Apery (~79)

Consider DE : $Ly = 0$ $\quad y = y(x)$ $\quad \partial = \frac{\partial}{\partial x}$

$$L = (x^4 - 34x^3 + x^2)\partial^4 + (10x^3 - 255x^2 + 5x)\partial^3 + (25x^2 - 418x + 4)\partial^2 + (15x - 117)\partial + 1$$

Space of analytical (in 0) solutions is 2-dimensional and is spanned by

$a(x) = 6x + a_2x^2 + a_3x^3 + \ldots$

and

$b(x) = 1 + 5x + b_2x^2 + b_3x^3 + \ldots$

"Crystal"

- $b_n \in \mathbb{Z}$
- $a_n \cdot \mathrm{lcm}(1, \ldots, n)^3 \in \mathbb{Z}$

- Symbol of L is $(x^2 - 34x + 1)$

  It has 2 roots $\alpha' < \alpha$

  $\alpha' = (1 - \sqrt{2})^4$

  $\alpha = (1 + \sqrt{2})^4$

$$\lim_{x \to \alpha'} \frac{a(x)}{b(x)} = \lambda$$

$c(x) := a(x) - \lambda b(x) = \sum c_n x^n$

$c(x)$ is regular for $|x| \leq \alpha'$

⇓

converges for $|x| < \alpha$

⇓

$\sqrt[n]{|a_n - \lambda b_n|} \to \alpha^{-1}$

$|a_n - \lambda b_n| < (\alpha^{-1} + \varepsilon)^n \leq$

$|\lambda - \frac{a_n}{b_n}| < \frac{\alpha^{-n}}{b_n} \leq \ldots (q_n)^{1+\varepsilon}$

$\frac{a_n}{b_n} = \frac{p_n}{q_n}$

$\Rightarrow \lambda = \zeta(3)$

On Apery constants of homogeneous varieties.

S. GALKIN

ABSTRACT. We do numerical computations of Apery constants for homogeneous varieties G/P for maximal parabolic groups P in Lie groups of type A_n, $n \leqslant 10$, B_n, C_n, D_n, $n \leqslant 7$, E_6, E_7 , E_8, F_4 and G_2. These numbers are identified to be polynomials in the values of Riemann zeta-function $\zeta(k)$ for natural arguments $k \geqslant 2$.

1. INTRODUCTION

The article is devoted to the computations of Apery numbers for the quantum differential equation of homogeneous varieties, so first we introduce these 3 notions.

Let X be a Fano variety of index r: $-K_X = rH$, and q be a coordinate on the anticanonical torus $\mathbb{Z} - K_X \otimes \mathbb{C}^* = G_m \in \mathrm{Pic}(X) \otimes \mathbb{C}^*$, and $D = q\frac{d}{dq}$ be an invariant vector field. Cohomologies $H^\bullet(X)$ are endowed with the structure of quantum multiplication $\star$, and associativity of $\star$ implies that first Dubrovin's connection given by

$$D\phi = H \star \phi \tag{1.1}$$

is flat.

If we replace in equation 1.1 quantum multiplication with the ordinary cup-product, then it's solutions are constant Lefschetz coprimitive (with respect to H) classes in $H^\bullet(X)$. Dimension μ of the space of homolorphic solutions of 1.1 is the same and equal to the number of admissible initial conditions (of the recursion on coefficients) modulo q, i.e. the rank of the kernel of cup-multiplication by H in $H^\bullet(X)$, that is the dimension of coprimitive Lefschetz cohomologies.

Solving equation 1.1 by Newton's method one obtains a matrix-valued few-step recursion reconstructing all the holomorphic solutions from these initial conditions.

Givental's theorem states that the solution $A = 1 + \sum_{n\geqslant 1} a^{(n)} q^n$ associated with the primitive class $1 \in H^0(X)$ is the I-series of the variety X (the generating function counting some rational curves of X). Choose a basis of other solutions $A_1, \ldots, A_{\mu-1}$ associated with homogeneous primitive classes of nondecreasing codimension [1]

Put $A = \sum_{n\geqslant 0} a^{(n)} t^n$ and $A_i = \sum_{n\geqslant 0} a_i^{(n)} t^n$. We call the number

$$\lim_{n\to\infty} \frac{a_i^{(n)}}{a^{(n)}}$$

i-th Apery constant after the renown work [2], where $\zeta(3)$ and $\zeta(2)$ were shown to be of that kind for some differential equations and such a presentation was used for proving the irrationality of these two numbers. If there is no choosen basis, for any coprimitive class γ one still may consider

[1]One could also consider other bases, e.g. it is often exists a base with ith element B_i determined by the condition $B_i = t^i (mod t^\mu)$. But the answer in this base looks worse. Finally one may reject to choose any basis and express everything invariantly in the dual space of primitive classes.

the solution $A_\gamma = \sum_{n\geqslant 1} a_\gamma^{(n)} q^n = Pr_0(\gamma + \sum_{n\geqslant 1} A_\gamma^{(n)} q^n)$ and the limit

$$Apery(\gamma) = \lim_{n\to\infty} \frac{a_\gamma^{(n)}}{a^{(n)}} \tag{1.2}$$

Defined in that way, *Apery* is a linear map from coprimitive cohomologies to $\mathbb{C}$. A linear map on coprimitive cohomologies is dual [2] to some (nonhomogeneous) primitive cohomology class with coefficients in $\mathbb{C}$. We name it *Apery characteristic class* $A(X) \in H^{\leqslant \dim X}(X, \mathbb{C})$.

Consider the homogeneous ring $R = \mathbb{Q}[c_1, c_2, c_3, \dots]$, $\deg c_i = i$ and a map $ev : R \to \mathbb{C}$ sending c_1 to Euler constant C [3], and c_i to $\zeta(i)$.

The main conjecture we verify is the following

Conjecture 1.3. *Let X be any Fano variety and $\gamma \in H^\cdot(X)$ be some coprimitive with respect to $-K_X$ homogeneous cohomology class of codimension n. Consider two solutions of quantum D-module: A_0 associated with 1 and A_γ associated with γ. Then Apery number for A_γ (i.e. $\lim_{k\to\infty} \frac{a_\gamma^{(k)}}{a_0^{(k)}}$) is equal to $ev(f_\gamma)$ for some homogeneous polynomial $f_\gamma \in R^{(n)}$ of degree n.*

Actually, in our case there is no Euler constant contributions, and the conjecture seems too strong to be true - it would imply that some of differential equations studied in [1] has non-geometric origin (at least come not from quantum cohomology), because their Apery numbers does not seem to be of the kind described in the conjecture (e.g. Catalan's constant, π^3, $\pi^3\sqrt{3}$).

From the other point of view, for toric varieties X the solutions of QDE are known to be pullbacks of hypergeometric functions, coefficients of hypergeometric functions are rational functions of Γ-values, and the Taylor expansion

$$\log \Gamma(1 + x) = Cx + \sum_{k\geqslant 2} \frac{\zeta(k)}{k} x^k \tag{1.4}$$

suggests all Apery constants would probably be rational functions in C and $\zeta(k)$. So whether one believes in toric degenerations or hypergeometric pullback conjecture, he would find natural to believe in 1.3. Also Apery limits like $\frac{91}{432}\zeta(3) - \frac{1}{216}\pi^3\sqrt{3}$ may appear as "square roots" or factors (convolutions with quadratic character or something) of geometric ones like $\frac{91^2}{432^2}\zeta(3)^2 - \frac{3}{216^2}\pi^6$.

This is not even the second paper (the computations of this paper were described by Golyshev 2-3 years ago) discussing the natural appearance of ζ-values in monodromies of QDEs. In case of fourfolds X the expression of monodromies in terms of $\zeta(3), \zeta(2k)$ and characteristic numbers of anticanonical section of X was given by van Straten [14], Γ-class for toric varieties appears in Iritani's work [9], and in general context in [10].

Let G be a (semi)simple Lie group, W be it's Weyl group, P be a (maximal) parabolic subgroup associated with the subset (or just one) of the simple roots of Dynkin diagram, and denote factor G/P by X. X is a homogeneous Fano variety with $\operatorname{rk}\operatorname{Pic} X$ equal to the number of chosen roots. In case when G is simple and P is maximal we have $\operatorname{Pic} X = \mathbb{Z}H$, where H is an ample generator, $K_X = -rH$.

For homogeneous varieties with small number of roots in Dynkin diagram (being more precise, with not too big total dimension of cohomologies) by the virtue of Peterson's version of Quantum

[2] One may choose between Poincare and Lefschetz dualities. We prefer the first one.

[3] $C = \lim_{n\to\infty}(\sum_{k=1}^{n} \frac{1}{k}) - \ln n$