The quark-antiquark potential in $\mathcal{N} = 4$ SYM from an open spin-chain

Nadav Drukker

Based on arXiv:1105.5144 - N.D. and V. Forini
arXiv:1203.1617 - N.D.
See also arXiv:1203.1913 - D. Correa, J. Maldacena and A. Sever

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Introduction and motivation

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- In gauge theories this is captured by a long rectangular Wilson loop, or a pair of antiparallel lines.
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• Explicit calculations at weak and at strong coupling:

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-\frac{\lambda}{4\pi L} + \frac{\lambda^2}{8\pi^2 L} \ln \frac{T}{L} + \cdots & \lambda \ll 1 \\
\frac{4\pi^2 \sqrt{\lambda}}{\Gamma\left(\frac{1}{4}\right)^4 L} \left(1 - \frac{1.3359 \ldots}{\sqrt{\lambda}} + \cdots\right) & \lambda \gg 1
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• Recently $O(\lambda^3)$ was calculated. \[\text{[Correa, Henn, Maldacena, Sever]}\]

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• Shouldn’t integrability allow us to calculate this for all values of the coupling (in the planar approximation)?
Outline

• Introduction and motivation

• Wilson loops
  – Cusp anomalous dimensions and the quark-antiquark potential
  – Local operator insertions

• Generalize quark-antiquark potential in $\mathcal{N} = 4$ SYM
  – Perturbative calculation
  – String calculation
  – Expansions in small angles

• Wilson loops and integrability
  – Operator insertions and open spin–chains
  – All loop reflection matrix and a twist
  – Wrapping effects and the quark-antiquark potential
Wilson loops

- In any gauge theory one can define Wilson loop operators
  \[ W = \text{Tr} \mathcal{P} \exp \left[ \oint iA_\mu \dot{x}^\mu \, ds \right] \]
  
- Can be defined for an arbitrary curve in spacetime.
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- For a pair of antiparallel lines

\[ \langle W \rangle \approx \exp \left[ -TV(L, \lambda) \right] \]

- The potential behaves like

\[ V(L, \lambda) = \begin{cases} 
  g(\lambda) & \text{screening} \\
  \frac{f(\lambda)}{L} & \text{conformal} \\
  \alpha' L & \text{confining} 
\end{cases} \]
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• In \( \mathcal{N} = 4 \) SYM the most natural Wilson loops includes a coupling to the scalar fields

\[ W = \text{Tr} \mathcal{P} \exp \left[ \oint (i A_\mu \dot{x}^\mu + |\dot{x}| n^I \Phi_I) \, ds \right] \]

\( n^I \) do not have to be constant.

• For a smooth loop and continuous \( |n^I| = 1 \), these are finite observables.
Cusp anomalous dimensions and quark-antiquark potential

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- It is simpler to control logarithmic divergences.
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- It is simpler to control \textit{logarithmic divergences}.
- Consider Wilson loops with cusps

\begin{center}
\begin{tikzpicture}
\draw[thick,-] (0,0) -- (4,0);
\draw[thick,dashed,blue] (-2,0) -- (0,0);
\draw[thick,dashed,red] (0,0) -- (2,0);
\draw[thick,dashed,green] (0,0) -- (3,0);
\end{tikzpicture}
\end{center}

- All but the black line will suffer from logarithmic divergences.
- Taking $\phi = i\varphi$ and $\varphi \to \infty$ gives the Lorenzian null cusp.
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$$\Phi_1 \quad \text{and} \quad \Phi_1 \cos \theta + \Phi_2 \sin \theta$$

• In a conformal theory, by the usual conformal Ward identity

$$\langle W \rangle \sim \frac{1}{d^2 \Delta}, \quad d = r \frac{\cos \frac{\phi}{2}}{1 - \sin \frac{\phi}{2}}$$

• $\Delta$ is the coefficient of the log divergence.
• By the inverse exponential map we get the gauge theory on $S^3 \times \mathbb{R}$.
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  Therefore $V(\phi, \theta, \lambda)$ is the same as $\Delta$, the coefficient of the log divergence.
• This $V(\phi, \theta, \lambda)$ is the generalization of $V(L, \lambda)$ — the quark-antiquark potential.
• For a conformal theory it has a pole at $\phi \to \pi$ and the residue is $LV(L, \lambda)$.
• More generally controls all log divergences of all Wilson loops.
• Needed for a proper renormalization program of Wilson loop operators (and to derive regularized loop equations).
Generalized quark-antiquark potential in $\mathcal{N} = 4$ SYM

- Crucial point: Calculations of $V(\phi, \theta, \lambda)$ are no harder than for the antiparallel case!
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- Expanding at weak coupling

$$V(\phi, \theta, \lambda) = \sum_{n=1}^{\infty} \left( \frac{\lambda}{16\pi^2} \right)^n V^{(n)}(\phi, \theta)$$

- And at strong coupling

$$V(\phi, \theta, \lambda) = \sqrt{\frac{\lambda}{4\pi}} \sum_{l=0}^{\infty} \left( \frac{4\pi}{\sqrt{\lambda}} \right)^l V_{AdS}^{(l)}(\phi, \theta)$$
**Perturbative calculation**

**1–loop**

- Just the exchange of a gluon and scalar field

- This graph is given by the integral

\[
\partial_\lambda \langle W \rangle \bigg|_{\lambda=0} = \int_{s<t} ds \, dt \langle (iA_\mu \dot{x}^\mu(s) + |\dot{x}| \Phi^I n^I(s)) (iA_\nu \dot{x}^\nu(t) + |\dot{x}| \Phi^J n^J(t)) \rangle
\]

\[
= \frac{\lambda}{8\pi^2} \int ds \, dt \frac{-\dot{x}_\mu(s) \dot{x}^\mu(t) + n^I(s) n^J(t)}{|x(s) - x(t)|^2}
\]

\[
= \frac{\lambda}{8\pi^2} \int ds \, dt \frac{-\cos \phi + \cos \theta}{s^2 + t^2 + 2st \cos \phi} = -\frac{\lambda}{8\pi^2} \frac{\cos \phi - \cos \theta}{\sin \phi} \phi \log \frac{R}{\epsilon}
\]

- Therefore

\[
V^{(1)}(\phi, \theta) = 2 \frac{\cos \phi - \cos \theta}{\sin \phi} \phi
\]
Higher order graphs

- Ladder graphs are relatively easy.
- They dominate a funny double-scaled limit where $\theta \to i\infty$ with $\lambda \theta$ fixed. \cite{Correa, Henn, Maldacena, Sever}
- They are given by harmonic polylogs apparently to all orders. \cite{Henn, Huber}
- Results at weak and strong coupling match.
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- Results at weak and strong coupling match.
- Interacting graphs are a bit more complicated.
- At two loops there are bubble graphs and the single cubic vertex graphs.
- they give
  \[
  V^{(2)}_{\text{int}}(\phi, \theta) = -\frac{2}{3}(\pi^2 - \phi^2)V^{(1)}(\phi, \theta)
  \]
- Full 3 loop answer was also calculated. [Correa, Henn Maldacena, Sever]
String calculation

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- At the leading order we should find the minimal surface ending on lines separated by $\pi - \phi$ on the boundary of $AdS$ and $\theta$ on $S^5$.
- All the string solutions fit inside $AdS_3 \times S^1$

\[ ds^2 = \sqrt{\lambda} \left( - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho \, d\varphi^2 + d\theta^2 \right) \]
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- Expand around $\phi = \theta = 0$ the answer is

\[
V_{AdS}^{(0)}(\phi, \theta) = \frac{1}{\pi} (\theta^2 - \phi^2) - \frac{1}{8\pi^3} (\theta^2 - \phi^2)(\theta^2 - 5\phi^2) \\
\quad + \frac{1}{64\pi^5} (\theta^2 - \phi^2)(\theta^4 - 14\theta^2\phi^2 + 37\phi^4) \\
\quad - \frac{1}{2048\pi^7} (\theta^2 - \phi^2)(\theta^6 - 27\theta^4\phi^2 + 291\theta^2\phi^4 - 585\phi^6) + O((\phi, \theta)^{10})
\]
1–loop determinant

- Complicated fluctuation problem.
- Can be done analytically (implicitly) for either $\phi = 0$ or $\theta = 0$.
- For $\theta = 0$ and small $\phi$ we can expand

$$
V^{(1)}_{AdS}(\phi, 0) = \frac{3}{2} \frac{\phi^2}{4\pi^2} + \left( \frac{53}{8} - 3 \zeta(3) \right) \frac{\phi^4}{16\pi^4} + \left( \frac{223}{8} - \frac{15}{2} \zeta(3) - \frac{15}{2} \zeta(5) \right) \frac{\phi^6}{64\pi^6}
+ \left( \frac{14645}{128} - \frac{229}{8} \zeta(3) - \frac{55}{4} \zeta(5) - \frac{315}{16} \zeta(7) \right) \frac{\phi^8}{256\pi^8} + O(\phi^{10})
$$
**\( \phi \to \pi \) limit**

- \( V^{(1)}, V^{(2)}, V_{AdS}^{(0)} \) and \( V_{AdS}^{(1)} \) all have poles at \( \phi = \pi \)
- In perturbation theory

\[
V(\phi, \theta) \to -\frac{\lambda}{8\pi} \frac{1 + \cos \theta}{\pi - \phi} + \frac{\lambda^2}{32\pi^3} \frac{(1 + \cos \theta)^2}{\pi - \phi} \log \frac{e}{2(\pi - \phi)} + O(\lambda^3)
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- In the case of \( \theta = 0 \) we get essentially the same as the antiparallel lines with \( L \to \pi - \phi \)

\[
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\frac{4\pi^2 \sqrt{\lambda}}{\Gamma(\frac{1}{4})^4 L} \left( 1 - \frac{1.3359 \ldots}{\sqrt{\lambda}} + \cdots \right) & \lambda \gg 1
\end{cases}
\]

- The strong coupling calculations also agree in the limit.
Expansions in small angles

- Consider the expansion of $V(\phi, \theta, \lambda)$ at small $\phi$ or $\theta$

$$
\frac{1}{2} \frac{\partial^2}{\partial \theta^2} V(\phi, \theta, \lambda) \bigg|_{\phi=\theta=0} = -\frac{1}{2} \frac{\partial^2}{\partial \phi^2} V(\phi, \theta, \lambda) \bigg|_{\phi=\theta=0} = \begin{cases} 
\frac{\lambda}{16\pi^2} - \frac{\lambda^2}{384\pi^2} + \cdots & \lambda \ll 1 \\
\sqrt{\lambda} - \frac{3}{8\pi^2} + \cdots & \lambda \gg 1 
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• This quantity was named the bremsstrahlung function $B(\lambda)$

[Correa, Henn Maldacena, Sever]

• Calculates the radiation of an accelerated quark.

• Is related to small deformations of BPS Wilson loops and can be calculated exactly

\[ B = \frac{1}{2 \pi^2} \lambda \partial_\lambda \langle W_\circ \rangle \]

\[ \langle W_\circ \rangle = \frac{1}{N} L_{N-1}^{1} \left( -\frac{\lambda}{4N} \right) e^{\frac{-\lambda}{8N}} \]
Result so far:

Explicit expressions for these families of Wilson loops at weak and strong coupling.
Wilson loops and integrability

- We want to apply the tools of integrability to the case of Wilson loops:
  - Find a spin-chain model.
  - Find the all loop scattering (and reflection) matrix
  - Try to solve it exactly.
- This will allow to derive the gauge theory perturbative results from world-sheet techniques.
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- This will allow to derive the gauge theory perturbative results from world-sheet techniques.

- Main trick will be to start with the Wilson loop with an arbitrary insertion in it, which will simplify the steps above and at the end remove the insertion.

- In the case of the straight line, after removing the insertion, the operator is $1/2$ BPS, so no anomalous dimension. So need to know how to treat the cusp.
Local operator insertions

- There is another source of log divergences in Wilson loops:
  Adjoint valued operators inserted into the Wilson loop.
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- For example, one operator in the straight line

\[ W = \text{Tr} \mathcal{P} \left[ \mathcal{O}(0) \exp \left( \int (iA_\mu \dot{x}^\mu + \Phi^I n^I |\dot{x}|) \, ds \right) \right] \]

\[ = \text{Tr} \left[ \mathcal{P} \exp \left( \int_{-\infty}^{0} (iA_\mu \dot{x}^\mu + \Phi^I n^I |\dot{x}|) \, ds \right) \mathcal{O}(0) \mathcal{P} \exp \left( \int_{0}^{\infty} (iA_\mu \dot{x}^\mu + \Phi^I n^I |\dot{x}|) \, ds \right) \right] \]

- \( \mathcal{O} \) is any adjoint operator, e.g., \( F_{23}, Z^L \), etc.
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- In a conformal theory, a Wilson loop with two operator insertions at a distance \(d\) will have a VEV

\[
\langle W \rangle \sim \frac{1}{d^{2\Delta}}
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- \(\Delta\) is the coefficient of the log divergences — the conformal dimension of the insertions.
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- Starting with and insertion of \( Z^J \) and replacing some of the \( Z \) by other fields, we will find a spin-chain model.
**string picture**

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- Study the spectrum of open string states all satisfying the same boundary conditions.
• An insertion of $Z^J$ is described by a string ending along the same curve on the boundary but in the bulk of space rotating around the equator of $S^5$ with momentum $J$.

• An excitation traveling along this string will not know that it’s an open string and not the usual $\text{Tr} Z^J$ vacuum.
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• Once it gets to the end of the string we should impose boundary conditions.
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**Gauge theory picture**

We take the cusped Wilson loop with an adjoint valued operator like $Z^J$ at the cusp.

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• Boundary interaction has to be studied separately.

• The two boundaries interact through wrapping effects at $O(g^{2(J+1)})$.

• For $J = 0$ this is at one-loop.
All loop reflection matrix and a twist

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- To do it to all loops we should use the symmetry:

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\begin{align*}
\text{psu}(2, 2|4) & \quad \longrightarrow \quad \text{psu}(2|2)_L \times \text{psu}(2|2)_R \\
\text{boundary} & \quad \downarrow \quad \downarrow \\
\text{osp}(4^*|4) & \quad \longrightarrow \quad \text{psu}(2|2)_D
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- To do it to all loops we should use the symmetry:

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\text{psu}(2, 2|4) & \xrightarrow{Z^J \text{ vacuum}} \text{psu}(2|2)_L \times \text{psu}(2|2)_R \\
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- A single boundary breaks the symmetry to a diagonal $\text{psu}(2|2)$.
- By the usual argument, the boundary reflection matrix should have the same matrix structure as the bulk one

$$\mathbb{R}_{a\dot{a}}^{bb}(p) = R_0(p)\hat{S}_{a\dot{a}}^{bb}(p, -p)$$

- It replaces $\text{psu}(2|2)_L \leftrightarrow \text{psu}(2|2)_R$ labels.
• Need to determine 
  \[ R_0(p) = \frac{\sigma_B(p)}{\sigma(p, -p)}. \]

• Like the crossing relation in the bulk, there is a boundary “crossing-unitarity equation”
  \[ \mathbb{R}(p) = S(p, -p)\mathbb{R}^c(p) \]
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• This translates to the conditions on \( \sigma_B \)
  \[ \sigma_B(p)\sigma_B(\bar{p}) = \frac{x^- + 1/x^-}{x^+ + 1/x^+}, \quad \sigma_B(p)\sigma_B(\bar{p}) = 1. \]

where the Joukowsky variables are a solution of
\[ e^{ip} = \frac{x^+}{x^-}, \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{1}{g}. \]
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• The solution which matches the all consistency requirements is
\[ \sigma_B(z) = \frac{1 + 1/(x^-)^2}{1 + 1/(x^+)^2} e^{-i\chi_B(x^+) + i\chi_B(x^-)} \]

where
\[ \chi_B(x) = -i \int \frac{dz}{2\pi i} \frac{1}{x-z} \log \frac{\sinh 2\pi g(z+1/z)}{2\pi g(z+1/z)}. \]
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• The left boundary is essentially the same.

• The choice of diagonal subgroup $\text{psu}(2|2)_L \times \text{psu}(2|2)_R \to \text{psu}(2|2)_{D'}$ may be different.

• Conjugate the reflection matrix by a twist matrix $G$ acting on the $\text{psu}(2|2)_L$ labels

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• But not the case $J = 0 \ldots$
Wrapping effects and the quark-antiquark potential

- One can derive a set of boundary thermodynamic Bethe ansatz equations for this open spin-chain.

- This can be simplified in the small angle limit, where the full answer was reproduced. 
  \[ \text{Correa, Maldacen, Sev, Gromov} \]

- They are the same as the usual TBA equations with several small modifications:
  - The $Y$ functions are related by reflection $Y_{a,s}(-u) = Y_{a,-s}(u)$
  - There are chemical potentials dependent on $\phi$ and $\theta$.
  - There is a complicated driving term for the massive $Y_{a,0}$ nodes (aka $Y_Q$).

- The $Y$-system equations are unmodified.
  - Analytic properties of the functions are different (determined by the asymptotic solution).
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by repeated use of the Yang-Baxter equation this simplifies to

• That is just the product of two twisted $\text{psu}(2|2)$ transfer matrices.
On the $Z^J$ vacuum this is

\[
T_Q^{\phi,\theta}(p) = s\text{Tr} \left[ R^{(R)}(p) R^{(L)}(\bar{p}) \right] = s\text{Tr} \left[ R^{(R)}(p) G R^{(R)}(-\bar{p}) G \right]
\]

\[
= \sigma_B(p) \sigma_B(-\bar{p}) \left( \frac{x^-}{x^+} \right)^2 \left( s\text{Tr} \ G \right)^2
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\]
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= \sigma_B(p) \sigma_B(-\bar{p}) \left( \frac{x^-}{x^+} \right)^2 (s\text{Tr} \mathbb{G})^2
\]

• Simple group theory gives
\[
(s\text{Tr} Q \mathbb{G})^2 = 4(\cos \phi - \cos \theta)^2 \frac{\sin^2 Q\phi}{\sin^2 \phi}
\]

And the Lüscher-Bajnok-Janik formula is
\[
\delta E \approx -\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{0}^{\infty} d\bar{p} \log \left( 1 + T_{Q}^{(\phi,\theta)}(\bar{p}) e^{-2J \hat{E}_Q} \right)
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• On the $Z^J$ vacuum this is

$$T_Q^\phi,\theta(p) = s\text{Tr} \left[ R^R(p) R^L(p) \right] = s\text{Tr} \left[ R^R(p) G R^R(-\bar{p}) G \right]$$

$$= \sigma_B(p) \sigma_B(-\bar{p}) \left( \frac{x^-}{x^+} \right)^2 (s\text{Tr} G)^2$$

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• Normally for small $g$ (or large $J$) can expand the logarithm

$$\delta E \approx \frac{1}{2\pi} \sum_{Q=1}^\infty \int_0^\infty d\tilde{p} \ T_Q^{(\phi,\theta)}(\tilde{p}) e^{-2J\tilde{E}_Q}$$

For $J = 0$ the answer will be proportional to $\frac{g^4(\cos \phi - \cos \theta)^2}{\sin^2 \phi} \ldots$
• Crucial fact is that the dressing factor has a double pole at $\tilde{p} = 0$

$$\sigma_B(\tilde{p})\sigma_B(-\tilde{p}) = e^{2i(\chi_B(x^+)+\chi_B(x^-))} \frac{(2\pi g)^2(x^+ + 1/x^+)(x^- + 1/x^-)}{\sinh(2\pi g(x^+ + 1/x^+))\sinh(2\pi g(x^- + 1/x^-))}$$

$$= e^{2i(\chi_B(x^+)+\chi_B(x^-))} \frac{(2\pi)^2(u^2 + Q^2/4)}{\sinh^2(2\pi u)} \sim \frac{Q^2}{\tilde{p}^2}$$

• Then using

$$\int_0^\infty d\tilde{p} \log \left( 1 + \frac{c}{\tilde{p}^2} \right) = \pi \sqrt{c},$$
• Crucial fact is that the dressing factor has a double pole at \( \tilde{p} = 0 \)

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= e^{2i(\chi_B(x^+) + \chi_B(x^-))} \left(\frac{2\pi}{\sinh(2\pi u)}\right)^2 \sim \frac{Q^2}{\tilde{p}^2}
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• Then using

\[
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\]

• The residue is

\[
\sqrt{T_Q^{\text{res}} e^{-2J\tilde{E}_Q}} = 2 \frac{\cos \phi - \cos \theta}{\sin \phi} \sin Q\phi (-1)^Q \left[ \frac{(4g^2)^{J+1}}{Q^{2J+1}} - 2(J + 2) \frac{(4g^2)^{J+2}}{Q^{2J+3}} + \cdots \right]
\]

• so

\[
\delta E \approx -(4g^2)^{J+1} \frac{\cos \phi - \cos \theta}{\sin \phi} \sum_{Q=1}^{\infty} \frac{(-1)^Q \sin Q\phi}{Q^{2J+1}} \\
= -(4g^2)^{J+1} \frac{\cos \phi - \cos \theta}{2i} \frac{1}{\sin \phi} \left( \text{Li}_{2J+1}(-e^{i\phi}) - \text{Li}_{2J+1}(-e^{-i\phi}) \right)
\]
For $J = 0$

\[
\delta E \approx -\frac{4g^2}{2i} \frac{\cos \phi - \cos \theta}{\sin \phi} \left( \text{Li}_1(-e^{i\phi}) - \text{Li}_1(-e^{-i\phi}) \right)
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= 2g^2 i \frac{\cos \phi - \cos \theta}{\sin \phi} \left( -\log(1 + e^{i\phi}) + \log(1 + e^{-i\phi}) \right)
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= 2g^2 \frac{\cos \phi - \cos \theta}{\sin \phi} \phi + O(g^4)
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- This integrability calculation is in exact agreement with the one loop perturbative calculation.
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- This integrability calculation is in exact agreement with the one loop perturbative calculation.

- For Konishi wrapping started at 4 loop order. The cusped Wilson loop is given purely by wrapping from one loop on.

- Is possible to solve iteratively to get higher orders.

- Numerics are hard, but people are working on it.

- Should also be possible to extract the strong coupling answer analytically.
Summary

When I talked about my paper with Valentina a year ago I would end with the question

Will there be a gauge theory derivation of the strong coupling potential:

\[ V(L, \lambda) = \frac{4\pi^2 \sqrt{\lambda}}{\Gamma(\frac{1}{4})^4 L} \]
Summary

When I talked about my paper with Valentina a year ago I would end with the question

Will there be a gauge theory derivation of the strong coupling potential:

$$V(L, \lambda) = \frac{4\pi^2 \sqrt{\lambda}}{\Gamma(\frac{1}{4})^4 L}$$

We are very close to answering Yes!
The end