Rare Dyon Decays

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- Based on:

  “Dyon death eaters”,
  Anindya Mukherjee, SM, Rahul Nigam, arxiv:0707.3035 [hep-th]

  “Kinematical analogy for marginal dyon decay”,
  Anindya Mukherjee, SM, Rahul Nigam, arXiv:0710.4533 [hep-th].

  “Constraints on “rare” dyon decays”,
  SM and Rahul Nigam, arXiv:0809.1157 [hep-th].

Outline

Motivation and background

- In 4d string compactifications having $\mathcal{N} = 4$ supersymmetry, the microscopic degeneracy is known very precisely through a general formula.
- It turns out that this degeneracy jumps at certain loci in parameter space, called curves of marginal stability.
- At these loci, a dyon decays into two or more other dyons.
- While certain decays cause the degeneracy to change, others do not. The latter kind, which have not been much studied in the literature, will be the main topic of this talk.
- Our goal will be a general picture of the dyon spectrum as a function of the moduli.

Outline

N=4 compactifications and marginal stability

- Compactifications of string theory to 4d having $\mathcal{N} = 4$ supersymmetry admit both $\frac{1}{4}$-BPS and $\frac{1}{2}$-BPS states.
- The desired compactification can be thought of as the type IIB string compactified on $K3 \times T^2$.
- It has other useful dual descriptions:

heterotic on T^6	type IIA on $K3 \times T^2$	M-theory on $K3 \times T^3$	M-theory on T^5/Z_2

- The resulting 4d theory has 28 $U(1)$ vector fields, so a general dyonic state has 28 electric and 28 magnetic charges $(\vec{Q}, \vec{P})$.
- Analysis of the supersymmetry algebra on these states reveals that they are:

$$\begin{array}{lll} \frac{1}{2}\text{-BPS} & \text{if} & \vec{Q} \parallel \vec{P} \\ \frac{1}{4}\text{-BPS} & \text{if} & \vec{Q} \nparallel \vec{P} \end{array}$$

- Notice that purely electric or purely magnetic states are automatically $\frac{1}{2}$-BPS.

- Marginal stability arises from the fact that at some points in the moduli space, a dyon can become degenerate in mass with a two-particle state.
- At such points a continuum spectrum opens up and the state-counting problem is effectively ill-defined.
- The moduli space for $\mathcal{N} = 4$ compactifications is:

$$SO(6,22,Z)\backslash SO(6,22,R)/SO(6,R) \times SO(22,R) \\ \times SL(2,Z)\backslash SL(2,R)/U(1)$$

- In the type IIB description, the second factor is labelled by the complex-structure modulus τ of the torus T^2.

- In $\mathcal{N}=4$ compactifications, inner products among the charge vectors involve the 28×28 matrices L and M, where:

$$L=\begin{pmatrix} 0 & \mathbf{I}_6 & 0 \\ \mathbf{I}_6 & 0 & 0 \\ 0 & 0 & -\mathbf{I}_{16} \end{pmatrix}$$

  and the symmetric matrix M satisfying $MLM=L$ encodes the $SO(6,22)$ moduli.

- It is convenient to write down the combination

$$L+M=\begin{pmatrix} G^{-1} & 1+G^{-1}(B+C) & G^{-1}A \\ 1+(-B+C)G^{-1} & (G-B+C)G^{-1}(G+B+C) & (G-B+C)G^{-1}A \\ A^T G^{-1} & A^T G^{-1}(G+B+C) & A^T G^{-1}A \end{pmatrix}$$

- The moduli appearing here are most familiar in the heterotic basis as the metric G_{ij}, B-field B_{ij} and gauge fields A_i^I in a T^6 compactification. We have defined $C_{ij}=A_i^I A_j^I$.

- The inner product between dyonic charge vectors that is relevant for us depends on the moduli:

$$\vec{P}\circ\vec{P} = \vec{P}^T(L+M)\vec{P},\ \vec{Q}\circ\vec{Q} = \vec{Q}^T(L+M)\vec{Q},\ \vec{Q}\circ\vec{P} = \vec{Q}^T(L+M)\vec{P}$$

- In the presence of a black hole, these moduli are understood to be the moduli at infinity.
- Another way to write the above product is to define the "right-projected" (hence moduli-dependent) charges:

$$\vec{P}_R = \sqrt{L+M}\,\vec{P}, \quad \vec{Q}_R = \sqrt{L+M}\,\vec{Q}$$

  where "right" refers to the right-moving part of the heterotic string. Then $\vec{P}\circ\vec{P} = \vec{P}_R\cdot\vec{P}_R = \vec{P}_R^2$ etc.

- An explicit form of the square root is provided by:

$$\sqrt{L+M} = \begin{pmatrix} E^{-1} & E^{-1}(G+B+C) & E^{-1}A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfying

$$\sqrt{L+M}^T \sqrt{L+M} = L+M$$

- Note that the projected charges $\vec{Q}_R, \vec{P}_R$ are not quantised.
- Also note that they are 6-component vectors.

- The BPS mass formula depends only on the projected charges:

$$M^2_{BPS}(\vec{Q},\vec{P}) = \frac{1}{\tau_2}\left|\vec{Q}_R - \bar{\tau}\vec{P}_R)\right|^2 + 2\sqrt{\Delta(\vec{Q}_R,\vec{P}_R)}$$

  where

$$\Delta(\vec{Q}_R,\vec{P}_R) \;\equiv\; \vec{Q}_R^2\vec{P}_R^2 - (\vec{P}_R\cdot\vec{Q}_R)^2$$

- The dependence on the torus parameter τ is explicit, but there is also an implicit dependence on the moduli M through $\vec{Q}_R, \vec{P}_R$.

- The mass formula has an amusing analogy with particle kinematics. It is analogous to:

$$E^2 = \vec{p}^{\,2} + m^2$$

where:

$$M_{BPS} \sim E, \qquad \frac{\vec{Q}_R - \tau \vec{P}_R}{\sqrt{\tau_2}} \sim \vec{p}, \qquad \sqrt{2}\,\Delta^{\frac{1}{4}} \sim m$$

- We can use this fact to find the conditions for marginal decay:

$$\begin{aligned} (\vec{Q}, \vec{P}) &= (\vec{Q}_1, \vec{P}_1) + (\vec{Q}_2, \vec{P}_2) \\ M_{BPS}(\vec{Q}, \vec{P}) &= M_{BPS}(\vec{Q}_1, \vec{P}_1) + M_{BPS}(\vec{Q}_2, \vec{P}_2) \end{aligned}$$

- In the kinematic analogy, conservation of charge is identified with conservation of momentum and the marginality condition is mapped to conservation of energy.

- Moreover $\frac{1}{2}$-BPS dyons, for which $\vec{Q} \parallel \vec{P}$, have

$$\Delta(\vec{Q}_R, \vec{P}_R) \equiv \vec{Q}_R^2 \vec{P}_R^2 - (\vec{Q}_R \cdot \vec{P}_R)^2 = 0$$

  and therefore they behave like massless particles.
- The decay of a massless particle is not meaningful. Thus only $\frac{1}{4}$-BPS particles can decay.
- As in normal relativistic kinematics, marginal dyon decay will be possible whenever the (analogue) energy-momentum conditions are satisfied.

- Consider a decay of a particle of mass m with a momentum $\vec{p}$ into two particles of masses m_1, m_2 and momenta $\vec{p}_1, \vec{p}_2$.
- Working in the rest frame, the invariant $p \cdot p_1$ is easily shown to be:

$$p \cdot p_1 = \frac{1}{2}(m^2 + m_1^2 - m_2^2)$$

The same invariant in the lab frame can be written:

$$p \cdot p_1 = m_1^2 + \sqrt{\vec{p}_1^2 + m_1^2}\sqrt{\vec{p}_2^2 + m_2^2} - \vec{p}_1 \cdot \vec{p}_2$$

- Equating the two, we have:

$$\sqrt{\vec{p}_1^2 + m_1^2}\sqrt{\vec{p}_2^2 + m_2^2} - \vec{p}_1 \cdot \vec{p}_2 = \frac{m^2 - m_1^2 - m_2^2}{2}$$

- Making the substitutions for the analogous dyon quantities, we find:

$$\sqrt{|\vec{Q}_R^{(1)} - \tau \vec{P}_R^{(1)}|^2 + 2\tau_2\sqrt{\Delta_1}}\sqrt{|\vec{Q}_R^{(2)} - \tau \vec{P}_R^{(2)}|^2 + 2\tau_2\sqrt{\Delta_2}}$$
$$-\mathrm{Re}(\vec{Q}_R^{(1)} - \tau \vec{P}_R^{(1)}) \cdot (\vec{Q}_R^{(2)} - \bar{\tau}\vec{P}_R^{(2)}) = \tau_2(\sqrt{\Delta} - \sqrt{\Delta_1} - \sqrt{\Delta_2})$$

where $\Delta_i = \Delta(\vec{Q}^{(i)}, \vec{P}^{(i)})$.

- For fixed charge vectors $\vec{Q}^{(i)}, \vec{P}^{(i)}$ and fixed moduli M, this is an equation for the torus parameter τ. This is called the curve of marginal stability.

Outline

Analysis of marginal stability curves

- To discuss marginal decays we first define the torsion I of the dyon charges, which is a duality invariant, by:

$$I = \text{g.c.d.}(Q_i P_j - Q_j P_i)$$

- For a dyon of torsion mn for some co-prime m, n, one can use $SL(2, Z)$ transformations of the 2-torus to bring the charge vectors into the form $(m\vec{Q}, n\vec{P})$ where:

$$\text{g.c.d.}(Q_i P_j - Q_j P_i) = 1$$

- If $m = n = 1$, the dyon has unit torsion. We will restrict ourselves to this case here.

- The general decay process involves a breakup of charges as:

$$\begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} = \begin{pmatrix} \vec{Q}^{(1)} \\ \vec{P}^{(1)} \end{pmatrix} + \begin{pmatrix} \vec{Q}^{(2)} \\ \vec{P}^{(2)} \end{pmatrix} + \cdots$$

  In the above formula all charges are quantised.

- The initial and final particles must all be mutually BPS, which requires that the projected charges $\vec{Q}_R^{(i)}, \vec{P}_R^{(i)}$ all lie in the same plane as Q_R, P_R.
- Thus we have:

$$\begin{pmatrix} \vec{Q}_R^{(i)} \\ \vec{P}_R^{(i)} \end{pmatrix} = \begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix} \begin{pmatrix} \vec{Q}_R \\ \vec{P}_R \end{pmatrix}$$

  where $\sum_i m_i = 1, \quad \sum_i n_i = 1, \quad \sum_i r_i = \sum_i s_i = 0$.

- As the projected charges are not quantised, m_i, n_i, r_i, s_i do not have to be integers. In fact, as we will soon see, they are moduli-dependent.

- Let us now focus on two-body decays.
- From the curve obtained using the kinematic analogy, by rearranging and squaring to remove the square roots, we find:

$$\left(\tau_1 - \frac{m_1 - n_1}{2s_1}\right)^2 + \left(\tau_2 + \frac{E}{2s_1}\right)^2 = \frac{1}{4s_1^2}\left((m_1 - n_1)^2 + 4r_1 s_1 + E^2\right)$$

where

$$E = -\frac{Q_R \wedge P_R}{\sqrt{\Delta}}$$

- In the above,

$$Q_R \wedge P_R = Q_{1R} \cdot P_R - P_{1R} \cdot Q_R$$

is the Saha angular momentum stored in the dyonic field of the first decay product relative to the original dyon, evaluated in the norm at infinity.

- Dependence of the marginal stability curve on the moduli M arises through E as well as m_1, r_1, s_1, n_1.
- Although it is not obvious, one can show that the RHS is positive definite.
- Therefore the curve is a circle in the torus moduli space with centre at:

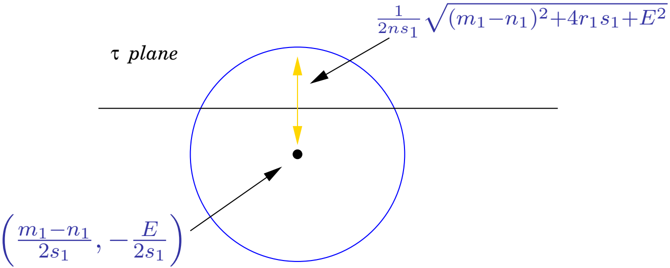
$$(\tau_1, \tau_2) = \left(\frac{m_1 - n_1}{2s_1}, -\frac{E}{2s_1}\right)$$

  and radius:

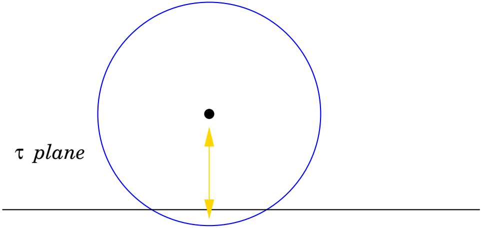
$$\frac{1}{2s_1}\sqrt{(m_1 - n_1)^2 + 4r_1 s_1 + E^2}$$

- The next step is to check whether this circle intersects the upper half-plane.
- There are two cases. If $\frac{E}{s_1} > 0$ then the centre of the circle is in the lower half plane.
- The circle will then intersect the upper half plane only if it intersects the real axis, which happens if:

$$(m_1 - n_1)^2 + 4r_1 s_1 > 0$$



- If $\frac{E}{s_1} < 0$ then the circle has its centre in the upper half plane, and therefore always has a finite region in the upper half-plane:



- One might conclude that all two-body decays satisfying the above conditions lead to walls of marginal stability.
- This would be wrong. The equation we have derived is a necessary but not sufficient condition for marginal decay.
- The reason is that rearranging and squaring to obtain the final equation introduces spurious solutions.
- Indeed one can check that the same curve arises for the reverse decays:

$$M_1 = M + M_2, \quad M_2 = M + M_1$$

in addition to the desired decay $M = M_1 + M_2$.
- Therefore, given a curve of marginal stability, the decay mode it describes can be one of the three modes above and and one needs to check which one is the case.

- Conversely, given a decay mode, one needs to check whether it is actually realised on the corresponding curve of marginal stability.
- In most cases these considerations put constraints on the moduli in M, which we will analyse in detail. Then the domain of marginal stability is a codimension > 1 surface in $M \times \mathrm{UHP}$.
- Such decays are therefore called "rare decays".
- In certain special decays there are no constraints on M. These are the "non-rare decays". In these cases there is a wall of marginal stability.

Outline

$\frac{1}{2}$-BPS two-body decays

- Let us now examine decays of a $\frac{1}{4}$-BPS dyon into two $\frac{1}{2}$-BPS states. In this case the final-state charges are already in the plane of the initial charges.
- Moreover, it is obvious from supersymmetry that an inverse decay is impossible.
- In this case one can show that the matrices:

$$\begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix}$$

have integer entries and vanishing determinant.

- In fact it is possible to write these integers as:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \begin{pmatrix} ad & -ab \\ cd & -bc \end{pmatrix}$$

where:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, Z)$$

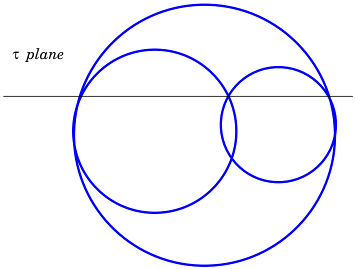
- Under these conditions the curve of marginal stability simplifies and can be written as:

$$\left(\tau_1 - \frac{ad + bc}{2cd}\right)^2 + \left(\tau_2 + \frac{E}{2cd}\right)^2 = \frac{1}{4c^2d^2}\left(1 + E^2\right)$$

where

$$E = -\frac{Q \wedge P}{\sqrt{\Delta}}$$

- This family of curves was originally derived by *Sen*. They are non-intersecting and divide the torus moduli space into "triangles" with circular boundaries.
- This is illustrated in the following figure:



- We found that these circles have an interesting number-theoretic significance, being related to Farey sequences and Ford circles which were invented to illustrate the structure of the group $SL(2, Z)$.

- Since no other moduli need to be adjusted, this curve is a domain wall in moduli space, dividing it into regions.
- Across a domain wall, physical quantities can jump discontinuously. And that is exactly what happens in this case – the dyon degeneracy jumps *[Dabholkar-Gaiotto, Sen]*.
- Instead of discussing these special decays in detail, we turn now to the more general decays.
- As indicated, these will turn out to be "rare" in the sense that they occur on codimension ≥ 2 loci in moduli space.

Outline

General two-body decays

- Let us now consider the general case:

$$\tfrac{1}{4}\text{-BPS} \;\rightarrow\; \tfrac{1}{4}\text{-BPS} + \tfrac{1}{4}\text{-BPS}$$

- As we have seen, the curve of marginal stability is:

$$\left(\tau_1 - \frac{m_1 - n_1}{2s_1}\right)^2 + \left(\tau_2 + \frac{E}{2s_1}\right)^2 = \frac{1}{4s_1^2}\Big((m_1 - n_1)^2 + 4r_1 s_1 + E^2\Big)$$

- In this general case, we do need to ensure that the R-projected charge vectors of all three dyons lie in a plane. We will now explore the implications of this.

- The quantity

$$\sqrt{\Delta} = \sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}$$

that appears in the BPS mass formula involves a square root, and we have taken all square roots to be positive.

- This has the following consequence. Observe that:

$$\Delta(m_i\vec{Q} + r_i\vec{P}, s_i\vec{Q} + n_i\vec{P}) = \det\begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix} \Delta(\vec{Q}, \vec{P})$$

- Positivity of Δ on both sides of the equation imposes the condition:

$$\det\begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix} > 0, \qquad i = 1, 2$$

- Since
$$\begin{pmatrix} m_2 & r_2 \\ s_2 & n_2 \end{pmatrix} = \begin{pmatrix} 1-m_1 & -r_1 \\ -s_1 & 1-n_1 \end{pmatrix}$$
we find two equations which can be summarised as:
$$m_1 n_1 - r_1 s_1 > \max\left(m_1 + n_1 - 1, 0\right)$$
- We now examine the quantities $\frac{M_1}{M}, \frac{M_2}{M}$ at some convenient point on the curve. To be on the correct branch, we need to ensure that both are less than 1.

- First consider the case $m_1 n_1 - r_1 s_1 > 1$. In this case it is possible to show that $\frac{M_1}{M} > 1$, so we are on the wrong branch.
- Next suppose $m_1 n_1 - r_1 s_1 = 1$. Now we find that $\frac{M_1}{M} = 1$. This means $M_2 = 0$ and therefore the charges associated to the second state are identically zero – a trivial case.
- Similar results hold on sending $1 \to 2$.
- That only leaves the case:

$$0 < m_1 n_1 - r_1 s_1 < 1, \quad 0 < m_2 n_2 - r_2 s_2 < 1$$

- Here we find $\frac{M_1}{M} < 1$, $\frac{M_2}{M} < 1$ and this indeed corresponds to the decay process that we were looking for.
- Clearly the above condition can only be satisfied for fractional coefficients.

- Fractional entries for the matrix:

$$\begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix}$$

  means that the decay process was into states with charges outside the $\vec{Q}, \vec{P}$ plane.
- Hence the moduli in M must be adjusted so that after R-projection, the final state charges do lie in the plane of the original charges.
- That means marginal stability occurs on a locus of co-dimension ≥ 2 in the full moduli space. This is why such decays are called rare decays.
- In particular, degeneracies cannot jump at such loci.

- We will now see how to explicitly characterise the loci in moduli space where such rare decays take place.
- We will also find explicit expressions for the numbers m_1, r_1, s_1, n_1 in terms of the quantised charge vectors $\vec{Q}, \vec{P}, \vec{Q}_1, \vec{P}_1$ and the moduli M.
- For this, define a quartic scalar invariant of four different vectors by:

$$\begin{aligned}\Delta(\vec{A},\vec{B};\vec{C},\vec{D}) &\equiv \det\begin{pmatrix}\vec{A}\circ\vec{C} & \vec{A}\circ\vec{D}\\ \vec{B}\circ\vec{C} & \vec{B}\circ\vec{D}\end{pmatrix}\\ &= (\vec{A}\circ\vec{C})(\vec{B}\circ\vec{D})-(\vec{A}\circ\vec{D})(\vec{B}\circ\vec{C})\end{aligned}$$

- The quartic invariant of two variables defined earlier is a special case of this new invariant:

$$\Delta(\vec{Q},\vec{P}) = \Delta(\vec{Q},\vec{P};\vec{Q},\vec{P})$$

- Now consider the equation:

$$\begin{pmatrix} \vec{Q}_R^{(i)} \\ \vec{P}_R^{(i)} \end{pmatrix} = \begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix} \begin{pmatrix} \vec{Q}_R \\ \vec{P}_R \end{pmatrix}$$

- Taking the first line for $i = 1$:

$$\vec{Q}_R^{(1)} = m_1 \vec{Q}_R + r_1 \vec{P}_R$$

and contracting successively with $\vec{Q}_R$ and $\vec{P}_R$ we find:

$$\begin{aligned} \vec{Q}_R^{(1)} \cdot \vec{Q}_R &= m_1 \vec{Q}_R^2 + r_1 \vec{Q}_R \cdot \vec{P}_R \\ \vec{Q}_R^{(1)} \cdot \vec{P}_R &= m_1 \vec{Q}_R \cdot \vec{P}_R + r_1 \vec{P}_R^2 \end{aligned}$$

- Multiplying the first equation by $\vec{P}_R^2$ and the second by $\vec{Q}_R \cdot \vec{P}_R$ and subtracting, we find:

$$m_1 \Delta(\vec{Q}_R, \vec{P}_R) = \Delta(\vec{Q}_R, \vec{P}_R; \vec{Q}_R^{(1)}, \vec{P}_R)$$

which enables us to solve for m_1 in terms of charges and moduli.

- Repeating this process we can solve for r_1, s_1, n_1 leading to the result:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \frac{1}{\Delta(\vec{Q}_R,\vec{P}_R)} \begin{pmatrix} \Delta(\vec{Q}_R,\vec{P}_R;\vec{Q}_R^{(1)},\vec{P}_R) & \Delta(\vec{Q}_R,\vec{P}_R;\vec{Q}_R,\vec{Q}_R^{(1)}) \\ \Delta(\vec{Q}_R,\vec{P}_R;\vec{P}_R^{(1)},\vec{P}_R) & \Delta(\vec{Q}_R,\vec{P}_R;\vec{Q}_R,\vec{P}_R^{(1)}) \end{pmatrix}$$

- Thus our original equation becomes:

$$\begin{pmatrix} \vec{Q}_R^{(1)} \\ \vec{P}_R^{(1)} \end{pmatrix} = \frac{1}{\Delta(\vec{Q}_R,\vec{P}_R)} \begin{pmatrix} \Delta(\vec{Q}_R,\vec{P}_R;\vec{Q}_R^{(1)},\vec{P}_R) & \Delta(\vec{Q}_R,\vec{P}_R;\vec{Q}_R,\vec{Q}_R^{(1)}) \\ \Delta(\vec{Q}_R,\vec{P}_R;\vec{P}_R^{(1)},\vec{P}_R) & \Delta(\vec{Q}_R,\vec{P}_R;\vec{Q}_R,\vec{P}_R^{(1)}) \end{pmatrix} \begin{pmatrix} \vec{Q}_R \\ \vec{P}_R \end{pmatrix}$$

- For fixed charge vectors $\vec{Q}, \vec{P}$ of the initial dyon and $\vec{Q}^{(1)}, \vec{P}^{(1)}$ of the first decay product, the above equation provides constraints on the moduli that must be satisfied for the $\frac{1}{4} \to \frac{1}{4} + \frac{1}{4}$ decay to be possible.
- These constraints together with the curve of marginal stability provide a necessary and sufficient set of kinematic conditions for marginal decay.

Outline

Solving the constraints

- In the above form, the constraints are rather implicit.
- Therefore we will consider some special cases.
- As a first check, the special case where the decay products are $\frac{1}{2}$-BPS should provide no constraints on the moduli.
- Inserting the $\frac{1}{2}$-BPS conditions:

$$\vec{P}^{(1)} = k_1 \, \vec{Q}^{(1)}, \quad \vec{P}^{(2)} = k_2 \, \vec{Q}^{(2)}$$

we find that:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = (k_2 - k_1) \frac{\Delta(\vec{Q}_R^{(1)}, \vec{Q}_R^{(2)})}{\Delta(\vec{Q}_R, \vec{P}_R)} \begin{pmatrix} k_2 & -1 \\ k_1 k_2 & -k_1 \end{pmatrix}$$

- We also have:

$$\Delta(\vec{Q}_R, \vec{P}_R) = (k_2 - k_1)^2 \Delta(\vec{Q}_R^{(1)}, \vec{Q}_R^{(2)})$$

- Substituting in the above equation, we find:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \frac{1}{k_2 - k_1} \begin{pmatrix} k_2 & -1 \\ k_1 k_2 & -k_1 \end{pmatrix}$$

- All moduli-dependence has disappeared from the matrix, and the equation indeed reduces to an identity.
- It is also easy to see that $k_1 - k_2$ divides the torsion of the original dyon, so in the unit-torsion case $k_1 - k_2 = 1$ and m_1, r_1, s_1, n_1 are all manifestly integral as expected.

- The next special case we will study has a restricted set of charges.
- Additionally, some of the background moduli are set to a specific value, namely zero in the chosen coordinates.
- We then examine the constraints on the remaining moduli.
- Introduce the notation:

$$\vec{Q} = \Big(\vec{Q}'(\text{6-comp}),\ \vec{Q}''(\text{6-comp}),\ \vec{Q}'''(\text{16-comp})\Big)$$

  and similarly for $\vec{P}$.
- Now we restrict ourselves to special initial-state charges given by:

$$\vec{Q}' = (Q'_1, 0, \cdots, 0), \quad \vec{Q}'' = (Q''_1, 0, \cdots, 0), \quad \vec{Q}''' = 0$$

  and

$$\vec{P}' = (0, P'_2, 0, \cdots, 0), \quad \vec{P}'' = (0, P''_2, 0, \cdots, 0), \quad \vec{P}''' = 0$$

- Next we set $B_{ij} = 0 = A_i^I$ as well as $G_{ij} = 0, \ i \neq j$.
- The above restrictions allow us to choose the orthonormal frames E_{ai} to be diagonal:

$$E_{ii} = R_i, \ i = 1, 2, \cdots, 6$$

with R_i the radii of the six compactified directions in the heterotic basis.

- In the restricted subspace of moduli space that we are considering here, the matrix $\sqrt{L+M}$ reduces to:

$$\sqrt{L+M} = \begin{pmatrix} E^{-1} & E & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Therefore the projected initial-state charge vectors are:

$$\vec{Q}_R = \begin{pmatrix} \frac{Q_1'}{R_1} + Q_1'' R_1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \qquad \vec{P}_R = \begin{pmatrix} 0 \\ \frac{P_2'}{R_2} + P_2'' R_2 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

- For this configuration we clearly have $\vec{Q}_R \cdot \vec{P}_R = 0$ and therefore the quartic invariant Δ is:

$$\Delta(Q_R, P_R) = \left(\frac{Q_1'}{R_1} + Q_1'' R_1\right)^2 \left(\frac{P_2'}{R_2} + P_2'' R_2\right)^2$$

- Then one can show that the constraint equation becomes:

$$\left(\frac{Q_1'}{R_1}+Q_1''R_1\right)\left(\frac{P_2'}{R_2}+P_2''R_2\right)\vec{Q}_R^{(1)} = \left(\frac{Q_1^{(1)'}}{R_1}+Q_1^{(1)''}R_1\right)\left(\frac{P_2'}{R_2}+P_2''R_2\right)\vec{Q}_R + \left(\frac{Q_1'}{R_1}+Q_1''R_1\right)\left(\frac{Q_2^{(1)'}}{R_2}+Q_2^{(1)''}R_2\right)\vec{P}_R$$

$$\left(\frac{Q_1'}{R_1}+Q_1''R_1\right)\left(\frac{P_2'}{R_2}+P_2''R_2\right)\vec{P}_R^{(1)} = \left(\frac{P_1^{(1)'}}{R_1}+P_1^{(1)''}R_1\right)\left(\frac{P_2'}{R_2}+P_2''R_2\right)\vec{Q}_R + \left(\frac{Q_1'}{R_1}+Q_1''R_1\right)\left(\frac{P_2^{(1)'}}{R_2}+P_2^{(1)''}R_2\right)\vec{P}_R$$

- These are $6+6$ equations.
- However, the first two components of each set are identically satisfied. These are the ones from which the numbers m_1, r_1, s_1, n_1 were determined.
- The remaining four components of each equation give the desired constraints on the moduli.

- Because of the way we have chosen $\vec{Q}, \vec{P}$, the RHS already vanishes on components 3 to 6, so the constraint is simply that the LHS vanishes.
- Thus we find the constraints:

$$\frac{Q_i^{(1)\prime}}{R_i} + Q_i^{(1)\prime\prime} R_i = 0, \quad i = 3,4,5,6$$
$$\frac{P_i^{(1)\prime}}{R_i} + P_i^{(1)\prime\prime} R_i = 0, \quad i = 3,4,5,6$$

- If the components of $\vec{Q}^{(1)}, \vec{P}^{(1)}$ are all nonvanishing, this implies that:

$$R_i = \sqrt{-\frac{Q_i^{(1)\prime}}{Q_i^{(1)\prime\prime}}} = \sqrt{-\frac{P_i^{(1)\prime}}{P_i^{(1)\prime\prime}}}, \quad i = 3,4,5,6$$

- In this special case the constraint equations have some particular features.
- For generic charge vectors $\vec{Q}^{(1)}$ and $\vec{P}^{(1)}$, there are no solutions. This simply means that our restricted moduli space fails to intersect the marginal stability locus in that case.
- To have any solutions at all, one must choose the charges of the decay products in such a way that the second equality in the above equation can be satisfied.
- Once this is done we find four constraints on the moduli, which fix the compactification radii R_3, R_4, R_5, R_6.
- For this special case, the numbers m_1, r_1, s_1, n_1 are given by:

$$
m_1 = \frac{\frac{Q_1^{(1)\prime}}{R_1} + Q_1^{(1)\prime\prime} R_1}{\frac{Q_1'}{R_1} + Q_1'' R_1}, \qquad r_1 = \frac{\frac{Q_2^{(1)\prime}}{R_2} + Q_2^{(1)\prime\prime} R_2}{\frac{P_2'}{R_2} + P_2'' R_2}
$$

$$
s_1 = \frac{\frac{P_1^{(1)\prime}}{R_1} + P_1^{(1)\prime\prime} R_1}{\frac{Q_1'}{R_1} + Q_1'' R_1}, \qquad n_1 = \frac{\frac{P_2^{(1)\prime}}{R_2} + P_2^{(1)\prime\prime} R_2}{\frac{P_2'}{R_2} + P_2'' R_2}
$$

- So far the decay products were taken to have generic charges.
- The situation changes if we choose less generic decay products.
- If we take $Q_i^{(1)'} = Q_i^{(1)''} = P_i^{(1)'} = P_i^{(1)''} = 0$ for any $i \in 3, 4, 5, 6$ then the corresponding constraint is trivially satisfied.
- As an example, if the above holds for $i = 4, 5, 6$ and if $\frac{Q_3^{(1)'}}{Q_3^{(1)''}} = \frac{P_3^{(1)'}}{P_3^{(1)''}}$ then there is only a single constraint coming from the above equations.
- The curve of marginal stability always provides one more constraint, so the decay will take place on a codimension-2 subspace of the restricted moduli space.

- If the charges $Q_i^{(1)'}, Q_i^{(1)''}, P_i^{(1)'}, P_i^{(1)''}$ vanish for all $i \in 3, 4, 5, 6$ then there are no constraints (beyond the curve of marginal stability).
- It is easily seen that this is the case where the final states are both $\frac{1}{2}$-BPS.

- Let us now consider initial dyons with the most general charges.
- Considerable simplification can be brought about in the formulae by using some known results on T-duality orbits *[Wall (1962), Banerjee-Sen (2007)]*.
- These results state that any pair of primitive charge vectors $\vec{Q}, \vec{P}$ can be brought via T-duality to the form:

$$\begin{aligned} \vec{Q}' &= (Q_1', 0, \cdots, 0), \quad \vec{Q}'' = (Q_1'', 0, \cdots, 0), \quad \vec{Q}''' = 0 \\ \vec{P}' &= (P_1', P_2', \cdots, 0), \quad \vec{P}'' = (P_1'', P_2'', \cdots, 0), \quad \vec{P}''' = 0 \end{aligned}$$

- This is close to our previous special case, but with P_1', P_1'' turned on. It is no longer a special case but represents the general case in a special basis.
- Next we restrict the moduli in the most minimal way consistent with finding a simple form of the constraint equation.

- The restriction will be a kind of "triangularity" condition:

$$(G+B+C)_{i1} = (G+B+C)_{i2} = 0, \quad i = 3,4,5,6$$

- Using this condition we find the constraint equations still in a relatively simple form:

$$\begin{aligned} Q_i^{(1)'} + (G+B+C)_{ij} Q_j^{(1)''} &= 0, \quad i = 3,4,5,6 \\ P_i^{(1)'} + (G+B+C)_{ij} P_j^{(1)''} &= 0, \quad i = 3,4,5,6 \end{aligned}$$

- These are the $4+4$ constraints on rare dyon decays into a pair of dyons.

- They must be supplemented by the curve of marginal stability, for which we need to know the numbers m_1, r_1, s_1, n_1.
- One finds that m_1 is given by:

$$m_1 = \left(P_2' + (G+B+C)_{2i} P_i''\right)^{-1} \left(Q_1' + (G+B+C)_{1i} Q_i''\right)^{-1} \times$$
$$\left[\left(Q_1^{(1)'} + (G+B+C)_{1i} Q_i^{(1)''}\right) \left(P_2' + (G+B+C)_{2i} P_i''\right) \right.$$
$$\left. - \left(Q_2^{(1)'} + (G+B+C)_{2i} Q_i^{(1)''}\right) \left(P_1' + (G+B+C)_{1i} P_i''\right) \right]$$

The other coefficients follow similarly.

- It remains to find the constraints in the completely general case (remove the “triangularity” condition).
- In this case we were no longer able to disentangle the constraints explicitly as we did above.
- It may be that a better choice of T-duality basis will allow us to handle the most general case. We leave such an investigation for the future.

Outline

Multi-particle decays

- So far we have written down conditions for decay of a dyon into two $\frac{1}{4}$-BPS final states.
- Let us now allow the final state to have n decay products of charges $(\vec{Q}^{(1)}, \vec{P}^{(1)}), (\vec{Q}^{(2)}, \vec{P}^{(2)}), \cdots (\vec{Q}^{(n)}, \vec{P}^{(n)})$.
- To find the conditions for this marginal decay, consider the collection of all marginal stability loci for the following decays:

$$\begin{pmatrix} \vec{Q}_R \\ \vec{P}_R \end{pmatrix} \to \begin{pmatrix} \vec{Q}_R^{(i)} \\ \vec{P}_R^{(i)} \end{pmatrix} + \begin{pmatrix} \vec{Q}_R - \vec{Q}_R^{(i)} \\ \vec{P}_R - \vec{P}_R^{(i)} \end{pmatrix}, \quad i = 1, 2, \cdots, n$$

- For each of these, the curve of marginal stability is $\mathcal{C}_i = 0$ where:

$$\mathcal{C}_i \equiv \left(\tau_1 - \frac{m_i - n_i}{2s_i}\right)^2 + \left(\tau_2 + \frac{E_i}{2s_i}\right)^2 - \frac{1}{4s_i^2}\left((m_i - n_i)^2 + 4r_i s_i + E_i^2\right)$$

where

$$E_i \equiv \frac{1}{\sqrt{\Delta}}\left(\vec{Q}^{(i)} \circ \vec{P} - \vec{P}^{(i)} \circ \vec{Q}\right)$$

- In addition we have the constraints on the remaining moduli as above.
- Those too can be expressed in terms of the single decay product labelled "i".
- Now to find the condition for a multi-dyon decay, we simply take the intersection of all these loci of marginal stability.

Outline

Discussion

- An enormous wealth of detail remains to be uncovered.
- The curves of marginal stability are an infinite set of subspaces, of varying codimension ≥ 1, of the $\mathcal{N} = 4$ moduli space.
- A number-theoretic interpretation has been found for only a small subset of these (decays to a pair of $\frac{1}{2}$-BPS states) in terms of Ford circles. What about the others?
- Curves of codimension $= 2$ can admit monodromies. What is the relevant physical observable (some sort of continuous-valued index) exhibiting this behaviour? And what about codimension > 2?
- What is the relation to multi-centred black holes?
- Marginal stability for black rings in $d = 5$?