Neumann

The fundamental theorem of algebra (FTA) tells us that every complex polynomial of degree n has precisely n complex roots. The first published proofs (including those of J. d'Alembert in 1746 and C. F. Gauss in 1799) of this conjecture from the seventeenth century had flaws, though Gauss's proof was generally accepted as correct at the time. Gauss later published three correct proofs of the FTA (two in 1816 and the last presented in 1849). It has subsequently been proved in a multitude of ways, using techniques from analysis, topology, and algebra; see [Bur 07], [FR 97], [Re 91], [KP 02], and the references therein for discussions of the history of FTA and various proofs. In the 1990s T. Sheil-Small and A. Wilmshurst proposed to extend FTA to a larger class of polynomials, namely, harmonic polynomials. (A complex polynomial $h(x, y)$ is called harmonic if it satisfies the Laplace equation $\triangle h = 0$, where $\triangle := \partial^2/dx^2 + \partial^2/dy^2$.)

A simple complex-linear change of variables $z = x + iy, \bar{z} = x - iy$ allows us to write any complex valued harmonic polynomial of two variables in the complex form

$$h(z) := p(z) - \overline{q(z)}$$

where p, q are analytic polynomials. While including terms in $\bar{z}$ looks harmless, the combination of these terms with terms in z can have drastic effects.

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Indeed, the harmonic polynomial $h(z) = z^n - \bar{z}^n$ has an infinite number of zeros (the zero set consists of n equally spaced lines through the origin). In 1992 Sheil-Small conjectured that if $n := deg\ p > m := deg\ q$, then h has at most n^2 zeros. In 1994 Wilmshurst found a more general sufficient condition for h to have a finite number of zeros and settled this conjecture using Bézout's theorem from algebraic geometry. While Wilmshurst's bound on the number of zeros is sharp, he also conjectured a smaller bound when the degrees of p and q differ by more than one.

In 2001 the first author and G. Świątek [KS 03] proved that the bound in Wilmshurst's conjecture held for the case of $f(z) = p(z) - \bar{z}$. Because this proof involves complex dynamics, it is natural to wonder whether this approach can be extended to find a bound on the number of zeros of the rational harmonic function $f(z) = p(z)/q(z) - \bar{z}$. The authors explored this question in 2003 [KN 06]. After posting a preprint, we learned that this bound settles a conjecture of S. H. Rhie concerning gravitational lensing. More precisely, it gives the maximal possible number of images of a light source (such as a star or a galaxy) that may occur due to the deflection of light rays by some massive obstacles. Even more surprising, Rhie had already shown that this bound is attained.

In this expository article we give a brief introduction to gravitational lensing. We also describe necessary background concerning harmonic polynomials and Wilmshurst's conjecture, as well as related results. We then discuss the ideas behind the proofs and also look at the question of sharpness. We close with a discussion of several possible directions for further study.

Properties of image magnification

Due to Liouville's theorem, the intensity obeys $I_\nu / \nu^3 = const.$

Achromaticity of gravity: $\nu = const.$ (cosmology ignored here)

Hence flux is proportional to solid angle, and the signed image magnification is

$$\mu = \frac{1}{det\, Jac(\eta)} \quad \text{where} \quad Jac(\eta) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix}$$

The sign defines image parity.

Properties of image magnification

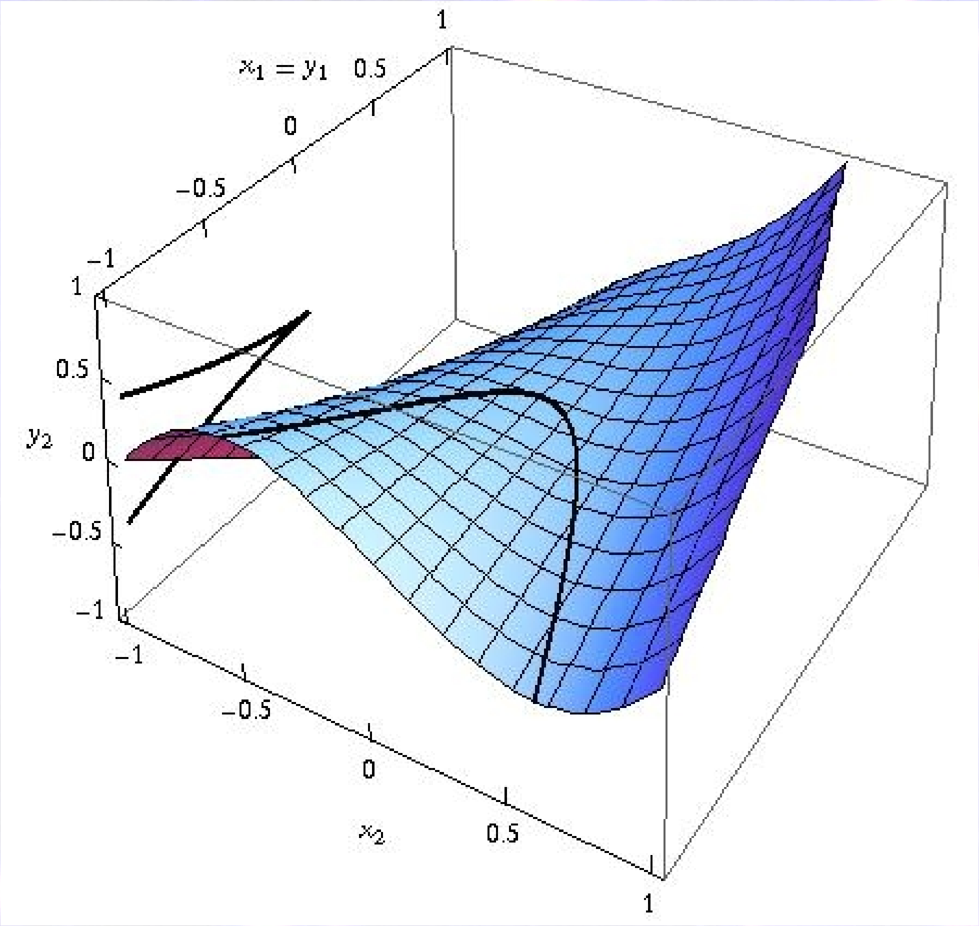
The critical set $Crit(\eta)$ of the lensing map is defined by $det\,Jac(\eta)=0$ in L , corresponding to infinite magnification μ .

The critical set is mapped to caustics in S by the lensing map, $Caustic(\eta)=\eta(Crit(\eta))$.

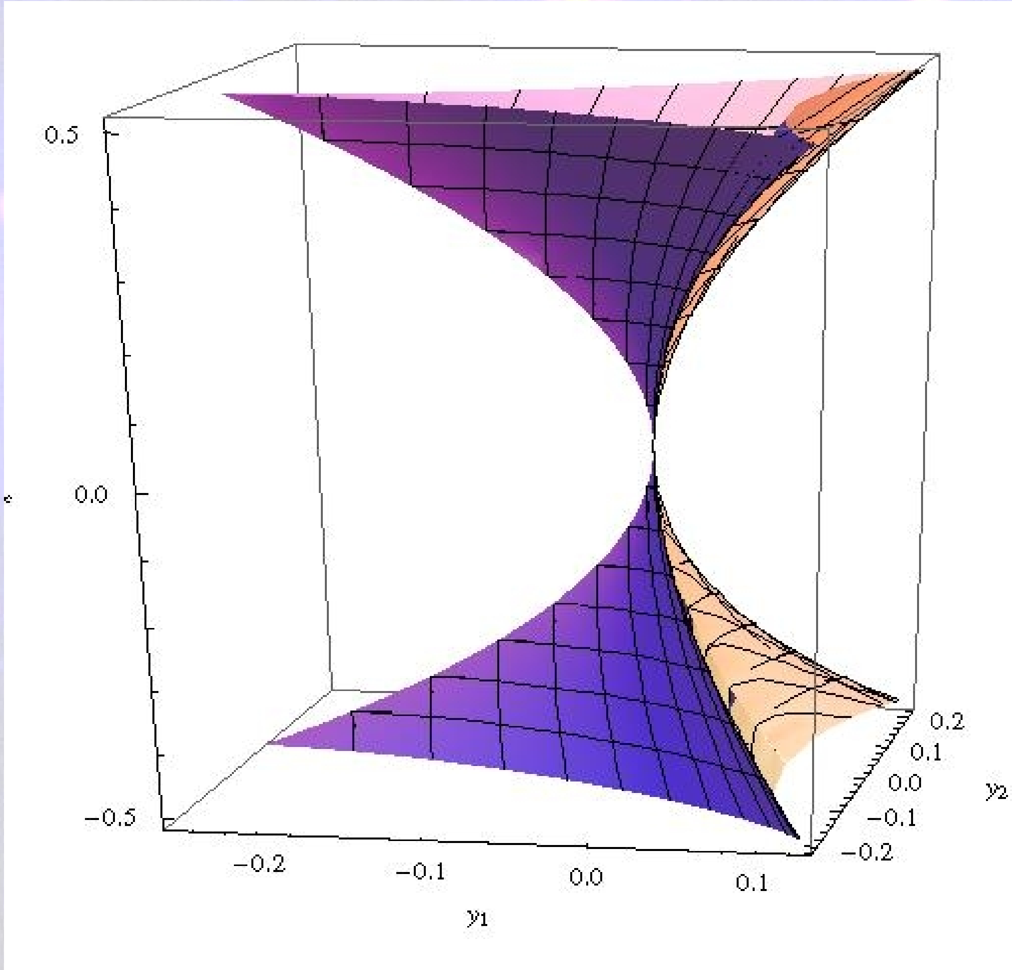
Caustics delimit domains of constant image number in S .

According to singularity theory, only certain types of caustics occur generically.

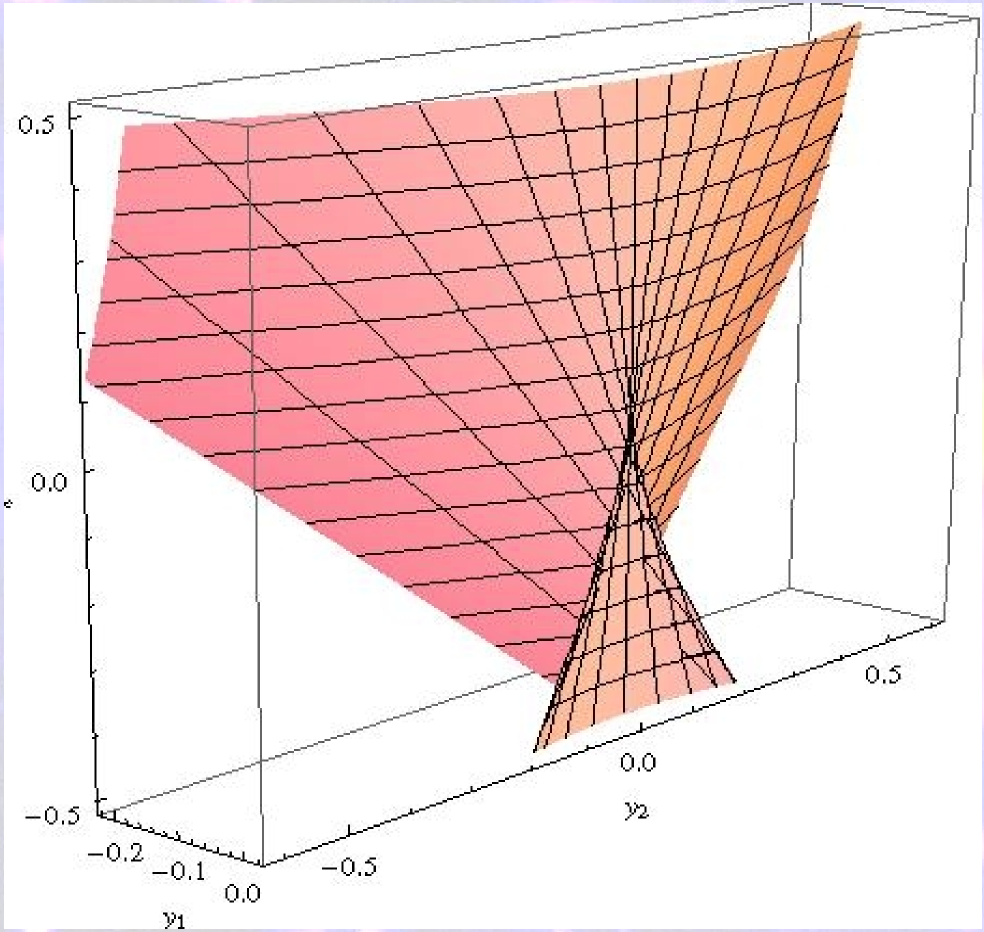
Singularities: the cusp caustic



Singularities: big caustic of the elliptic umbilic



Singularities: big caustic of the swallowtail



The flux ratio anomaly: an astronomical problem...

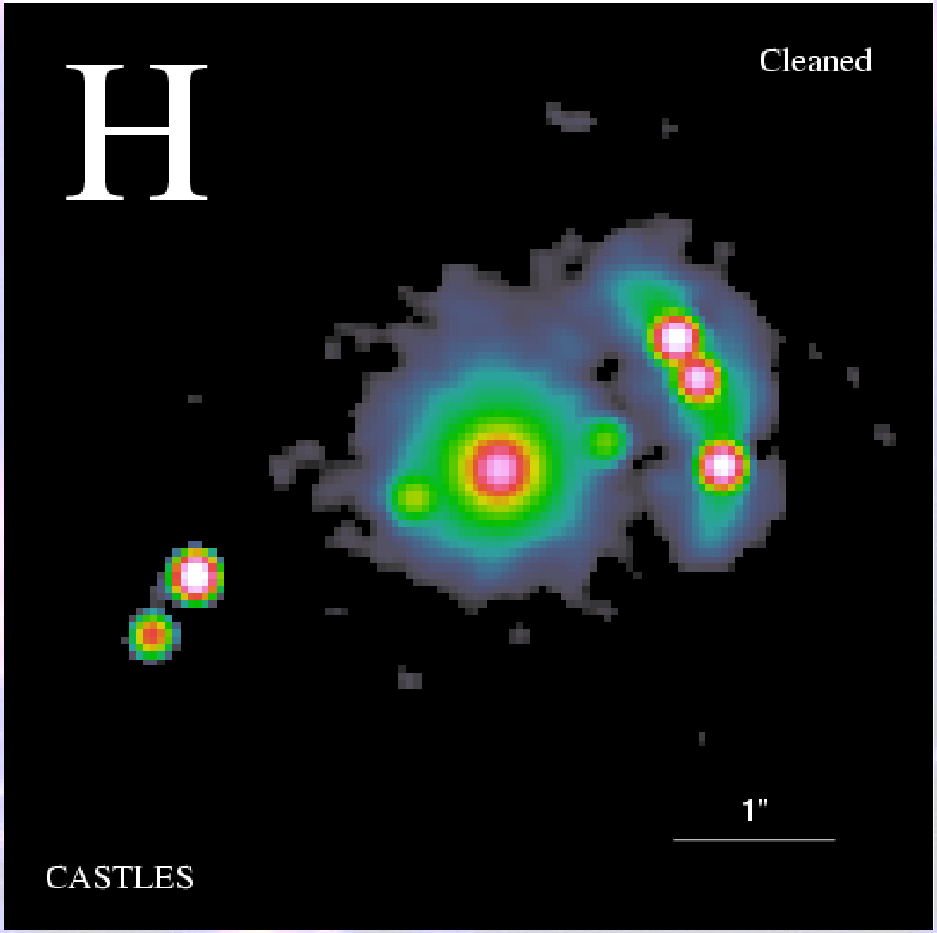
Gravitational lens system CLASS B2045+265:

Quasar at z=1.28 lensed by galaxy at z=0.87, in H band.

$$\frac{\mu_A+\mu_B+\mu_C}{|\mu_A|+|\mu_B|+|\mu_C|}=0.51$$

Indicative of substructure?

[Cf. Fassnacht et al. (1999), Koopmans et al. (2003), Keeton et al. (2003), McKean et al. (2007). Image: CASTLES lensing database]



...inspires a mathematical question: What are magnification invariants?

A constant sum of signed image magnifications, for

> Source near caustic, in maximal caustic domain

> Subset of n images of the caustic multiplet

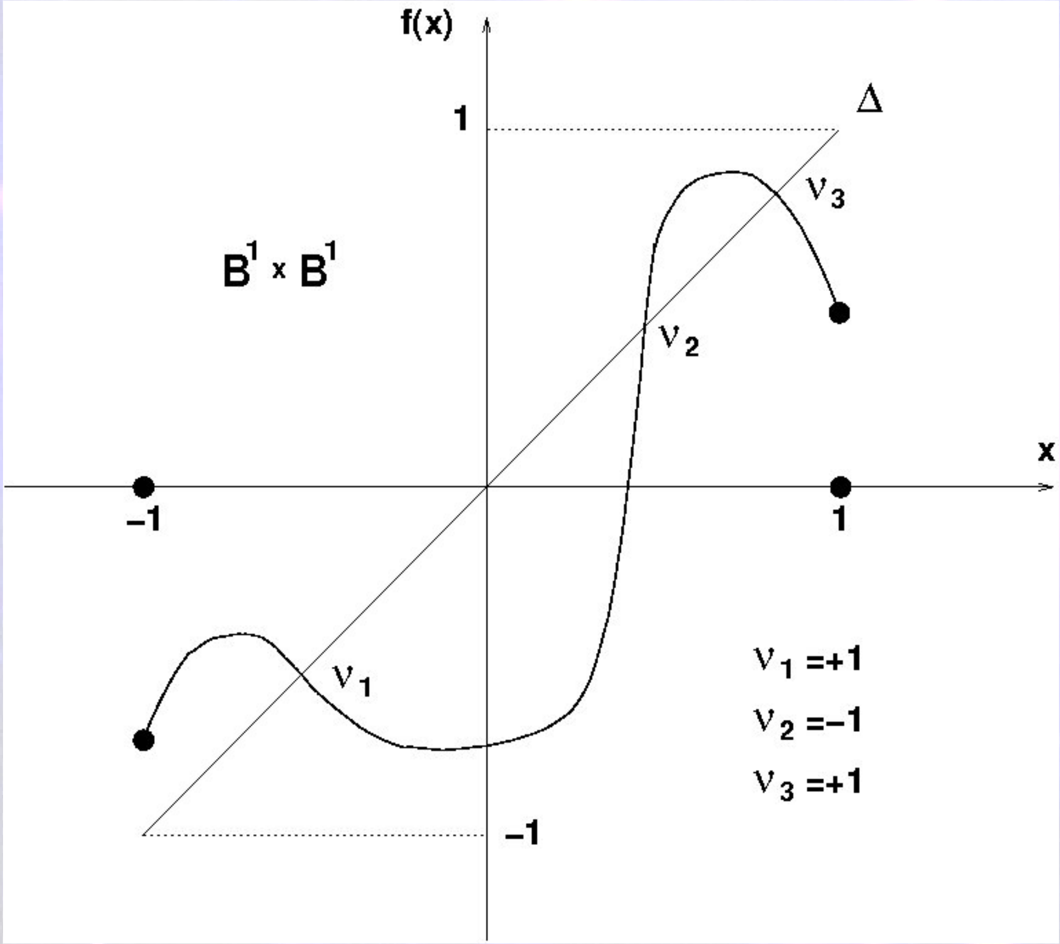
> Independent of lens model: genericity of caustics

Well-known for folds and cusps, recently extended to higher singularities by Aazami and Petters:

$$\sum_{i=1}^{n} \mu_i = 0$$

[Blandford & Narayan (1986), Schneider & Weiss (1992), Aazami & Petters (2009, 2010), Aazami, Petters & Rabin (2011). Application of Lefschetz: Werner (2009)]

Is there a topological interpretation? Lefschetz fixed point formalism



Lefschetz and lensing

Given $f: M \to M$, then Lefschetz fixed point theory connects local fixed point indices ν with a global property of f on M, the Lefschetz number (a homotopy invariant):

$$L = \sum_{Fix(f)} \nu$$

Recast the lens equation as a holomorphic map $f: \mathbb{C}^2 \to \mathbb{C}^2$ such that $Fix(f)$ in the maximal caustic domain are the real (physical) images. Then it turns out that the fixed point index

$$\nu = \frac{1}{det(I - D(f))} = \mu$$

Homogenize to extend f to a map $F: \mathbb{C}P^2 \to \mathbb{C}P^2$ such that

$$Fix(F_{\mathbb{C}^2}) = Fix(f)$$

Explaining (some) magnification invariants

Applying the holomorphic Lefschetz fixed point formula,

$$1 = L_{hol}(F) = \sum_{Fix(F)} \frac{1}{det(I - D(F))}$$

$$= \sum_{Fix(F_{\mathbb{C}^2})} \frac{1}{det(I - D(F))} + \sum_{Fix(F_{\mathbb{C}P^1})} \frac{1}{det(I - D(F))}$$

$$= \sum_{Fix(f)} \frac{1}{det(I - D(f))} + 1$$

$$= \sum_{i=1}^{n} \mu_i + 1$$

yields the result [Werner 2009, currently working on extension].

So what have we learnt with this?

> A successful topological explanation of a subset of the currently known magnification invariants, and the first application of Lefschetz fixed point theory in astronomy

> Connecting magnification invariants with topology will help understanding an astronomically important question: under which perturbations of the lens model do the invariants continue to hold (i.e., are applicable in a real situation)

> Can this approach be extended to all magnification invariants? Connecting geometrical quantities at critical points with topology: an interesting problem in algebraic geometry and topology

Solomon Lefschetz

Born 1884 in Moscow, died 1972 in Princeton, NJ.

Engineering in Paris, then emigration to USA at 21.

Turned to mathematics after accident. PhD at Clark, MA, in 1911.

A founding father of algebraic topology, in *Topology* (AMS, 1930).

[Picture: from the St. Andrews, UK, History of Mathematics Site]

Optical geometry

Also called Fermat geometry and optical reference geometry:
Metric manifold whose geodesics are the spatial projections of spacetime null geodesics, by Fermat's principle

Useful for the study of

> Inertial forces in general relativity [e.g., Abramowicz, Carter & Lasota (1988)]

> Gravitational lensing: deflection angle, multiple images and topology, using Gauss-Bonnet [c.f. Gibbons 1993, Gibbons & Werner (2008), Werner (2012)]

Static spacetime:	Riemannian optical geometry
Stationary spacetime:	Finslerian optical geometry

Optical geometry of static spacetimes

Consider a static spacetime (M, g) with chart $x^{\mu}=(t, x^{a})$ and line element

$$ds^{2}=g_{\mu\nu}\,dx^{\mu}\,dx^{\nu}$$

The coordinate time along spatial projections of null curves obeys

$$dt^{2}=g^{opt}_{ab}\,dx^{a}\,dx^{b}$$

with optical metric $g^{opt}_{ab}=g_{ab}/(-g_{tt})$

whose geodesics are spatial light rays, by Fermat's principle.

Optical geometry of Schwarzschild

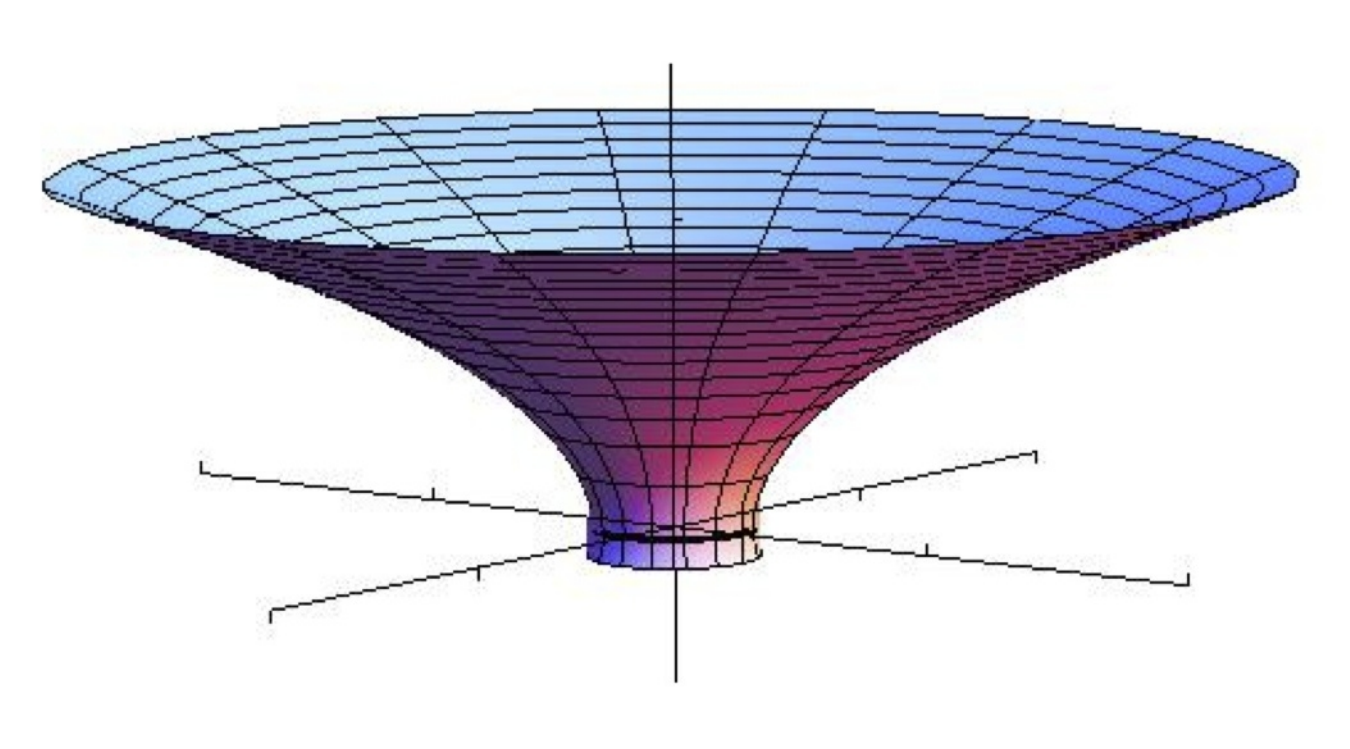
Given the line element of the Schwarzschild solution in Schwarzschild coordinates,

$$ds^2 = -\left(1-\frac{2\mu}{r}\right)dt^2 + \left(1-\frac{2\mu}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

the metric of the optical geometry can be read off from

$$dt^2 = \left(1-\frac{2\mu}{r}\right)^{-2} dr^2 + \left(1-\frac{2\mu}{r}\right)^{-1} r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Optical geometry of Schwarzschild



Isometric embedding of the equatorial plane $\theta = \pi/2$ in $\mathbb{R}^3$, thick line indicates the photon sphere at $r = 3\mu$.

Lensing in this optical geometry

Geodesics on this surface correspond to spatial light rays. However, the Gaussian curvature at every point

$$K < 0$$

so geodesics must locally diverge.

Then how can two light rays from a light source refocus at the observer, so that the two images of the Schwarzschild lens are obtained?

Gravitational lensing and Gauss-Bonnet

Consider a region A of a totally geodesic surface (e.g., the equatorial plane of Schwarzschild) in the optical geometry, with Euler characteristic $\chi(A)$, Gaussian curvature K, the boundary curve ∂A with geodesic curvature κ and exterior jump angles ϵ_i at vertices.

Then the Gauss-Bonnet theorem says

$$\iint_A K\,dA + \int_{\partial A} \kappa\,dt + \sum_i \epsilon_i = 2\pi\chi(A)$$

Gravitational lensing and Gauss-Bonnet

Consider a region A_L in A bounded by two geodesics γ_1, γ_2 from light source S to the observer O, enclosing the lens L surrounded by the closed photon orbit γ_L as inner boundary. The geodesics intersect at angles $\delta_S > 0$ at S and $\delta_O > 0$ at O. The exterior jump angles at the vertices S and O are $\epsilon_S = \pi - \delta_S$ and $\epsilon_O = \pi - \delta_O$ respectively.

Also, consider a region A_R in A bounded by γ_1 and a circle segment γ_R between O and S centered on L, such that $r(S) = r(O) = R$ and $A_L \cap A_R = \gamma_1$.

Gravitational lensing and Gauss-Bonnet

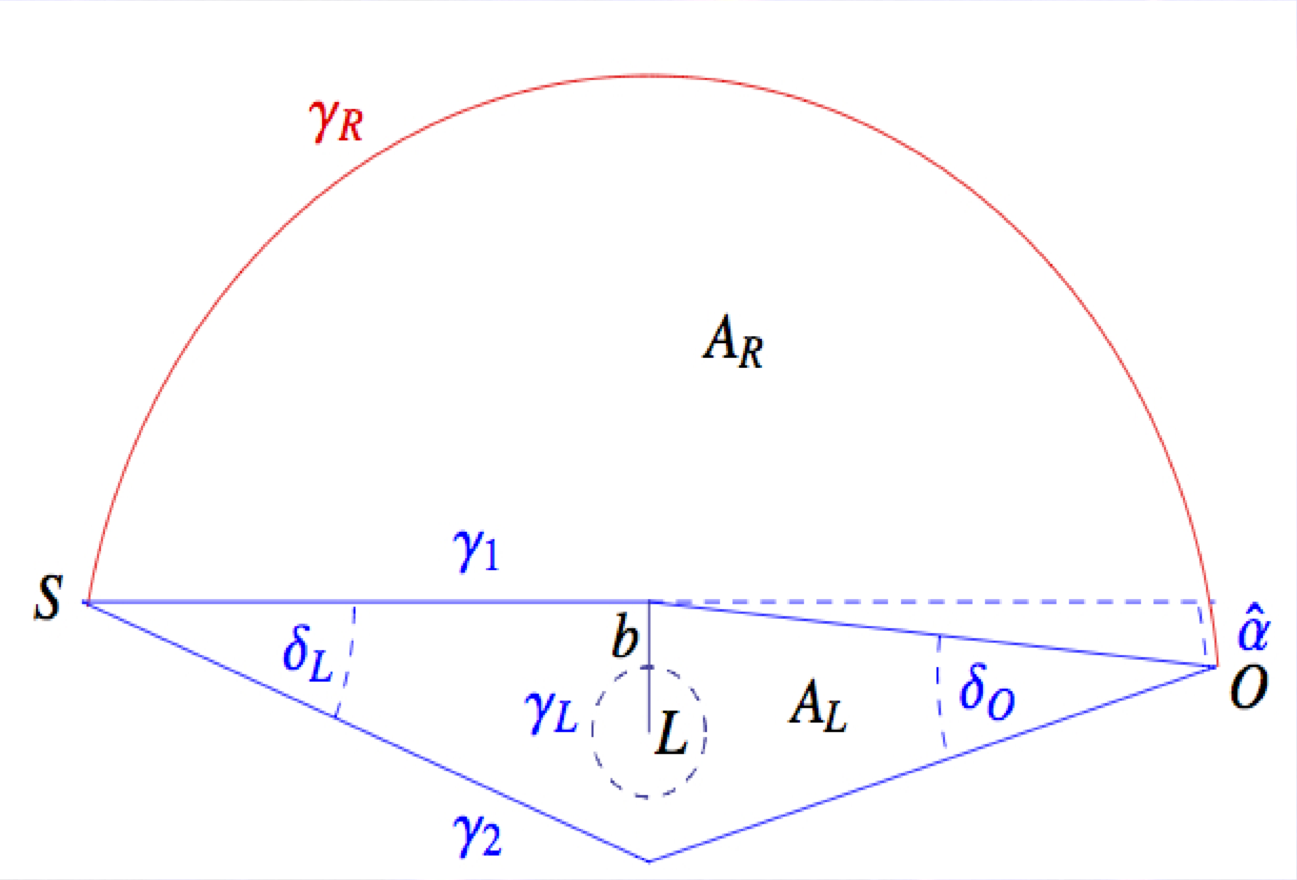


Image multiplicity and Gauss-Bonnet

Suppose, for the moment, that the lens $L \in A_L$ is non-singular so that A_L is topologically simply connected with $\chi(A_L) = 1$.

Since $\kappa(\gamma_1) = \kappa(\gamma_2) = 0$, Gauss-Bonnet implies that

$$\delta_S + \delta_O = \iint_{A_L} K \, dA$$

However, if $K < 0$ then $\delta_S + \delta_O > 0$ is impossible.
Hence, the occurence of multiple images requires either
> non-trivial topology of the surface, or
> a region with positive Gaussian curvature.

Image multiplicity and Gauss-Bonnet

In fact, the Schwarzschild lens is singular, with $\chi(A_L)=0$.
Since $\kappa(\gamma_1)=\kappa(\gamma_2)=\kappa(\gamma_L)=0$,

$$\delta_S+\delta_O=2\pi+\iint_{A_L} K\,dA>0$$

is possible even if $K<0$.

Hence, the non-trivial topology of A_L is essential for two lensed images to occur.

Deflection angle and Gauss-Bonnet

Now consider region A_R with $\chi(A_R)=1$ and $\kappa(\gamma_1)=0$ on its geodesic boundary.

As the radius of the circular perimeter $R\to\infty$, the exterior jump angles at source and observer become $\epsilon_S=\epsilon_O\to\pi/2$. Hence,

$$\int_0^{\pi+\alpha}\kappa(\gamma_R)\frac{dt}{d\phi}d\phi-\pi=-\iint_{A_R}K\,dA$$

where α is the asymptotic deflection angle. Since $\kappa(\gamma_R)\to 1/R$ we obtain

$$\alpha=-\iint_{A_\infty}K\,dA$$

where A_∞ is the infinite region bounded by γ_1 and excluding the lens.

Application to Schwarzschild

Computing the Gaussian curvature of the equatorial plane in the optical geometry,

$$K\,dA = -\frac{2\mu}{r^2}\left(1-\frac{2\mu}{r}\right)^{-3/2}\left(1-\frac{3\mu}{2r}\right) dr\, d\phi$$

To evaluate the leading term of the asymptotic deflection angle, take the line $r(\phi)=b/\sin\phi$ as first approximation of γ_1 bounding A_∞. Hence,

$$\alpha = -\iint_{A_\infty} K\,dA \approx \int_0^{\pi}\int_{\frac{b}{\sin\phi}}^{\infty} \frac{2\mu}{r^2}\,dr\,d\phi = \frac{4\mu}{b}$$

as required. Higher order terms can be computed iteratively.

Application to the singular isothermal sphere

The singular isothermal sphere with mass density

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

is a simple (non-relativistic) model for a galaxy with velocity dispersion σ. Computing the optical metric, the Gaussian curvature of the equatorial plane is $K(r>0)=0$.

An isometric embedding in $\mathbb{R}^3$ with cylindrical coordinates (R, z, ϕ), i.e. setting

$$dt^2 = dz(r)^2 + dR(r)^2 + R(r)^2 d\phi^2$$

yields a cone

$$z(R) = \frac{\sqrt{8\sigma^2 - 36\sigma^4}}{1-6\sigma^2} R$$

Application to the singular isothermal sphere

Using the Gauss-Bonnet method as before, the leading term of the asymptotic deflection angle is

$$\alpha \approx 4\pi\sigma^2$$

which is constant, and half the cone's deficit angle $\varepsilon \approx 8\pi\sigma^2$.

Notice the similarity with gravitational lensing by cosmic strings, with mass per unit length μ_S . The deflection angle is likewise constant,

$$\alpha \approx 4\pi G\mu_S$$

and also half the deficit angle of the conic spacetime, $\varepsilon \approx 8\pi G\mu_S$

Optical geometry of Kerr

In Boyer-Lindquist coordinates $(t, x^i), x^i = (r, \theta, \phi)$, the Kerr solution is

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)\,d\phi - a\,dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

defining as usual $\Delta = r^2 - 2\mu r + a^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$. Solving for the optical geometry, one finds

$$dt = F(x, dx) = \sqrt{a_{ij}(x)\,dx^i dx^j} + b_i(x)\,dx^i$$

where a_{ij} is a Riemannian metric and b_i a one-form.

Hence, the optical geometry here is not Riemannian.

Finsler geometry

A Finsler manifold (M, F), writing $x \in M$ and $X \in T_x M$, has a real, non-negative, smooth function $F: TM \to \mathbb{R}_0^+$ which is positively homogeneous of degree one in X and convex such that the Hessian

$$g_{ij}(x, X) = \frac{1}{2} \frac{\partial^2 F^2(x, X)}{\partial X^i \partial X^j}$$

is positive definite. Hence, by homogeneity,

$$F^2(x, X) = g_{ij}(x, X) X^i X^j$$

The deviation from Riemann can be characterized with the Cartan tensor

$$C_{ijk}(x, X) = \frac{1}{2} \frac{\partial g_{ij}(x, X)}{\partial X^k}$$

Kerr-Randers optical geometry

The Kerr optical geometry is defined by a Finsler metric of Randers type,

$$F(x, dx) = \sqrt{a_{ij}(x)\, dx^i dx^j} + b_i(x)\, dx^i$$

provided $a^{ij}(x) b_i(x) b_j(x) < 1$ for non-negativity and convexity, translating to the condition

$$\frac{(2\mu a r \sin\theta)^2}{\Delta \rho^4} < 1$$

which holds precisely outside the ergoregion.

The equatorial plane $\theta = \pi/2$ is geodesically complete, with optical metric

$$F\left(r, \phi, \frac{dr}{dt}, \frac{d\phi}{dt}\right) = \sqrt{\frac{r^4}{\Delta(\Delta - a^2)}\left(\frac{dr}{dt}\right)^2 + \frac{r^4 \Delta}{(\Delta - a^2)^2}\left(\frac{d\phi}{dt}\right)^2} - \frac{2\mu a r}{\Delta - a^2}\frac{d\phi}{dt}$$

Geodesics in Finsler geometry

The Hessian can be used to define vector duals,

$$v_i = g_{ij}(x, X) X^j$$

an inverse such that

$$g_{ij}(x, X) g^{jk}(x, v) = \delta_i^k$$

and hence formal Christoffel symbols

$$\Gamma^i_{jk}(x, X) = \frac{1}{2} g^{il}(x, v) \left(\frac{\partial g_{lj}(x, X)}{\partial x^k} + \frac{\partial g_{lk}(x, X)}{\partial x^j} - \frac{\partial g_{jk}(x, X)}{\partial x^l} \right)$$

which reduce to Levi-Civita connection components if the Cartan tensor vanishes.

Then arc-length parametrized geodesics γ_F in (M, F), so that $dt = F(x, dx)$, can be written

$$\ddot{x}^i + \Gamma^i_{jk}(x, \dot{x}) \dot{x}^j \dot{x}^k = 0$$

Gauss-Bonnet method for Kerr-Randers: applying Nazım's construction

In (M, F), choose a non-zero smooth vector field $\bar{X}(x)$ such that $\bar{X}(\gamma_F)=\dot{x}$. Then the Hessian can be converted to a Riemannian metric thus,

$$\bar{g}_{ij}(x)=g_{ij}(x, \bar{X}(x))$$

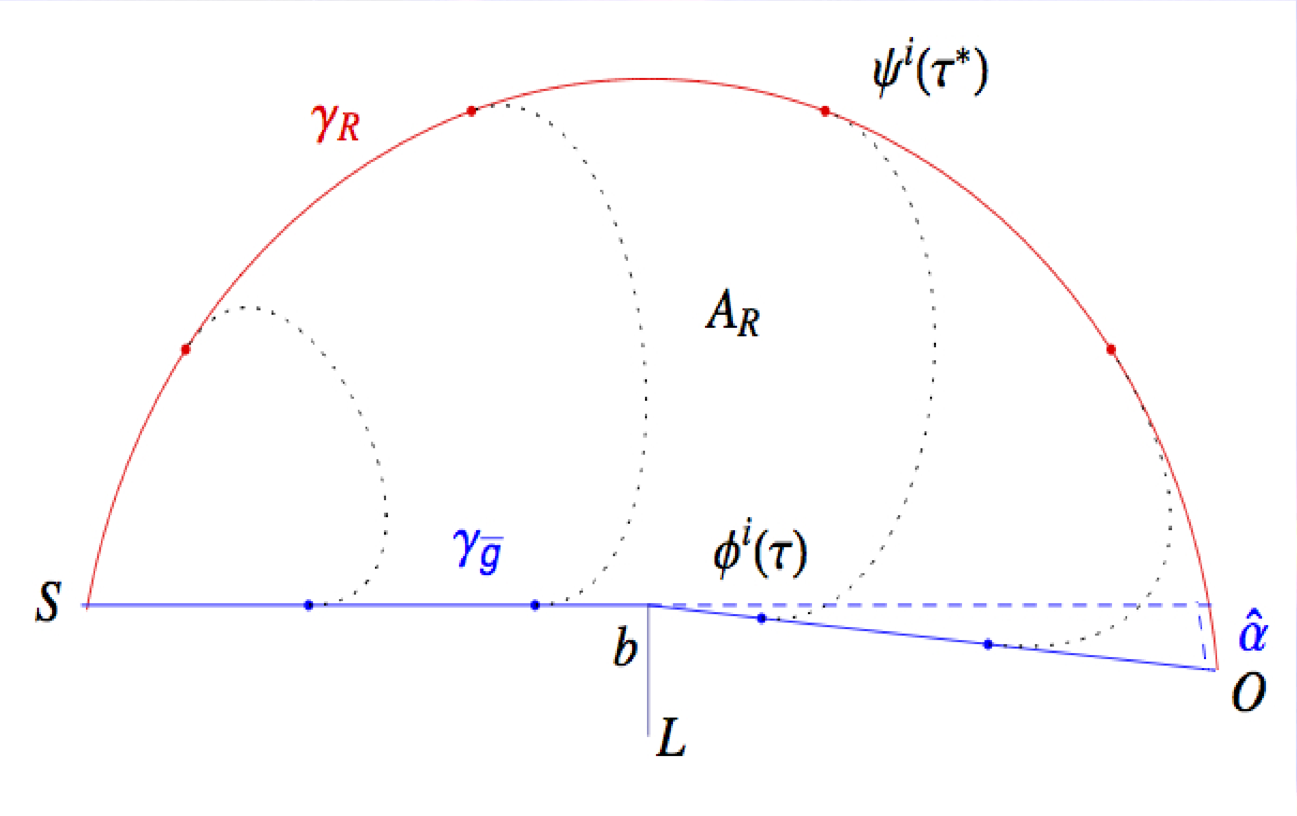
with compatible Levi-Civita connection $\bar{\Gamma}^i_{jk}$. While geometric quantities depend on the choice of vector field, γ_F is also a geodesic of $(M, \bar{g})$ since

$$0=\ddot{x}^i+\Gamma^i_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k=\ddot{x}^i+\bar{\Gamma}^i_{jk}(x)\dot{x}^j\dot{x}^k$$

so angles defined along it do not depend on that choice.

Hence the deflection angle of γ_F can be computed in $(M, \bar{g})$, the osculating Riemannian manifold, using the Gauss-Bonnet method as discussed above. [Cf. Nazım (1936), Werner (2012)]

Lensing and Gauss-Bonnet in Kerr-Randers



Concluding remarks

Topology plays an important role in gravitational lensing, e.g.

> constraining image number: Morse theory and the odd number theorem, optical geometry and Gauss-Bonnet

> explaining certain magnification invariants in terms of Lefschetz fixed point theory

Some open problems:

> Can the Lefschetz fixed point formalism be applied to all magnification invariants? Are there spacetime versions of magnification invariants

> Find an intrinsically Finslerian description of lensing in the Kerr-Randers optical geometry, e.g. with a Finsler version of Gauss-Bonnet

Opening remarks

> Math-Astro Seminars, a new joint series. In 2011/12:
5 introductory lectures on lensing theory and geometry in the fall semester, 4 specialized lectures on research topics in lensing theory in the spring semester.
To be continued and broadened in 2013

> New postdoc Amir Aazami (Mathematics, Duke University) arriving in January 2013: mathematical theory of lensing

> A symposium on “Gravity and Light in Non-Lorentzian Geometries” planned for 30 September- 3 October 2013. Invited speakers include Gary Gibbons (Cambridge)