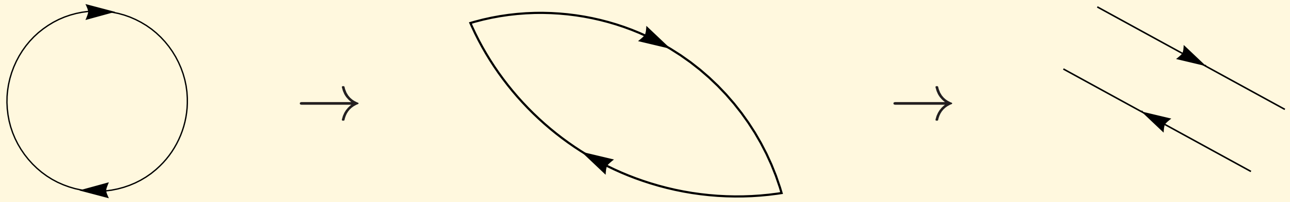


The quark-antiquark potential in $\mathcal{N} = 4$ SYM from an open spin-chain

Nadav Drukker

KING'S College LONDON

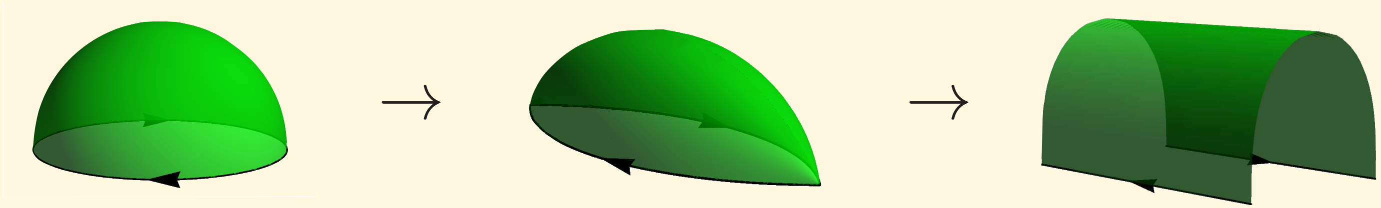
Based on `arXiv:1105.5144` - N.D. and V. Forini

`arXiv:1203.1617` - N.D.

See also `arXiv:1203.1913` - D. Correa, J. Maldacena and A. Sever

Kavli IPMU

January 29, 2013.



Introduction and motivation

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- In gauge theories this is captured by a long rectangular Wilson loop, or a pair of antiparallel lines.

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- Shouldn't integrability allow us to calculate this for all values of the coupling (in the planar approximation)?

Outline

- Introduction and motivation
- Wilson loops
  - Cusp anomalous dimensions and the quark-antiquark potential
  - Local operator insertions
- Generalize quark-antiquark potential in $\mathcal{N} = 4$ SYM
  - Perturbative calculation
  - String calculation
  - Expansions in small angles
- Wilson loops and integrability
  - Operator insertions and open spin–chains
  - All loop reflection matrix and a twist
  - Wrapping effects and the quark-antiquark potential

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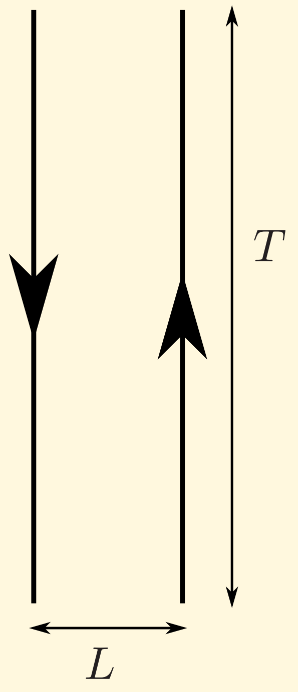

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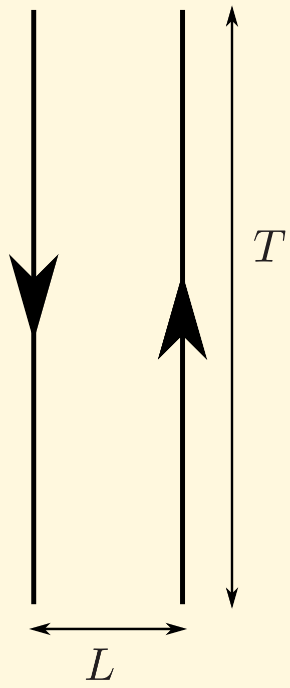

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- In $\mathcal{N}=4$ SYM the most natural Wilson loops includes a coupling to the scalar fields

$$W = \mathrm{Tr}\,\mathcal{P}\exp\left[\oint \big(iA_\mu \dot{x}^\mu + |\dot{x}| n^I \Phi_I\big) ds\right]$$

  n^I do not have to be constant.

- For a smooth loop and continuous $|n^I| = 1$, these are finite observables.

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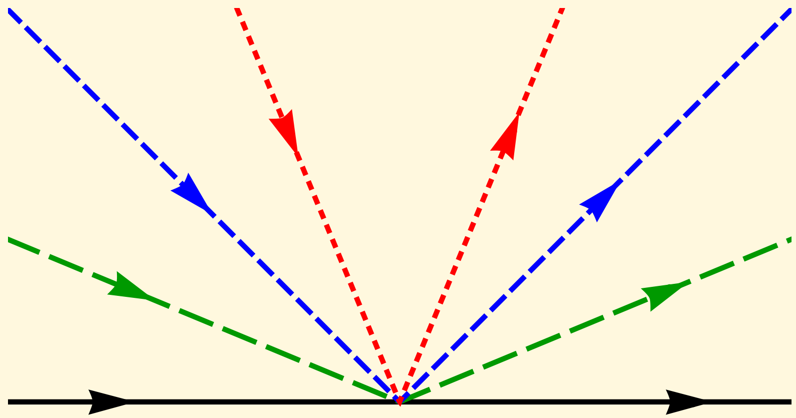
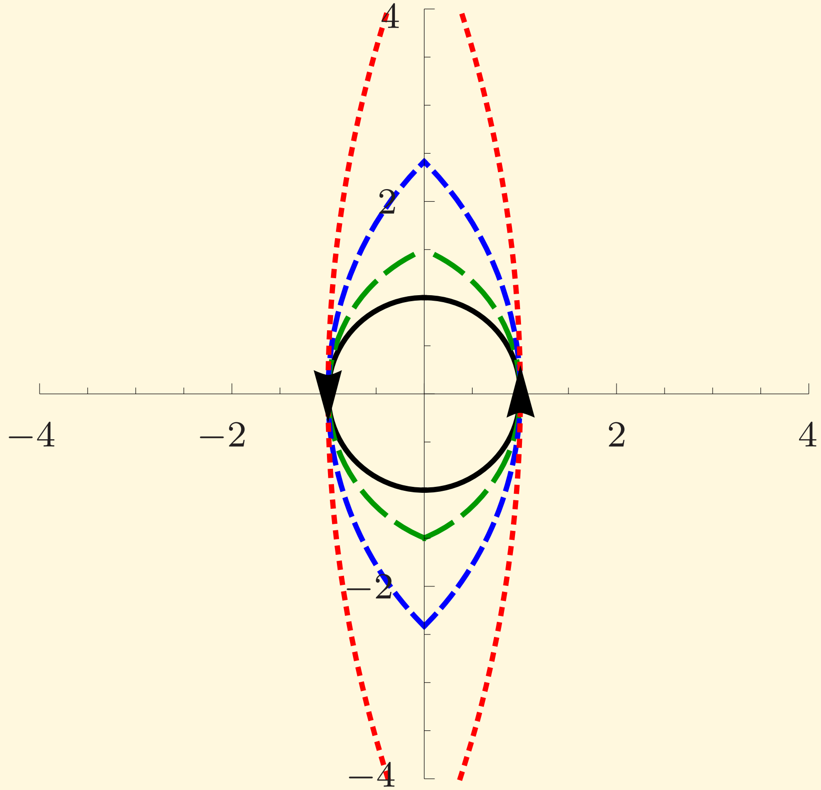
- It is simpler to control logarithmic divergences.
- Consider Wilson loops with cusps



- All but the black line will suffer from logarithmic divergences.
- Taking $\phi = i\varphi$ and $\varphi \to \infty$ gives the Lorenzian null cusp.

- A compact versions of cusped loops.
- No gauge-invariance subtleties!



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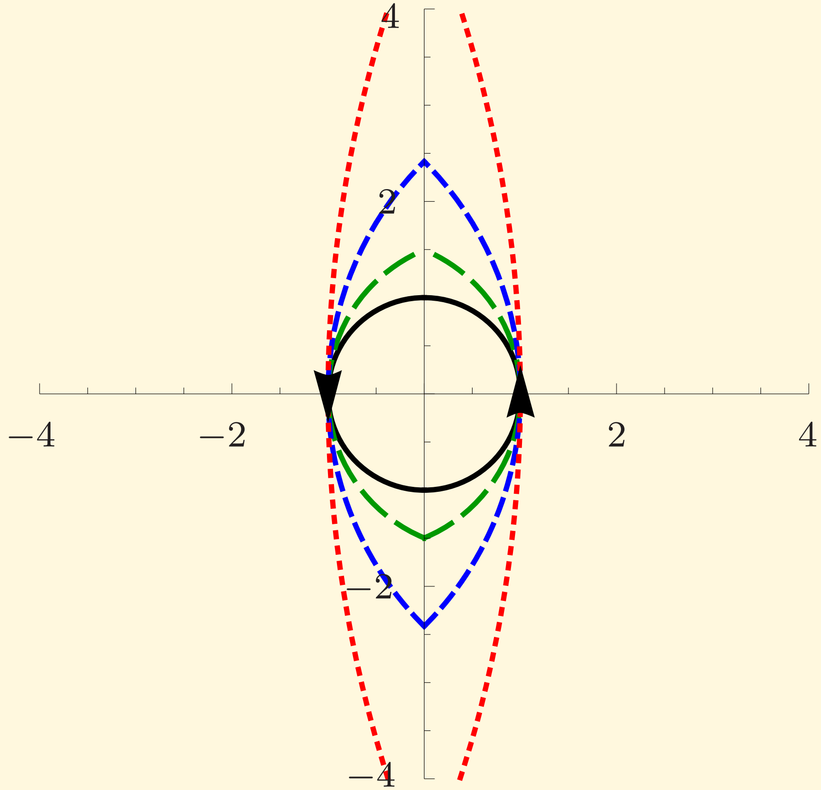

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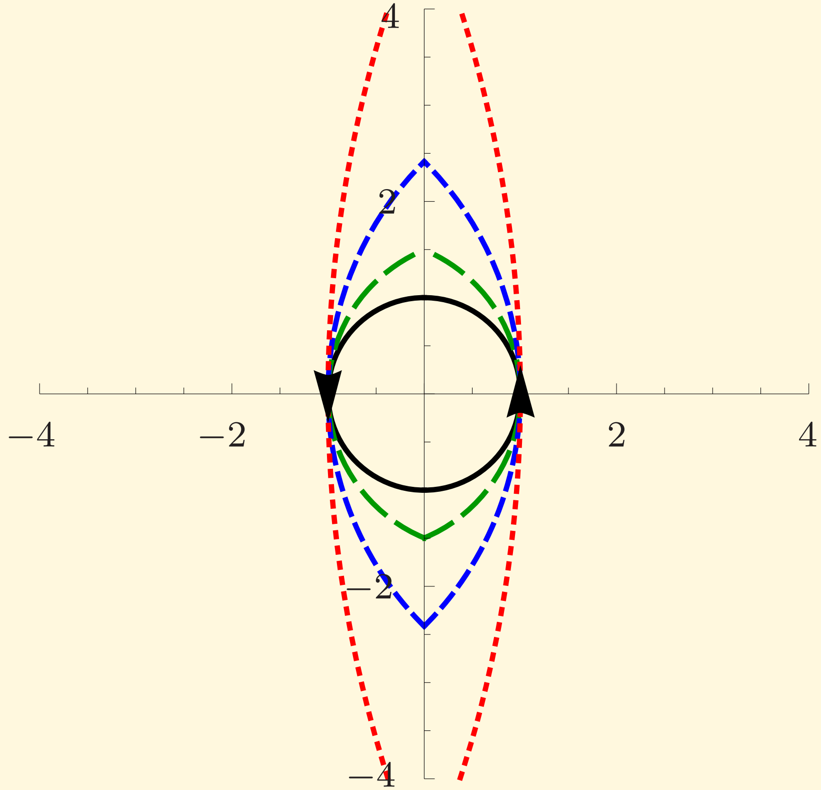

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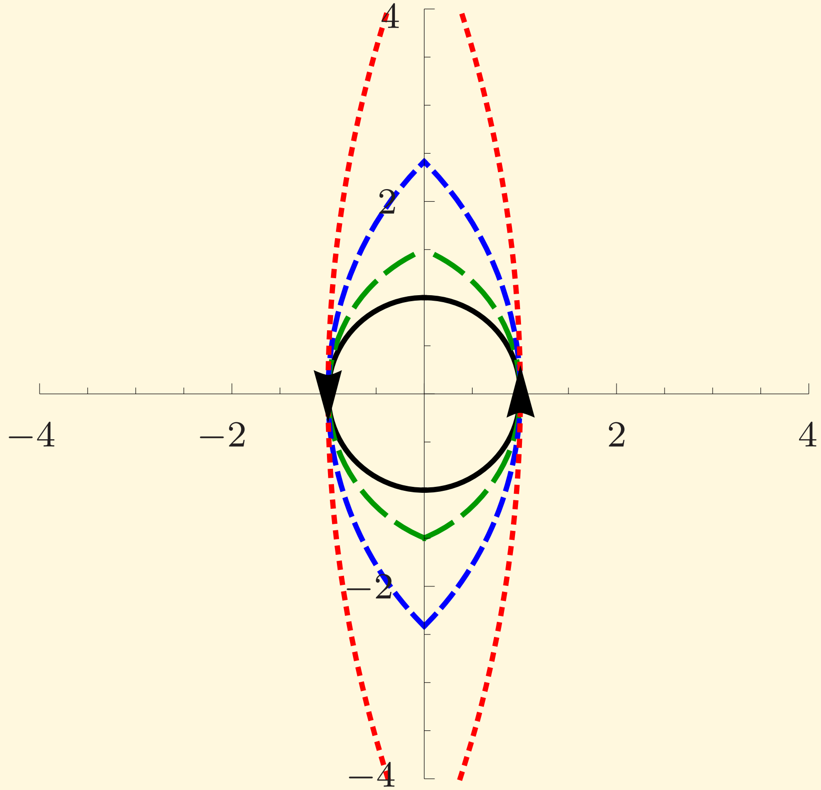
- Can couple the two arcs to two different scalar fields

$$\Phi_1 \qquad \text{and} \qquad \Phi_1 \cos\theta + \Phi_2 \sin\theta$$



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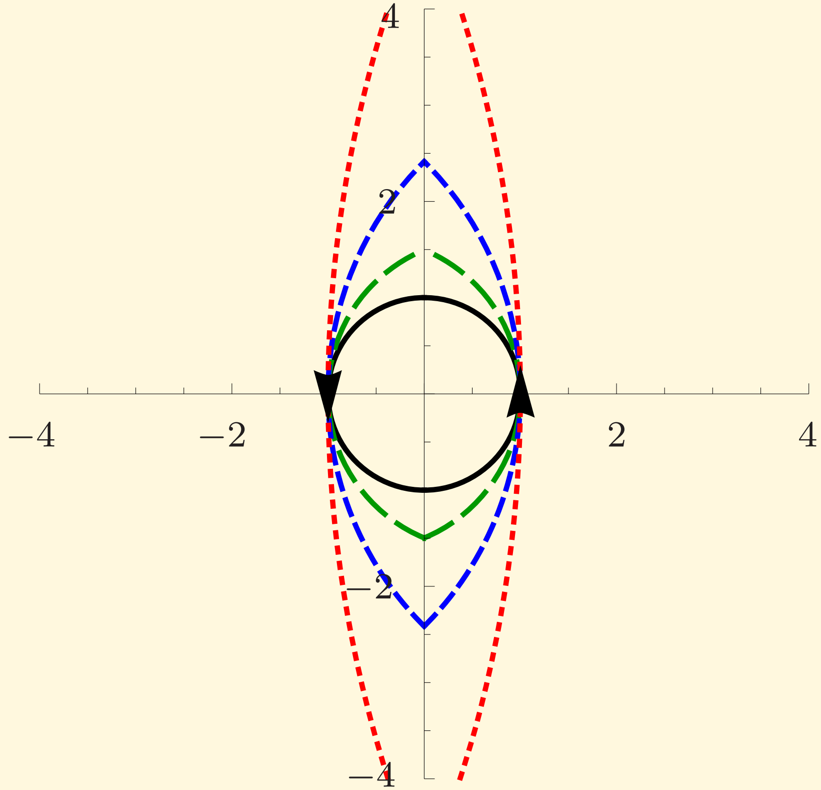

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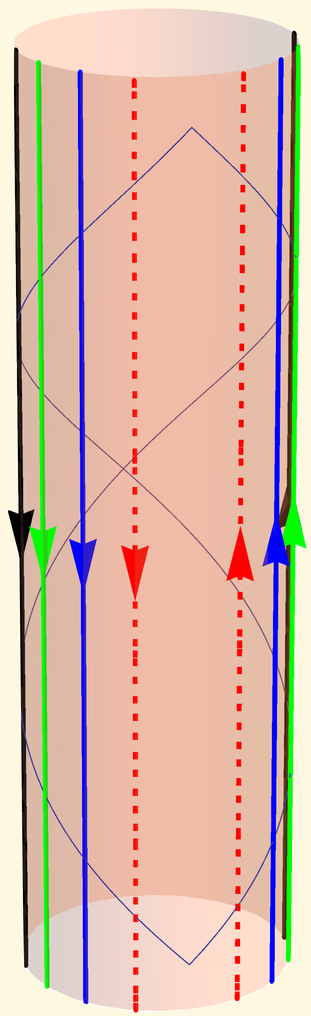
- In a conformal theory, by the usual conformal Ward identity

$$\langle W \rangle \sim \frac{1}{d^{2\Delta}}\,, \qquad\qquad d = r\,\frac{\cos\frac{\phi}{2}}{1-\sin\frac{\phi}{2}}$$

- Δ is the coefficient of the log divergence.

- By the inverse exponential map we get the gauge theory on $\mathbb{S}^3 \times \mathbb{R}$
- These are parallel lines on $\mathbb{S}^3 \times \mathbb{R}$.



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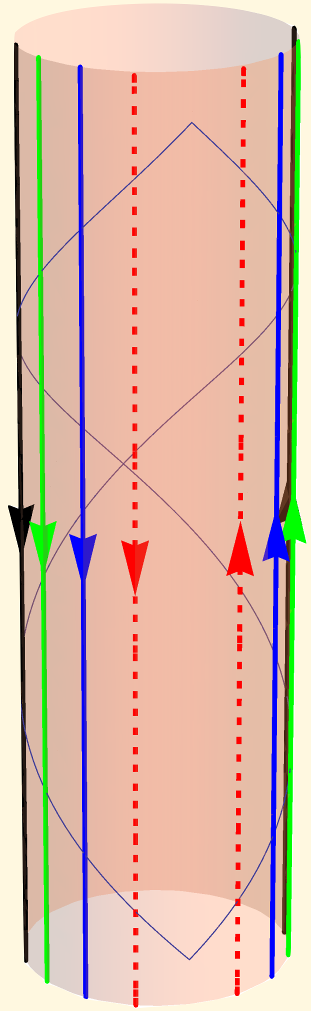
- From this last picture we expect

$$\langle W \rangle \approx \exp\Big[-T\,V(\phi,\theta,\lambda)\Big]$$

- In a conformal theory T is related to divergence at the cusp by the exponential map

$$T = \log \frac{\Lambda_{\mathrm{IR}}}{\Lambda_{\mathrm{UV}}}$$

- Therefore $V(\phi,\theta,\lambda)$ is the same as Δ, the coefficient of the log divergence.



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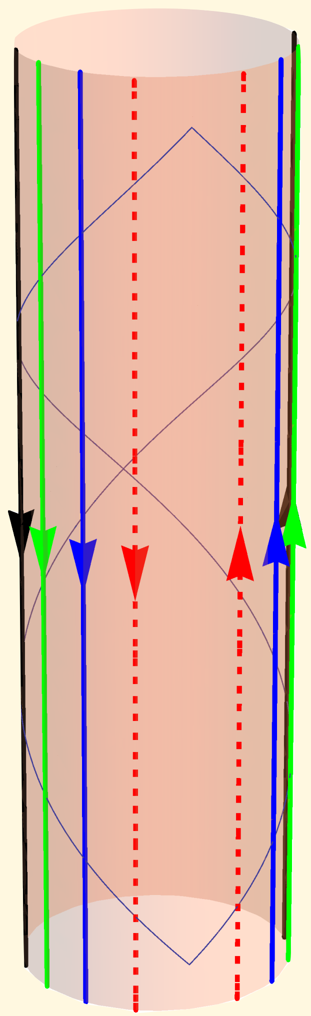
- Therefore $V(\phi, \theta, \lambda)$ is the same as Δ, the coefficient of the log divergence.
- This $V(\phi, \theta, \lambda)$ is the generalization of $V(L, \lambda)$ — the quark-antiquark potential.
- For a conformal theory it has a pole at $\phi \to \pi$ and the residue is $LV(L, \lambda)$.
- More generally controls all log divergences of all Wilson loops.
- Needed for a proper renormalization program of Wilson loop operators (and to derive regularized loop equations).



Generalized quark-antiquark potential in $\mathcal{N} = 4$ SYM

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- Expanding at weak coupling

$$V(\phi, \theta, \lambda) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{16\pi^2} \right)^n V^{(n)}(\phi, \theta)$$

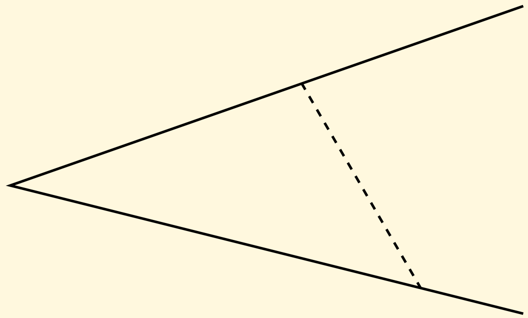
- And at strong coupling

$$V(\phi, \theta, \lambda) = \frac{\sqrt{\lambda}}{4\pi} \sum_{l=0}^{\infty} \left(\frac{4\pi}{\sqrt{\lambda}} \right)^l V_{AdS}^{(l)}(\phi, \theta)$$

Perturbative calculation

1–loop

- Just the exchange of a gluon and scalar field



- This graph is given by the integral

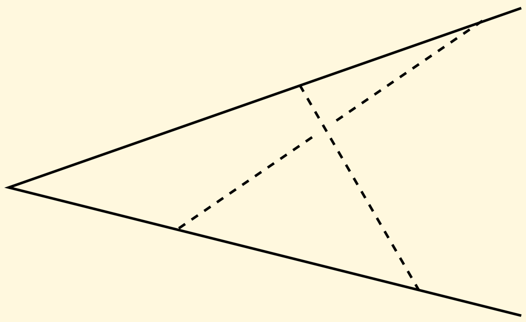
$$\partial_\lambda \langle W\rangle\Big|_{\lambda=0} = \int_{s<t} ds\,dt\,\Big\langle\left(iA_\mu \dot x^\mu(s) + |\dot x|\Phi^I n^I(s)\right)\left(iA_\nu \dot x^\nu(t) + |\dot x|\Phi^J n^J(t)\right)\Big\rangle$$
$$= \frac{\lambda}{8\pi^2}\int ds\,dt\,\frac{-\dot x_\mu(s)\dot x^\mu(t) + n^I(s)n^I(t)}{|x(s)-x(t)|^2}$$
$$= \frac{\lambda}{8\pi^2}\int ds\,dt\,\frac{-\cos\phi + \cos\theta}{s^2+t^2+2st\cos\phi} = -\frac{\lambda}{8\pi^2}\,\frac{\cos\phi-\cos\theta}{\sin\phi}\,\phi\log\frac{R}{\epsilon}$$

- Therefore

$$V^{(1)}(\phi,\theta) = 2\,\frac{\cos\phi-\cos\theta}{\sin\phi}\,\phi$$

Higher order graphs

- Ladder graphs are relatively easy.
- They dominate a funny double-scaled limit where $\theta \to i\infty$ with $\lambda\theta$ fixed. [Correa, Henn Maldacena, Sever]
- They are given by harmonic polylogs apparently to all orders. [Henn, Huber]
- Results at weak and strong coupling match.



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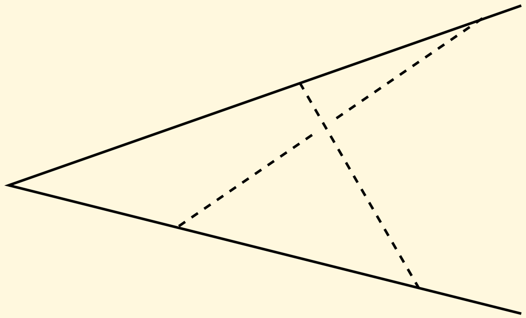
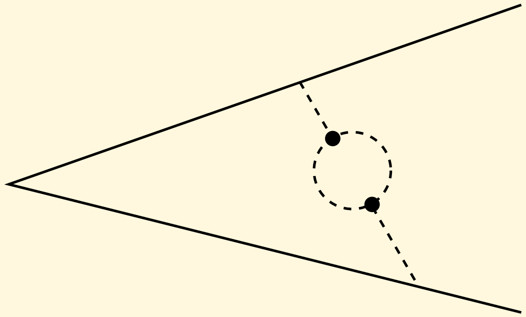
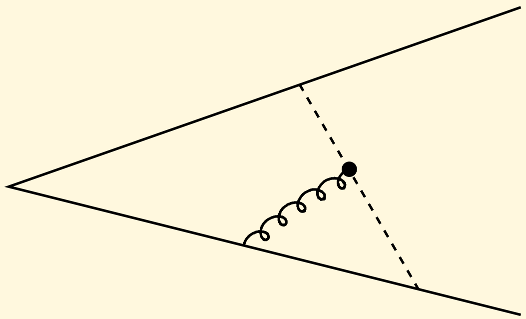


- Interacting graphs are a bit more complicated.
- At two loops there are bubble graphs and the single cubic vertex graphs.
- they give

$$V^{(2)}_{\rm int}(\phi,\theta) = -\frac{2}{3}(\pi^2-\phi^2)V^{(1)}(\phi,\theta)$$



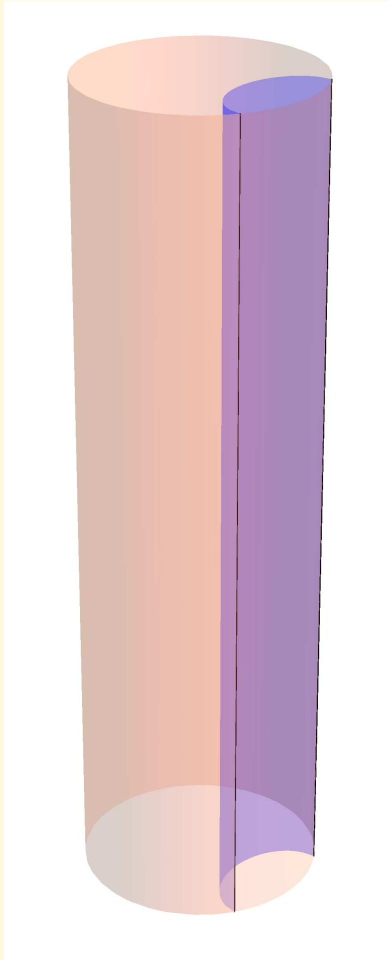


- Full 3 loop answer was also calculated. [Correa, Henn Maldacena, Sever]

String calculation

[Maldacena][Rey, Yee][Drukker Gross, Ooguri]

- Within the AdS/CFT correspondence Wilson loops are calculated by an infinite open string extending to the boundary of AdS.



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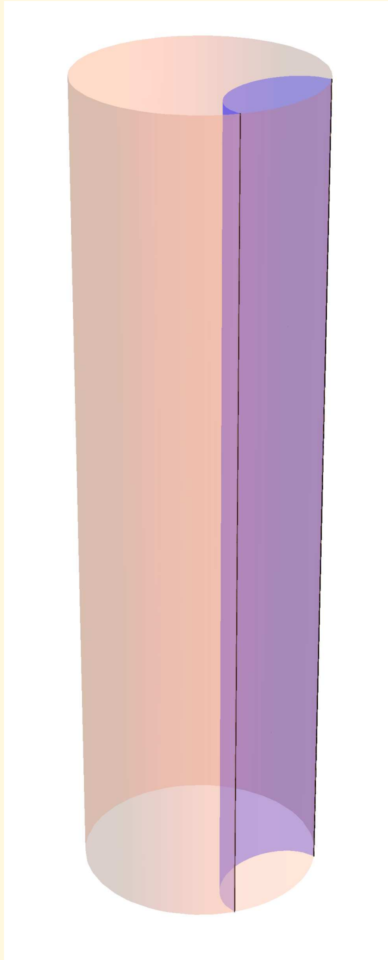

- At the leading order we should find the minimal surface ending on lines separated by $\pi - \phi$ on the boundary of AdS and θ on $\mathbb{S}^5$.
- All the string solutions fit inside $AdS_3 \times \mathbb{S}^1$

$$ds^2 = \sqrt{\lambda}\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\varphi^2 + d\vartheta^2\right)$$

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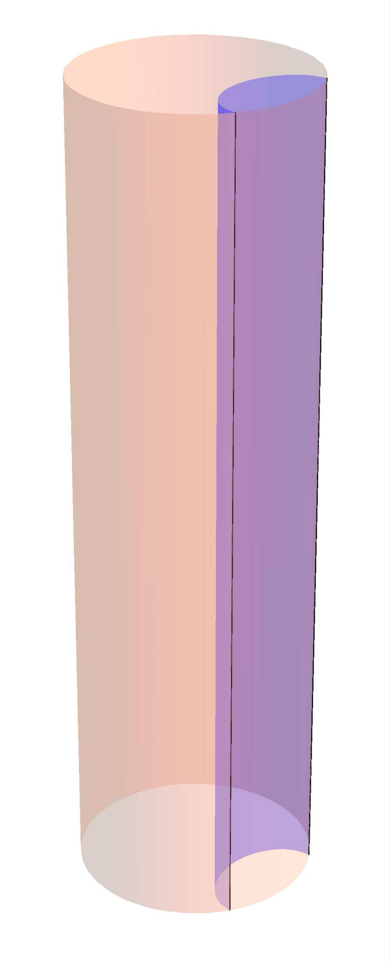

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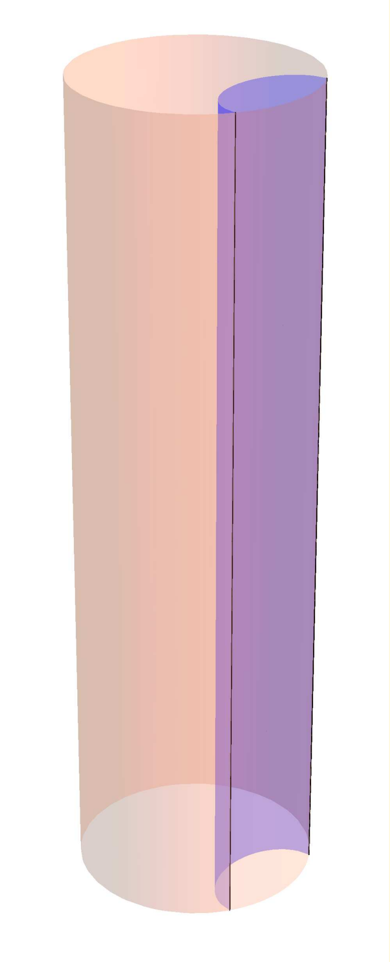

- The equations of motion and classical action can be solved by elliptic integrals.
- $V^{(0)}_{AdS}$ given by a solution of a transcendental equation
- Expand around $\phi = \theta = 0$ the answer is

$$\begin{aligned} V^{(0)}_{AdS}(\phi,\theta) = &\frac{1}{\pi}(\theta^2 - \phi^2) - \frac{1}{8\pi^3}(\theta^2-\phi^2)\left(\theta^2 - 5\phi^2\right) \\ &+ \frac{1}{64\pi^5}(\theta^2-\phi^2)\left(\theta^4 - 14\theta^2\phi^2 + 37\phi^4\right) \\ &- \frac{1}{2048\pi^7}(\theta^2-\phi^2)\left(\theta^6 - 27\theta^4\phi^2 + 291\theta^2\phi^4 - 585\phi^6\right) + O((\phi,\theta)^{10}) \end{aligned}$$

1–loop determinant

- Complicated fluctuation problem.
- Can be done analytically (implicitly) for either $\phi = 0$ or $\theta = 0$.
- For $\theta = 0$ and small ϕ we can expand

$$V^{(1)}_{AdS}(\phi, 0) = \frac{3}{2}\frac{\phi^2}{4\pi^2} + \left(\frac{53}{8} - 3\,\zeta(3)\right)\frac{\phi^4}{16\pi^4} + \left(\frac{223}{8} - \frac{15}{2}\zeta(3) - \frac{15}{2}\zeta(5)\right)\frac{\phi^6}{64\pi^6}$$
$$+ \left(\frac{14645}{128} - \frac{229}{8}\zeta(3) - \frac{55}{4}\zeta(5) - \frac{315}{16}\zeta(7)\right)\frac{\phi^8}{256\pi^8} + O(\phi^{10})$$

$\phi \to \pi$ limit

- $V^{(1)}$, $V^{(2)}$, $V^{(0)}_{AdS}$ and $V^{(1)}_{AdS}$ all have poles at $\phi = \pi$
- In perturbation theory

$$V(\phi,\theta) \to -\frac{\lambda}{8\pi}\frac{1+\cos\theta}{\pi-\phi} + \frac{\lambda^2}{32\pi^3}\frac{(1+\cos\theta)^2}{\pi-\phi}\log\frac{e}{2(\pi-\phi)} + O(\lambda^3)$$

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- In the case of $\theta = 0$ we get essentially the same as the antiparallel lines with $L \to \pi - \phi$

$$V(L,\lambda) = \begin{cases} -\dfrac{\lambda}{4\pi L} + \dfrac{\lambda^2}{8\pi^2 L}\ln\dfrac{T}{L} + \cdots & \lambda \ll 1 \\ \dfrac{4\pi^2\sqrt{\lambda}}{\Gamma(\frac{1}{4})^4 L}\left(1 - \dfrac{1.3359\ldots}{\sqrt{\lambda}} + \cdots\right) & \lambda \gg 1 \end{cases}$$

- The strong coupling calculations also agree in the limit.

Expansions in small angles

- Consider the expansion of $V(\phi,\theta,\lambda)$ at small ϕ or θ

$$\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0}=-\frac{1}{2}\frac{\partial^2}{\partial\phi^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0}=\begin{cases}\dfrac{\lambda}{16\pi^2}-\dfrac{\lambda^2}{384\pi^2}+\cdots & \lambda\ll 1\\[2ex] \dfrac{\sqrt{\lambda}}{4\pi^2}-\dfrac{3}{8\pi^2}+\cdots & \lambda\gg 1\end{cases}$$

Expansions in small angles

- Consider the expansion of $V(\phi,\theta,\lambda)$ at small ϕ or θ

$$\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0}=-\frac{1}{2}\frac{\partial^2}{\partial\phi^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0}=\begin{cases}\dfrac{\lambda}{16\pi^2}-\dfrac{\lambda^2}{384\pi^2}+\cdots & \lambda\ll 1\\[2ex] \dfrac{\sqrt{\lambda}}{4\pi^2}-\dfrac{3}{8\pi^2}+\cdots & \lambda\gg 1\end{cases}$$

- This quantity was named the bremsstrahlung function $B(\lambda)$ [Correa, Henn Maldacena, Sever]

- Calculates the radiation of an accelerated quark.

- Is related to small deformations of BPS Wilson loops and can be calculated exactly

$$B=\frac{1}{2\pi^2}\lambda\partial_\lambda\langle W_\circ\rangle$$

$$\langle W_\circ\rangle=\frac{1}{N}\,L^1_{N-1}\left(-\frac{\lambda}{4N}\right)e^{\frac{\lambda}{8N}}$$

Result so far:

Explicit expressions for these families of Wilson loops at weak and strong coupling.

Wilson loops and integrability

- We want to apply the tools of integrability to the case of Wilson loops:
  - Find a spin–chain model.
  - Find the all loop scattering (and reflection) matrix
  - Try to solve it exactly.
- This will allow to derive the gauge theory perturbative results from world-sheet techniques.

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  - Find the all loop scattering (and reflection) matrix
  - Try to solve it exactly.
- This will allow to derive the gauge theory perturbative results from world-sheet techniques.
- Main trick will be to start with the Wilson loop with an arbitrary insertion in it, which will simplify the steps above and at the end remove the insertion.
- In the case of the straight line, after removing the insertion, the operator is $1/2$ BPS, so no anomalous dimension. So need to know how to treat the cusp.

Local operator insertions

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  Adjoint valued operators inserted into the Wilson loop.

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\begin{aligned}
W &= \mathrm{Tr}\,\mathcal{P}\left[\mathcal{O}(0)\exp\left(\int(iA_\mu\dot{x}^\mu+\Phi^I n^I|\dot{x}|)ds\right)\right] \\
&= \mathrm{Tr}\left[\mathcal{P}\exp\left(\int_{-\infty}^{0}(iA_\mu\dot{x}^\mu+\Phi^I n^I|\dot{x}|)ds\right)\mathcal{O}(0)\mathcal{P}\exp\left(\int_{0}^{\infty}(iA_\mu\dot{x}^\mu+\Phi^I n^I|\dot{x}|)ds\right)\right]
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- Δ is the coefficient of the log divergences — the conformal dimension of the insertions.
- Starting with and insertion of Z^J and replacing some of the Z by other fields, we will find a spin-chain model.

string picture

- The string dual of a Wilson loop with an insertion is an excited state of the open string describing the Wilson loop.

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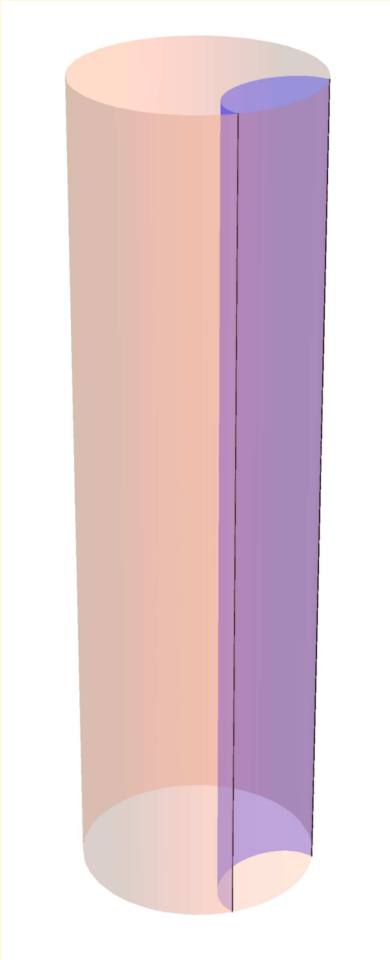

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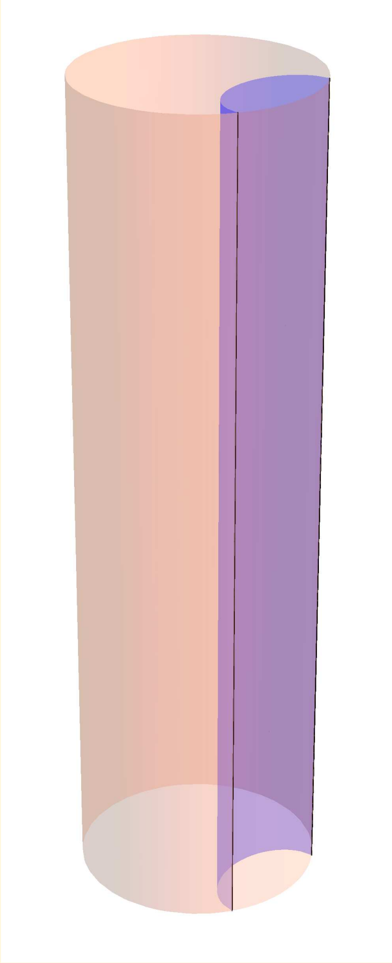
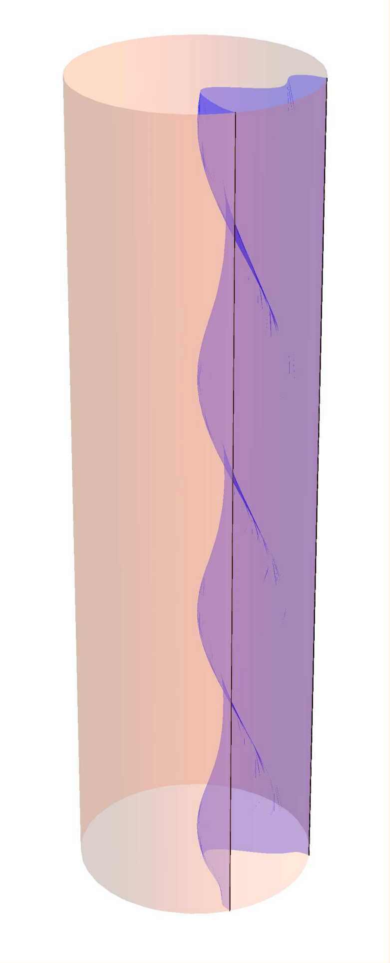


$\Longrightarrow$



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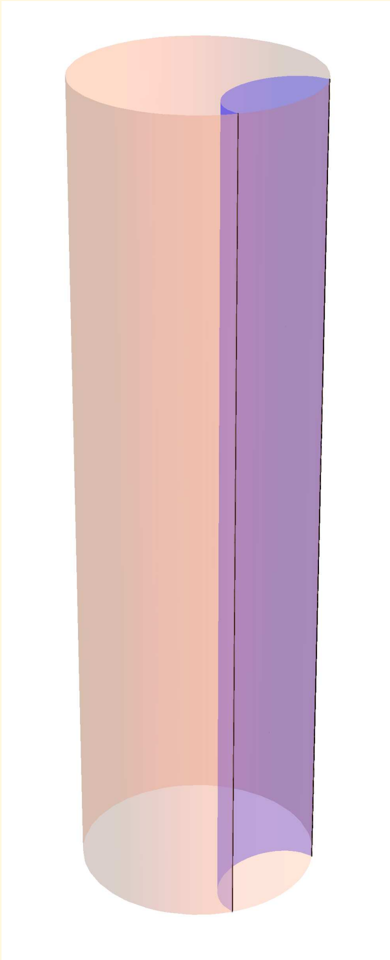

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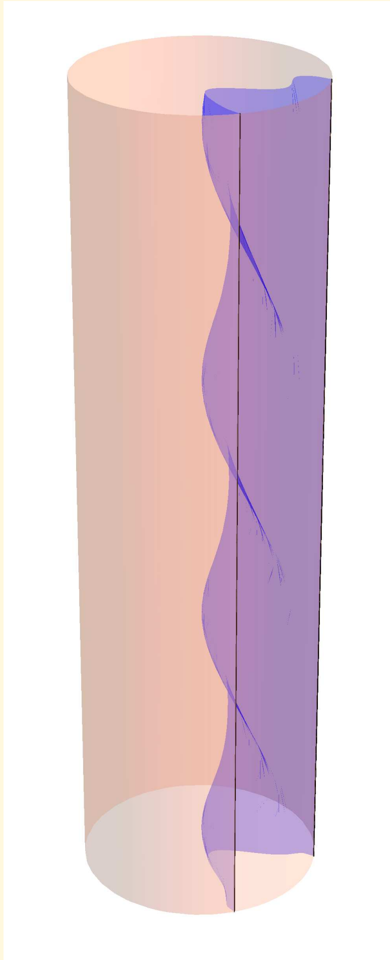

- Study the spectrum of open string states all satisfying the same boundary conditions.

- An insertion of Z^J is described by a string ending along the same curve on the boundary but in the bulk of space rotating around the equator of $\mathbb{S}^5$ with momentum J.
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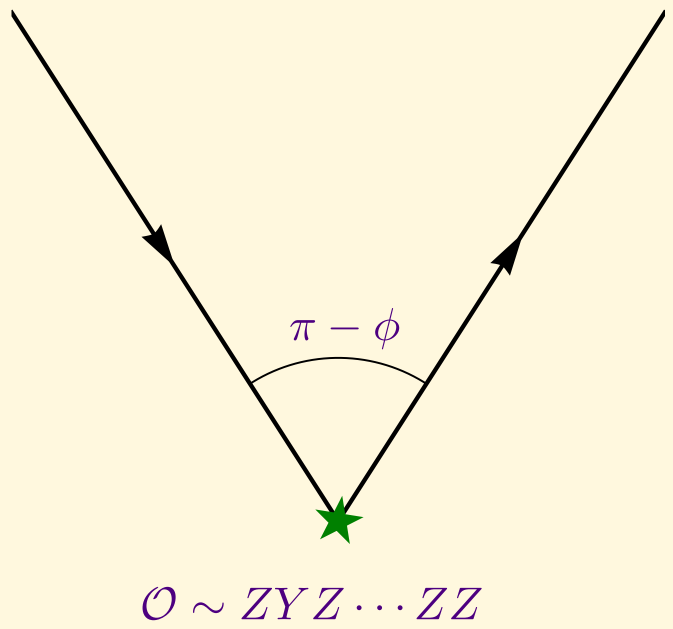
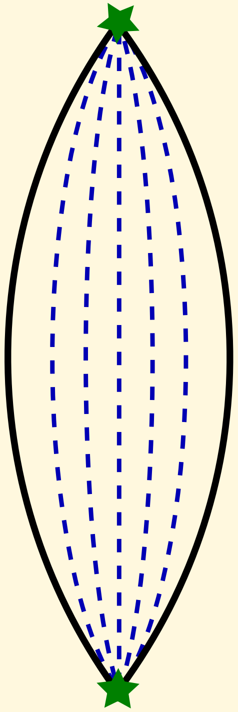
Gauge theory picture

We take the cusped Wilson loop with an adjoint valued operator like Z^J at the cusp.



- It is clear how to see the appearance of the spin–chain by considering the compact operator in the gauge theory



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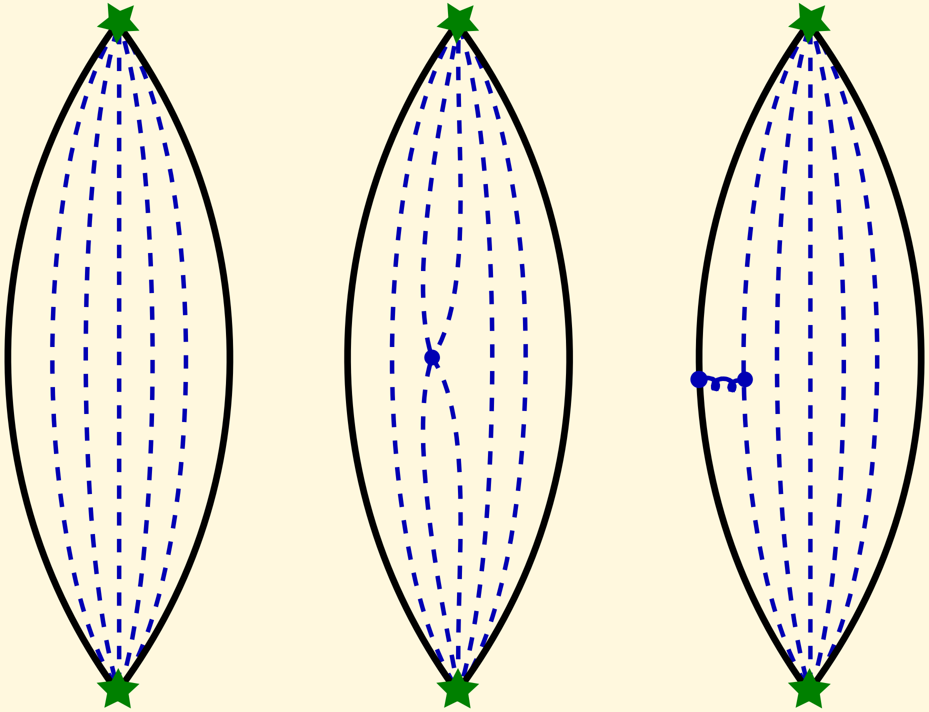

- In this case the classical dimension is 5.
- The bulk hamiltonian is like the usual Minahan-Zarembo, Staudacher spin–chain (Beisert S-matrix $\mathbb{S}_{ab}^{cd}(p_1, p_2) \otimes \mathbb{S}_{\dot{a}\dot{b}}^{\dot{c}\dot{d}}(p_1, p_2)$).
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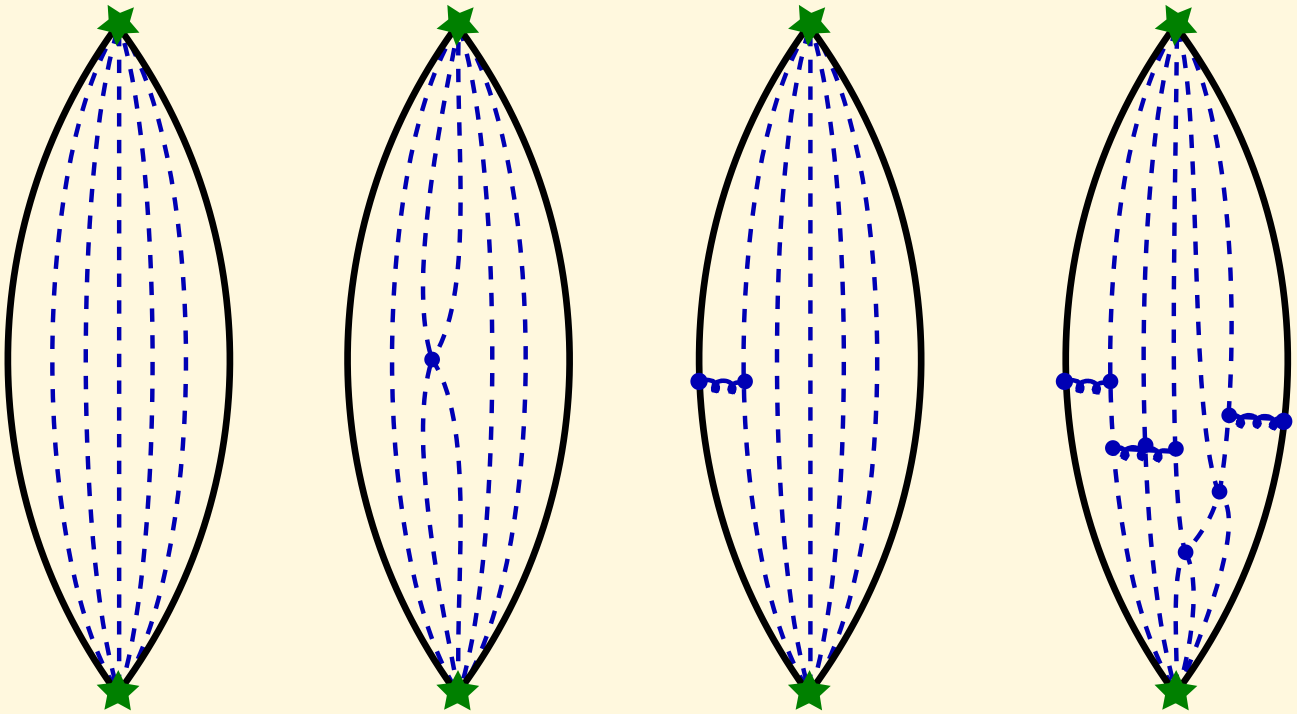

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- Boundary interaction has to be studied separately.
- The two boundaries interact through wrapping effects at $O(g^{2(J+1)})$.
- For $J=0$ this is at one-loop.

All loop reflection matrix and a twist

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- To do it to all loops we should use the symmetry:

$$\begin{array}{ccc} \mathfrak{psu}(2,2|4) & \underset{Z^J \text{ vacuum}}{\longrightarrow} & \mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R \\ \text{boundary}\downarrow & & \downarrow \\ \mathfrak{osp}(4^*|4) & \longrightarrow & \mathfrak{psu}(2|2)_D \end{array}$$

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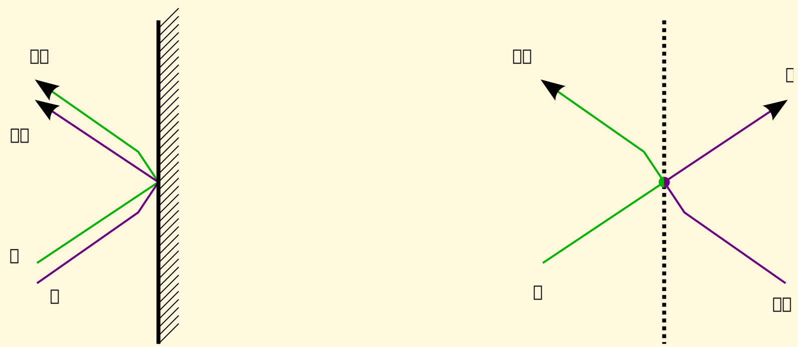
- A single boundary breaks the symmetry to a diagonal $\mathfrak{psu}(2|2)$.
- By the usual argument, the boundary reflection matrix should have the same matrix structure as the bulk one

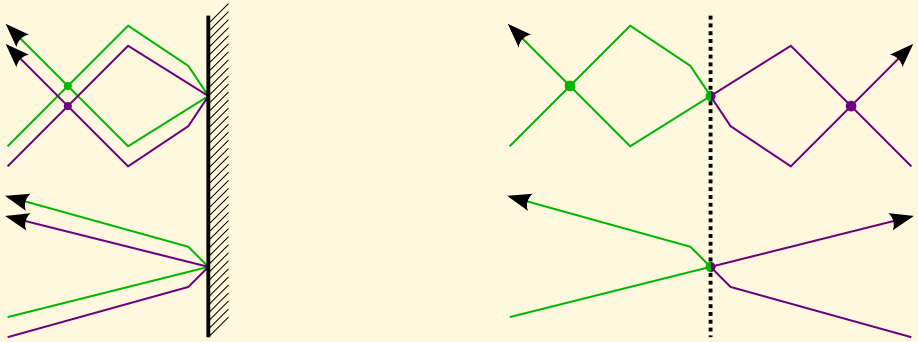
$$\mathbb{R}_{a\dot{a}}^{b\dot{b}}(p) = R_0(p)\hat{\mathbb{S}}_{a\dot{a}}^{b\dot{b}}(p,-p)$$



- It replaces $\mathfrak{psu}(2|2)_L \leftrightarrow \mathfrak{psu}(2|2)_R$ labels.

- Need to determine
  $R_0(p) = \sigma_B(p)/\sigma(p, -p)$.
- Like the crossing relation in the bulk, there is a boundary "crossing-unitarity equation"

$$\mathbb{R}(p) = \mathbb{S}(p, -p)\mathbb{R}^c(\bar{p})$$



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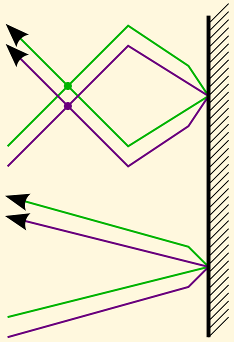

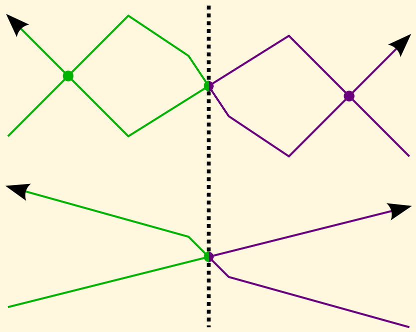

- This translates to the conditions on σ_B

$$\sigma_B(p)\sigma_B(\bar{p}) = \frac{x^- + 1/x^-}{x^+ + 1/x^+}\,, \qquad \sigma_B(p)\sigma_B(\bar{p}) = 1\,.$$

where the Joukowsky variables are a solution of

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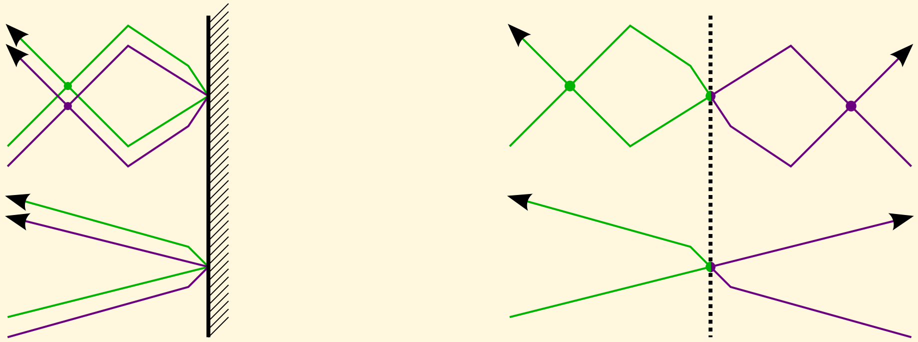

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- The solution which matches the all consistency requirements is

$$\sigma_B(z) = \frac{1 + 1/(x^-)^2}{1 + 1/(x^+)^2}\, e^{-i\chi_B(x^+) + i\chi_B(x^-)}$$

where

$$\chi_B(x) = -i \oint \frac{dz}{2\pi i}\, \frac{1}{x - z}\, \log \frac{\sinh 2\pi g(z + 1/z)}{2\pi g(z + 1/z)}\,.$$

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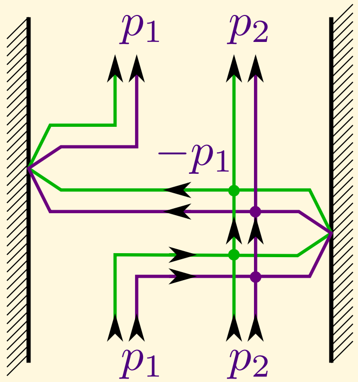
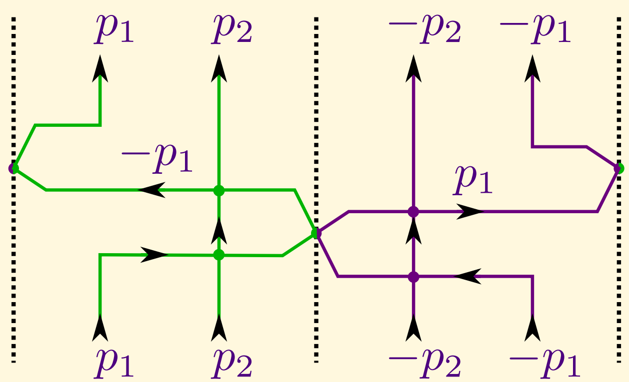
- This is all the information needed to understand the spectrum of asymptotically large insertions into the Wilson loop.





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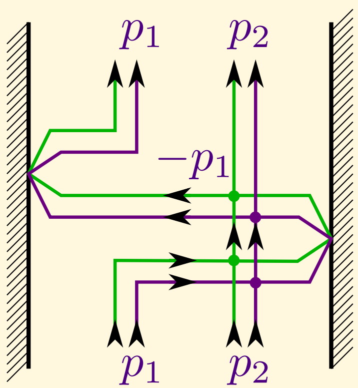
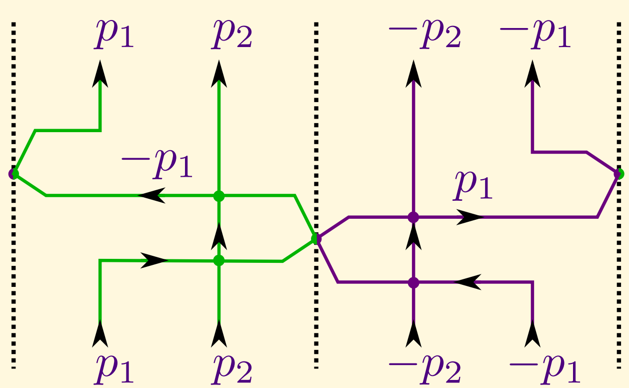


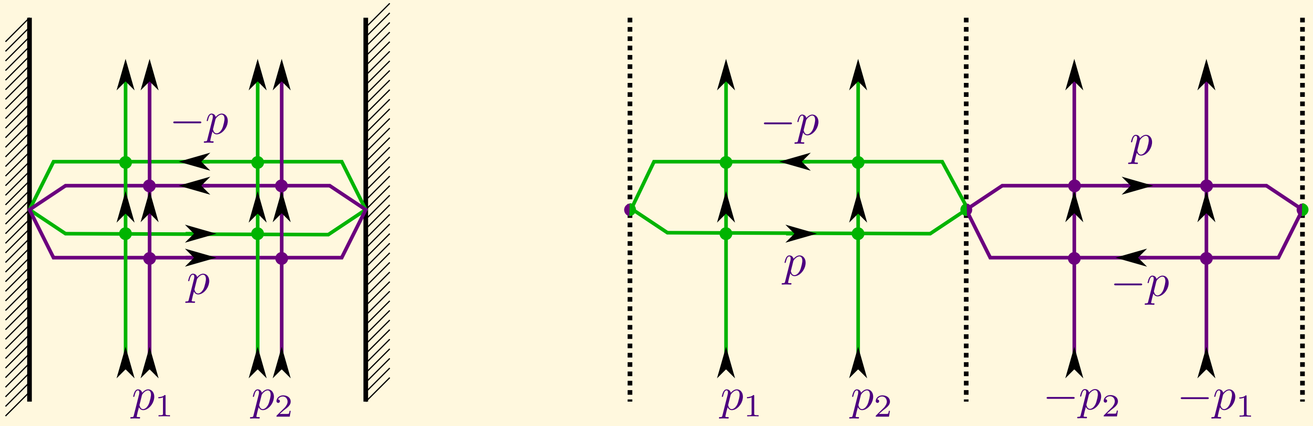


- But not the case $J = 0$ ...

Wrapping effects and the quark-antiquark potential

- One can derive a set of boundary thermodynamic Bethe ansatz equations for this open spin–chain.
- This can be simplified in the small angle limit, where the full answer was reproduced. [Correa, Maldacen, Sever] [Gromov Sever]
- They are the same as the usual TBA equations with several small modifications:
  - The Y functions are related by reflection $Y_{a,s}(-u) = Y_{a,-s}(u)$
  - There are chemical potentials dependent on ϕ and θ.
  - There is a complicated driving term for the massive $Y_{a,0}$ nodes (aka Y_Q).
- The Y-system equations are unmodified.
  - Analytic properties of the functions are different (determined by the asymptotic solution).

- To reproduce the one loop answer it is enough to consider Lüscher-like corrections.
- This requires to calculate the eigenvalues of the transfer matrix



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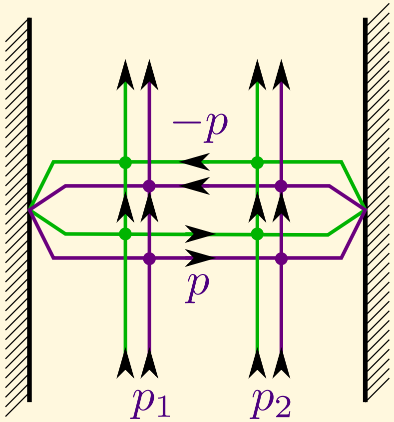
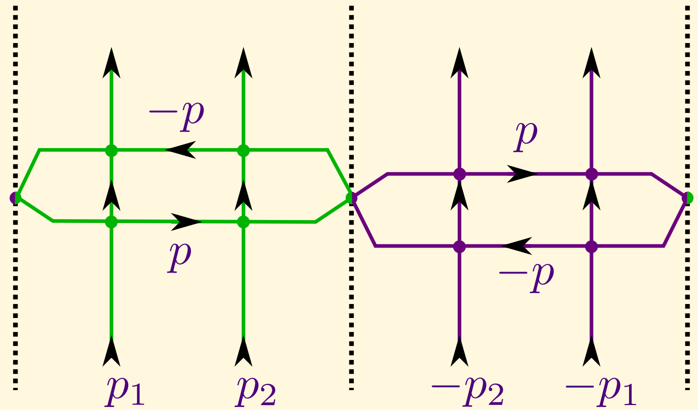
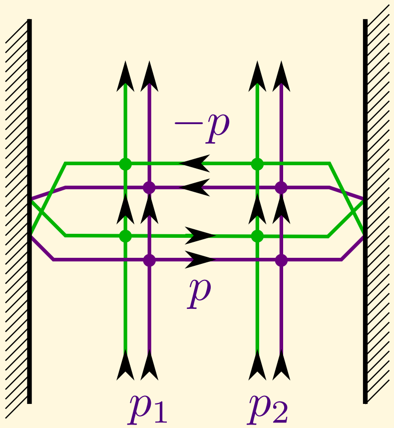
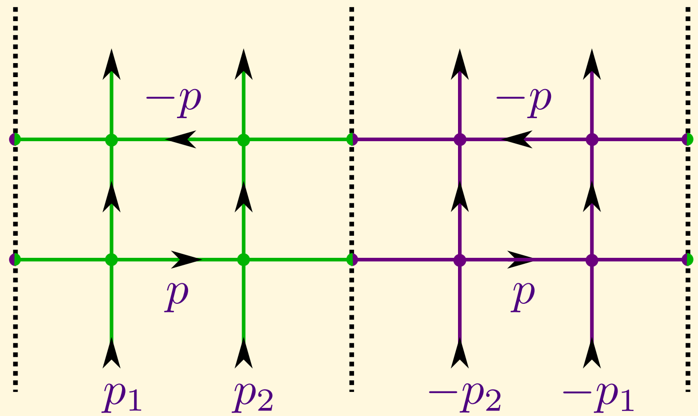




- by repeated use of the Yang-Baxter equation this simplifies to





- That is just the product of two twisted $\mathfrak{psu}(2|2)$ transfer matrices.

- On the Z^J vacuum this is

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T_Q^{\phi,\theta}(p) &= \mathrm{sTr}\left[\mathbb{R}^{(R)}(p)\,\mathbb{R}^{(L)^c}(\bar{p})\right] = \mathrm{sTr}\left[\mathbb{R}^{(R)}(p)\,\mathbb{G}\,\mathbb{R}^{(R)^c}(-\bar{p})\,\mathbb{G}\right] \\
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- Simple group theory gives

$$(\mathrm{sTr}_Q\,\mathbb{G})^2 = 4(\cos\phi - \cos\theta)^2\frac{\sin^2 Q\phi}{\sin^2\phi}$$

And the Lüscher-Bajnok-Janik formula is

$$\delta E \approx -\frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_0^{\infty} d\tilde{p}\,\log\left(1 + T_Q^{(\phi,\theta)}(\tilde{p})e^{-2J\tilde{E}_Q}\right)$$

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- Normally for small g (or large J) can expand the logarithm

$$\delta E \approx \frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_0^{\infty} d\tilde{p}\;T_Q^{(\phi,\theta)}(\tilde{p})e^{-2J\tilde{E}_Q}$$

For $J = 0$ the answer will be proportional to $\dfrac{g^4(\cos\phi - \cos\theta)^2}{\sin^2\phi}\ldots$

- Crucial fact is that the dressing factor has a double pole at $\tilde{p} = 0$

$$\sigma_B(\tilde{p})\sigma_B(-\bar{\tilde{p}}) = e^{2i(\chi_B(x^+)+\chi_B(x^-))} \frac{(2\pi g)^2(x^+ + 1/x^+)(x^- + 1/x^-)}{\sinh(2\pi g(x^+ + 1/x^+))\sinh(2\pi g(x^- + 1/x^-))}$$

$$= e^{2i(\chi_B(x^+)+\chi_B(x^-))} \frac{(2\pi)^2(u^2 + Q^2/4)}{\sinh^2(2\pi u)} \sim \frac{Q^2}{\tilde{p}^2}$$

- Then using

$$\int_0^\infty d\tilde{p}\, \log\left(1 + \frac{c}{\tilde{p}^2}\right) = \pi\sqrt{c}\,,$$

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- Then using

$$\int_0^\infty d\tilde{p}\,\log\left(1+\frac{c}{\tilde{p}^2}\right) = \pi\sqrt{c}\,,$$

- The residue is

$$\sqrt{T_Q^{\mathrm{res}}e^{-2J\tilde{E}_Q}} = 2\frac{\cos\phi-\cos\theta}{\sin\phi}\,\sin Q\phi\,(-1)^Q\left[\frac{(4g^2)^{J+1}}{Q^{2J+1}} - 2(J+2)\frac{(4g^2)^{J+2}}{Q^{2J+3}}+\cdots\right]$$

- so

$$\delta E \approx -(4g^2)^{J+1}\frac{\cos\phi-\cos\theta}{\sin\phi}\sum_{Q=1}^{\infty}\frac{(-1)^Q\sin Q\phi}{Q^{2J+1}}$$

$$= -\frac{(4g^2)^{J+1}}{2i}\frac{\cos\phi-\cos\theta}{\sin\phi}\left(\mathrm{Li}_{2J+1}(-e^{i\phi})-\mathrm{Li}_{2J+1}(-e^{-i\phi})\right)$$

For $J = 0$

$$\begin{aligned}\delta E &\approx -\frac{4g^2}{2i}\,\frac{\cos\phi-\cos\theta}{\sin\phi}\,\big(\mathrm{Li}_1(-e^{i\phi})-\mathrm{Li}_1(-e^{-i\phi})\big)\\ &= 2g^2 i\frac{\cos\phi-\cos\theta}{\sin\phi}\,\big(-\log(1+e^{i\phi})+\log(1+e^{-i\phi})\big)\\ &= 2g^2\,\frac{\cos\phi-\cos\theta}{\sin\phi}\,\phi+O(g^4)\end{aligned}$$

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- This integrability calculation is in exact agreement with the one loop perturbative calculation.

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- This integrability calculation is in exact agreement with the one loop perturbative calculation.

- For Konishi wrapping started at 4 loop order. The cusped Wilson loop is given purely by wrapping from one loop on.
- Is possible to solve iteratively to get higher orders.
- Numerics are hard, but people are working on it.
- Should also be possible to extract the strong coupling answer analytically.

Summary

When I talked about my paper with Valentina a year ago I would end with the question

Will there be a gauge theory derivation of the strong coupling potential:

$$V(L,\lambda)=\frac{4\pi^2\sqrt{\lambda}}{\Gamma(\frac{1}{4})^4\,L}$$

Summary

When I talked about my paper with Valentina a year ago I would end with the question

Will there be a gauge theory derivation of the strong coupling potential:

$$V(L,\lambda) = \frac{4\pi^2\sqrt{\lambda}}{\Gamma(\frac{1}{4})^4\,L}$$

We are very close to answering Yes!

The end