Wall-Crossing & Quiver Invariants

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with

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prototype : D=4 N=2 SU(2) → U(1) Seiberg-Witten

$$W = (0, 2)$$

$$\bar{D}_n = (-1, 2n)$$

$$M \equiv D_0 = (1, 0)$$

$$\bar{D} \equiv \bar{D}_1 = (-1, 2)$$

$$\Omega(\bar{W}) = \Omega(W) = -2$$

$$\Omega(\bar{D}_n) = \Omega(D_n) = 1$$

with the 2nd helicity trace, for BPS states in D=4 N=2 SUSY

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2$$

$$\to (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

or, more generally, the protected spin character

$$SO(4) = SU(2)_{\mathrm{rotation}} \times SU(2)_{\mathrm{R-symmetry}}$$

$$J \qquad\qquad\qquad I$$

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 \quad \underset{y=1}{\Leftarrow} \quad \Omega(y) = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 y^{2I_3+2J_3}$$

2$^{\text{nd}}$ helicity trace

protected spin character

Gaiotto, Moore, Neitzke 2010 / Maldacena 2010

$$\to (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

wall-crossing of BPS states with 4 (or less) supersymmetries

marginal stability wall

γ_1 $\quad$ $\gamma_1 + \gamma_2$

γ_2 $\quad$ $\gamma_2 + \gamma_3$

γ_3 $\quad$ $\gamma_1 + \gamma_3$

$\vdots$ $\quad$ $\vdots$

$\gamma_1 + \gamma_2 + \gamma_3$

$\vdots$

$\sum k_i \gamma_i$

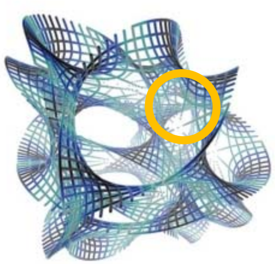
$\vdots$

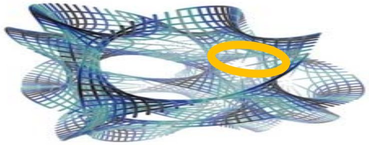
γ_1

γ_2

γ_3

$\vdots$





wall-crossing examples

calibrated geometry : special Lagrange submanifolds in CY3

type II string theories on CY3 : wrapped D-branes

D=4 N=2 supergravity : BPS black holes

D=4 N=2 Seiberg-Witten theory : BPS dyons

⋮

D=4 N=4 …… ¼ BPS ……

⋮

D=2 N=2 Landau-Ginzburg : BPS kinks

Kontsevich-Soibelman, 2008

Schwinger product

$$[V_\alpha, V_\beta] = (-1)^{\langle \alpha,\beta \rangle} \langle \alpha, \beta \rangle V_{\alpha+\beta} \qquad K_\gamma \equiv \exp\left(\sum_n \frac{V_{n\gamma}}{n^2} \right)$$

a marginal stability wall

$+$ side $$\prod_\gamma K_\gamma^{\Omega^+(\gamma)} = \prod_{\gamma'} K_{\gamma'}^{\Omega^-(\gamma')}$$ $-$ side

$$[V_\alpha, V_\beta] = (-1)^{\langle \alpha,\beta\rangle}\langle \alpha,\beta\rangle V_{\alpha+\beta}$$

$$\cancel{K_\gamma \equiv \exp\left(\sum_n \frac{V_{n\gamma}}{n^2}\right)}$$

a marginal stability wall

$$\bar\Omega(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

+ side

$$\prod_\gamma e^{\bar\Omega^+(\gamma)V_\gamma} = \prod_{\gamma'} e^{\bar\Omega^-(\gamma')V_{\gamma'}}$$

- side

true, in general ?

how to see from BPS state building/counting ?

& why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

input data ?

$$\Omega^+(\gamma) = \Omega^-(\gamma)$$

true, in general ?

how to see from BPS state building/counting ?

& why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

or, more generally, what about the cases

$$\Omega^+(\gamma) \neq 0 \neq \Omega^-(\gamma)$$

real space wall-crossing

BPS quivers

quiver invariants

1998 Lee + P.Y.

N=4 SU(n) ¼ BPS states via multi-center classical dyon solitons

1999 Bak + Lee + Lee + P.Y.

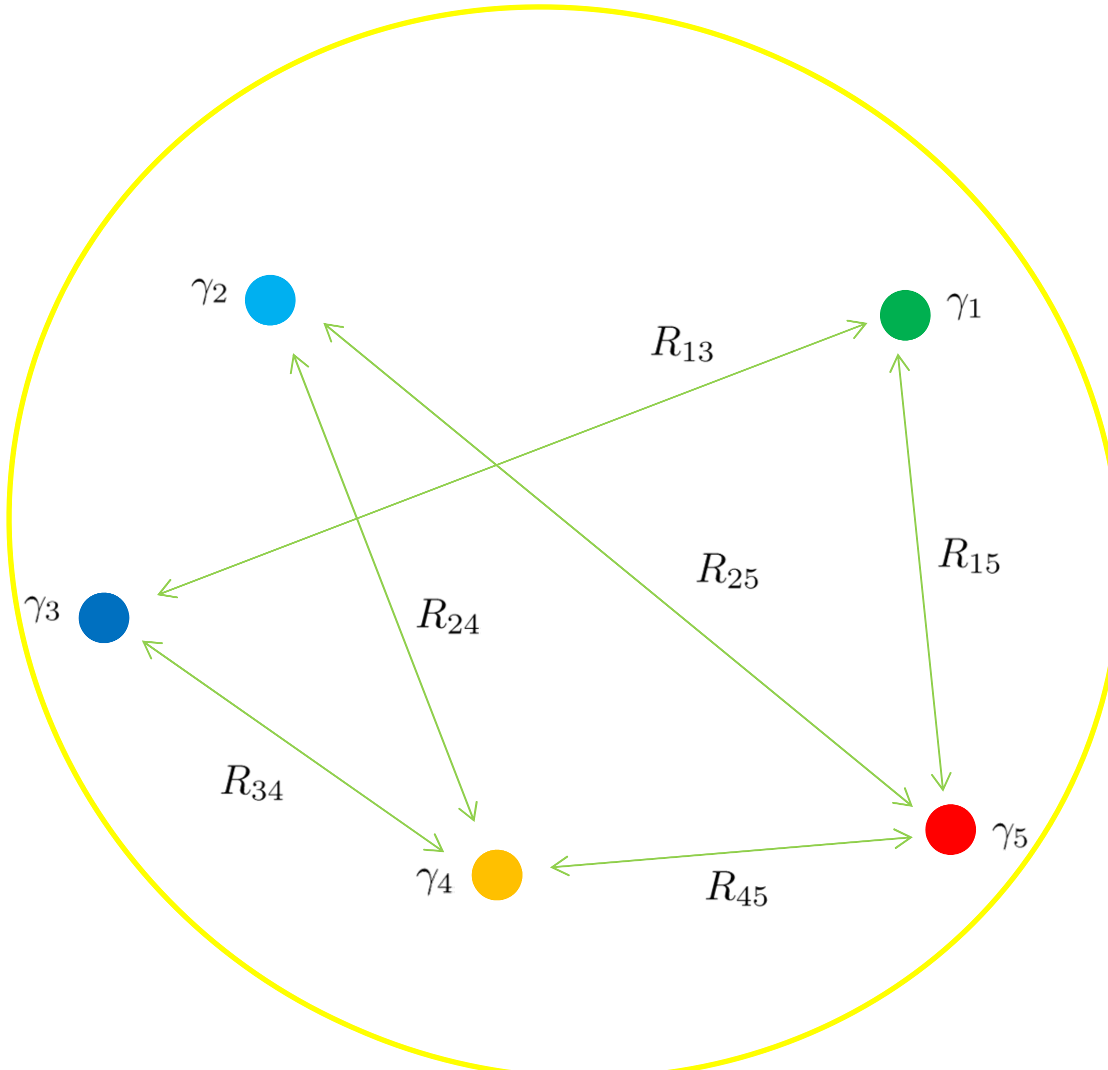
N=4 SU(n) ¼ BPS states via multi-center monopole dynamics

1999-2000 Gauntlett + Kim + Park + P.Y. / Gauntlett + Kim + Lee + P.Y. / Stern + P.Y.

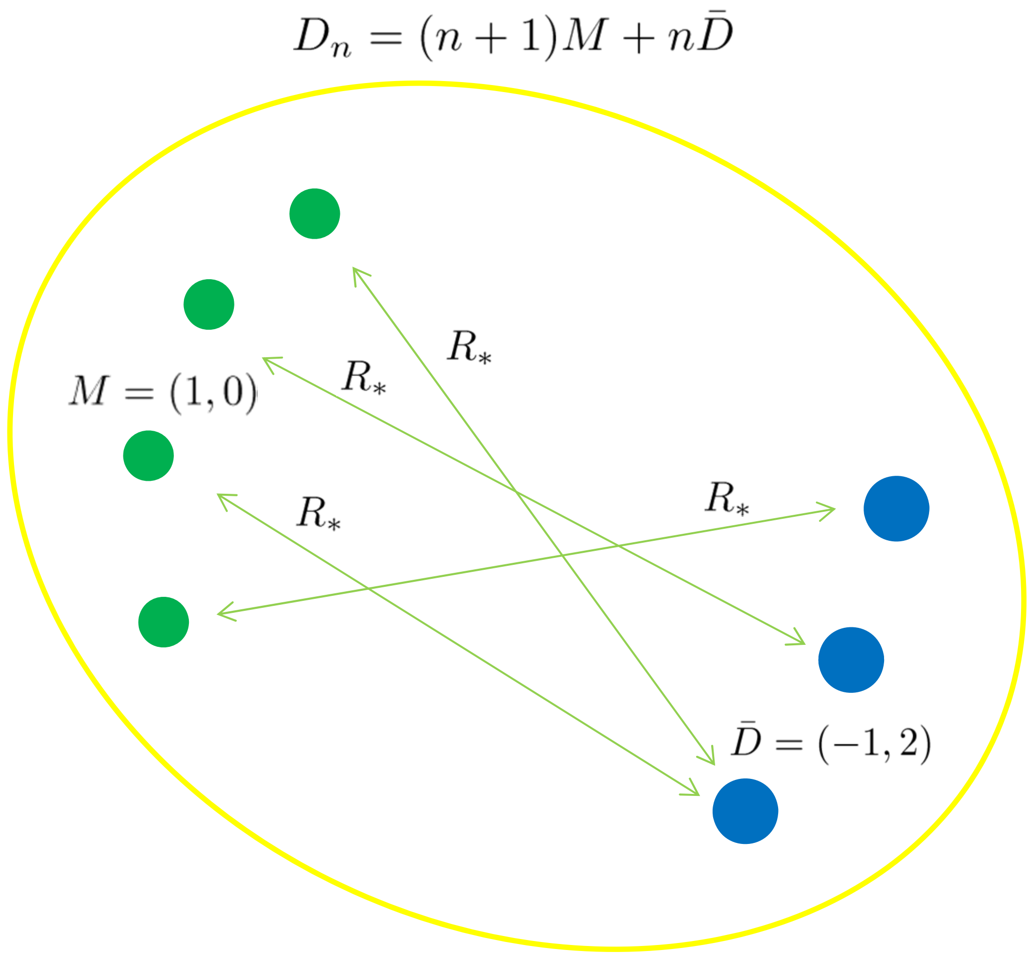
N=2 SU(n) BPS states via multi-center monopole dynamics

generic BPS "particles" are loose bound states of charge centers

$R^3 = \{\vec{X}\}$

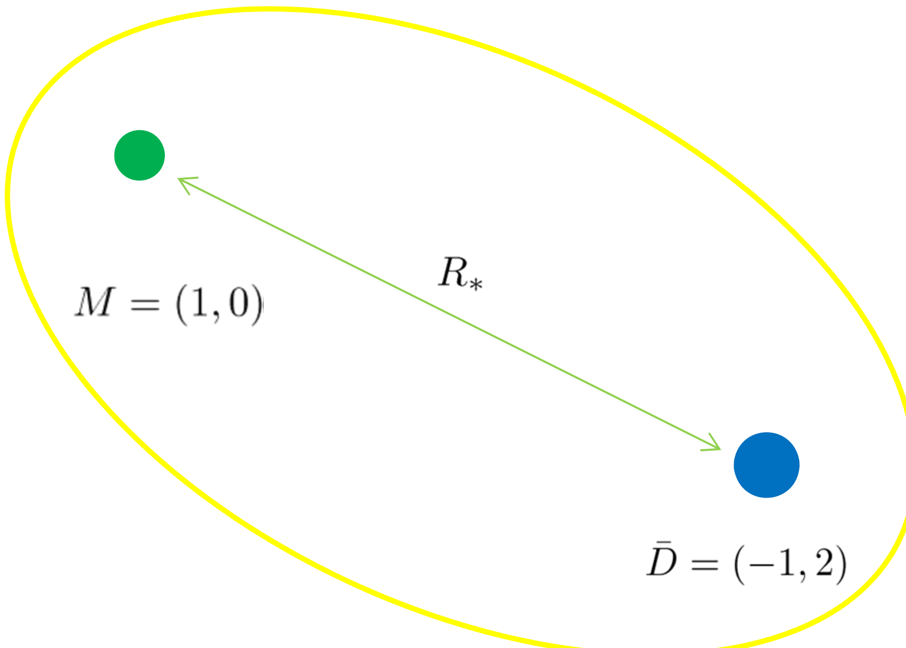


in particular, for SU(2) Seiberg-Witten

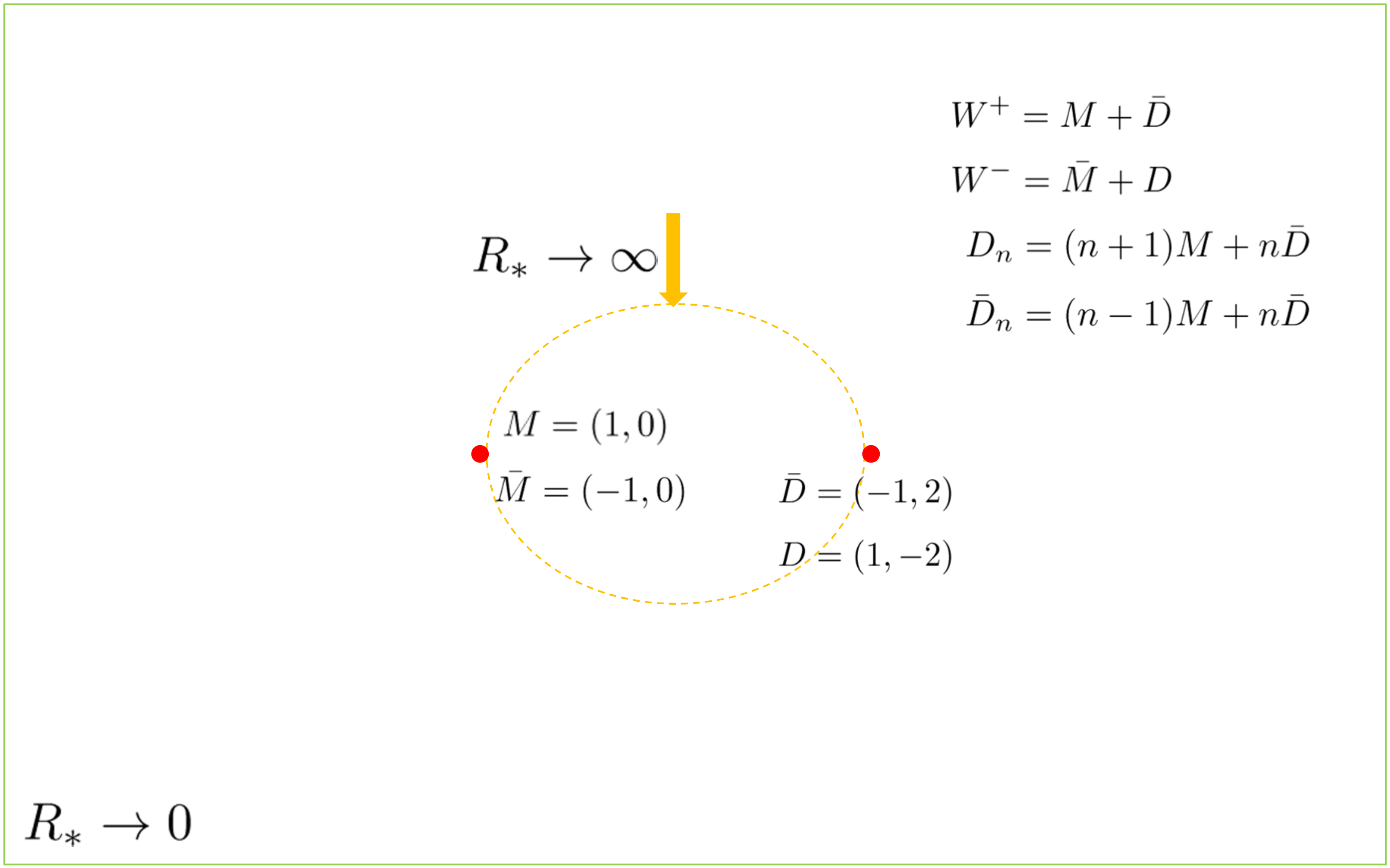


in particular, for SU(2) Seiberg-Witten

$$W^+ = M + \bar{D}$$



wall-crossing ← dissociation of supersymmetric bound states



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N=2 SU(n) BPS states via multi-center monopole dynamics

2001 Denef

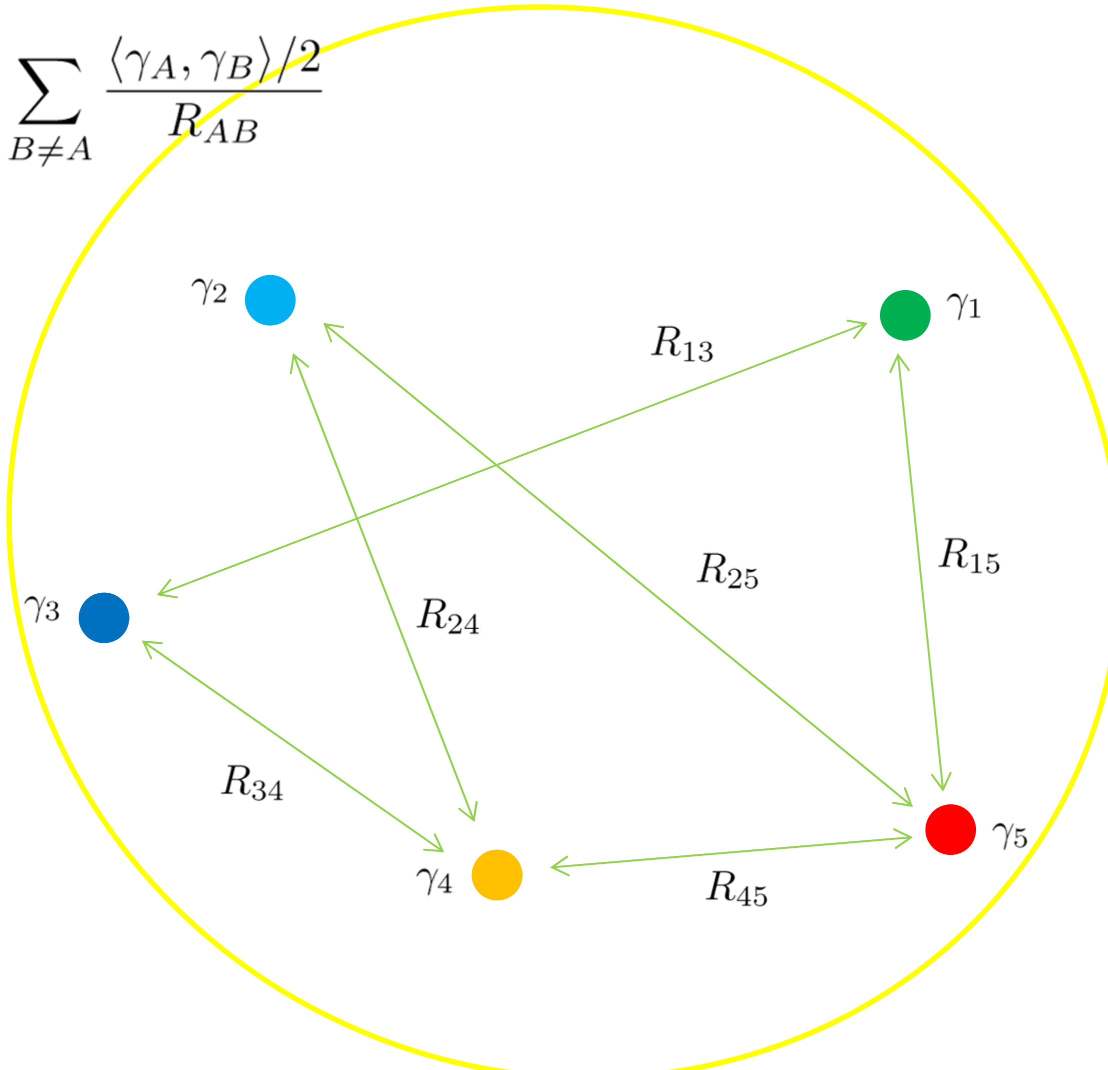
N=2 supergravity via classical multi-center black holes attractor solutions

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}} \qquad \zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

generic BPS "particles" are loose bound states of charge centers

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}}$$

$$R^3 = \{\vec{X}\}$$



wall-crossing problem
= how to count & classify such many-body bound states

index theorem & low energy dynamics of BPS states

also, effectively, index theorem for the Coulomb phase of BPS quivers

from Seiberg-Witten BPS dyons as semiclassical & asymptotic solutions

with electric charges n^i and magnetic charges m^i

$$\mathcal{E} \;=\; |Z| \qquad\qquad Z \equiv \left\langle m^i \phi_D^i + n^i \phi^i \right\rangle = |Z|\zeta$$

$$\boxed{\begin{aligned} F_a^i &= i\zeta^{-1}\partial_a \phi^i \\ (F_D)_a^i &= i\zeta^{-1}\partial_a \phi_D^i \end{aligned}}$$

$$F_a^i \equiv B_a^i + iE_a^i \qquad \mathrm{Re}\int_{S^2} F^i = 4\pi m^i$$

$$(F_D)_a^i \equiv \tau^{ij} F_j^a \qquad \mathrm{Re}\int_{S^2} F_D^i = -4\pi n^i$$

two-body interactions

$$\gamma = (p, 2q) \qquad \Gamma = (m, 2n)$$

$\mathcal{Z}_{(p,2q)}$: the central charge function of the probe dyon in the background of other dyons

$$-\partial_a \mathrm{Im}[\zeta^{-1}\partial_a \mathcal{Z}_{(p,2q)}]$$

$$= \mathrm{Re}\left(q^i \partial_a F^i_a + p^i \partial_a (F_D)^i_a\right) = \sum_A \boxed{(q^i m^i_A - p^i n^i_A)} 4\pi\delta^3(\vec{x} - \vec{x}_A)$$

$$= \langle \gamma, \Gamma \rangle / 2$$

$$\mathrm{Im}[\zeta^{-1}\mathcal{Z}_{\gamma_h}] = \mathrm{Im}[\zeta^{-1} Z_{\gamma_h}] - \sum_A \frac{\langle \gamma_h, \gamma_A \rangle / 2}{|\vec{x} - \vec{x}_A|} \quad \rightarrow \quad \text{Lorentz Force !!!}$$

3n bosons + 4n fermions in two different superspaces

4n N=1 supermultiplets

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa} \qquad \Lambda^A = i\lambda^A + i\theta b^A \qquad A = 1, 2, \ldots, n$$

position of A-th dyon

n N=4 supermultiplet ~ n D=4 N=1 vector multiplets

$$\hat{\Phi}^{Aa} = -\frac{i}{4}(\epsilon\sigma^a)^{\alpha\beta}\Phi^A_{\alpha\beta}\ ; \qquad \Phi^A_{\alpha\beta} = (D_\alpha\bar{D}_\beta + \bar{D}_\beta D_\alpha)V^A$$

(1) ab initio, real space N=4 susy quantum mechanics for n dyons

Kim+Park+P.Y.+Wang 2011

$$\int dt\, \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

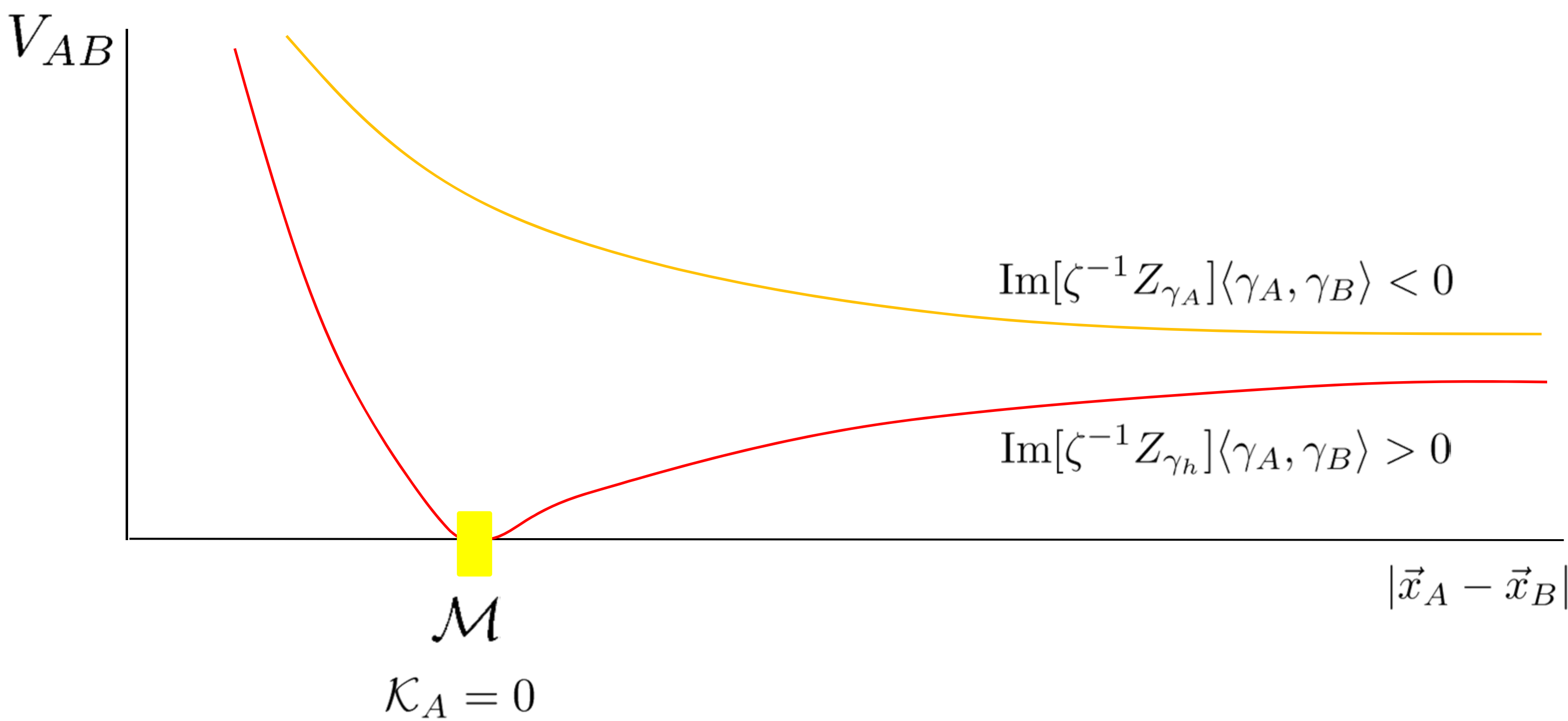
$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta \;\left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B\rangle/2}{|\vec{x}_A - \vec{x}_B|}$$

$$\vec{W}_A = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B\rangle}{2}\; \vec{W}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)$$

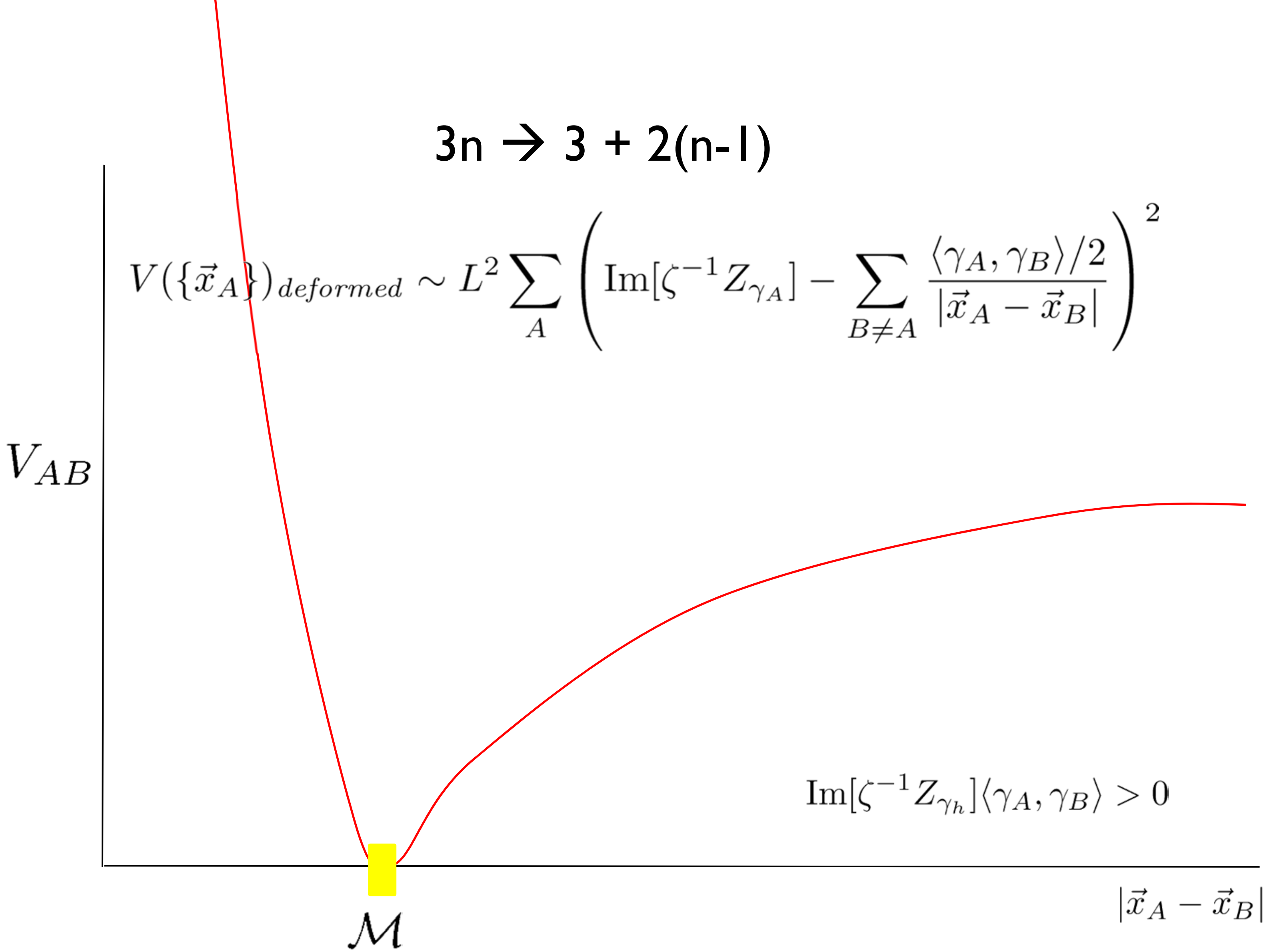
$$F(\vec{x}_A) \simeq \sum_A |Z_A||\vec{x}_A|^2 = \sum_A m_A |\vec{x}_A|^2 \qquad \text{asymptotically}$$

$$V(\{\vec{x}_A\}) \sim \sum_A \left(\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{|\vec{x}_A - \vec{x}_B|} \right)^2$$



3n → 3 + 2(n-1)

$$V(\{\vec{x}_A\})_{deformed} \sim L^2 \sum_A \left(\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{|\vec{x}_A - \vec{x}_B|} \right)^2$$



the counting problem reduces to a N=1 Dirac index of a nonlinear sigma model on the manifold $\mathcal{K}_A = 0$

Kim+Park+P.Y.+Wang 2011

3n bosons + 4n fermions → 2(n-1) bosons + 2(n-1) fermions

$$\mathcal{L}_{deformed}^{for\ index\ only}\Big|_{L\to\infty} \quad \rightarrow \quad \mathcal{L}_{index}$$

$$\mathcal{L}_{index} \simeq \frac{1}{2} g_{\mu\nu}\dot{z}^{\mu}\dot{z}^{\nu} - \dot{x}^{\mu}\cdot\mathcal{A}_{\mu} + \frac{i}{2}g_{\mu\nu}\psi^{\mu}\left(\dot{\psi}^{\nu} + \dot{z}^{\alpha}\Gamma^{\nu}_{\alpha\beta}\psi^{\beta}\right) + i\mathcal{F}_{\mu\nu}\psi^{\mu}\psi^{\mu}$$

$$\mathcal{F} = d\mathcal{A} \equiv \sum_{A} dW_A\Big|_{\mathcal{K}_A=0}$$

(2) basic state counting index

Manschot, Pioline, Sen 2010/2011

Kim+Park+P.Y.+Wang 2011

$$I_n(\{\gamma_A\}) = \mathrm{tr}\,\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr}\,\left[(-1)^F e^{-\beta Q^2}\right]$$

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}/R^3} ch(\mathcal{F}) \wedge \boxed{\mathbf{A}(\mathcal{M})}$$

trivial for a complete intersection
in flat ambient space

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}/R^3} ch(\mathcal{F})$$

$$= \frac{1}{(2\pi)^{n-1}(n-1)!}\int_{\mathcal{M}_n} \mathcal{F}^{n-1}$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle \gamma_A, \gamma_B\rangle}{2}\, \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

(3) protected spin character = equivariant index on $\mathcal{M}$

Kim+Park+P.Y.+Wang 2011

$$\Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I_3)}\right]$$

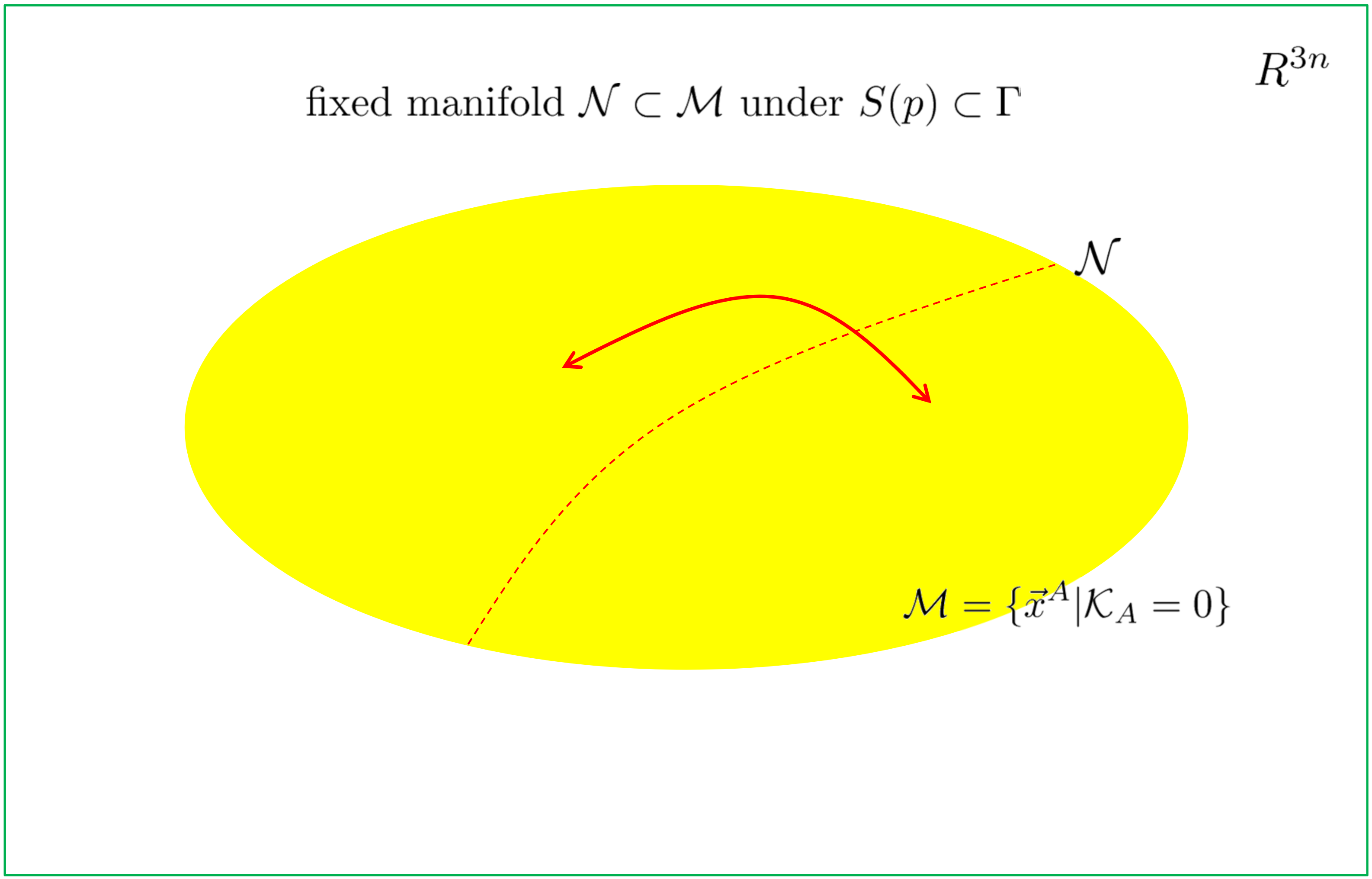
$$H = H_{\text{center of mass}} \otimes H_{\text{reduced}}$$

$$\Omega = \mathrm{tr}_{H_{\text{reduced}}}\left[(-1)^{2L_3+2(S_3-I_3)}(-1)^{2I_3}y^{2(J_3+I_3)}\right]$$

reduction to $\mathcal{M}$

$$\Omega = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F y^{2\mathcal{J}_3})$$

with quantum statistics taken into account



with quantum statistics taken into account

$$\text{fixed manifold } \mathcal{N} \subset \mathcal{M} \text{ under } S(p) \subset \Gamma$$

$$\Gamma' = \Gamma / S(p) \qquad \mathcal{P}' = \sum_{\sigma \in \Gamma'} (\pm 1)^{|\sigma|} \sigma$$

$$\mathrm{tr}\,(-1)^F e^{-\beta H} \mathcal{P}$$

$$= \mathrm{tr}_{\mathcal{M}/\Gamma - \mathcal{N}}\,(-1)^F e^{-\beta H} \mathcal{P} + \boxed{\Delta_{\mathcal{N}}}\,\mathrm{tr}_{\mathcal{N}/\Gamma' - \mathcal{N}'}\,(-1)^F e^{-\beta H} \mathcal{P}' + \cdots$$

with quantum statistics taken into account for a pair

$$\mathcal{P}_2^{(\pm)} \;:\; x \to -x,\; \psi \to -\psi$$

P.Y. 1997

$$\Delta_{\mathcal{N}}^{(\pm)}\Big|_{p=2} \leftarrow \lim_{\beta\to 0} \mathrm{tr}_{R^d;n_f}\left[(-1)^{F^\perp} e^{\beta\partial^2/2}\mathcal{P}_2^{(\pm)}\right]$$

$$= \lim_{\beta\to 0}\int_{R^d} d^dx\,\langle -x|e^{\beta\partial^2/2}|x\rangle \times (\pm 2^{n_{fermion}/2-1})$$

$$= \lim_{\beta\to 0}\frac{\pm 2^{n_{fermion}/2-1}}{(2\pi\beta)^{d/2}}\int_{R^d} d^dx\; e^{-(x+x)^2/2\beta}$$

$$= \lim_{\beta\to 0}\frac{\pm 2^{n_{fermion}/2-1}}{2^d}$$

$$\rightarrow \quad \frac{\pm 1}{2^2}$$

$$\begin{array}{cccccc} n_f & = & 2 & 4 & 8 & 16 \\ d & = & 2 & 3 & 5 & 9 \end{array}$$

with quantum statistics taken into account for p identical particles

fixed manifold $\mathcal{N} \subset \mathcal{M}$ under $S(p) \subset \Gamma$

$$\Delta_{\mathcal{N}}^{(\pm)} = \mathrm{tr}_{\mathcal{N}^\perp} \left[(-1)^{F^\perp} e^{-\beta H^\perp} \mathcal{P}_{S(p)}^{(\pm)} \right] \quad = \quad \frac{\pm 1}{p^2}$$

P.Y. 1997 / Green+Gutperle 1997

(4) wall-crossing formula from real space dynamics

Manschot, Pioline, Sen 2010/2011

Kim+Park+P.Y.+Wang 2011

$$\Omega^-\left(\sum \gamma_A\right) - \Omega^+\left(\sum \gamma_A\right) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\}+n-1} \frac{\prod_A \bar{\Omega}^-(\gamma_A)}{|\Gamma|} \int_{\mathcal{M}} ch(\mathcal{F})$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\}+n'-1} \frac{\prod_{A'} \bar{\Omega}(\gamma'_{A'})}{|\Gamma'|} \int_{\mathcal{M}'} ch(\mathcal{F}')$$

$$\vdots$$

$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\}+n''-1} \frac{\prod_{A''} \bar{\Omega}(\gamma''_{A''})}{|\Gamma''|} \int_{\mathcal{M}''} ch(\mathcal{F}'')$$

$$\vdots$$

$$\sum_{A=1}^{n} \gamma_A = \sum_{A'=1}^{n'} \gamma'_{A'} = \sum_{A''=1}^{n''} \gamma''_{A''} = \cdots$$

$$\boxed{\bar{\Omega}(\gamma) = \sum_{p|\gamma} \Omega(\gamma/p)/p^2}$$

(5) partial equivalence to Kontsevich-Soibelman

with all charges γ_A on a single plane of charge lattice

& in the absence of "scaling regime"

this has been shown to be equivalent to the results from Kontsevich-Soibelman

(Ashoke Sen, December 2011)

BPS quivers

wall-crossing examples

calibrated geometry : special Lagrange submanifolds in CY3

type II string theories on CY3 : wrapped D-branes

D=4 N=2 supergravity : BPS black holes

D=4 N=2 Seiberg-Witten theory : BPS particles

⋮

D=4 N=4 …… ¼ BPS ……

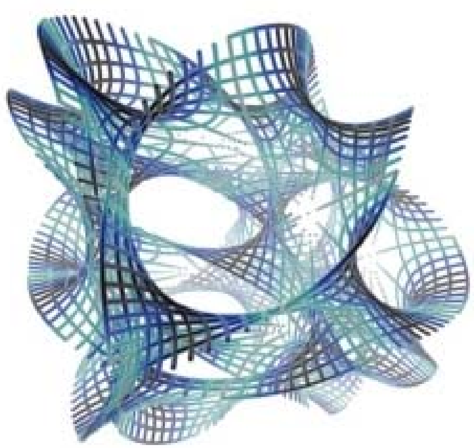
⋮

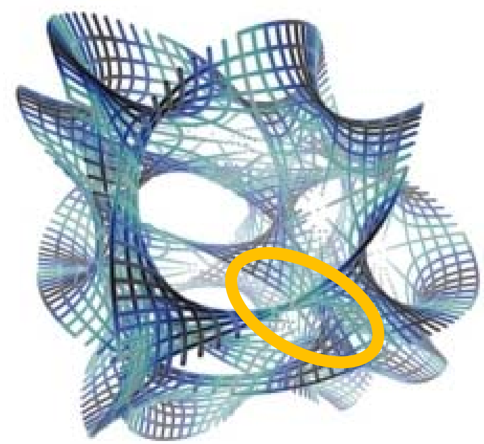
D=2 N=2 Landau-Ginzburg : BPS kinks

calibrated 3-cycles in Calabi-Yau 3-fold

$$J^{(1,1)} \qquad\qquad J^{(1,1)}\Big| \quad = 0$$

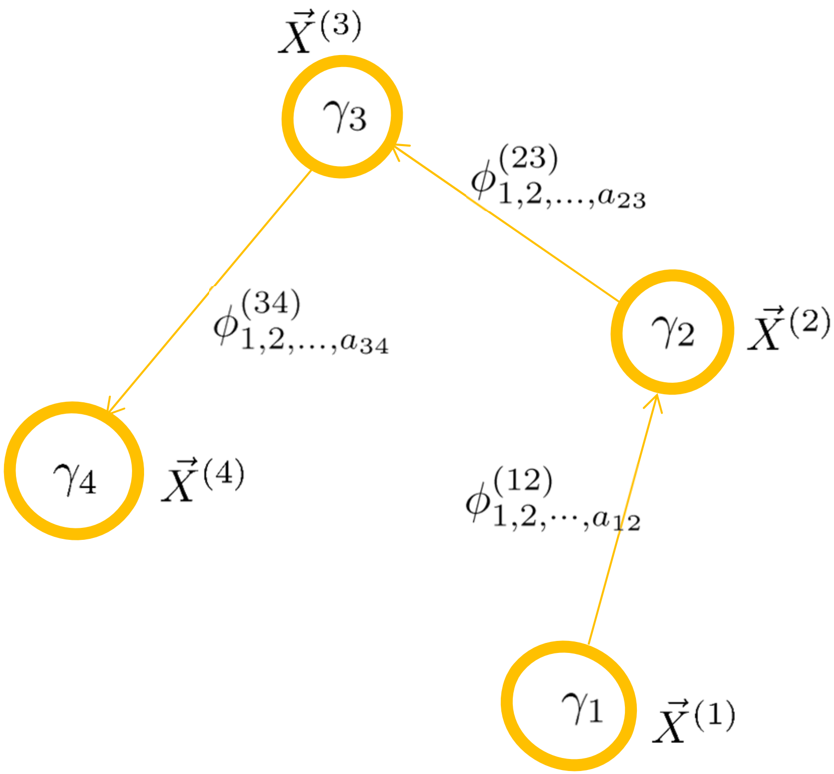
$$\Omega^{(3,0)} \qquad\qquad \zeta^{-1}\Omega^{(3,0)}\Big| \quad = \text{volume density of}$$





D3 on SL 3-cycles in CY3 → BPS quiver quantum mechanics

Denef 2002

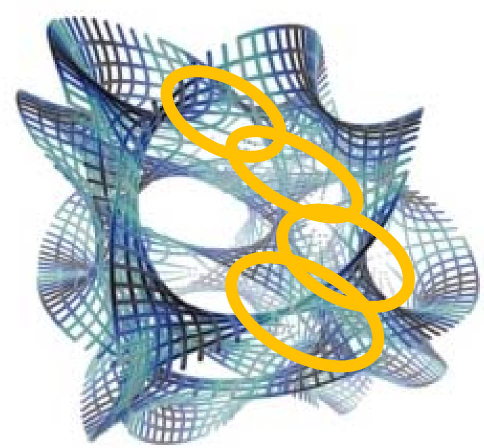


$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(1) \times U(1) \times U(1) \times U(1)$$

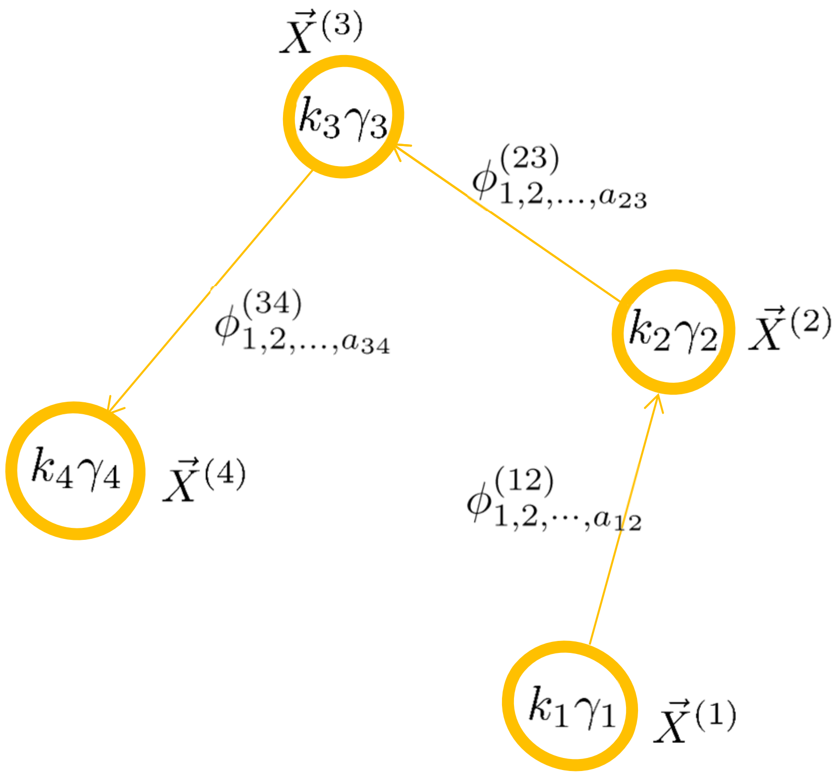
$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



D3 wrapped on 3-cycles in CY3 → BPS quiver quantum mechanics

Denef 2002

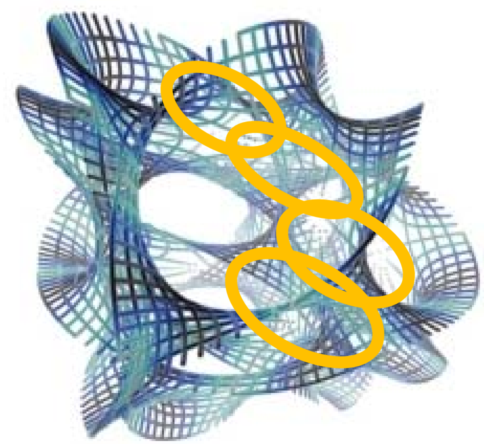


$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

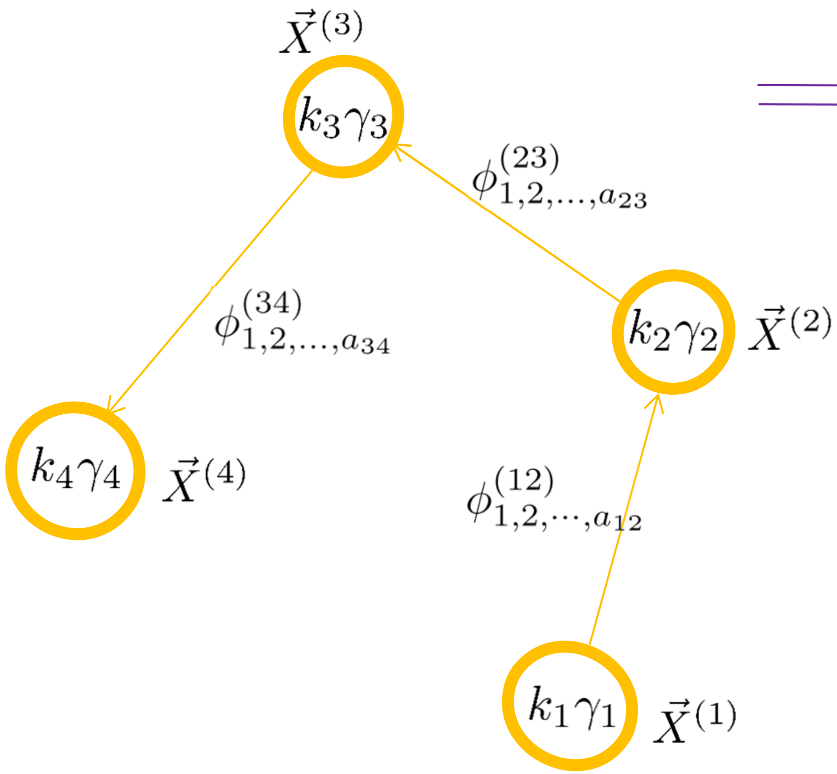
$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



Coulomb "phase"

Denef 2002

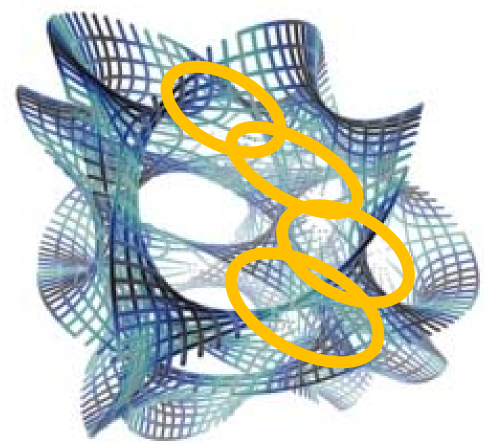


$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

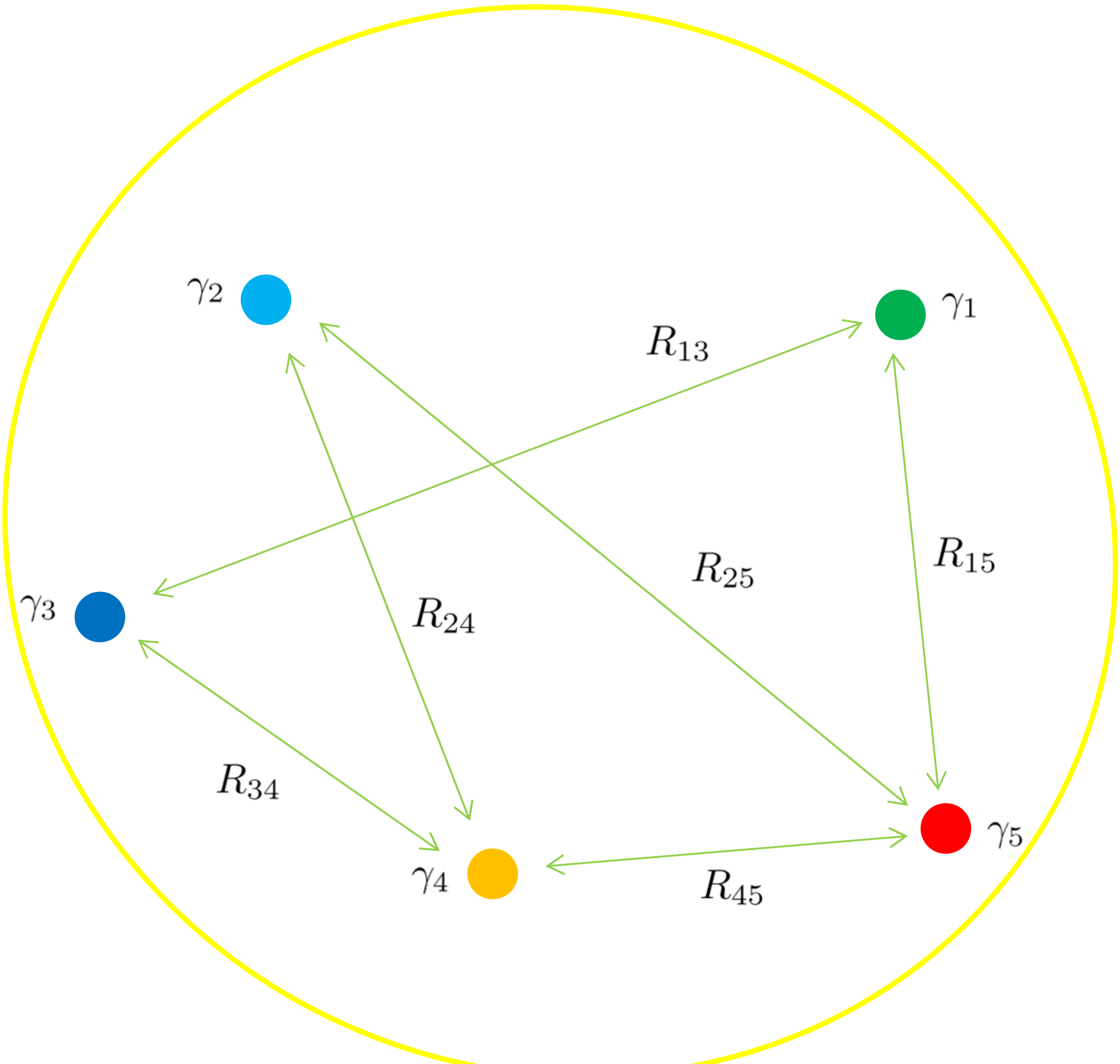
$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



which was, in effect, addressed in the first half of this talk

$R^3 = \{\vec{X}\}$



since real space N=4 susy interactions for n dyons
= Coulomb phase interactions of the corresponding BPS quivers

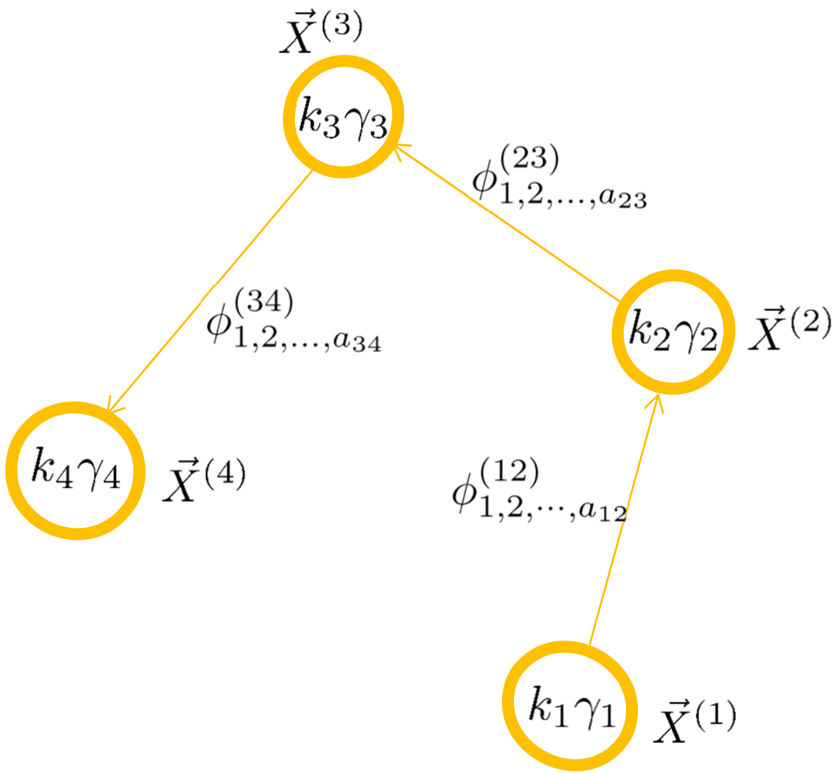
$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta\ \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{|\vec{x}_A - \vec{x}_B|}$$

$$\vec{W}_A = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{2}\ \vec{W}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)$$

= the Coulomb phase interaction of BPS quivers

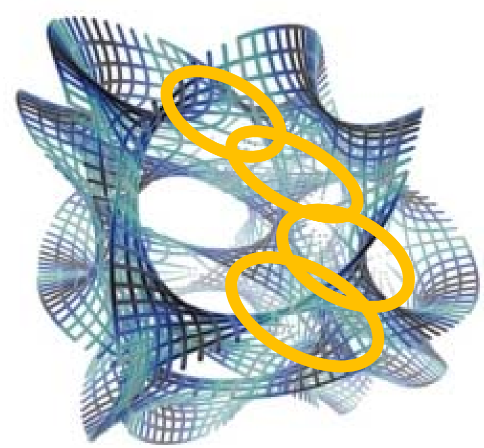
Higgs "phase"



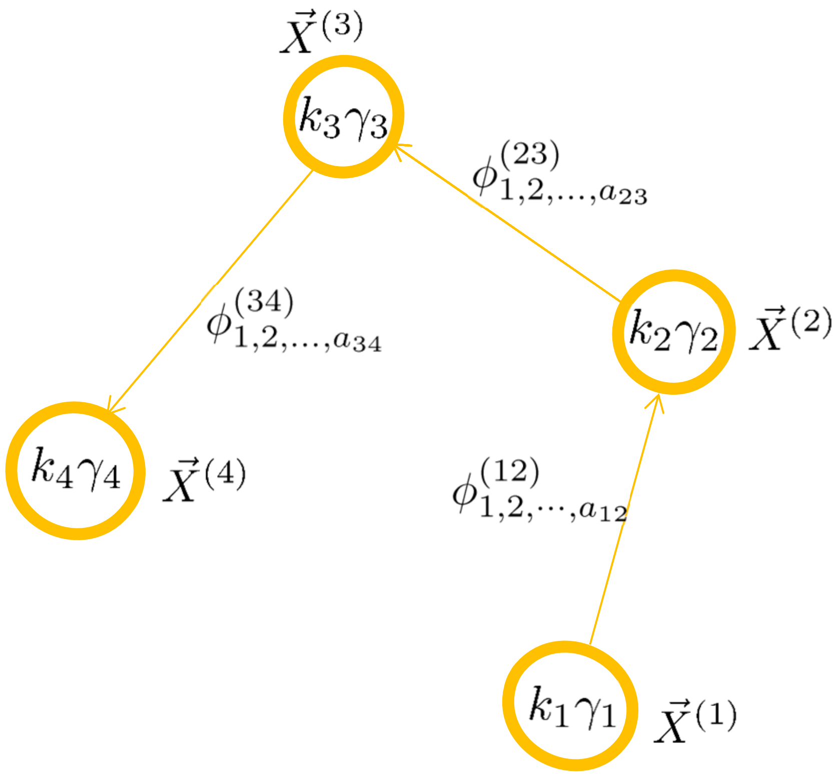
$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



Higgs phase : $\mathcal{M}_{\mathrm{H}} = \{\phi^{(12)}, \ldots | D^{(i)} = \xi^{(i)} 1_{k_i \times k_i}\} / \prod_i U(k_i)$



$$\Omega_{\mathrm{Higgs}}\left(\sum_i k_i\gamma_i; \xi^{(i)}\right)$$

$$\sim \chi\left(\mathcal{M}_{\mathrm{H}}\right)$$

$$= \sum_l (-1)^l \dim\left[H^l\left(\mathcal{M}_{\mathrm{H}}\right)\right]$$

Higgs phase : $\mathcal{M}_{\rm H} = \{\phi^{(12)}, \ldots | D^{(i)} = \xi^{(i)} 1_{k_i \times k_i}\} / \prod_i U(k_i)$

marginal stability wall

$$\xi^{(1)} > \xi^{(2)} > \xi^{(3)} > \xi^{(4)}$$

$$\chi\left(\mathcal{M}_{\rm H}\right) = 0$$

$$\chi\left(\mathcal{M}_{\rm H}\right) = a_{12} \times a_{23} \times a_{34}$$

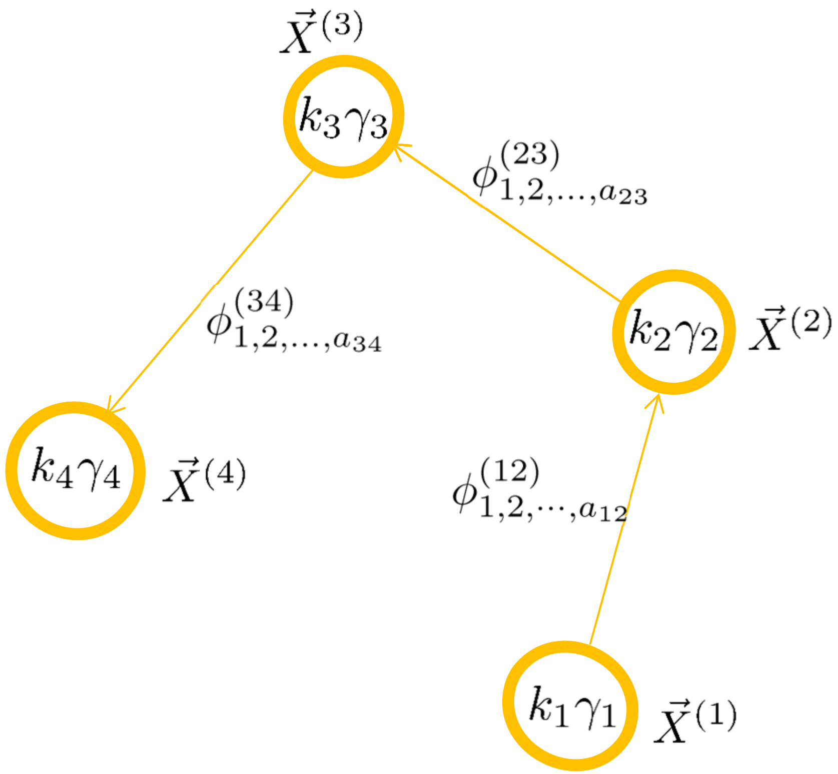
$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$

$$\mathcal{M}_{\rm H}\left(\sum_i \gamma_i; \xi^{(i)}\right) = 0$$

$$\mathcal{M}_{\rm H}\left(\sum_i \gamma_i; \xi^{(i)}\right)$$

$$= CP^{a_{12}-1} \times CP^{a_{32}-1} \times CP^{a_{34}-1}$$

can be elevated to the equivariant index → Hirzebruch character



$$\Omega_{\text{Higgs}}[y]\left(\sum_i k_i\gamma_i;\xi^{(i)}\right)$$

$$= \mathrm{tr}(-1)^{p+q-d}y^{2p-d}$$

$$= \sum_{p,q}(-1)^{p+q-d}y^{2p-d}h^{(p,q)}$$

$$h^{(p,q)} = \dim H^{(p,q)}(M)$$

$$\Omega_{\mathrm{Higgs}} = \Omega_{\mathrm{Coulomb}}$$

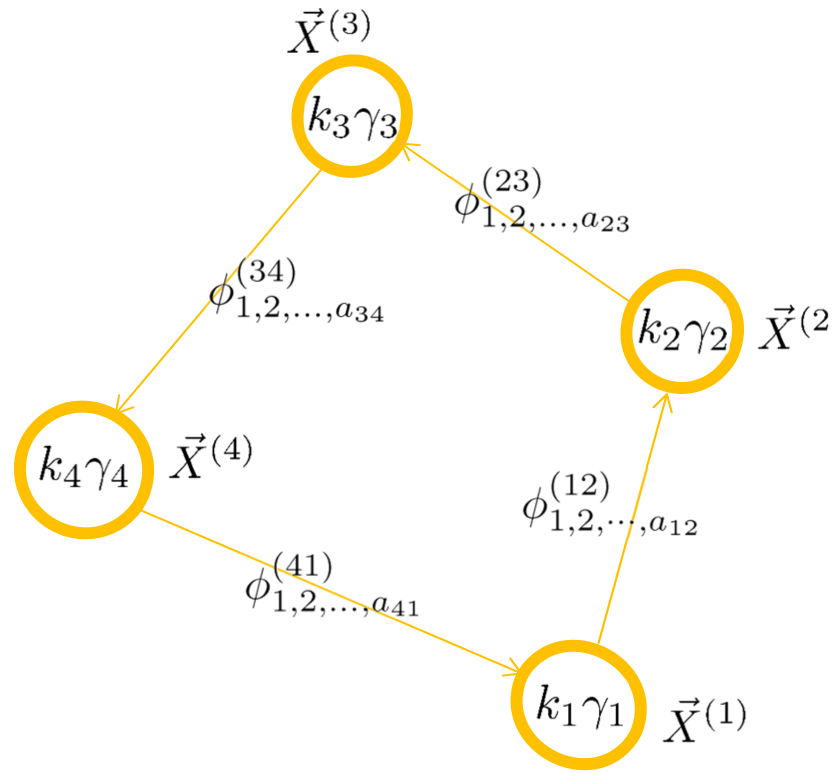
F. Denef 2002 + A. Sen 2011

which apparently fails for some quivers with loops

Denef + Moore 2007

$$|\Omega_{\text{Higgs}}| \;\geq\; |\Omega_{\text{Coulomb}}|$$

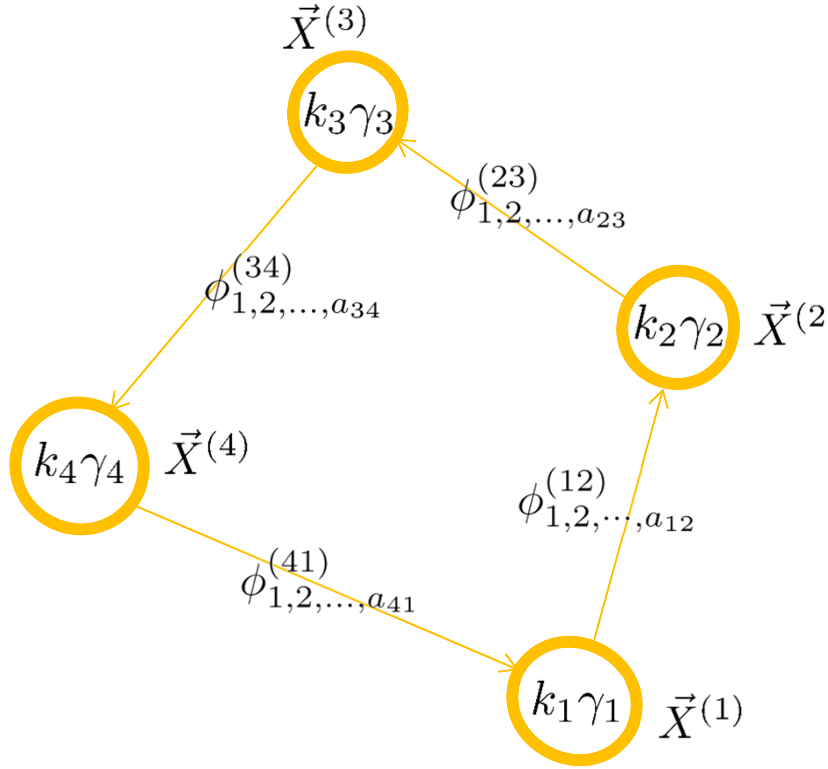
$$(\,|\Omega_{\text{Higgs}}| \;>>\; |\Omega_{\text{Coulomb}}|\,)$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

what physical & mathematical properties characterize these extra BPS states in the Higgs phase ?

$$|\Omega_{\rm Higgs}| \;-\; |\Omega_{\rm Coulomb}| \;=\; ???$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

also, all known wall-crossing formulae need input data

$$\Omega^{+}(\gamma) = \Omega^{-}(\gamma)$$

how to count & figure out these wall-crossing safe states ?

example 1 : elementary objects such as certain 2r+f hypermultiplet dyons in Seiberg-Witten theory of rank r and f flavors

$$\Omega^{+} = 1$$

$$\Omega^{-} = 1$$

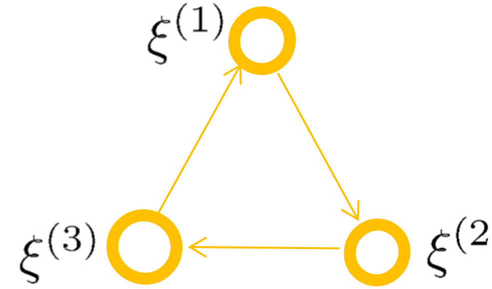
example 1 : elementary objects such as certain 2r+f hypermultiplet dyons in Seiberg-Witten theory of rank r and f flavors

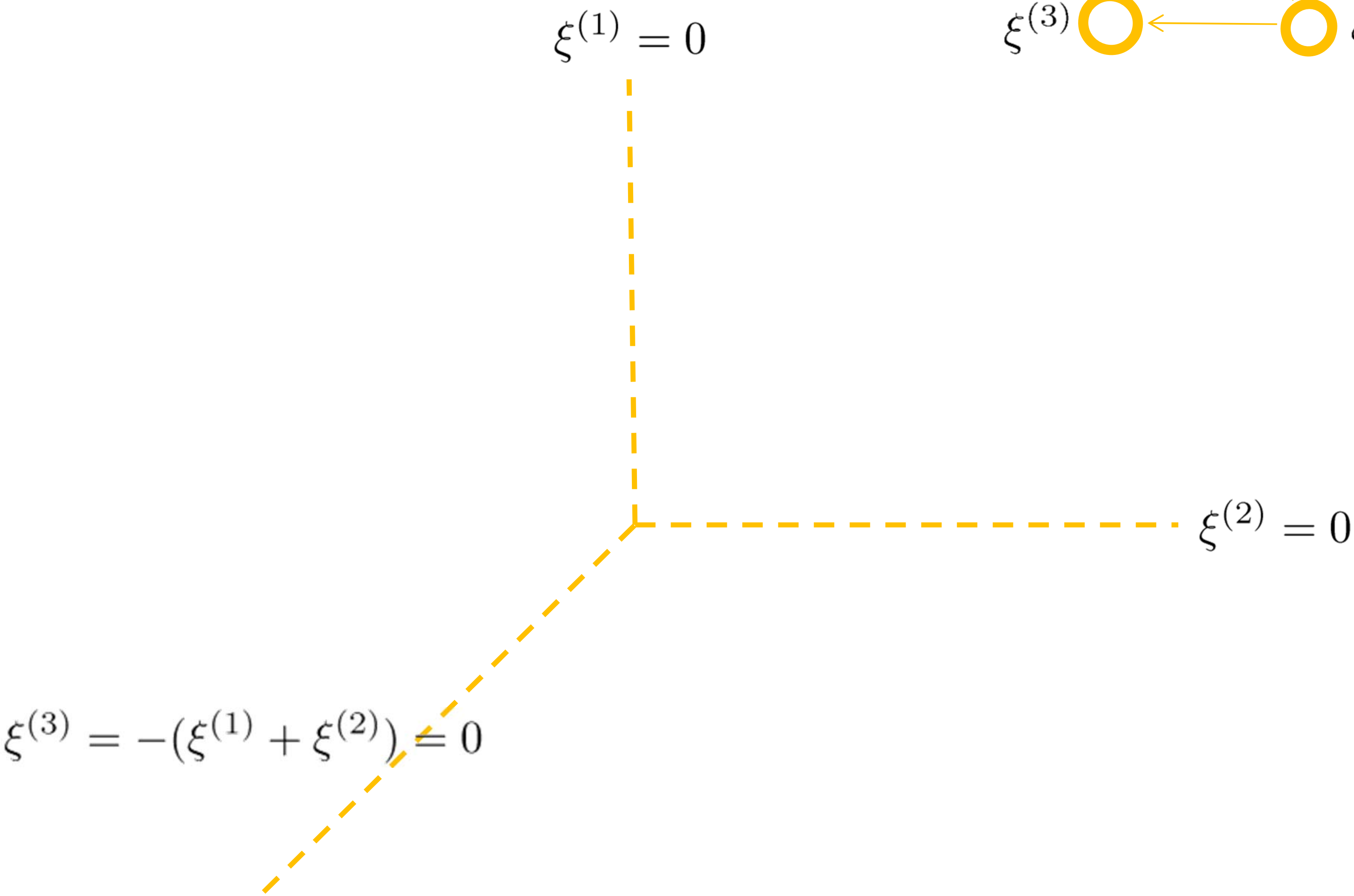
example 2 : single-center black holes

$$\Omega^{+} = \boxed{\Omega_{\text{Invariant}}} + \Omega^{+}_{\text{Coulomb}}$$

$$\Omega^{-} = \boxed{\Omega_{\text{Invariant}}} + \Omega^{-}_{\text{Coulomb}}$$

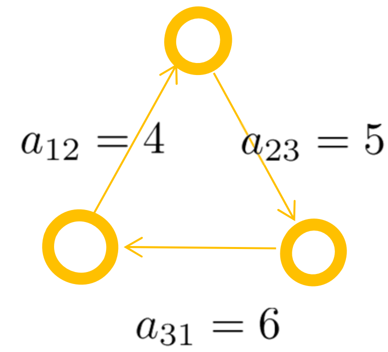
an example





an example

$$\dim\, H^{(p,q)}(\mathcal{M}_H)$$



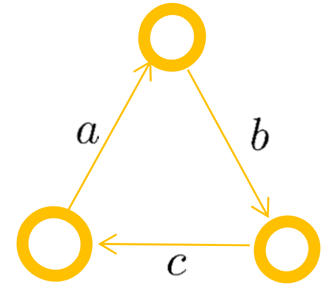
$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 3 & & 0 & & 0 & \\
0 & & 0 & & 26 & & 26 & & 0 & & 0 \\
 & 0 & & 0 & & 3 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

$$\begin{array}{ccc}
 & 1 & \\
26 & & 26 \\
 & 1 &
\end{array}$$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 2 & & 0 & \\
0 & & 26 & & 26 & & 0 \\
 & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

an example

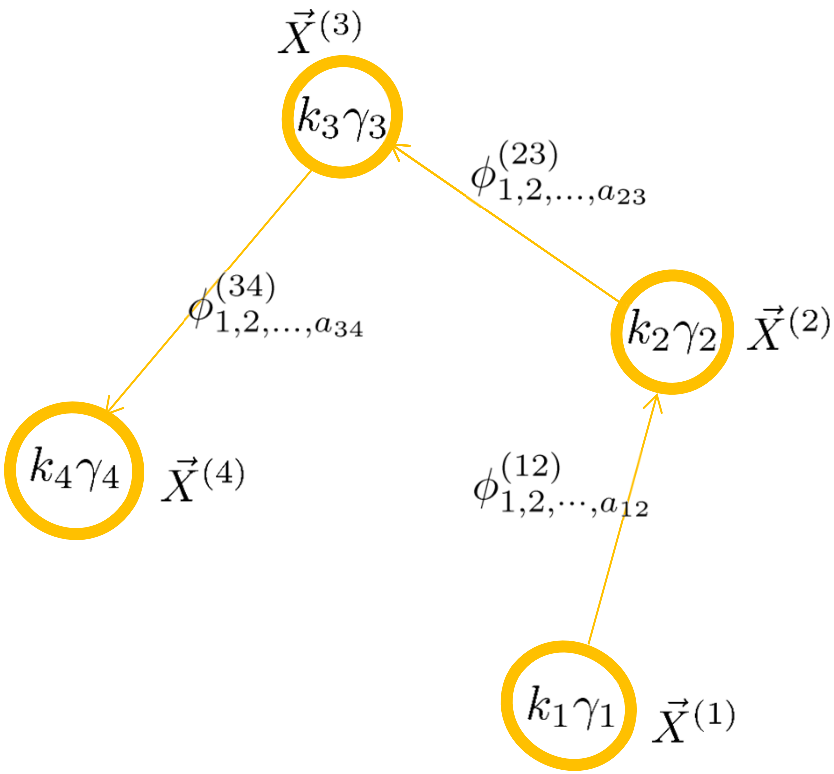
Denef + Moore 2007



$$\chi(\mathcal{M}_H) \sim \underbrace{\boxed{a\cdot(c-b)}}_{=\,\Omega_{\text{Coulomb}}} + \# \cdot 2^{(a+b+c)/2} + \cdots$$

quiver invariants

Higgs phase of BPS quiver quantum mechanics

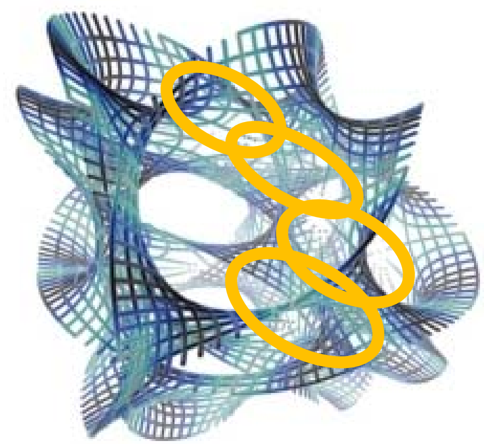


$$\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)} \quad \vec{X}^{(4)}$$

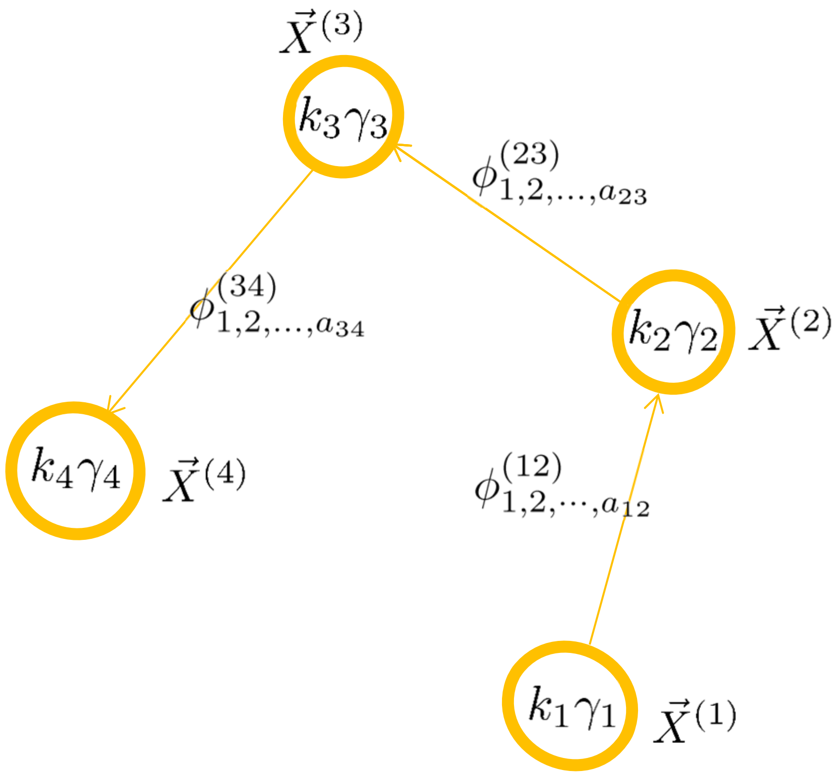
$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



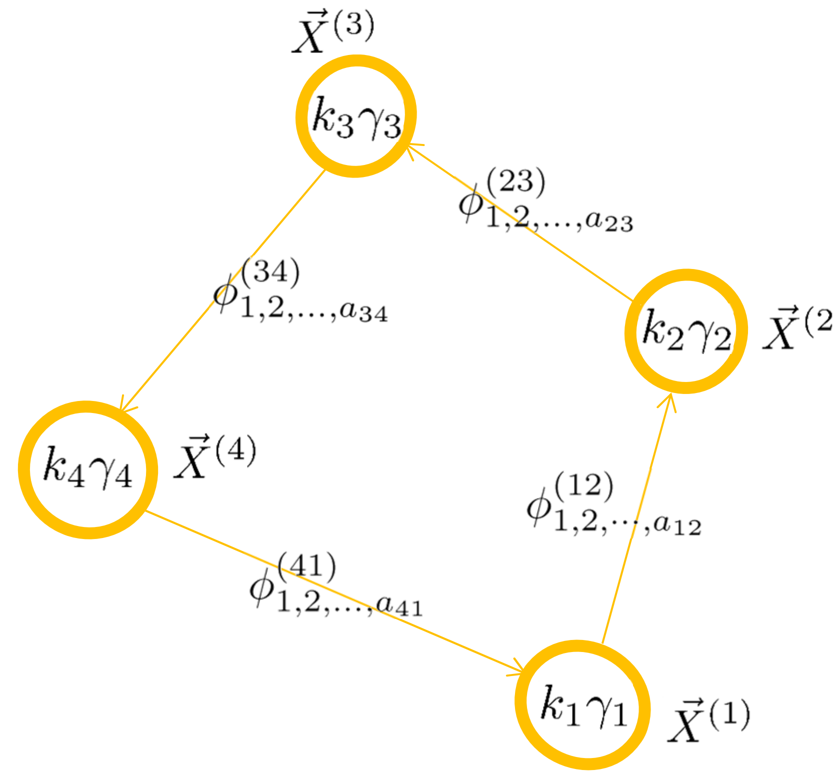
$$\Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}}$$



$$\{\phi^{(12)}, \ldots\} \xrightarrow{\;D \;\sim\; / \prod_i U(k_i)_C\;} \mathcal{M}_{\text{H}}$$

$$\left|\Omega_{\text{Higgs}}\right| \;\geq\; \left|\Omega_{\text{Coulomb}}\right|$$

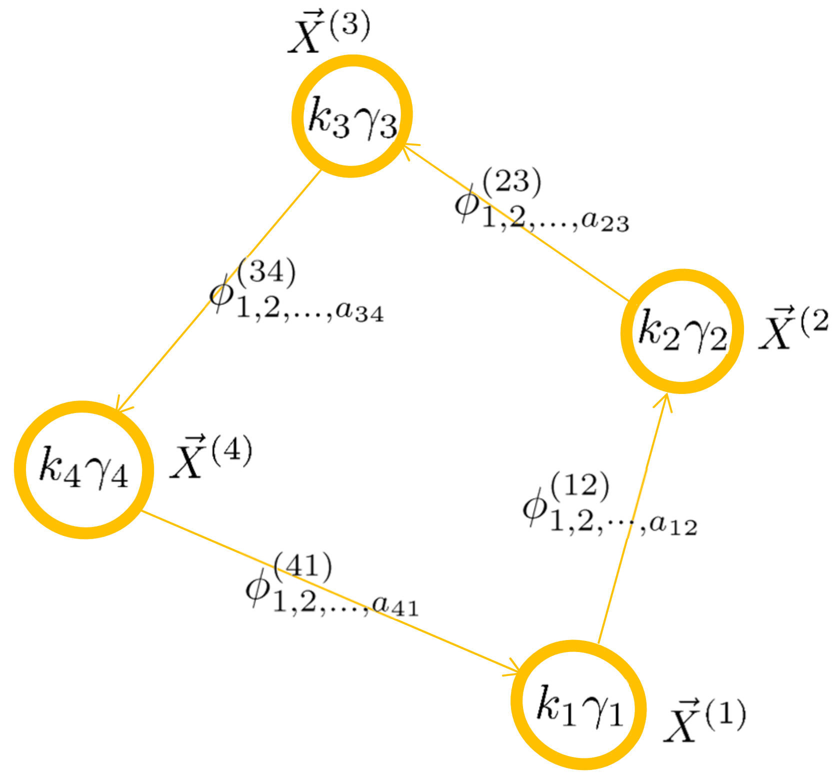
$$\begin{tikzcd} \{\phi^{(12)},\ldots\} \arrow[d, "D\ \sim\ /\prod_i U(k_i)_C"] \\ X_{\mathrm{H}} \arrow[d, "F\ \sim\ \partial_\phi W = 0"] \\ \mathcal{M}_{\mathrm{H}} \end{tikzcd}$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

$$|\Omega_{\text{Higgs}}| \;\geq\; |\Omega_{\text{Coulomb}}|$$

$$\begin{tikzcd}
\{\phi^{(12)}, \ldots\} \arrow[d, "D \;\sim\; /\prod_i U(k_i)_C"] \\
X_{\mathrm{H}} \\
\mathcal{M}_{\mathrm{H}} \arrow[u, "i \;\sim\; \partial_\phi W = 0"', "\text{embedding map}"]
\end{tikzcd}$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

two conjectures

$$\begin{tikzcd} \{\phi^{(12)},\ldots\} \arrow[d, "D"] \\ X_{\mathrm{H}} \\ \mathcal{M}_{\mathrm{H}} \arrow[u, "i"', "\text{embedding map}"] \end{tikzcd}$$

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

S.L. Lee + Z.L. Wang + P.Y., 2012

Bena + Berkooz + de Boer +
El-Showk + d. Bleeken, 2012

two conjectures

S.L. Lee + Z.L. Wang + P.Y., 2012

$$\{\phi^{(12)},\dots\} \qquad H^*(\mathcal{M}_H) = \sum H^{(p,q)}(\mathcal{M}_H)$$

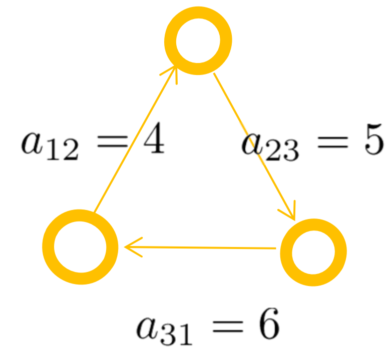
$$\downarrow D \qquad = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$X_{\mathrm{H}} \qquad \mathrm{tr}_{i^*(H(X))}(-1)^{p+q-d}y^{2p-d} \qquad \mathrm{tr}_{\mathrm{Intrinsic}}(-1)^{p+q-d}y^{2p-d}$$

embedding map $\uparrow i$ $\qquad \Updownarrow \qquad \Updownarrow$

$$\mathcal{M}_{\mathrm{H}} \qquad \Omega_{\mathrm{Coulomb}} \qquad \Omega_{\mathrm{Invariant}}$$

back to the simplest example



$$\dim H^{(p,q)}(\mathcal{M}_H)$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 3 & & 0 & & 0 & \\
0 & & 0 & & 26 & & 26 & & 0 & & 0 \\
 & 0 & & 0 & & 3 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

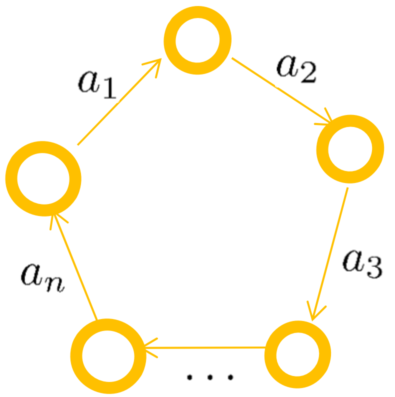
$$\begin{array}{ccc}
 & 1 & \\
26 & & 26 \\
 & 1 &
\end{array}$$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 2 & & 0 & \\
0 & & 26 & & 26 & & 0 \\
 & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

the total equivariant index of a cyclic Abelian quiver is

$$\Omega^{(k)}(y)\Big|_{\mathrm{Higgs}} = \mathrm{tr}_{H^*(\mathcal{M}_H^{(k)})}(-1)^{2J_3} y^{2J_3+2I} = \sum (-1)^{p+q-d} y^{2p-d} h^{(p,q)}(\mathcal{M}_H^{(k)})$$

$$= (-y)^{-d_k} \chi_{t=-y^2}(\mathcal{M}_H^{(k)})$$



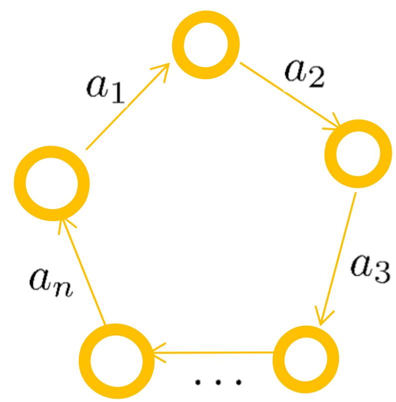
which is easily computable via

$$\chi_t(\mathcal{M}_H^{(k)}) = \frac{1}{(1+t)^n} \int_{X_H^{(k)}} \left[\prod_{i \neq k} \left(J_i \frac{1 + te^{-J_i}}{1 - e^{-J_i}} \right)^{a_i} \right] \cdot \left(\frac{1 - e^{-\sum_{i \neq k} J_i}}{1 + te^{-\sum_{i \neq k} J_i}} \right)^{a_k}$$

$$X_{\mathrm{H}}^{(k)} = \prod_{i \neq k} CP^{a_i - 1}$$

embedding map

$$\begin{tikzcd} X_{\mathrm{H}}^{(k)} \\ \mathcal{M}_{\mathrm{H}}^{(k)} \arrow[u, "i"'] \end{tikzcd}$$



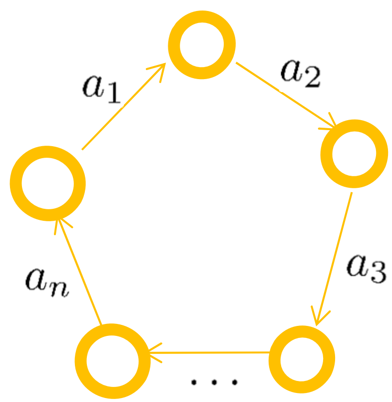
and can be decomposed into two parts

$$
\begin{aligned}
\Omega^{(k)}_{\text{Higgs}}(y) \;&=\; \boxed{(-1)^{d_k} y^{a_k} \prod_{i\neq k} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}} + \Delta\Omega_{\{a_i\}}(y)} \\
&+\; \boxed{\frac{(-y)^{n+2-\sum_i a_i}}{(y^2-1)^n} \prod_i \oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left[\prod_i \left(\frac{1-y^2\omega_i}{1-\omega_i}\right)^{a_i}\right] \cdot \frac{1}{1-y^2\prod_i \omega_i} - \Delta\Omega_{\{a_i\}}(y)}
\end{aligned}
$$

$$
X^{(k)}_{\text{H}} = \prod_{i\neq k} CP^{a_i-1}
$$

embedding map $\Big\uparrow\, i$

$$
\mathcal{M}^{(k)}_{\text{H}}
$$



→ complete proof / explicit counting formulae

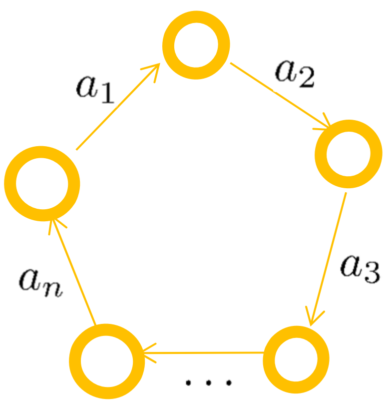
S.L. Lee + Z.L. Wang + P.Y., 2012
Manschot + Pioline + Sen, 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\mathrm{tr}(-1)^{p+q-d}y^{2p-d}$$

$$\Omega_{\text{Coulomb}} \qquad\qquad \Omega_{\text{Invariant}}$$

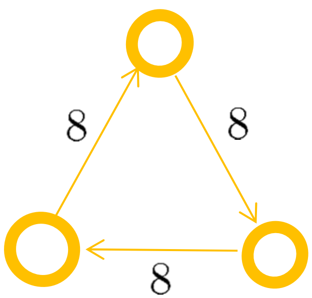
$$= \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$



Lefschetz
hyperplane
theorem !!!

wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$



$\dim H^{(p,q)}(\mathcal{M}_H)$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & & 322 & & 6803 & & 4+18212 & & 6803 & & 322 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$

wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\dim H^{(p,q)}(\mathcal{M}_H)$$

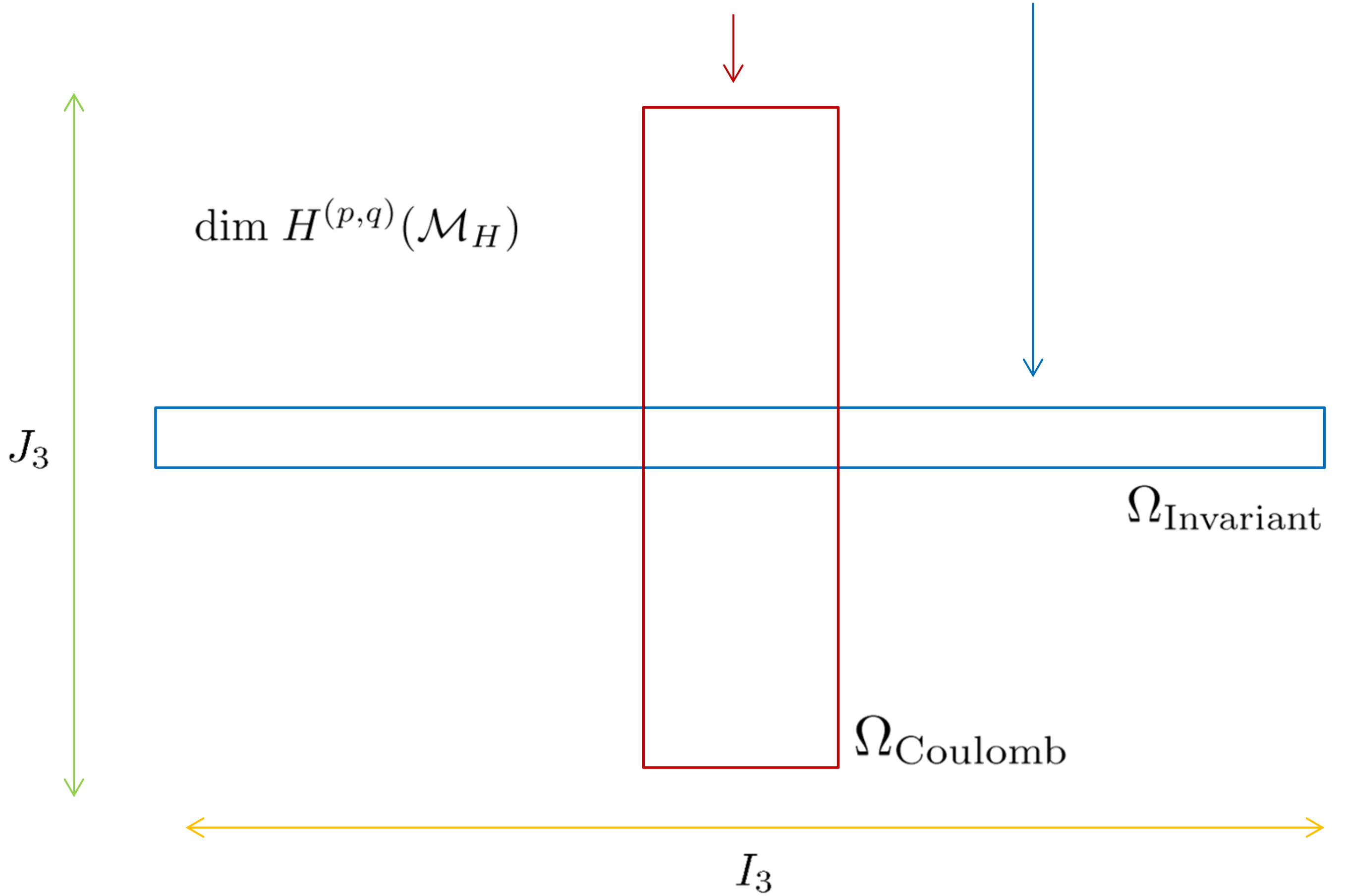
$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & & 322 & & 6803 & & 4+18212 & & 6803 & & 322 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}
$$

J_3

I_3

wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*[H^*(X_H)] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$



wall-crossing states vs. wall-crossing-safe states

Seung-Joo Lee + Zhao-Long.Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d}y^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\text{Coulomb}} \qquad\qquad \Omega_{\text{Invariant}} = \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$

many body bound states
wall-crossing

single-center black holes?
wall-crossing-safe

vertical in the Hodge diamond

horizontal middle
in the Hodge diamond

wall-crossing states vs. wall-crossing-safe states

Seung-Joo Lee + Zhao-Long Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$\Big\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d}y^{2p-d} \qquad \Big\Updownarrow$$

$$\Omega_{\mathrm{Coulomb}} \qquad\qquad \Omega_{\mathrm{Invariant}} = \Omega_{\mathrm{Higgs}} - \Omega_{\mathrm{Coulomb}}$$

many body bound states
wall-crossing

single-center black holes?
wall-crossing-safe

angular momentum

R-charge

wall-crossing states vs. wall-crossing-safe states

Seung-Joo Lee + Zhao-Long Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d} y^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\text{Coulomb}} \qquad\qquad \Omega_{\text{Invariant}} = \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$

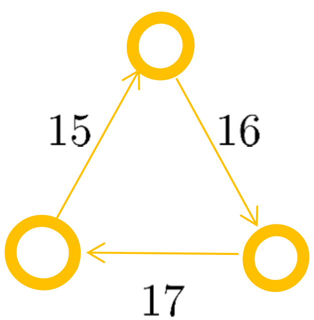
many body bound states wall-crossing	single-center black holes? wall-crossing-safe
angular momentum (R-charge = 0 for SW ?)	R-charge angular momentum = 0

wall-crossing-safe states of cyclic Abelian quivers

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Omega(y)\Big|_{\text{Intrinsic}}^{\{a_{i,i+1}\}=(15,16,17)} = \mathrm{tr}_{\text{Intrinsic}}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p.q)}_{\text{Intrinsic}}$$

$$\begin{aligned}
&= 1665y^{-12}\\
&+ 724674y^{-10}\\
&+ 60686563y^{-8}\\
&+ 1523273844y^{-6}\\
&+ 13886938949y^{-4}\\
&+ 50685934038y^{-2}\\
&+ 77668453887\\
&+ 50685934038y^{2}\\
&+ 13886938949y^{4}\\
&+ 1523273844y^{6}\\
&+ 60686563y^{8}\\
&+ 724674y^{10}\\
&+ 1665y^{12}
\end{aligned}$$

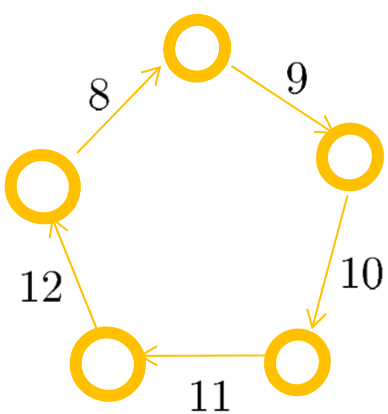


wall-crossing-safe states of cyclic Abelian quivers

$$H^*(\mathcal{M}_H) \ = \ i^*\left[H^*(X_H)\right] \ \oplus \ H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Omega(y)\Big|_{\text{Intrinsic}}^{\{a_{i,i+1}\}=(8,9,10,11,12)} = \mathrm{tr}_{\text{Intrinsic}}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p,q)}_{\text{Intrinsic}}$$

$$\begin{aligned}
= \ & 32294250/y^{22} + 58872952926/y^{20} + 23086762587054/y^{18} \\
& +3146301650299568/y^{16} + 186529800766285403/y^{14} \\
& +5480846262397291070/y^{12} + 86780383421802203555/y^{10} \\
& +783408269154731872224/y^{8} + 4192271239441338802849/y^{6} \\
& +13657486692285216220742/y^{4} + 27560691162972524163666/y^{2} \\
& +34791235315880411958041 + 27560691162972524163666y^{2} \\
& +13657486692285216220742y^{4} + 4192271239441338802849y^{6} \\
& +783408269154731872224y^{8} + 86780383421802203555y^{10} \\
& +5480846262397291070y^{12} + 186529800766285403y^{14} \\
& +3146301650299568y^{16} + 23086762587054y^{18} \\
& +58872952926y^{20} + 32294250y^{22}
\end{aligned}$$



summary

wall-crossing formulae from direct index computation

  for SW BPS dyons with ab initio low energy dynamics

  for the Coulomb phase of quiver descriptions

subtleties with index theorems in the Coulomb phase

equivalence to Kontsevich-Soibelman (when $\Omega_{\mathrm{Invariant}} = 0$),
and rational invariants from statistics orbifolding

quiver invariants, or wall-crossing safe BPS states in the Higgs phase

non-Abelian quivers / stringy realizations ?