Aspects of black hole time-irreversibility in AdS_3 and AdS_5

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IPMU, September 15, 2008

Based on:

Bhattacharyya, Hubeny, Loganayagam, GM, Minwalla, Morita, Rangamani, Reall (B8), hep-th 0803.2526 *Local fluid dynamical entropy from gravity*
Bhattacharyya, Hubeny, Minwalla, Rangamani (BHMR), hep-th 0801.1435 *Nonlinear Fluid Dynamics from Gravity*
Das, GM, 0810.xxxx *Microstate dependence of scattering from the D1-D5 system*
Dhar, GM, Wadia, hep-th/9605234 *Absorption versus decay of black holes in string theory and T symmetry*
GM, Minwalla, Morita, Wadia,... in progress

PLAN

- **I. Motivation: T-irreversibility of gravity**
- **II. Gravity and fluid dynamics**
- **III. Absorption by the D1-D5 black hole**
- **IV. Conclusions**

I. MOTIVATION

- **An interesting issue of black hole physics is the classical time-irreversibility associated with the black hole horizon. Waves (or particles) can only enter the horizon but cannot come out. Also, the area of the horizon does not decrease.**

- **This is famously similar to another familiar arrow of time: the second law of thermodynamics.**

- **Both of these arrows exist even though basic laws of physics are T-symmetric (at least at the relevant scales).**

- **We will revisit the question of (i) relation between the two arrows of time and (ii) compatibility with the T-symmetry with the basic laws,
in two contexts:**

  (a) gravity/fluid dynamics correspondence in asymptotically AdS_5

  (b) absorption of waves by a D1-D5 black hole

II. T-irreversibility of black brane horizon and of hydrodynamics

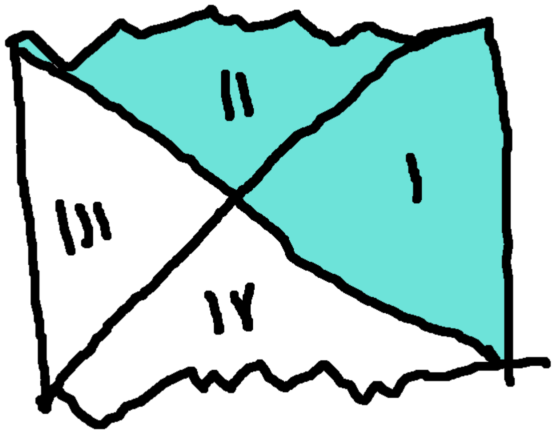
Consider a static black brane solution in AdS_5:

$$ds^2 = -dt^2 f(r) r^2 + \frac{dr^2}{f(r) r^2} + r^2 d\vec{x}^2$$

where

$$f(r) = 1 - \frac{r_0^4}{r^4}$$

The extended Penrose diagram consists of the four regions I,II,III,IV.



The above coordinates cover only region I. An ingoing wave from $r = \infty$ is expected to enter region II without encountering any singularity at the horizon $r = r_0$. To describe them we can use the "ingoing Eddington-Finkelstein" coordinates $(v, r, \vec{x})$ where v is defined by

$$v = t + r^*, r^* = \int \frac{dr}{f(r) r^2} \approx \frac{\ln(r - r_0)}{(r^2 f)'(r_0)}$$

In these coordinates, the metric looks like

$$ds^2 = 2dvdr - r^2 f(r)dv^2 + r^2 d\vec{x}^2$$

which is non-singular in I+II.

Horowitz and Hubeny[1] considered a scalar wave satisfying the covariant Laplace equation in the above background. They showed that the wave must be purely ingoing at the horizon $r = r_0$ if it is regular in regions I+ II:

$$\text{Ingoing} \ : \ \phi \sim e^{-iw(t+r^*)} = e^{-iwv} = \ \textbf{analytic for finite } v$$

$$\text{Outgoing} \ : \ \phi \sim e^{-iw(t-r^*)} = e^{-iwv} e^{2iwr^*} \approx e^{-iwv}(r - r_0)^{2iw/(r^2 f)'(r_0)}$$

(a) Im w turns out to be negative, so that the power of $r - r_0$ is positive; hence the outgoing mode vanishes at the horizon. However,
(b) this power also turns out to be non-integer, hence the outgoing mode is actually singular at $r = r_0$.

The same conclusion holds if the scalar field is replaced by a graviton.

Now, BHMR[2] constructed *fluctuating black brane solutions* which permit a perturbative derivative expansion around the above background geometry. The solutions are regular in regions I+II and are found to be dual to solutions of fluid dynamics.

The solutions, consistently with the above reasoning, are ingoing at the horizon.

[1] Horowitz, Hubeny, hep-th/9909056
[2] Bhattacharyya, Hubeny, Minwalla, Rangamani hep-th 0801.1435

Some details of the Horowitz-Hubeny argument:

Write

$$\phi = e^{-iwv}\ r^{-3/2}\ \psi(r)\ X(\vec{x}).$$

Define $F(r) = r^2 f(r)$**.**

$$F\psi'' + (f' - 2iw)\psi' - V\psi = 0$$

$$\int_{r_0}^{\infty} dr\ (F|\psi'|^2 + 2iw\bar{\psi}\psi' + V|\psi|^2) = 0$$

$$\int_{r_0}^{\infty} dr\ \bar{\psi}\psi' = \frac{w}{\bar{w} - w}|\psi(r_0)|^2$$

$$\int_{r_0}^{\infty} dr\ (F|\psi'|^2 V|\psi|^2) = -\frac{|w|^2}{(w - \bar{w})/(2i)}|\psi(r_0)|^2$$

Hence Im $w = (w - \bar{w})/(2i)$ **must be negative. (Im** $w = 0$ **can be ruled out).**

The BHMR metric:

$$ds^2 = -2u_\mu(x)dx^\mu dr - r^2 f(r/r_0(x))u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu$$
$$+\frac{2}{r_0}r^2 F(r/r_0(x))\sigma_{\mu\nu}dx^\mu dx^\nu + \frac{2}{3}ru_\mu u_\nu \partial_\lambda u^\lambda - ru^\lambda \partial_\lambda (u_\mu u_\nu)dx^\mu dx^\nu + ...$$

The first line represents the black brane in a boosted frame where the boost parameters and r_0 are made into functions of x^μ (collective coordinates). The second line represents the leading correction in a derivative expansion.

$$P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu}$$
$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\beta\nu}\partial_{(\alpha}u_{\beta)} - \frac{1}{3}P^{\mu\nu}\partial_\lambda u^\lambda$$

Stress tensor:

$$T^\mu_\nu = -2\ \lim_{r\to\infty}\ r^4(K^\mu_\nu - \delta^\mu_\nu)$$
$$= \rho(\eta^{\mu\nu} + 4u^\mu u^\nu) - 2\eta\sigma^{\mu\nu} + ...$$

where

$$\rho = (\pi T)^4, \eta = (\pi T)^3, T = r_0/\pi$$

Note that the energy density ρ and the viscosity η are determined by gravity.

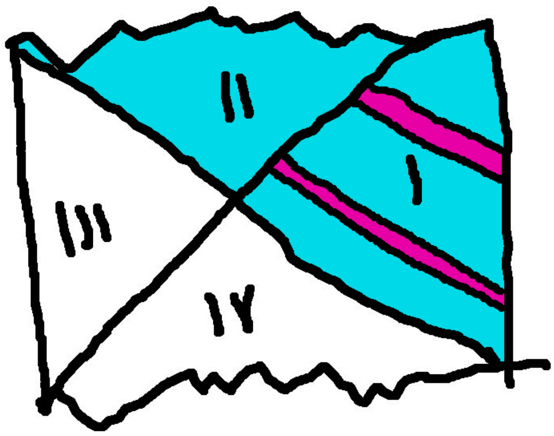
Validity of the perturbative expansion in powers of derivative requires $L \gg 1/T$ where L= length scale of variation of $u_\mu(x), T(x)$.

In the region of validity of the solutions, the segment of the horizon is the future horizon $\mathcal{H}^+$.

B8[3] observed the following: There is a standard argument in black hole physics which shows that the expansion parameter

$$\theta = \frac{1}{a_H}\frac{da_H}{d\lambda}$$

is nonnegative on a future horizon. Here a_H is the (three-dimensional) cross-sectional area of a geodesic congruence on the future horizon.



By using a "tubewise" map

$$f : \text{boundary}(\tau, \vec{y}) \rightarrow \text{horizon}(v, \vec{x}),\ \tau = v, \vec{y} = \vec{x}$$

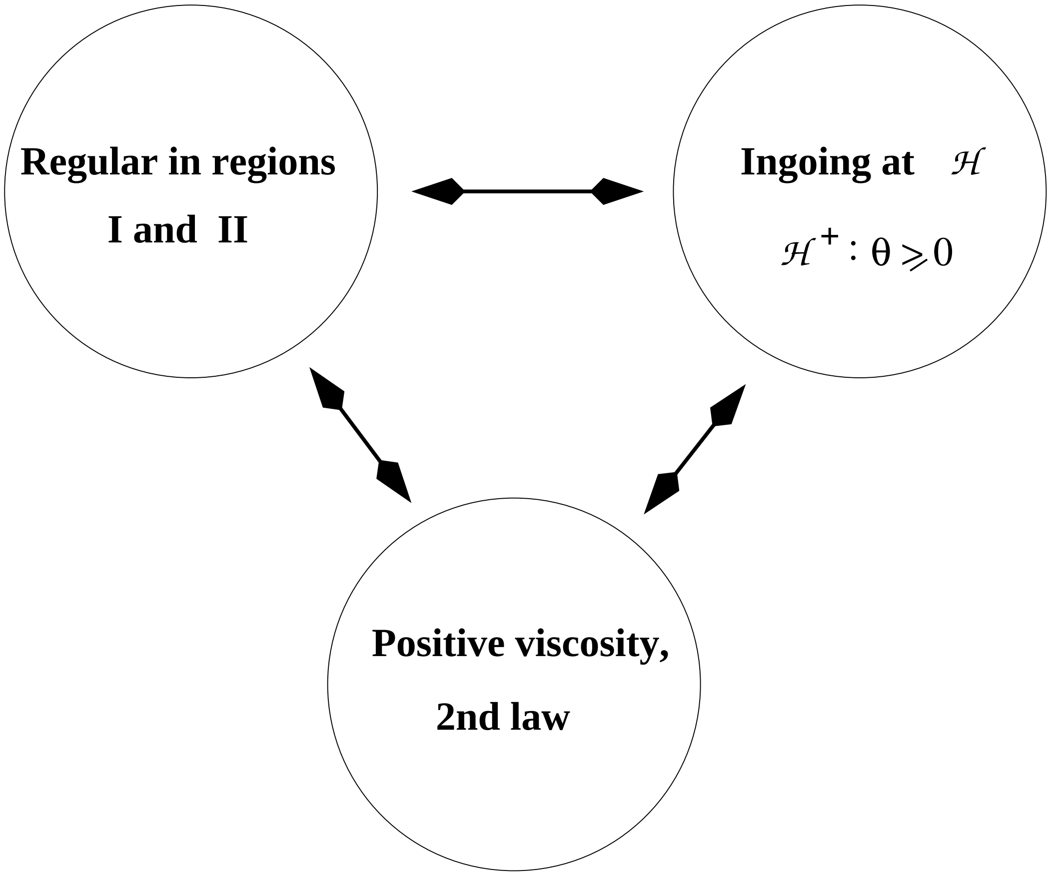
and defining an entropy current $\mathrm{s} = *f_*(a_H)$, it follows that

$$\partial_\mu s^\mu = *d * \mathrm{s} \equiv *df_*(a_H) = *f_*(da_H) \geq 0$$

which shows nonnegative entropy production.

[3] Bhattacharyya, Hubeny, Loganayagam, GM, Minwalla, Morita, Rangamani, Reall, hep-th 0803.2526

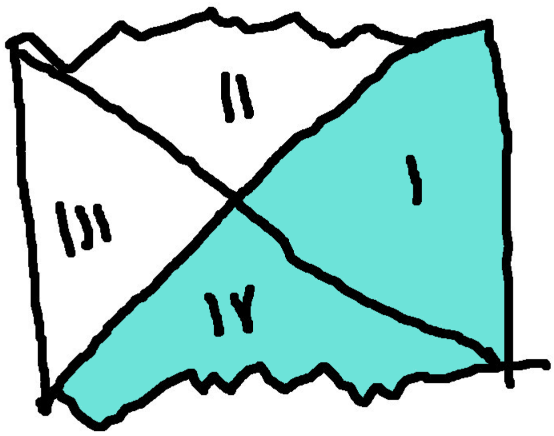
Summary so far:



These solutions were regular in I+II, are ingoing at the horizon and correspond to a fluid configuration which satisfies the second law.

We now ask the reverse question[4]:
If we remove the regularity condition in I+II, do they correspond to some fluid configuration which violates the second law of thermodynamics?

To address this, let us consider wave solutions are regular in the outgoing E-F coordinates $(u, r, \mathbf{x})$ where $u = t - r^*$, which cover I+IV.



The static black brane in these coordinates looks like

$$ds^2 = -2dudr - r^2 f(r)du^2 + r^2 d\vec{x}^2$$

where the sign of the first term is flipped.

[4]GM, Minwalla, Morita 2008

$$\text{Ingoing} \; : \; \phi \sim e^{-iw(t+r^*)} = e^{-iwu} e^{-2iwr^*} \approx e^{-iwv} (r - r_0)^{-2iw/(r^2 f)'(r_0)}$$
$$\text{Outgoing} \; : \; \phi \sim e^{-iw(t-r^*)} = e^{-iwu} = \text{ analytic for finite } u$$

Because of the flipped sign of $dudr$, now Im w turns out to be *positive*. Since the power of $(r-r_0)$ has also flipped sign, the function of r is, again, non-analytic at $r = r_0$.

The behaviour of the outgoing mode as $u \to \infty$ will be discussed in detail later.

Details of the Im $w > 0$ argument:

Write

$$\phi = e^{-iwv} \; r^{-3/2} \; \psi(r) \; X(\vec{x}).$$

Define $F(r) = r^2 f(r)$.

$$F\psi'' + (f' + 2iw)\psi' - V\psi = 0$$
$$\int_{r_0}^{\infty} dr \; (F|\psi'|^2 - 2iw\bar{\psi}\psi' + V|\psi|^2) = 0$$
$$\int_{r_0}^{\infty} dr \; \bar{\psi}\psi' = \frac{w}{\bar{w} - w} |\psi(r_0)|^2$$
$$\int_{r_0}^{\infty} dr \; (F|\psi'|^2 V|\psi|^2) = -\frac{|w|^2}{(w - \bar{w})/(-2i)} |\psi(r_0)|^2$$

Hence $-$Im $w = (w - \bar{w})/(-2i)$ must be negative. (Im $w = 0$ can be ruled out).

Fluctuating gravity solutions in derivative expansion:

In terms of an arbitrary boosted frame, the flipped sign of $dudr$ corresponds to a sign flip $u^\mu \to -u^\mu$ in the solution.

By following a procedure similar to the earlier one we now get

$$ds^2 = +2u_\mu(x)dx^\mu dr - r^2 f(r/r_0(x))u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu$$
$$-\frac{2}{r_0}r^2 F(r/r_0(x))\sigma_{\mu\nu}dx^\mu dx^\nu - \frac{2}{3}r u_\mu u_\nu \partial_\lambda u^\lambda + r u^\lambda \partial_\lambda(u_\mu u_\nu)dx^\mu dx^\nu + ...$$

The essential change is that terms containing an odd number of u^μ's has flipped sign (we have explicitly checked that the above metric is a solution of Einstein equation up to first order in derivative expansion).

Stress tensor:

$$\begin{aligned} T^\mu_\nu &= -2\ \lim_{r\to\infty}\ r^4\ (K^\mu_\nu - \delta^\mu_\nu) \\ &= (\pi T)^4(\eta^{\mu\nu} + 4u^\mu u^\nu) + 2(\pi T)^3\sigma^{\mu\nu} + ... \end{aligned}$$

which shows that

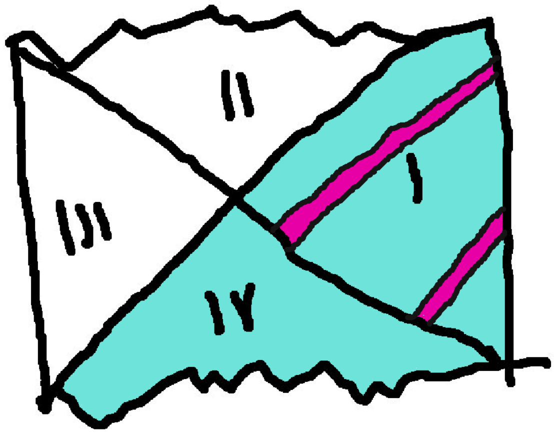
$$\rho = (\pi T)^4, \eta = -(\pi T)^3, T = r_0/\pi$$

We get negative viscosity!

Again, by using the result that for a conformal fluid the divergence of the entropy current is viscosity times a sum of squares, we deduce that

$$\partial_\mu s^\mu \leq 0!$$

This is consistent with the following reasoning from gravity, similar to B8. Since we are considering here solutions regular in I + IV, the solutions are outgoing at $r = r_0$ which happens at the past horizon $\mathcal{H}^-$.



The expansion parameter θ is non-positive on the past horizon. Therefore by using a similar argument as before, we get non-increasing entropy of the boundary fluid:

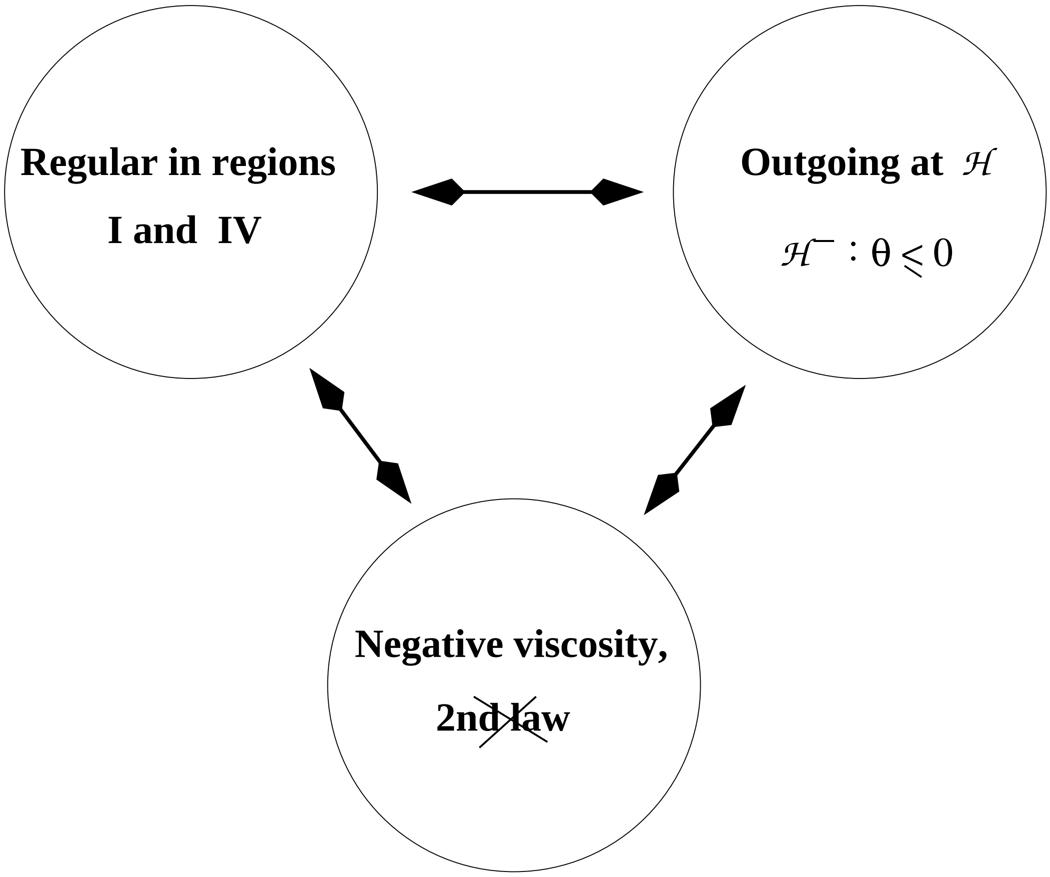
$$\partial_\mu s^\mu \leq 0!$$

Here we use the new "tubewise" map

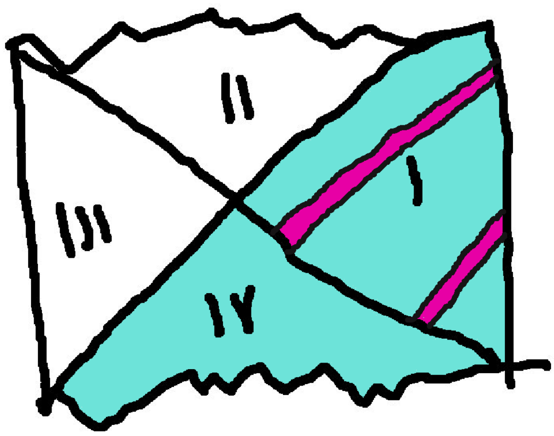
$$f : \text{boundary}(\tau, \vec{y}) \to \text{horizon}(u, \vec{x}),\ \tau = u, \vec{y} = \vec{x}$$

where u becomes the boundary time τ.

To summarise,



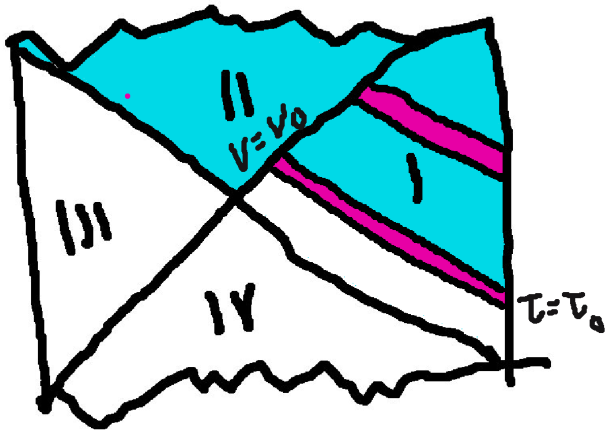
Possible interpretation of decreasing entropy in the boundary theory



Now consider the outgoing modes e^{-iwu} again which we found to be regular.

Since $w = r+iq$, $q > 0$, the outgoing mode $e^{-iwu} \propto e^{qu} = e^{q\tau}$ grows infinitely large in the boundary time τ.

This is the time-reversed situation of the previously discussed solution which behaved as $e^{-iwv} \propto e^{-q\tau}$, $q = -\mathrm{Im}w > 0$. This solution decayed in time, however it blew up in the past. In order to correctly specify the behaviour in the past, we must specify in that case an initial condition at $\tau = \tau_0$ such that the graviton mode vanishes for all $\tau < \tau_0$. (This can only be achieved by some external "stirring" device, e.g. an additional field).



At the horizon, a general combination of outgoing modes is

$$\phi(v, \vec{x}) = \int dw \ A(w, \vec{x}) \ e^{-iwv}$$

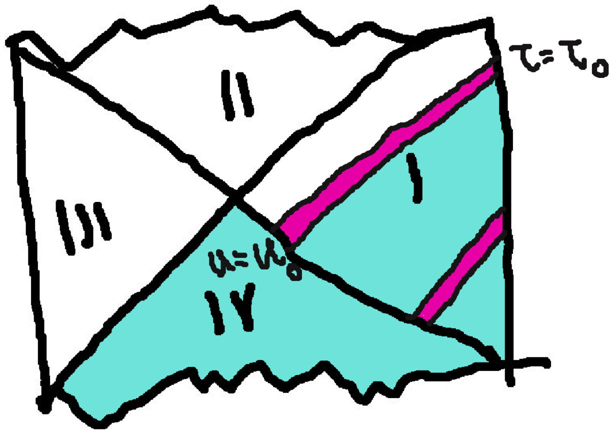
Equivalently, at the boundary, by the "tubewise" map

$$\phi(\tau, \vec{x}) = \int dw \ A(w, \vec{x}) \ e^{-iw\tau}$$

In order that we have $\phi(\tau, \vec{x}) = 0, \tau < \tau_0$, we must have a special set of Fourier components $A(w, \vec{x})$ such as those of a step function.

The initial condition is specified by these special Fourier components; eventually, because of non-linearities, the Fourier components become generic.

In the present case, in order that the solution, $e^{q\tau}$, does not blow up in the future we must specify a *final condition* at some finite time $\tau = \tau_0$ such that it vanishes for all $\tau > \tau_0$.



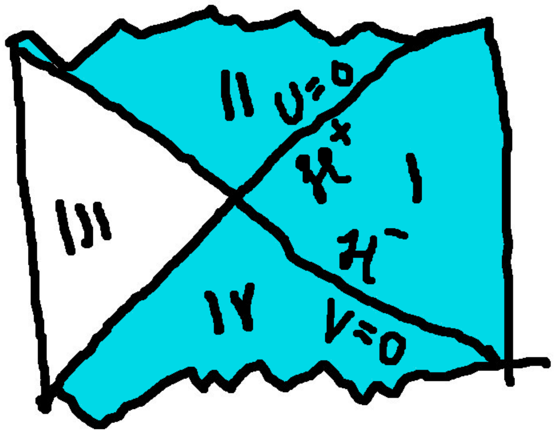
In this case, the *final condition* is specified by these specific Fourier modes similar to the ones discussed above.

Now, in order for a non-linear evolution, which allows for mixing of Fourier modes, to lead to these specific Fourier modes at a final time, we need a *fine-tuned* initial condition at an earlier time. (Similar to that of reversing the velocity of each gas molecule in a room).

Such a situation can decrease the available phase space, and hence decrease entropy.

Can we take a combination of these two cases, e.g. waves that are outgoing at $\mathcal{H}^-$ and ingoing at $\mathcal{H}^+$?[5]

We show here that if a field satisfies the derivative expansion and is regular in I+II+IV, it is static.



Consider a scalar field ϕ which satisfies the above regularity condition.
Then, in Kruskal coordinates

$$U = -e^{-\kappa u}, V = e^{\kappa v}$$

the field has a Taylor series expansion

$$\phi(U, V) = \sum_{m,n \geq 0} \phi_{mn} U^m V^n$$

In (v, r) coordinates

$$U^m V^n = \exp[\kappa(n-m)v] \exp[2n\kappa r^*]$$

The derivative with respect to v is $(n-m)\kappa \sim (m-n)T$ which is large unless $m = n$ in which case the field is a function only of r. Such fields give rise to a static solution. [Derivative expansion requires $\partial_v / T \sim 1/(LT) \ll 1$.]

The result can be easily generalized to gravitons, by correcting taking into account the Jacobian of the coordinate transformation $(u, v) \to (U, V)$.

[5]Bak, Gutperle, Karch, hep-th 0708.3691

A question:

It appears that in gauge/gravity correspondence, the second law of thermodynamics in the boundary can be explained by the presence of a black hole/black brane.

How about situations in which there is no black hole?

III. Absorption by the D1-D5 black hole

Consider a supersymmetric D1-D5 black hole obtained by wrapping Q_5 D5-branes on $T^4 \times S^1$ and Q_1 D1 branes on the S^1. The system can carry a right- (*or* left-) moving momentum P along the S^1. The near-horizon limit is an extremal BTZ black hole with $M = L = P$.

In terms of the conformal group, $L_0 = P, \bar{L}_0 = 0$. The $P = 0$ black hole corresponds to the $M = L = 0$ BTZ black hole ($L_0 = \bar{L}_0 = 0$).

It was shown in 1996[6], in the context of this black hole, that the time irreversibility associated with the BH horizon is equivalent to the time-irreversibility of the second law of thermodynamics, *provided we represent the black hole as a microcanonical ensemble.*

The dual CFT$_2$ is based on (a marginal deformation of) a symmetric product orbifold with Ramond boundary condition in the S^1 direction. A ground state with $L_0 = \bar{L}_0 = 0$ is characterized by a set of integers $N_n^i, n = 1, ..., N, i = 1, 2, ...8$ satisfying

$$\sum_{n,i} n N_n^i = N \equiv Q_1 Q_5$$

The number of ground states is given by the number Ω_i of solutions of this equation.

Roughly, the N_n^i represents the number of CFT modes ("long strings") with effective winding length n (and "type" i which we will ignore henceforth).

[6] Dhar, GM, Wadia 1996

Setup for an absorption process

A wave of energy E (some mode of the closed string, say a minimally coupled scalar) is incident on the CFT system, exciting it from one of the ground states to an excited state.

Suppose that the number of ground states (initial states) $= \Omega_i$, and the number of available final (excited) states $= \Omega_f$.

Provided we identify the BTZ BH (both the initial BH and the final BH) with microcanonical ensembles in the CFT **, we get**

$$\text{(prob. of absorption)/(prob. of decay)} = \Omega_f/\Omega_i$$

Since for a large mass black hole Ω increases very fast with energy, $\Omega_f \gg \Omega_i$. Hence, classically "absorption" is overwhelmingly the more dominant process.

However this depends crucially on representing the BH as a microcanonical ensemble.

If we take individual CFT states to represent the initial and final states of the BTZ black hole, we get (not surprisingly)

$$\text{(prob. of absorption)/(prob. of decay)} = 1$$

Absorption by a pure CFT state

$$\begin{aligned} P(t) &= \sum_f |\int_0^t dt' \langle f| e^{iwt'} \partial X \bar{\partial} X |i\rangle|^2 \\ &= \kappa_5^2 \sum_n N_n \sum_{m=1}^{\infty} \frac{m^2}{w} \frac{\sin^2(w-2m/n)t}{(w-2m/n)^2} \end{aligned}$$

We find that $P(t)$ oscillates and does not grow linearly in t. In other words, the system does not "absorb".

This situation is familiar from the context of Fermi Golden Rule. We find that in stead of a monochromatic wave $\exp[-iwt]$, we must use a wave-packet $\rho(w)$, of width Δw. For $t \ll 1/\Delta w$[7],

$$\sigma \equiv \frac{P(t)}{t} = \kappa_5^2 \sum_n N_n \sum_{m=1}^{\infty} m\rho(2m)$$

For $\Delta w \gg 1/(Rn_{typ})$ where n_{typ} is the typical length of a long string, we get, after a long calculation

$$\sigma = \sigma_{classical} + \text{corrections}$$

where $\sigma_{classical} = \pi^3 R^4 w$ is the absorption cross-section by the classical black hole (without α' corrections).

The correction terms are damped by $\exp(-R\ \Delta w\ n_{typ})$; further, *they are also microstate dependent.*

[7] Das, GM 2008

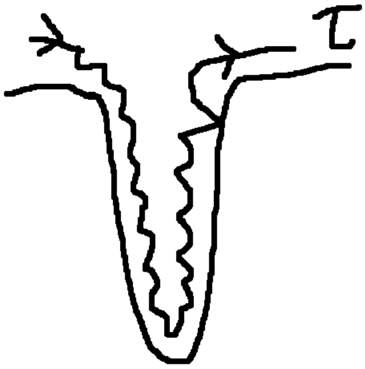
Interpretation

Thus, we found classical absorption if

$$t \ll \frac{1}{\Delta w} \ll R n_{typ}$$

For $t \gg 1/\Delta w$ we found oscillatory behaviour.

A bulk calculation similar to this was performed by Lunin and Mathur[8] in a "microstate" geometry (fuzzball). These geometries have throats that are finite and reflects waves back and out, eventually all of it!



However, there is a certain trapping time $\tau = R n_{typ}$ such that the incident wave-packet stays inside the throat for $t < \tau$. Thus, for $t \ll \tau$, the wave travels in and does not feel the presence of the "cap". (The Lunin-Mathur calculations took all long strings of equal length n_{typ}). We arrive at this result from a CFT calculation which is exact.

[8]Lunin, Mathur hep-th/0107113

Absorption by a two-state system

A related issue is the following. A 2-state system cannot absorb a photon. Its interaction with radiation of energy E is oscillatory in time:

Consider a hamiltonian

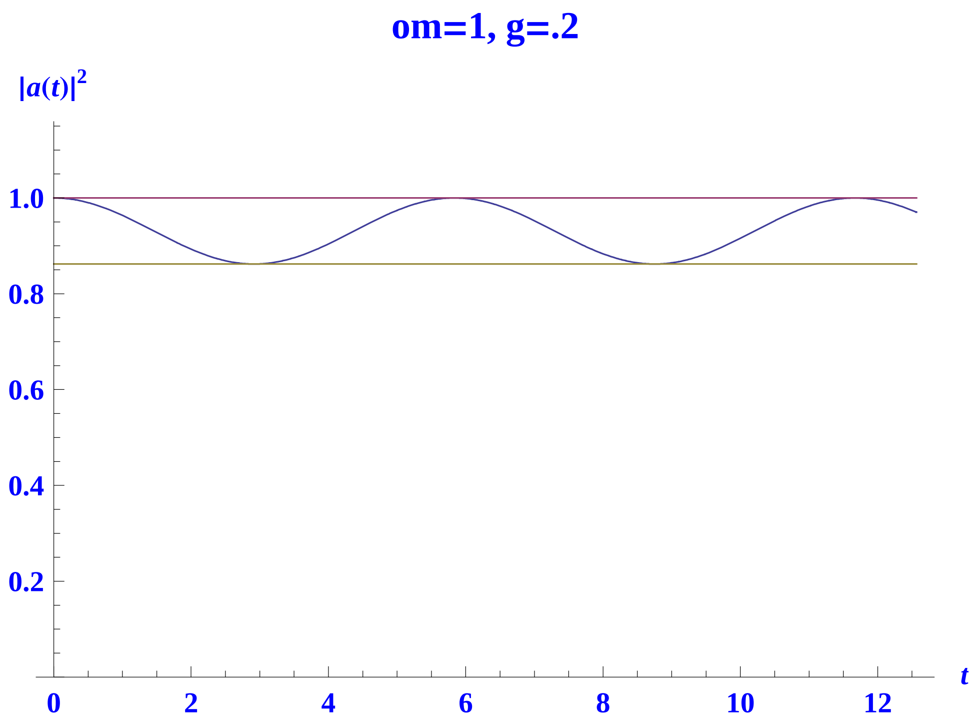
$$H = \begin{pmatrix} E_1 & ge^{-iwt} \\ ge^{iwt} & E_2 \end{pmatrix}$$

The Schrodinger equation is given by

$$i\dot{\psi} = H\psi$$
$$\psi = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

Suppose $\psi = (1 \quad 0)$ at $t = 0$. The solution is:

$$|a(t)|^2 = A^2 + (1-A)^2 + 2A(1-A)\cos(\kappa t)$$
$$A = \frac{1}{2} - \frac{\Omega}{2\kappa},\ \Omega = w + E_2 - E_1,\ \kappa = \sqrt{\Omega^2 + 4g^2}$$



An absorption would have to mean that the lower energy state must steadily lose probability. Instead, we find that the probability oscillates.

There is a certain "trapping time" τ such that at $t = \tau$ such that the probability $|b(t)|^2$ of occupying the higher state reaches a maximum (before falling again).

IV. CONCLUSION

- **We discussed two instances of T-irreversibility in gravity.**

- **In the context of gauge-gravity correspondence T-irreversibility of a black brane horizon and the associated positive entropy production owes to the "ingoing" boundary condition at the horizon (equivalently the imposition of regularity in the ingoing E-F coordinates).**

- **We showed that the choice of the other boundary condition (outgoing) leads to entropy decrease for the boundary fluid. We argued, after analyzing the appropriate regularity condition, that the fluid should start from a fine-tuned initial condition. This is a possible interpretation of the reduction in entropy.**

- **We addressed the issue of absorption of waves by a black hole from the viewpoint of T-irreversibility. We showed that a pure state does not absorb, much as a two-state system keeps oscillating when exposed to radiation. However if one takes the "time of measurement" to be less than a certain "trapping time" one effectively has an absorption, which agrees with the BH absorption cross-section in a suitable limit.**