# Borcherds-Kac-Moody symmetry of N=4 Dyons


**Atish Dabholkar**

TIFR, Mumbai

CNRS/LPTHE, Paris


IPMU

- ***Miranda Cheng, A. D.,***

  arXiv:0809.nnnn
- ***A.D., Joao Gomez, Sameer Murthy***

  arXiv:0802.0761
- ***A. D., Davide Gaiotto,Suresh Nampuri***

  hep-th/0702150
- ***A. D., Davide Gaiotto,*** hep-th/0612011
- ***A. D., Suresh Nampuri,*** hep-th/0603066

# References

- **Dijkgraaf, Verlinde, Verlinde;**
- **Cardoso, de Wit, Kappelli , Mohaupt**
- **Kawai;**
- **Gaiotto, Strominger, Xi, Yin**
- **David, Jatkar, Sen; Banerjee,Srivastava**
- **Dabholkar, Gaiotto, Nampuri,**
- **Cheng and Verlinde**

# Plan

- Exact dyon degeneracies in N=4
- Puzzles and their Resolutions
- Synopsis of Results
- Explanation

# Quarter-BPS dyons

- In the past few years there has been considerable progress in understanding the *exact* spectrum of quarter-BPS dyons in four-dimensional compactifications with N=4 supersymmetry starting with the work of Dijkgraaf, Verlinde, Verlinde.
- The spectrum has revealed a surprisingly rich structure.

# Two Surprises

- The first surprise is of course that the exact spectrum is *computable at all*

  for complicated bound states of KK, NS5, D-branes, strings and momenta.
- Now known *at all points* in the moduli space *for all duality orbits* despite a complicated structure of wall crossing. Not only at weakly coupled corners but the middle of the moduli space.

- Such an exact nonperturbative information about all dyons  at all points in the moduli is  sure tp illuminate the structure of the theory in unexpcted ways.
- One can address a number of physical questions in far greater detail than is possible for N=2 dyons.

- For example, from the *microscopic* analysis, one can compute very precisely the *structure of the walls of marginal stability*.
- One can compute the *subleading corrections to Wald entropy* of the corresponding dyonic black holes
- Both in beautiful agreement with the independent *macroscopic* supergravity analysis

# Possible New Symmetries

- The symmetry of string theory is probably a vast generalization of coordinate and gauge invariance since by duality all membrane charges are on the same footing. Exact information about the dyons might give us some clues about such a symmetry.
- For example, the spectrum of dyons prompted Montonen-Olive conjecture. In string theory, half-BPS branes suggested the web of dualities. We will see some evidence for a hyperbolic Kac-Moody symmetry.

# Quantum Black Holes

- We can now understand the sub-leading corrections to quantum Wald entropy in some cases exactly.
- We can hope to define a full *quantum macroscopic partition function* because we know *quantum microscopic partition function* (index) exactly. (*OSV, Sen*)
- This will help us to understand some of the subtle questions about this comparison.

# Heterotic on $T^4$ £ $T^2$

- Total rank of the four dimensional theory is

**16** ( $\mathbf{E_8 \times E_8}$ ) + **12** ( $\mathbf{g_{\mu m}}, \mathbf{B_{\mu m}}$ ) = **28**

- **N=4** supersymmetry in **D=4**

- Duality group

$$\mathbf{SO(22,6;Z) \times SL(2;Z)}$$

# Charges and T-duality Invariants

- A charge vector is specified as

$$\Gamma = \begin{pmatrix} Q \\ P \end{pmatrix}$$

- Transforms as a vector of the T-duality group and a doublet of S-duality group.
- Define the T-duality invariants

$$\Lambda_{Q,P} \equiv \Gamma \cdot \Gamma^t = \begin{pmatrix} Q \cdot Q & Q \cdot P \\ Q \cdot P & P \cdot P \end{pmatrix}$$

# CHL Orbifolds

- Orbifolds of the type $T^5$ £ $S^1 = Z_N$
- Orbifold symmetry generated by $\alpha T$

o $\alpha$ is a twist acting on left-movers only.

o T is an order N shift along the circle,

$$X \; ! \; X + 2\pi R/N$$

# Same SUSY but Reduced Rank

- Supersymmetry is carried by right-moving fermions which are not affected by the left-moving twist so we still have $N = 4$ susy as for the unorbifolded theory.
- Some vectors from untwisted sector projected out by the left-moving twist. Twisted states are massive because of T, so no twisted vectors.

# Allowed Models

- Possible left-moving symmetries exist for

$$N = 1; 2; 3; 5; 7; 11$$

- For example, for N=2, we can take the $E_8$ £ $E_8$ string. The generator $\alpha$ just flips these two factors. So the rank of gauge group reduces by eight.

# Why CHL Orbifolds?

- S-duality group is a **subgroup of SL(2, Z)** so counting of dyons is quite different.
- **Wald entropy is modified** in a nontrivial way and is calculable.
- Nontrivial but tractable generalization with interesting physical differences for the spectrum of black holes and dyons.

# S-duality group = $\Gamma_1(N)$

Because of the shift, there are 1/N quantized electric charges (winding modes). For example for N=2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/2 \\ n \end{pmatrix} = \begin{pmatrix} 1/2 \\ n \end{pmatrix}$$

This requires that c = 0 mod 2

which gives $\Gamma_0(2)$ subgroup of SL(2, Z).

# Spectrum of Dyons

- For a **$Z_N$** orbifold, the dyonic degeneracies are encapsulated by a Siegel modular form $\Phi_k(\Omega)$ of **Sp(2, Z)** of **level N** and **index k** as a function of period matrices **$\Omega$** of a ***genus two Riemann surface***..

$$k = \frac{24}{N+1} - 2$$

N = 1; 2; 3; 5; 7; 11

# Sp(2, Z)

- 4 £ 4 matrices ***g*** of integers that leave the symplectic form invariant:

$$\mathbf{g} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}$$

$$\mathbf{g^t J g = J}$$

where A, B, C, D are 2£ 2 matrices.

# Genus Two Period Matrix

- Like the $\tau$ parameter at genus one

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}.$$

# Siegel Modular Forms

- $\Phi_k(\Omega)$ is a Siegel modular form of weight k and level N if

$$\Phi_{\mathbf{k}}(\Omega') = \{\det(C\Omega + D)\}^k \Phi_{\mathbf{k}}(\Omega)$$

$$\Omega' = (A\Omega + B)(C\Omega + D)^{-1},$$

under elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $G_0(N)$.

# Spectrum of Half-BPS States

- Right-moving ground states of heterotic string.
- Partition function of left-moving oscillators.
- Degeneracy given by Fourier coefficients of this partition function, a genus-one modular form.

$$Z(q) = \frac{1}{\Delta_k(q)} = \sum c(N)\, q^N$$

$$d(Q) = c(Q^2/2)$$

- For example,

$$\Delta_k(q) = \eta^{24}(q)$$

# Spectrum of Quarter-BPS States

- Electric charge Q, magnetic charge P
- Degeneracy given in terms of Fourier coefficients of genus two *Siegel Modular forms* (almost!!)

$$Z(q,p,y) = \frac{1}{\Phi_k(q,p,y)} = \sum c(n,m,l)\, q^n p^m y^l$$

$$d(Q,P) = c(Q^2/2, P^2/2, Q\cdot P)$$

Comparison of ***S= k log (d)*** is impressive. Both macroscopic and microscopic entropy obtained by the minimum value of ¾the *same* function F of two variables **a** and

$$F = \frac{¼}{2}\left[\frac{a^2 + ¾^2}{¾} P^2 + \frac{1}{¾} Q^2 ¡ \frac{2a}{¾} Q ¢ P + 128¼ Á(a;¾) + :::\right]$$

$$Á(a;¾) = ¡ \frac{1}{64¼^2} f (n+2) \log ¾ + \log jf^{(n)}(a + i¾)j^2 g$$

$$f^{(n)}(\tau) \equiv \eta(\tau)^{n+2}\, \eta(N\tau)^{n+2}$$

# Irreducibility Criteria

- For electric and magnetic charges Q & P,

$$I = \gcd(Q \wedge P)$$

- This is an additional *discrete* duality invariant, there can be more.

*A. D., Gaiotto, Nampuri*

# Counting All possible Dyons

- One can show that in this context, ***I*** is the only additional duality invariant to worry about. The exact degeneracy for all values of I and hence for all duality orbits *Banerjee,Sen,Srivastava; A.D,Gomez,Murthy)*
- In this talk we will restrict to I=1

# Puzzles

- Which contour to choose? Siegel forms have an intricate structure of zeros in the Siegel upper half plane. Poles of partition function signify ``phase transitions''. So different contours will give different answers.
- What about the moduli-dependence? Quarter-BPS states are expected to  decay on walls of marginal stability but in the formula before there is no apparent moduli dependence.
- Degeneracies (apparently)  ***not***  S-duality invariant.

# Answers

- For fixed charges, different choices of contours give the degeneracy in different regions of the moduli space. *Partition function is unchanged.*
- Moduli space is divided into regions separated by walls. *Crossing a wall corresponds to crossing a pole.*
- *Jump in degeneracy equals the residue at the pole*. Resulting answers are *S-duality invariant*.

*Sen; A. D., Gaiotto, Nampuri; Cheng & Verlinde*

# Lorentzian Lattice

- Duality symmetry ***SL(2)*** can be viewed as ***SO(1, 2)*** Lorentz symmetry.
- The charge matrix $\Lambda_{Q,P}$ transforms as vector of this Lorentz symmetry and defines a point in Lorentzian lattice.
- Under duality, it transforms as

$$\Lambda \rightarrow \gamma(\Lambda_{Q,P}) := \gamma \Lambda_{Q,P} \gamma^t$$

- Symmetric, real matrices

$$X = \begin{pmatrix} x^+ & x \\ x & x^- \end{pmatrix}; \quad X = X^t; \quad x^{\pm} = t \pm y$$

with metric $(X, X) = -2 \det X = 2(-t^2 + x^2 + y^2)$

invariant under $X \to \gamma(X) := \gamma X \gamma^t$

- Charges $\Lambda_{Q,P}$, chemical Potentials $\Omega$ and a moduli dependent `central charge matrix' $Z$ all live in this X space.
- Define $|X|^2 = \det X$ for later usage.

# Moduli

- Axion-dilaton S-moduli are

$$¸ = a + i e^{¡ 2Á}$$

1

- The T-moduli on Narain space are vierbeins

$$Q_L^i = ¹^i_l Q^l ; \quad i = 1; : : : ; 22$$

$$Q_R^m = ¹^m_l Q^l ; \quad m = 1; : : : ; 6$$

# Central Charge Vector

- S-moduli vector

$$S = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}$$

- T-moduli vector

$$\Lambda_R = \begin{pmatrix} Q_R \cdot Q_R & P_R \cdot Q_R \\ P_R \cdot Q_R & P_R \cdot P_R \end{pmatrix} \qquad T_R = \frac{\Lambda_R}{|\Lambda_R|}$$

- Central charge vector

$$Z = |\Lambda_R|^{\frac{1}{2}} (T_R + S)$$

# Attractor Equations

- BPS mass formula $M_{Q;P} = |Z|$

- Moduli at the horizon align such that the charge vectors lie in the right-moving plane

$$P = P_R; \quad Q = Q_R$$

- Similarly for axion-dilaton. In other words,

$$S = T = \frac{¤}{|¤|}$$

# Two-centered Solution

- Attractor flow is a gradient flow in the mass. The attractor value of the magnitude of the central charge vector determines the entropy of single centered black hole.
- The direction of this vector determines whether a two-centered solution exists in a given a region of moduli space.

# Contour and Moduli dependence

The contour $C(Q;P;¸;¹)$

is specified by

$f \, \mathrm{Im} - = "¡1 Z; 0 \cdot Re½; \frac{1}{N} Re¾; Rev < 1g$

Note that the *entire* moduli dependence comes from the choice of the contour. This would be far more complicated in $N = 2$ context. *Cheng and Verlinde*

# Quarter-BPS Dyon Degeneracies

- Let us write the degeneracies as

$$d_N(Q;P;\lambda;\bar\lambda) \approx \oint_C d\Omega \, \frac{e^{-i\pi(\Lambda;\Omega)}}{\Phi_k(\Omega)}$$

- The contour depends in a precise way on the moduli. For fixed charges, *all* dependence on moduli $(\lambda;\bar\lambda)$ is captured by this contour dependence.

# Illustration in N=1 model

- Consider a charge configuration
  $(Q^2; P^2; Q \cdot P) = (-2; -2; N)$

- Effective partition function for these states can be obtained from the full partition fn that goes as $p^{-1} q^{-1}$.

$$(y^{1/2} - y^{-1/2})^{-2}$$

# Contour = Moduli dependence

- Different expansion for different regions
- Our partition function has $Z_2$ symmetry.

$$\frac{y^{-1}}{(1-y^{-1})^2} = \sum_{N=1}^{\infty} N y^{-N} \qquad N > 0; a > 0$$

$$\frac{y}{(1-y)^2} = \sum_{N=1}^{\infty} N y^{N} \qquad N > 0; a < 0$$

- Parity takes *N* to *–N* and ***a*** to ***–a***
- Degeneracy = |*N*| if ***N a >0***

# Walls of marginal stability

- Wall of marginal stability at *a =0.*
- The **Z$_2$** can be identified with the Weyl group of an SU(2). It can be used to map charges and moduli to the Weyl chamber. ®
- Simple root of SU(2) is ® ! ¡ ® and Weyl symmetry takes
- Weyl vector is just half the sum over simple roots which is here ½ = $\frac{1}{2}$ ®

# Partition function as a denominator

- One can view this partition function as a square of the denominator of the character of SU(2)

$$\chi_j = \mathrm{Tr}\, y^{J_3} = y^{-j} + \ldots y^{j}$$

$$= \frac{y^{-j-\frac{1}{2}} - y^{j+\frac{1}{2}}}{y^{-\frac{1}{2}} - y^{\frac{1}{2}}}$$

This formula has a powerful generalization.

# Walls and Weyl

- Walls of marginal stability associated with real roots of a would-be algebra

$$(Z, \alpha) = 0$$

- Fundamental Weyl Chamber defined by

$$\Lambda_{P,Q} \in W = \{ X \mid (X, \alpha) < 0 \text{ for all positive roots} \}$$

- Stable two-centered solution exists if

$$(\alpha, Z)(\alpha, \Lambda_{P,Q}) < 0$$

# Some Group Theory

- For a finite, simple Lie algebra, we have Cartan subalgebra of mutually commuting matrices.
- In a given representation, states are defined by eigenvalues of the Cartan generators. In particular for the adjoint representation each generator a definite eigenvalue. The zero eigenvalue has multiplicity ***r*** which equals the rank of the Lie algebra. All nonzero eigenvalues have unit multiplicity.

- The trace allows us to defines a natural Euclidean (Cartan) metric on this space.
- Plotting these eigenvalues defines a Euclidean lattice Euclidean space of dimension ***r***
- One can identify ***r*** simple roots. Their matrix of inner product defines the Cartan matrix.
- Cartan classification gives a complete list of allowed Cartan matrices.

# Weyl-Kac Character Formula

- One of the most important tools in the theory of affine Lie algebras is the Weyl character formula. Consider a representation V with highest weight ¤
- Weyl group W and Weyl vector %
- The character for this is given in terms of positive roots, their multiplicities, Weyl group, Weyl vector, and the weight.

$$\mathrm{ch}(V) = \frac{\sum_{w \in W} \det(w)\, w\left(e(\lambda + \rho)\, S\right)}{e(\rho) \prod_{\alpha > 0} \left(1 - e(-\alpha)\right)^{\mathrm{mult}\,\alpha}}$$

- S is contribution from imaginary simple roots. For finite and affine algebras S=1.
- Denominator identity

$$\sum_{w \in W} \det(w)\, w\left(e(\rho)\, S\right) = e(\rho) \prod_{\alpha > 0} \left(1 - e(-\alpha)\right)^{\mathrm{mult}\,\alpha}$$

# Hyperbolic Superalgebras

- Most of these concepts generalize naturally to hyperbolic algebras even though in that case the classification and representation theory is much less understood. Root multiplicities not known.
- Cartan matrix is hyperbolic (has exactly one negative eigenvalue).
- Both fermionic and bosonic generators.

# BKM Algebras for the CHL models

- The cases for $N < 4$ and $N \geq 4$ are qualitatively very different.
- Only for $N < 4$ the moduli space is divided into regions bounded by *finite number* of walls. In these cases, one can identify a Weyl group, Weyl vector, Cartan matrix, a hyperbolic BKM algebra and the partition function is a denominator of this algebra.
- *The other cases remain mysterious.*

# 1) Simple (Real) Roots

- we identify a special set of lattice vectors $\{\alpha_i^{(N)}\}$, $i = 1, \ldots, r_{(N)}$ with $r_{(N)} = 3, 4, 6$ where $r_{(N)}$ are the number of walls of Weyl chamber for $N = 1, 2, 3$ respectively.
- These lattice vectors will eventually be identified as the real simple roots of the Borcherds-Kac-Moody algebra.

# 2) Cartan Matrices

- We can define Cartan matrices as usual as the inner product matrix of the simple roots. $A^{(N)}_{ij} = (\alpha^{(N)}_i, \alpha^{(N)}_j), \quad i, j = 1, \ldots, r^{(N)}$

- In Lorentzian algebras one has real roots that have spacelike norm and imaginary roots that have timelike or lightlike norm. The imaginary roots are complicated.

# N=1

- Weyl chamber has three walls.

$$A^{(1)}_{ij} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

- Compare with SU(2) finite and affine Cartan matrices.

# N=2

- Weyl chamber has four walls.

$$A^{(2)}_{ij} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}$$

- It still has rank three and exactly one negative eigenvalue. It is a hyperbolic, rank three, generalized Cartan matrix.

# N=3

- Weyl Chamber has six walls.

$$A^{(3)}_{ij} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix}$$

# Classification

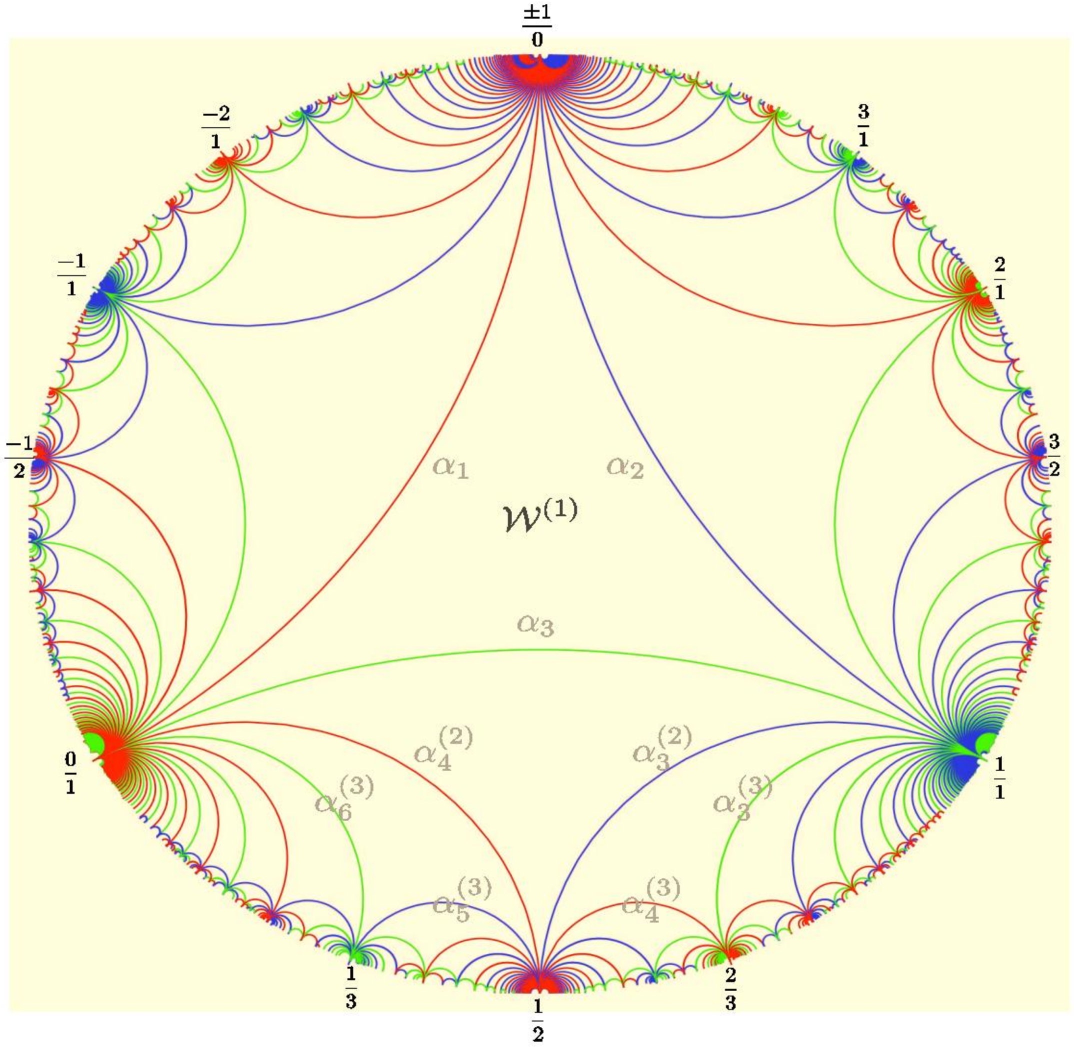
- Gritsenko and Nikulin have classified hyperbolic, rank three Cartan matrices of elliptic type that admit Weyl vector.
- Our matrices $A_{II,1}, A_{II,2}, A_{II,3}$ $A^{(1)}, A^{(2)}, A^{(3)}$ are in this classification.
- Moreover, they are the unique ones that have vertices of Weyl chamber at infinity.

# 3) Weyl Group

$$\tilde{\Gamma}_1(N) = W_{(N)} \rtimes \mathrm{Sym}(W_{(N)})$$

- LHS is physical duality group and RHS is the semi-direct product of Weyl group and the symmetry of the fundamental Weyl chamber.
- Physically, it maps any chamber of the moduli space to the `attractor region' where only immortal single-centered black holes exist.

# Walls



$\frac{\pm 1}{0}$
$\frac{-2}{1}$
$\frac{3}{1}$
$\frac{-1}{1}$
$\frac{2}{1}$
$\frac{-1}{2}$
$\frac{3}{2}$
$\alpha_1$
$\alpha_2$
$\mathcal{W}^{(1)}$
$\alpha_3$
$\alpha_4^{(2)}$
$\alpha_3^{(2)}$
$\frac{0}{1}$
$\frac{1}{1}$
$\alpha_6^{(3)}$
$\alpha_3^{(3)}$
$\alpha_5^{(3)}$
$\alpha_4^{(3)}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{1}{2}$

# 4) Lattice Weyl Vector

- For all models, a lattice vector exists and is given by

$$\rho_{(N)} = \begin{pmatrix} 1/N & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

- Physically it is related to the ``ground state energy" in level matching condition.

# 5) Denominator Identity

- The dyon partion function is a square of a Siegel form. In all cases they admit a product representation.
- With the identification of the lattice Weyl vector denominator identity is satisfied

# $\Phi_k$ is a complicated beast

- **Sum representation (Maass lift)**

  **Makes integrality of d(Q) manifest**

- **Product representation (Borcherds lift)**

  **Relates to 5d elliptic genus of D1D5P**

  **Equality of the two can be interpreted as a denominator identity.**

# 6) Root Multiplicities

- Hyperbolic algebras have made their appearance in other contexts in string theory. For practical applications, one stumbling block is that the root multiplicities are not known.
- In these examples, root multiplicities of real and imaginary roots can be computed explicitly.

# Conclusions

- Certain hyperbolic Borcherds-Kac-Moody algebras seem to appear as a symmetry of the spectrum of N=4 quarter dyons.
- Weyl vector exists only for N<4. The Weyl group plays a physical role in determining the structure of wall-crossings.
- Product formula of the partition function can be interpreted as denominator identity.

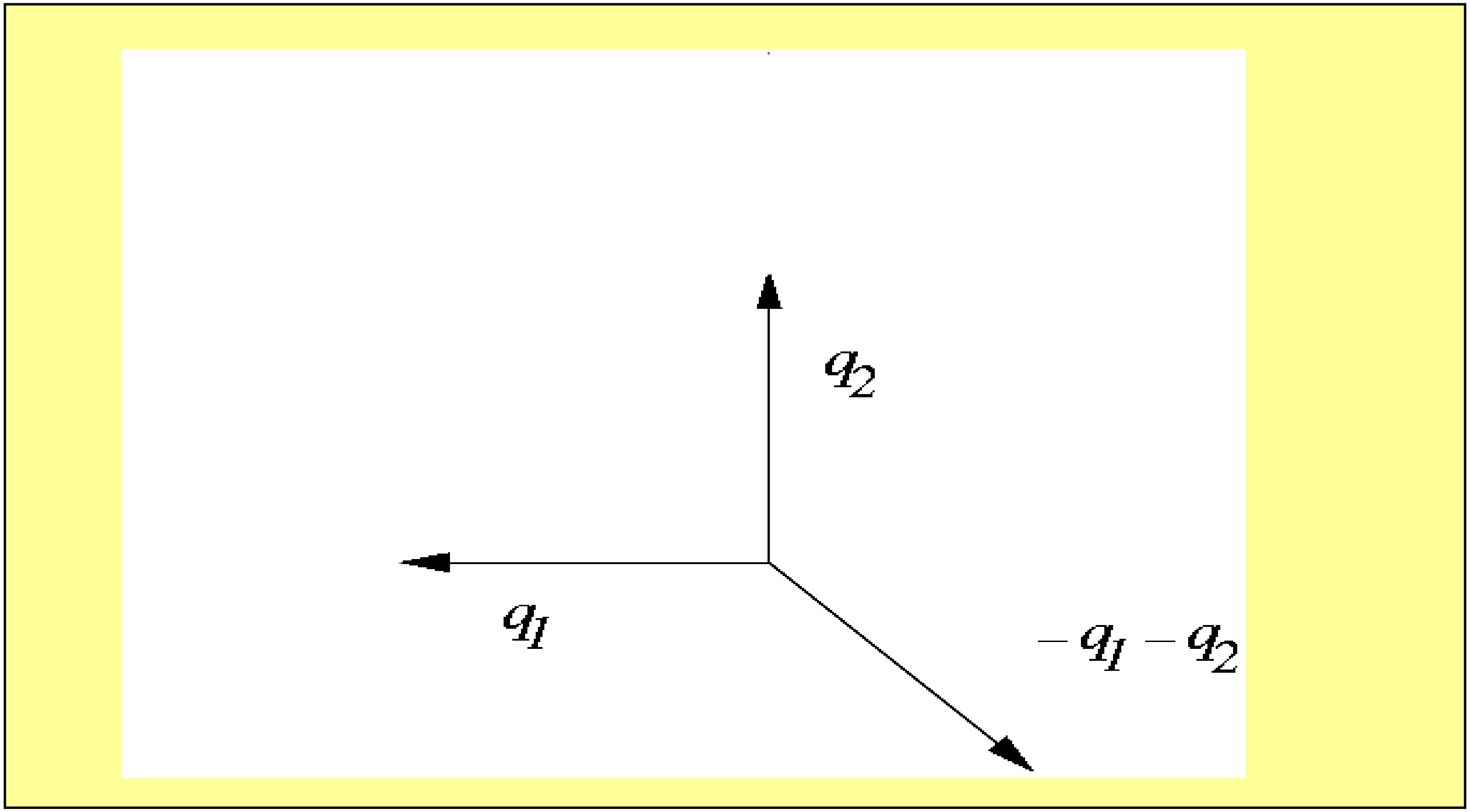
# Why genus-two and Sp( 2, Z)?

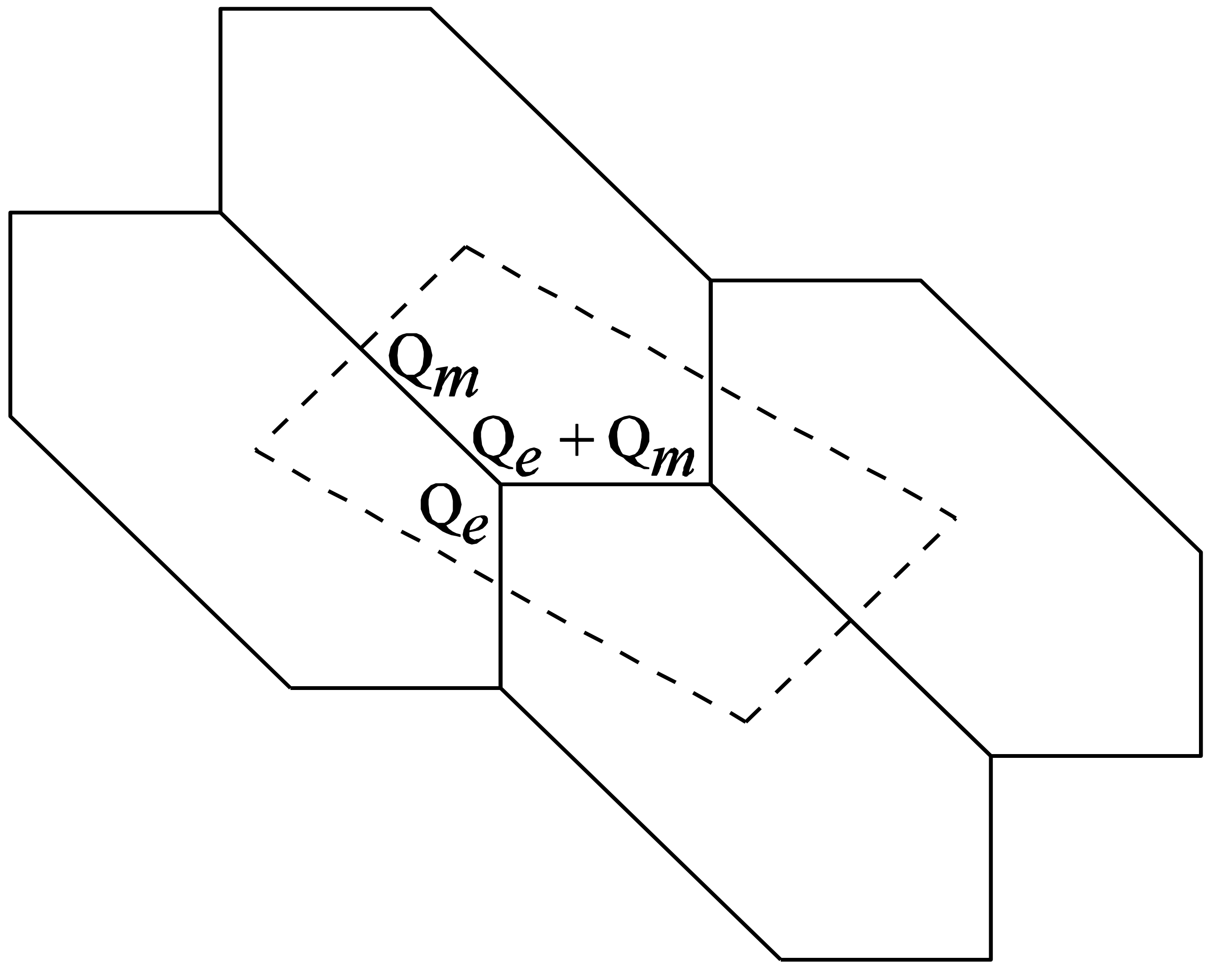
- Dyon partition function can be mapped by duality to genus-two partition function of the left-moving heterotic string on CHL orbifolds.
- Makes modular properties under subgroups of **Sp(2, Z)** manifest.
- Suggests a new derivation of the formulae.

# String Webs

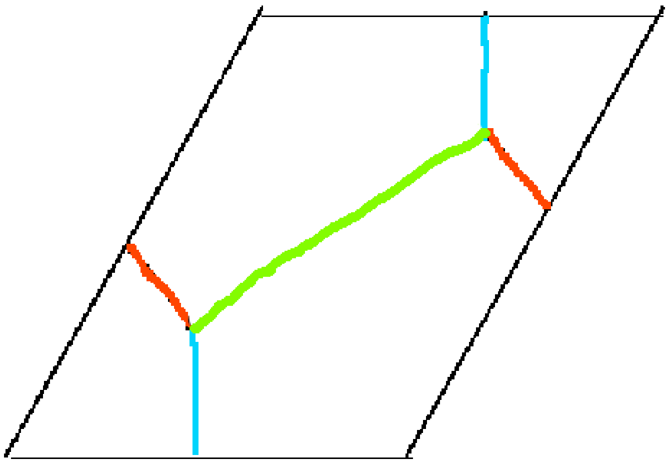
- Quarter BPS states of heterotic on **$T^4$ £ $T^2$** is described as a string web of (p, q) strings wrapping the **$T^2$** in Type-IIB string on **K3 £ $T^2$** and left-moving oscillations.
- The strings arise from wrapping various D3, D5, NS5 branes on cycles of **K3**

# String junction tension balance

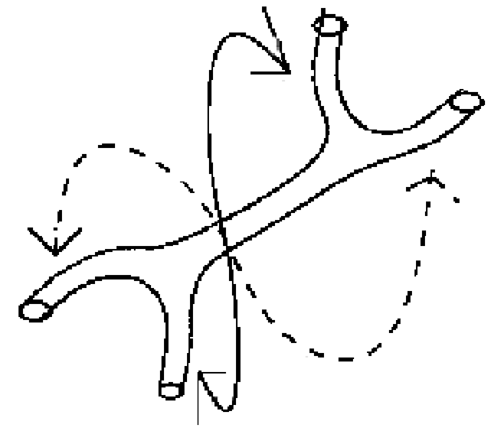



$Q_m$
$Q_e + Q_m$
$Q_e$

# M-lift of String Webs



$\equiv$



- Genus-2 worldsheet is worldvolume of Euclidean M5 brane with various fluxes turned on wrapping **K3 £ $T^2$**. The **$T^2$** is holomorphically embedded in **$T^4$** by Abel map. It can carry left-moving oscillations.
- K3-wrapped M5-brane is the heterotic string. So genus-2 chiral partition fn of heterotic counts its left-moving BPS oscillations.