$\mathcal{N} = 2$ S-dualities from M5-branes

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based on works in collaboration with

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F. Benini,
S. Benvenuti, D. Gaiotto

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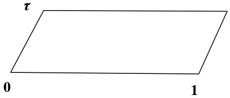
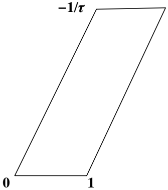
Contents

Montonen-Olive duality

- $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM at coupling $\tau = \theta/(2\pi) + (4\pi i)/g^2$ equivalent to the same theory coupling $\tau' = -1/\tau$
- One way to 'understand' it: start from 6d $\mathcal{N} = (2,0)$ theory, i.e. the theory on N M5-branes, put on a torus





- Low energy physics depends only on the complex structure
  → S-duality!

S-dualities in $\mathcal{N} = 2$ theories

- You can wrap N M5-branes on a more general Riemann surface, possibly with punctures, to get $\mathcal{N} = \mathbf{2}$ superconformal field theories
- Different limits of the shape of the Riemann surface gives different weakly-coupled descriptions, giving S-dualities among them
- Anticipated by [Witten,9703166], but not well-appreciated until [Gaiotto,0904.2715]

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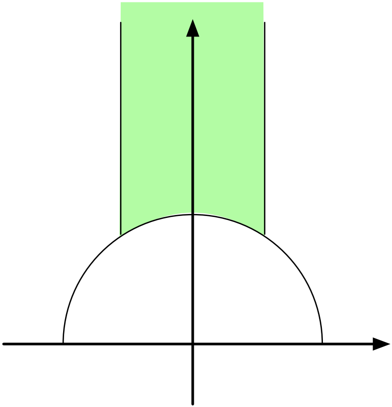
S-duality in $\mathcal{N} = 2$

SU(2) with $N_f = 4$

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$$

$$\tau \to \tau + 1, \qquad \tau \to -\frac{1}{\tau}$$

- Exchanges monopoles and quarks
- Comes from S-duality of Type IIB



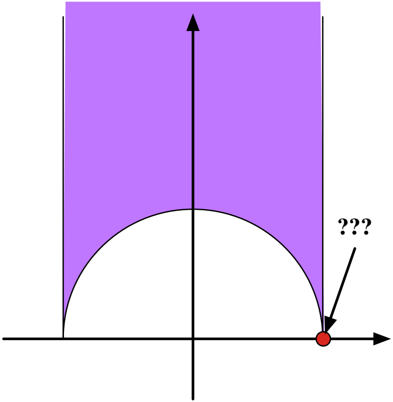
S-duality in $\mathcal{N} = 2$

$SU(3)$ with $N_f = 6$

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$$

$$\tau \to \tau + 2, \qquad \tau \to -\frac{1}{\tau}$$

- Exchanges monopoles and quarks
- Infinitely Strongly coupled at $\tau = 1$



Argyres-Seiberg duality

$\mathrm{SU}(3) + 6$ flavors

at coupling τ

$\updownarrow$

$\mathrm{SU}(2) + 1$ flavor $+ \mathrm{SCFT}[E_6]$

at coupling $\tau' = \dfrac{1}{1-\tau}$, $\mathrm{SU}(2) \subset E_6$ is gauged

[Argyres-Seiberg,0711.0054]

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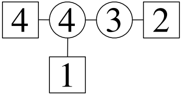
[Argyres-Seiberg,0711.0054]

What??? Huh???

Gaiotto's construction

Consider the quiver 

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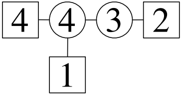
Consider the quiver 

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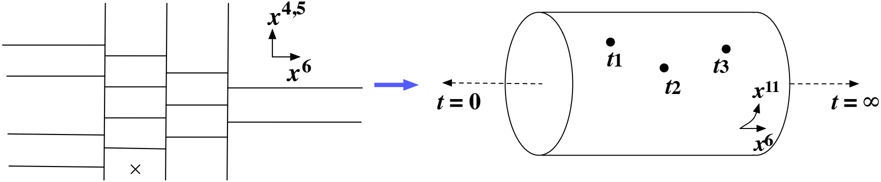

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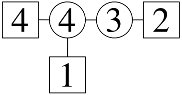
Consider the quiver 

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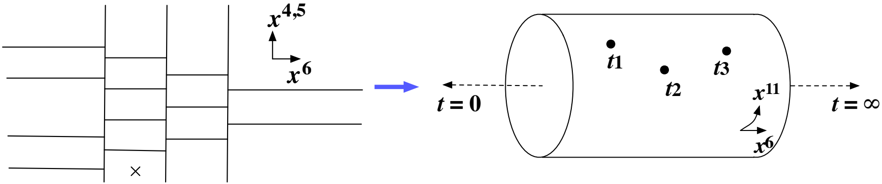
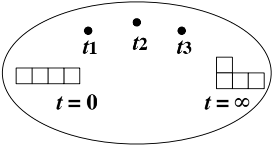


[Gaiotto,0904.2715] rewrote it further, when the quiver is conformal:



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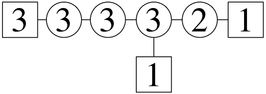

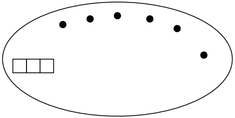

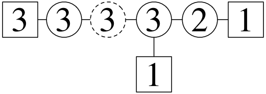

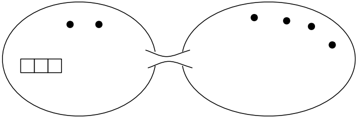

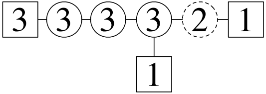

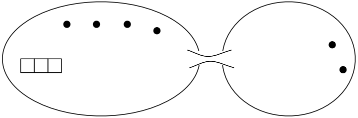

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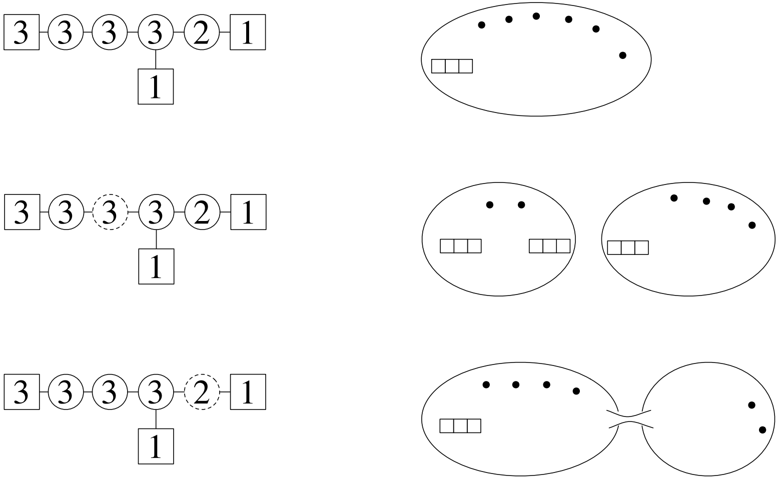

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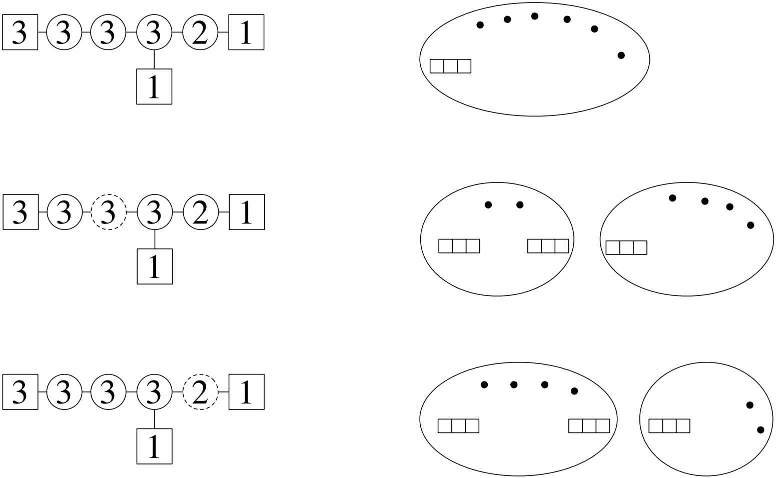

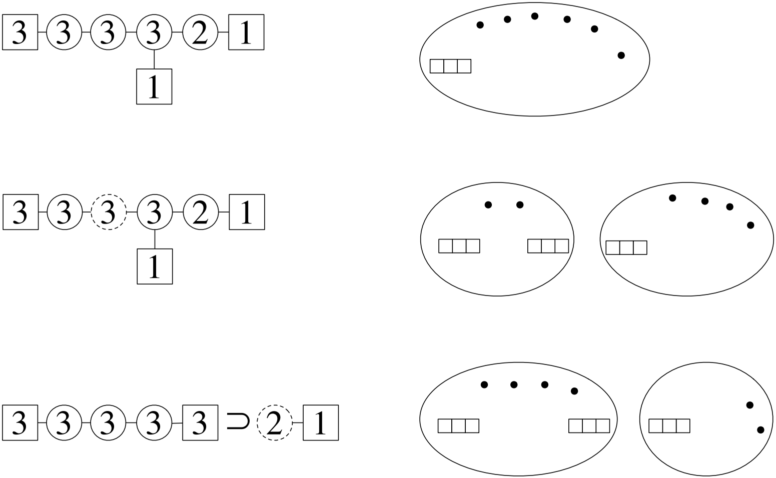
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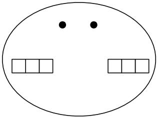
Argyres-Seiberg from Gaiotto



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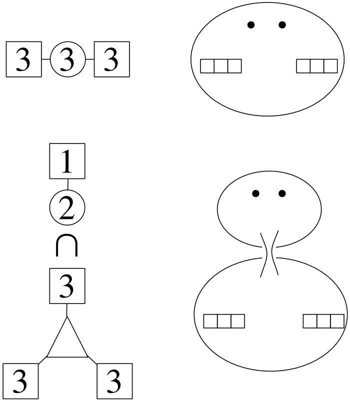

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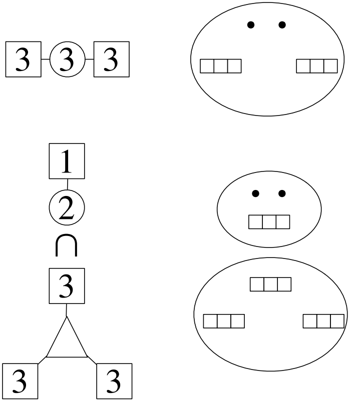
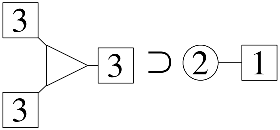


Argyres-Seiberg from Gaiotto

$\mathbf{SU(3)}$ with $\boldsymbol{N_f = 6}$

is S-dual to $\mathbf{SU(2)}$ with $\boldsymbol{N_f = 1}$,
coupled to a strange theory with $\mathbf{SU(3)^3}$ flavor symmetry



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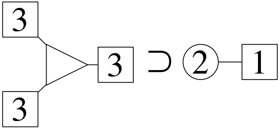

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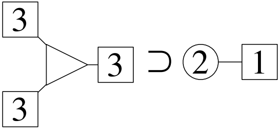

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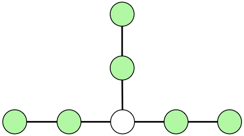
⟶ $\boldsymbol{E_6}$ flavor symmetry!



Effective number of multiplets

- Basic quantities for CFT: central charges
- a and c in 4d $\sim n_v$ and n_h if $\mathcal{N} = 2$
- $\mathbf{SU(3)}$ with $N_f = 6$:

$$n_v = 8, \qquad n_h = 18$$

- $\mathbf{SU(2)}$ with $N_f = 1$ and SCFT[E_6]

$$n_v = 3 + ??, \qquad n_h = 2 + ??$$

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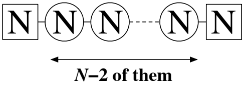
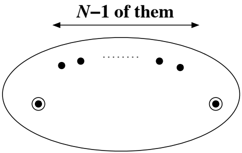
- SCFT[E_6] has

$$n_v = 5, \qquad n_h = 16$$

- agrees with other independent calculations [Aharony-YT]

The T_N theory





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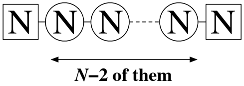
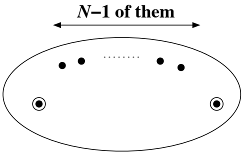
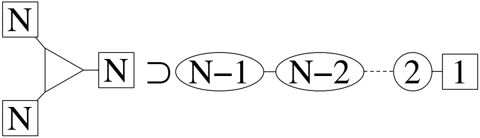
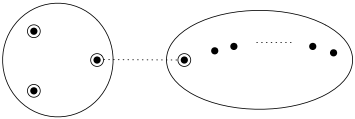
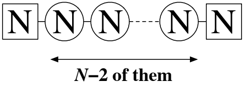
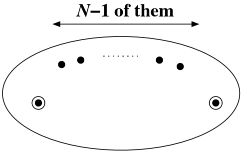


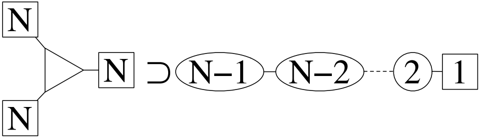


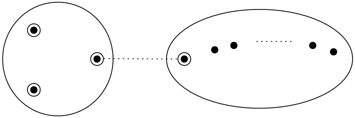








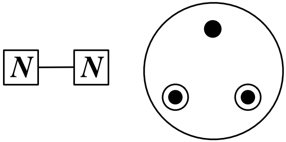




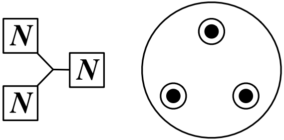
$$n_v = \frac{2}{3}N^{3} - \frac{3}{2}N^2 - \frac{N}{6} + 1, \qquad n_h = \frac{2}{3}N^{3} - \frac{2}{3}N.$$

The T_N theory

- the bifundamental,
  $\mathbf{SU}(\boldsymbol{N}) \times \mathbf{SU}(\boldsymbol{N}) \times \mathbf{U(1)}$

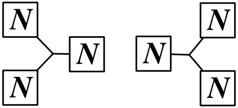


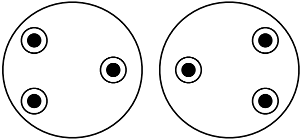
- the $\boldsymbol{T_N}$ theory,
  $\mathbf{SU}(\boldsymbol{N}) \times \mathbf{SU}(\boldsymbol{N}) \times \mathbf{SU}(\boldsymbol{N})$



Fun with T_N

a)





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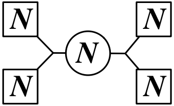
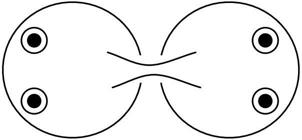
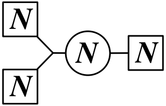
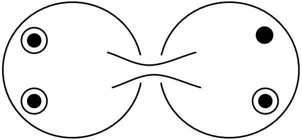




c)





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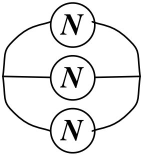

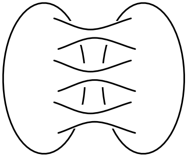

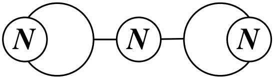

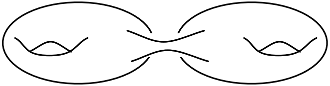

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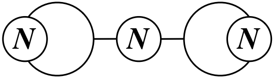
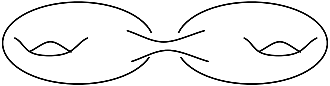
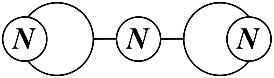
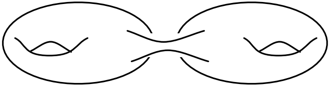




- $\mathbf{2}(g - \mathbf{1})$ copies of T_N, $\mathbf{3}(g - \mathbf{1})$ copies of $\mathrm{SU}(N)$
  → $\mathbf{3}(g - \mathbf{1})$ marginal couplings!

Fun with T_N

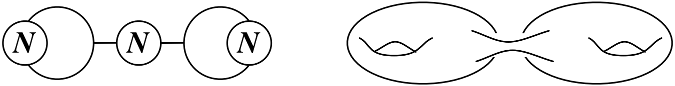




- $2(g-1)$ copies of T_N, $3(g-1)$ copies of $\mathrm{SU}(N)$
  → $3(g-1)$ marginal couplings!

- $n_v = (g-1)\left[\frac{4}{3}N^{3} - \frac{N}{3} - 1\right]$,
- $n_h = (g-1)\left[\frac{4}{3}N^{3} - \frac{4N}{3}\right]$.

Fun with T_N



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- agree with the central charge of the gravity sol'n found in [Maldacena-Nuñez,hep-th/0007018]
- also agree with the info contained in the 6d anomaly [Harvey-Minasian-Moore,hep-th/9808060]

Triskelion and Sicily

Gaiotto called these theories “generalized quiver theories,”
but we [Benini-YT-Wecht] didn’t like it.

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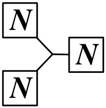
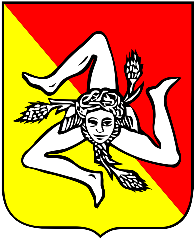
Siciliy's flag



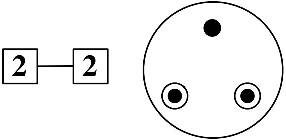


has in it a triskelion: tri+ skelios (Gk. leg).
We adopted the terminology "Sicilian gauge theories." Please do.

Contents

The T_2 theory

The bifundamental,



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The T_2 theory

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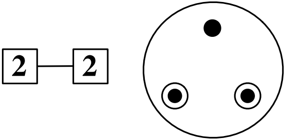

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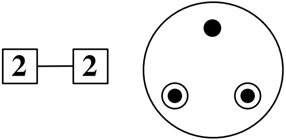

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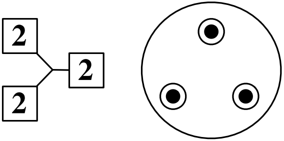
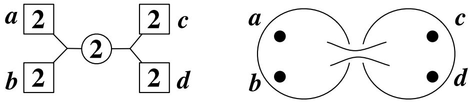
$=$ the T_2 theory with $\mathbf{SU(2)^3}$ symmetry



No distinction between $\bullet$ and $\odot$.

- two quark pairs = two (doublets + anti-doublets) = four doublets
  $\mathbf{SO(4)}$ flavor symmetry + $\mathbf{SU(2)}$ gauge symmetry
- Consider $\mathbf{SU(2)}$ with four quark pairs



- $\mathbf{SO(8)}$ →
  $\mathbf{SO(4)} \times \mathbf{SO(4)} = \mathbf{SU(2)}_a \times \mathbf{SU(2)}_b \times \mathbf{SU(2)}_c \times \mathbf{SU(2)}_d$

$$\mathbf{8}_V \to \mathbf{2}_a \otimes \mathbf{2}_b \quad \oplus \quad \mathbf{2}_c \otimes \mathbf{2}_d$$

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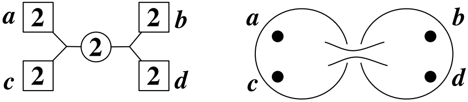

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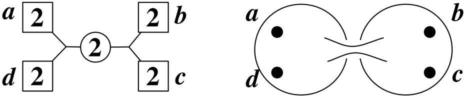

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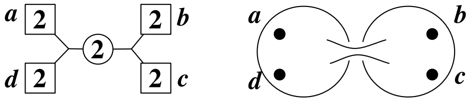

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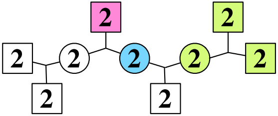
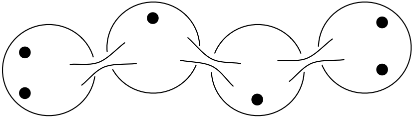
- S-duality induces triality of $\mathbf{SO(8)}$! [Seiberg-Witten]

S-duality with T_2





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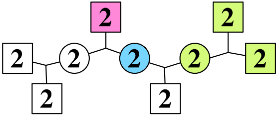
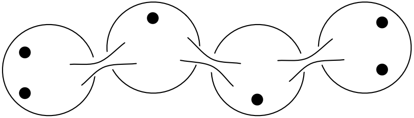
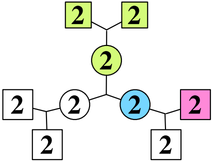
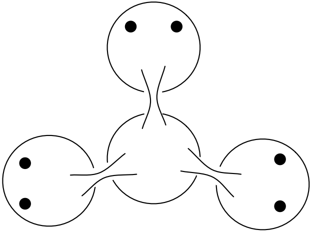




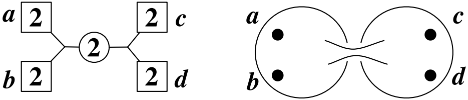




Mass parameters

- N quark pairs $Q_i, \tilde{Q}^j$ ⟶ Mass terms $m^i_j Q_i \tilde{Q}^j$ ⟶ $\sum_i m_i Q_i \tilde{Q}^i$.

  Mass parameters ∼ (Cartan part of) the flavor symmetry.
- $\mathbf{SU(2)}$ with four quark pairs with mass $m_{1,2,3,4}$



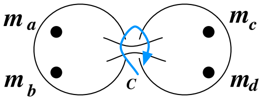
- $\mathbf{SO(8)} \to \mathbf{SU(2)^4}$ symmetry:

  $m_a = (m_1 + m_2)/2, \quad m_b = (m_1 - m_2)/2,$

  $m_c = (m_3 + m_4)/2, \quad m_d = (m_3 - m_4)/2.$

Seiberg-Witten curve

- $\mathrm{SU}(2)$ with four quark pairs with mass parameters $m_{a,b,c,d}$

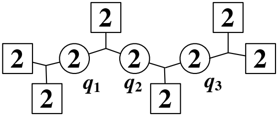


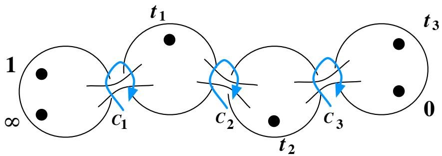
- The SW curve is $y^2 = \phi_2(z)$ where
  - z is the coordinate of the base sphere
  - SW differential is ydz
  - $\phi_2(z)(dz)^2$ is a quadratic differential with (for $i = a, b, c, d$)

$$\phi_2(z)(dz)^2 \sim m_i^2 dz^2/(z - z_i)^2$$

  - $\exp(-2\pi i \tau_{UV})$ is the cross-ratio of $z_{a,b,c,d}$
  - Mass of the W-boson is $\int_C ydz$

Seiberg-Witten curve





- SW curve $y^2 = \phi_2(z)$, $\phi_2(dz)^2$, with double poles $\sim m^2$ for each of the $SU(2)$ flavor symmetry
- UV couplings $q_i = \exp(-2\pi i \tau_i) = t_i/t_{i-1}$
- W-boson masses $a_i = \int_{C_i} y dx$

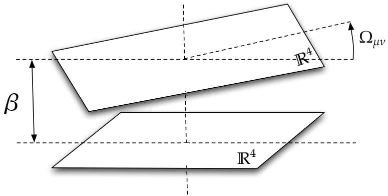
2d CFT vs 4d CFT

- Any calculation on the 4d side gives something about punctured Riemann surface.

2d CFT vs 4d CFT

- Any calculation on the 4d side gives something about punctured Riemann surface.
- Nekrasov's instanton partition function = the Virasoro conformal block.
- Full partition function = the Liouville correlation function.

Nekrasov's partition function



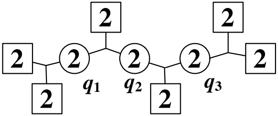
- Start from 5d, take 4d limit keeping Ω/β fixed, call two angular velocities $\epsilon_{1,2}$
- Angular rotation generates potential. Take partition function $Z(\epsilon_1, \epsilon_2)$.

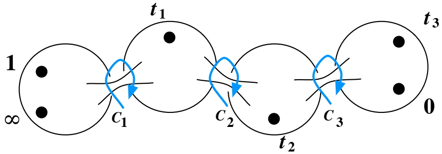
$$\log Z(\epsilon_1, \epsilon_2; a_i) \sim \frac{F(a_i)}{\epsilon_1 \epsilon_2} + \cdots$$

- $F(a_i)$ is the prepotential: $S = \int d^4x d^4\theta F(a) + c.c.$
- $Z = Z_{\text{1-loop}} Z_{\text{inst}}$
- Explicit formula known, as a summation over pairs of Young tableaux.

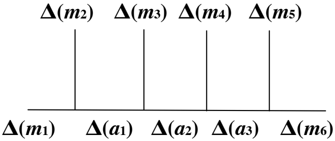
Nekrasov vs. Conformal block

Nekrasov partition function $Z_{\text{inst}}(\epsilon_1, \epsilon_2)$ of this theory





computes the conformal block $\mathcal{F}$

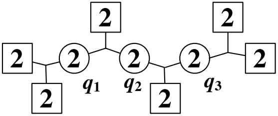


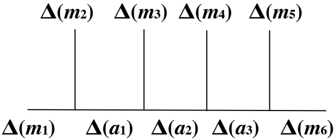
where

$$c = 1 + 6Q^2, \qquad \Delta(\alpha) = (\frac{Q}{2})^2 - \frac{\alpha^2}{\epsilon_1 \epsilon_2}$$

with Q determined via $Q = b + 1/b$, $\qquad (b^2 = \epsilon_1/\epsilon_2)$.

Nekrasov vs. Conformal block



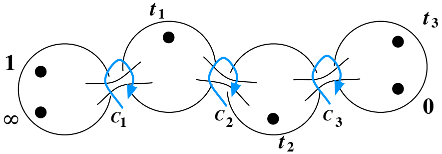


- Both are power series in q_i, coefficients rational in m_i, a_i and ϵ_i:

$$\sum q_1^{n_1} q_2^{n_2} q_3^{n_3} \dots$$

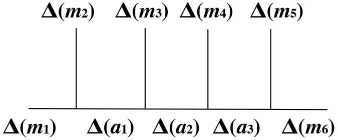
- Nekrasov's side:
  n_i is the instanton number for the i-th $SU(2)$ gauge group
- Conformal block's side:
  n_i is the level of the descendant on the primary with dimension $\Delta(a_i)$

Nekrasov vs. Conformal block



- Both depend on the decomposition of the Riemann surface into pairs of pants
- Nekrasov's side:
  decomposition determines the S-duality frame
- Conformal block's side:
  decomposition determines the channel
- S-duality is the s-t channel duality!

Duality invariant objects



- Correlation functions of Liouville theory

$$S = \frac{1}{\pi} \int d^2x \left(|\partial_\mu \phi|^2 + \mu e^{2b\phi} \right)$$

are duality invariant:

$$\langle V_{m_1} V_{m_2} \cdots \rangle = \int da_1 \cdots da_n$$
$$\times\, C_{m_1,m_2,a_1} C_{a_1,m_3,a_2} \cdots C_{a_3,m_5,m_6} |\mathcal{F}|^2$$

where

- $V_a(z) = e^{2a\phi(z)}$ and $C_{\alpha_1,\alpha_2,\alpha_3}$: DOZZ 3pt functions

Duality invariant objects

$$\langle V_{m_1} V_{m_2} \cdots \rangle = \int \prod da_i$$
$$\times C_{m_1,m_2,a_1} C_{a_1,m_3,a_2} \cdots C_{a_3,m_5,m_6} |\mathcal{F}|^2$$

- Conformal block $\mathcal{F}$ is Nekrasov's instanton partition function Z_{inst}
- Product of C's happens to be $|Z_{\text{1-loop}}|^2$

$$\langle V_{m_1} V_{m_2} \cdots \rangle = \int \prod (a_i^2 da_i) |Z_{\text{1-loop}} Z_{\text{inst}}|^2$$

- When $b = 1$ (i.e. $\epsilon_1 = \epsilon_2$, $c = 25$)
  the RHS is the partition function on S^4. [Pestun]

WHY???

Summary

- M5-branes provides a systematic understanding of S-duality of $\mathcal{N} = 2$ superconformal theories first found by Argyres-Seiberg. [Gaiotto]

- Constructed theories with $E_{6,7,8}$ flavor symmetry. [Benini-Benvenuti-YT]

- Reviewed the T_N theory. [Gaiotto-Maldacena]

- $\mathrm{SU}(2)$ and Liouville. [Alday-Gaiotto-YT]