SO群とUSp群のケーラー商及び ハイパーケーラー商とランプ解の構成

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Outline

Motivations

- The low energy effective theory on the Higgs branch in $\mathcal{N} = 2$ sQCD is hyperKähler
- $\ldots \mathcal{N} = 1$ sQCD is Kähler
- Topology, metric
  $\rightarrow$ low energy effective action
- Topology
  $\rightarrow$ topological excitations in the gauge theories
  $\rightarrow$ non-perturbative effects
- In extension of the line of research of Nitta et.al. (BPS domain walls) and Eto et.al. (BPS domain walls and SO/USp vortices)

A simple model : $O(3)$

Let us consider a simple model with a **scalar field** $\Phi = (\phi_1, \phi_2, \phi_3)$ and target space S^2:

$$\mathcal{L} = \frac{1}{4} \left|\partial_\mu \Phi\right|^2 + \nu \left(1 - |\Phi|^2\right) ,$$

with a constraint by means of a **Lagrange multiplier**. This is a **conformal field theory** with equation of motion

$$\Box\Phi - (\Phi \cdot \Box\Phi)\,\Phi = 0 ,$$

which bears the name **non-linear sigma model**. Finite energy solutions are constant at spatial infinity, hence symmetry breaking occurs

$$O(3) \underset{\mathrm{SSB}}{\rightarrow} O(2) .$$

Lumps : topological solitons

Topological solutions to this equation exist and are characterized by the map

$$\Phi : S^2 \mapsto S^2 \ , \qquad \pi_2(S^2) = \mathbb{Z} \ ,$$

and this integer can be calculated as the pull-back of the standard area-form on S^2.

In this model there exists a Bogomol'nyi bound

$$E \geq 2\pi |N| \ .$$

The $\mathbb{C}P^1$-model

Rewriting in terms of the Riemann sphere coordinate on the target space the $O(3)$ model

$$R(z,\bar{z}) = \frac{\phi_1 + i\phi_2}{1+\phi_3} , \qquad \mathcal{L} = \frac{\partial_\mu R \partial^\mu \bar{R}}{\left(1+|R|^2\right)^2} ,$$

which is indeed the $\mathbb{C}P^1$**-model** in which the Bogomol'nyi equation is

$$\bar{\partial} R = 0 \quad \Rightarrow \quad R = R(z) ,$$

[A.A.Belavin & A.M.Polyakov, JETP Lett.22:245-248 (1975)]

which is also known as the Cauchy-Riemann equation. In fact, the solution is every holomorphic function which can be written as a rational map

$$R(z) = \frac{p(z)}{q(z)} .$$

Non-Abelian Vortices

In non-Abelian gauge theory $U(1) \times G'$ with a fully Higgsed vacuum

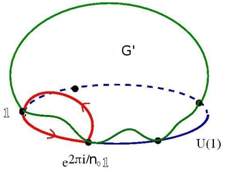
$$\langle H \rangle \sim \mathbb{1} \; ,$$

vortices are supported by

$$\pi_1 \left(U(1) \times G' \right) \simeq \mathbb{Z} \; .$$

The true gauge group should be considered as

$$G = \frac{U(1) \times G'}{\mathbb{Z}_{n_0}} \quad \Rightarrow \quad T_G = \frac{T_{\text{ANO}}}{n_0} \; .$$



Color-flavor symmetry

The vortex vacuum $\langle H \rangle \sim \mathbb{1}$ is color-flavor symmetric G'_{c+f}. However, it is broken down by the vortex solution giving rise to **orientational moduli**:

$$\frac{G'_{c+f}}{H_{c+f}} \ .$$

The $U(N)$ non-Abelian vortex

A $U(N)$ gauge theory with N_{F} flavors of squarks in the fundamental representation

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}^2 + D_\mu H(D^\mu H)^\dagger - \frac{g^2}{4}\left(HH^\dagger - \xi\mathbb{1}_N\right)^2\right],$$

which has to be considered as the **bosonic sector** of an $\mathcal{N}=2$ supersymmetric theory.

[Eto et.al., Phys.Rev.Lett.96:161601,2006 [hep-th/0511088].

Hanany & Tong, JHEP 0307:037,2003 [hep-th/0306150]; JHEP 0404:066,2004 [hep-th/0403158].

Shifman & Yung, Phys.Rev.D70:045004,2004 [hep-th/0403149].

The FI parameter $\xi > 0$ puts the theory on the **Higgs branch** and it has the moduli space $\mathcal{M} = Gr_{N,N_{\mathrm{F}}}$. The (BPS) vortices are supported by a **non-trivial** $\pi_1(U(N)) = \mathbb{Z}$ and they have a $SU(N_{\mathrm{F}})$ **flavor symmetry** acting from the right.

The fundamental $U(N)$ vortex

The fundamental vortex in $U(N)$ with N flavors is the embedding of the ANO vortex

$$H = U \begin{pmatrix} H_{\text{ANO}} & 0 \\ 0 & \mathbb{1}_{N-1} \end{pmatrix} U^\dagger \ , \qquad U \in SU(N)_{c+f} \ ,$$

where all internal zero modes are Nambu-Goldstone modes generated by the symmetry breaking of the vortex

$$\mathbb{C}P^{N-1} = \frac{SU(N)_{c+f}}{SU(N-1)_{c+f} \times U(1)_{c+f}} \ .$$

BPS equations

Performing a Bogomol'nyi type completion

$$T = \int_{\mathbb{C}} \mathrm{Tr} \left[\frac{1}{g^2} \left| F_{12} - \frac{g^2}{2} \left(H H^\dagger - \xi \mathbb{1}_N \right) \right|^2 + 4 |\bar{D} H|^2 - \xi F_{12} \right]$$
$$> -\xi \int_{\mathbb{C}} \mathrm{Tr}\, F_{12} \ .$$

we obtain the Bogomol'nyi bound.
To **solve the BPS equations** we take the ansatz

$$H \equiv S^{-1}(z, \bar{z}) H_0(z) \ , \qquad \bar{W} = -i S^{-1}(z, \bar{z}) \bar{\partial} S(z, \bar{z}) \ ,$$

with $z = x^1 + i x^2$, while the other BPS equation is rewritten to the **master equation**

$$\partial \left(\Omega^{-1} \bar{\partial} \Omega \right) = \frac{g^2}{4} \left(\Omega^{-1} H_0 H_0^\dagger - \xi \mathbb{1}_N \right) \ , \ \Omega \equiv S(z, \bar{z}) S(z, \bar{z})^\dagger \ ,$$

where $H_0(z)$ is an $N \times N_{\mathrm{F}}$ holomorphic matrix **containing all the moduli** of the BPS vortex.

Supersymmetric non-linear σ-model

We now want to write down the low energy effective theory of a supersymmetric model.

Taking the Kähler potential describing the theory in question in terms of chiral superfields :

$$K = K(\Phi, \bar{\Phi}) \; ,$$

the bosonic part of the non-linear σ-model can be written as

$$\mathcal{L} = -g_{i\bar{\imath}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{\imath}} \; , \qquad \text{with} \qquad g_{i\bar{\imath}} \equiv \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \bar{\phi}^{\bar{\imath}}} K \; ,$$

$g_{i\bar{\imath}}$ being the metric on the Kähler manifold.

Kähler potential for a gauge theory

Consider an $\mathcal{N} = 1$ gauge theory with **a simple** gauge group G with N_{F} chiral superfields in the fundamental representation $\square$ of the gauge group along with a vector superfield V'

$$K = \mathrm{Tr}\left[QQ^{\dagger}e^{-V'}\right]$$

notice that the invariance of the Lagrangian is $G^{\mathbb{C}}$, e.g. for $SU(N)$ it is $SU(N)^{\mathbb{C}} = SL(N, \mathbb{C})$ or for $U(N)$ it is $U(N)^{\mathbb{C}} = GL(N, \mathbb{C})$

$$\begin{aligned} Q &\to e^{i\Lambda}Q \\ e^{V'} &\to e^{i\Lambda}e^{V'}e^{-i\Lambda^{\dagger}} \ , \qquad \text{with } e^{i\Lambda} \in G^{\mathbb{C}} \end{aligned}$$

we would now like to calculate the Kähler quotient i.e. integrating out the vector multiplet

Roads to a low energy effective theory

[M. A. Luty and W. Taylor, Phys. Rev. D 53, 3399 (1996)]

- 1^{st} road
  - fix the gauge, i.e. Wess-Zumino gauge $\leftrightarrow$ resolve the D-term constraints
  - take the limit $g \to \infty$
  - mod out the gauge symmetry
- 2^{nd} road
  - take the limit $g \to \infty$
  - mod out the **full complexified** gauge symmetry
- 3^{rd} road
  - write down the **holomorphic invariants**
  - identify algebraic relations (e.g. the Plücker relation)

In fact, these ways are all identical.
A comment in store is that the validity is strongly dependent on being in the **full Higgs** phase, i.e. full rank of Q

A quick example : $SU(N)$ gauge theory

Let us consider a quick example of an $SU(N)$ gauge theory.

$$\begin{aligned} K &= \mathrm{Tr}\left[QQ^\dagger e^{-V'}\right] \\ &= N\left[\det QQ^\dagger\right]^{\frac{1}{N}} \qquad 2^{\text{nd}} \text{ road} \\ &= N\left[\sum_{\langle A\rangle}\left|B^{\langle A\rangle}\right|^2\right]^{\frac{1}{N}} \qquad 3^{\text{rd}} \text{ road} \end{aligned}$$

where $\det QQ^\dagger = 0$ resembles the Coulomb branch. This is a point on the target space which is a $\mathbb{Z}_N$ **conifold singularity**. As the name suggests, gauge fields become massless at this point, which gives rise to the singularity.

Lifting the singularity : $U(N)$ gauge theory

- $U(N)$: the singularity can be lifted by an FI parameter $\xi > 0$

$$K = \mathrm{Tr}\left[QQ^\dagger e^{-V'} e^{-V_e}\right] + \xi V_e$$
$$= \frac{\xi}{N} \log \det QQ^\dagger \qquad 2^{\mathrm{nd}} \text{ road}$$
$$= \frac{\xi}{N} \log \left[\sum_{\langle A \rangle} \left| B^{\langle A \rangle} \right|^2 \right] \qquad 3^{\mathrm{nd}} \text{ road}$$

with V_e being a $U(1)$ vector superfield

$\mathbb{C}P^1$ revisited

Using the last result, we can re-obtain the $\mathbb{C}P^1$ model. $U(1)$ gauge theory with $N_{\mathrm{F}} = 2$, gauge fixing $Q \sim (1, b)$.

$$K = \xi \log QQ^\dagger = \xi \log \left(1 + |b|^2\right) \ ,$$

which yields the Lagrangian

$$\mathcal{L} = \frac{\partial_\mu b \partial^\mu \bar{b}}{(1 + |b|^2)^2} \ ,$$

as we expected.

$\mathcal{N}=1, SO(N)$ and $USp(2M=N)$ gauge theories

The Kähler potential can be written as

$$K = \mathrm{Tr}\left[QQ^{\dagger}e^{-V'}\right] ,$$

where the following constraints on the gauge fields kick in

$$V'^{T}J + JV' = 0 \quad \Leftrightarrow \quad e^{-V'^{\mathrm{T}}}J\,e^{-V'} = J ,$$

with $J = \begin{pmatrix} \mathbf{0} & \mathbf{1}_M \\ \epsilon\mathbf{1}_M & \mathbf{0} \end{pmatrix}$ where $\epsilon = \pm 1$ for the SO, USp gauge group respectively.

D-flatness conditions

The D-flatness conditions in Wess-Zumino gauge read

$$D^A = \mathrm{Tr}_{\mathrm{F}} Q^\dagger_{\mathrm{WZ}} T^A Q_{\mathrm{WZ}} = 0 \; ,$$

with $T^A \in \{\mathfrak{so}(N), \mathfrak{usp}(N=2M)\}$. These conditions fix $\{SO(N)^{\mathbb{C}}, USp(2M)^{\mathbb{C}}\}$ to $\{SO(N), USp(2M)\}$

These conditions are difficult and as far as we know they have not been solved in the literature.

The moduli space of vacua : SO case

The VEVs can after fixing global and local symmetries, be put on the form

$$\langle Q_{\mathrm{WZ}} \rangle = \mathrm{diag}\big(a_1, \ldots, a_N, \underbrace{0, \ldots, 0}_{N_{\mathrm{F}} - N}\big) ,$$

and the classical moduli space of vacua in a generic point (that is $a_i \neq a_j$ for $i \neq j$) is

$$\mathcal{M}_{SO(N)} \simeq \mathbb{R}_+^N \times \frac{U(N_{\mathrm{F}})}{U(N_{\mathrm{F}} - N) \times (\mathbb{Z}_2)^{N-1}} \simeq \text{quasi-NGs} \times \text{NGs} .$$

While in the most symmetric vacuum $a_i = a_1, \forall i$

$$\mathcal{M}_{SO(N)} \simeq \mathbb{R}_+^{\frac{1}{2}N(N+1)} \ltimes \frac{U(N_{\mathrm{F}})}{U(N_{\mathrm{F}} - N) \times SO(N)_{\text{color+flavor}}} .$$

The moduli space of vacua : USp case

The VEVs can after fixing global and local symmetries, be put on the form ($N = 2M$)

$$\langle Q_{\mathrm{WZ}} \rangle = \mathbb{1}_2 \otimes \mathrm{diag}\big(a_1, \ldots, a_M, \underbrace{0, \ldots, 0}_{M_{\mathrm{F}}-M}\big) \ ,$$

and the classical moduli space of vacua in a generic point (that is $a_i \neq a_j$ for $i \neq j$) is

$$\mathcal{M}_{USp(2M)} \simeq \mathbb{R}_+^M \times \frac{U(N_{\mathrm{F}})}{U(N_{\mathrm{F}} - 2M) \times USp(2)^M}$$
$$\simeq \text{quasi-NGs} \times \text{NGs} \ .$$

While in the most symmetric vacuum $a_i = a_1, \forall i$

$$\mathcal{M}_{USp(2M)} \simeq \mathbb{R}_+^{M(2M-1)} \ltimes \frac{U(N_{\mathrm{F}})}{U(N_{\mathrm{F}} - 2M) \times USp(2M)_{\text{color+flavor}}} \ .$$

Key point : enlarging the algebra

Let's take the vector multiplet to be in $SU(N) \ni e^{-V'}$

$$K = \mathrm{Tr}\left[QQ^\dagger e^{-V'} + \lambda\left(e^{-V'^{\mathrm{T}}} J e^{-V'} - J\right)\right]$$

- e.o.m.s for λ solve the constraint of the algebra of SO/USp
- e.o.m.s for V' include λ, but λ can be eliminated from these

The solution is obtained as

$$X = \sqrt{QQ^\dagger} e^{-V'} \sqrt{QQ^\dagger}$$

$$X^2 = \left(Q^{\mathrm{T}} J \sqrt{QQ^\dagger}\right)^\dagger \left(Q^{\mathrm{T}} J \sqrt{QQ^\dagger}\right)$$

$$K = \mathrm{Tr}\ X \qquad\qquad 2^{\text{nd}} \text{ road}$$

The solution in terms of holomorphic invariants

Using another trick, i.e. rewriting a color trace as a flavor trace

$$\mathrm{Tr}_{\mathrm{C}}\sqrt{AA^\dagger} = \mathrm{Tr}_{\mathrm{F}}\sqrt{A^\dagger A}$$

we can write the solution as

$$K = \mathrm{Tr}_{\mathrm{F}}\sqrt{MM^\dagger} \qquad\qquad 3^{\mathrm{rd}} \text{ road}$$

with $M = Q^{\mathrm{T}}JQ$ being the **meson field**

Target spaces

- *USp*

$$\mathcal{M}_{USp} = \{M \,|\, M \in \mathbb{C}^{N_{\rm F}} \times \mathbb{C}^{N_{\rm F}},\ M^{\rm T} = -M,\ {\rm rank}M = 2M_C\}$$

  - reflects the fact that there are no independent baryons in this *USp* gauge theory

- *SO*

$$\mathcal{M}_{SO} = \{M, B^{\langle A\rangle} \,|\, M \in \mathbb{C}^{N_{\rm F}} \times \mathbb{C}^{N_{\rm F}},\ M^{\rm T} = M,\ \det M^{\langle A\rangle\langle B\rangle} = (\det J)\, B^{\langle A\rangle} B^{\langle B\rangle},\ N_C - 1 \le {\rm rank}M \le N_C\}$$

Extending the gauge group with an overall $U(1)$

Thus, we consider the theories in the class

$$G = U(1) \times G' \ , \qquad \text{with } G' = \{SO(N), USp(2M)\} \ .$$

Using the result from previously together with the e.o.m. for V_e, the Kähler potential can now be written as

$$\begin{aligned} K &= \mathrm{Tr}\left[QQ^\dagger e^{-V'} e^{-V_e}\right] + \xi V_e \\ &= \xi \log \mathrm{Tr}_{\mathbf{F}} \sqrt{MM^\dagger} \end{aligned}$$

with V_e being a $U(1)$ vector superfield

VEVs as expansion point

For large N (N_{F}) it is a hard task to compute the Kähler potential. It will prove convenient to develop an expansion formula.
We expand around the VEVs which can be written as

$$M^{SO}_{\mathrm{vev}} \equiv uMu^{\mathrm{T}} = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_N, 0, \cdots) \ ,$$
$$M^{USp}_{\mathrm{vev}} \equiv uMu^{\mathrm{T}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_M, 0, \cdots) \ .$$

These can be written as

$$(M_{\mathrm{vev}})_{ij} = \mu_i (J)_{ij} = (J)_{ij}\mu_j \ , \qquad (1)$$

with J the invariant tensor of SO, USp.

Expansion formula

Considering a small fluctuation

$$\phi = M - M_{\mathrm{vev}} \ ,$$

we can write a trace of a function f as

$$\begin{aligned}\mathbf{Tr}[f(X_0+\delta X)] &= \frac{1}{2\pi i}\oint_{\mathcal{C}} d\lambda\, f(\lambda)\mathbf{Tr}\left[\frac{\mathbf{1}}{\lambda\mathbf{1}-X_0-\delta X}\right] \\ &= \mathbf{Tr}[f(X_0)] \\ &+ \sum_{n=1}^{\infty}\frac{1}{2\pi n\, i}\oint_{\mathcal{C}} d\lambda\, f'(\lambda)\mathbf{Tr}\left[\left(\frac{\mathbf{1}}{\lambda\mathbf{1}-X_0}\delta X\right)^n\right] ,\end{aligned}$$

where

$$X = MM^\dagger \ , \quad X_0 = \mathrm{diag}(\mu_1^2, \cdots, \mu_{N_{\mathrm{C}}}^2) \ ,$$
$$\delta X = M_{\mathrm{vev}}\phi^\dagger + \phi M_{\mathrm{vev}}^\dagger + \phi\phi^\dagger \ .$$

Integrations

We just need to evaluate the line integrals around the positive real axis **excluding the origin** as it is a branch point of $f(x) = \sqrt{x}$

$$A_n(\mu_1, \cdots, \mu_n) \equiv \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{d\lambda}{\sqrt{\lambda}} \prod_{i=1}^{n} \frac{1}{\lambda - \mu_i^2} \ ,$$

which can easily be done, then it's just summing up the terms to arrive at the expanded Kähler potential

The expanded Kähler potential for $SO(N)$

$$K_{SO} = \frac{1}{2}\sum_{i,j}\frac{\phi_{ij}\phi^\dagger_{ji}}{\mu_i+\mu_j}$$

$$-\frac{1}{2}\sum_{i,j,k}\frac{\mu_i\,\phi_{ij}\phi^\dagger_{jk}\phi_{ki}}{(\mu_i+\mu_j)(\mu_j+\mu_k)(\mu_k+\mu_i)} + \text{c.c.}$$

$$+\frac{1}{2}\sum_{i,j,k,l}\frac{\mu_j\mu_k C^{(1)}_{ijkl}}{P_{ijkl}}\phi_{ij}\phi_{jk}\phi_{kl}\phi^\dagger_{li} + \text{c.c.}$$

$$+\frac{1}{2}\sum_{i,j,k,l}\frac{\mu_j\mu_l C^{(1)}_{ijkl}}{P_{ijkl}}\phi_{ij}\phi_{jk}\phi^\dagger_{kl}\phi^\dagger_{li} - \frac{1}{4}\sum_{i,j,k,l}\frac{C^{(3)}_{ijkl}}{P_{ijkl}}\phi_{ij}\phi^\dagger_{jk}\phi_{kl}\phi^\dagger_{li}$$

$$+\text{Kähler trf.} + \mathcal{O}(\phi^5)\ ,$$

with P and C being standard symmetric polynomials

Scalar curvature for $SO(N)$

From the expanded potential we can compute the curvature

$$
\begin{aligned}
R|_{\phi=0} &= -2g^{I\bar{J}}\partial_I\partial_{\bar{J}} \log\det g\Big|_{\phi=0} \\
&= 2\sum_{i>j}\left(\frac{1}{\mu_i+\mu_j}+\sum_k \frac{\mu_k}{(\mu_k+\mu_i)(\mu_k+\mu_j)}\right) > 0\,,
\end{aligned}
$$

Note that a singularity emerges at **2 vanishing** eigenvalues, that is rank $M \leq N-2$

Scalar curvature for $USp(2M)$

For the USp case

$$R|_{\phi=0} = 4\sum_{i>j}^{M_{\mathrm{C}}}\left(\frac{1}{\mu_i+\mu_j}+\sum_{k}^{M_{\mathrm{C}}}\frac{4\mu_k}{(\mu_k+\mu_i)(\mu_k+\mu_j)}\right) > 0\ .$$

Note that **again** a singularity emerges at **2 vanishing** eigenvalues, however, **this time** we have
rank $M \leq N-4$
We would naively expect it to arise **already** at
rank $M \leq N-2$, that is, we need full rank to have a regular solution

Deformed Kähler potential

Inspired by the conical singularity, we consider deforming to Kähler potential to detect a singularity arising **already** at rank $M = N - 2$, we consider the deformation

$$K_{USP,\text{deformed}} = \text{Tr}\ \sqrt{MM^\dagger + \varepsilon^2}\ .$$

Taking now only one eigenvalue, $\mu_1 \to 0$ we find a term in the scalar curvature

$$\lim_{\mu_1 \to 0} R|_{\phi=0} \supset \frac{2}{\varepsilon}\ ,$$

which shows the presence of a singularity for **one vanishing** eigenvalue, that is corresponding to an unbroken $USp(2) \simeq SU(2)$ symmetry.
For $USp(4)$ we have found by direct computation **an orbifold** singularity emerging at rank $M = 2$

$\mathcal{N} = 2$ SUSY : hyperKähler quotient for $SO(N)$ and $USp(2M = N)$

Considering the $\mathcal{N} = 2$ gauge theory, we can write the Kähler potential as

$$\tilde{K} = \mathrm{Tr}\left[QQ^\dagger e^{-V'} + \tilde{Q}^\dagger \tilde{Q} e^{V'} + \lambda\left(e^{-V'^{\mathrm{T}}} J e^{-V'} - J\right)\right]$$

with the superpotential

$$\mathcal{W} = \mathrm{Tr}\left[Q\tilde{Q}\Sigma' + \chi\left(\Sigma'^{\mathrm{T}} J + J\Sigma'\right)\right]$$

Using the algebra of SO, USp, $e^{V'^{\mathrm{T}}} = J^{\mathrm{T}} e^{-V'} J$ the Kähler potential can be written as

$$\tilde{K} = \mathrm{Tr}\left[QQ^\dagger e^{-V'} + J^{\mathrm{T}} e^{-V'} J (\tilde{Q}^\dagger \tilde{Q})^{\mathrm{T}} + \lambda\left(e^{-V'^{\mathrm{T}}} J e^{-V'} - J\right)\right]$$

Defining now a field with $2N_\mathrm{F}$ flavors

$$\mathcal{Q} = \left(Q,\ J\tilde{Q}^\mathrm{T}\right)$$

the Kähler potential can be written as

$$\tilde{K} = \mathrm{Tr}\ \left[\mathcal{Q}\mathcal{Q}^\dagger e^{-V'} + \lambda\left(e^{-V'^\mathrm{T}} J e^{-V'} - J\right)\right]$$

then the $\mathcal{N}=1$ solution readily applies!

$$\tilde{K} = \mathrm{Tr}_F \sqrt{\mathcal{M}\mathcal{M}^\dagger}\ ,$$

where the meson field now is

$$\mathcal{M} = \mathcal{Q}^\mathrm{T} J \mathcal{Q}\ .$$

Constraint coming from the superpotential

The following constraint

$$\mathcal{Q}\tilde{J}\mathcal{Q}^{\mathrm{T}} = 0 \,, \qquad \text{where } \tilde{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_M \\ -\epsilon\mathbf{1}_M & \mathbf{0} \end{pmatrix} \,,$$

that is

$$\mathcal{M}^{\mathrm{T}} = \epsilon\mathcal{M} \,, \qquad \mathcal{M}\tilde{J}\mathcal{M} = 0 \,,$$

and $N_{\mathrm{C}} - 2 < \mathrm{rank}\mathcal{M} \leq N_{\mathrm{C}}$ which shows the well-known result, that

- An $SO(N_C)$ gauge theory has a $USp(2N_{\mathrm{F}})$ flavor symmetry
- A $USp(2M_C)$ gauge theory has an $SO(2N_{\mathrm{F}})$ flavor symmetry

Resultant spaces for the hyper-Kähler quotients

$USp(2N_{\mathrm{F}})$ and $SO(2N_{\mathrm{F}})$ isometries act on the $SO(N_{\mathrm{C}})$ and $USp(2M_{\mathrm{C}})$ spaces, respectively. Locally in generic points we have

$$\begin{aligned}\mathcal{M}^{\mathrm{HK}}_{SO(N_{\mathrm{C}})} &\simeq \mathbb{R}^{N_{\mathrm{C}}}_{>0} \times \frac{USp(2N_{\mathrm{F}})}{USp(2N_{\mathrm{F}}-2N_{\mathrm{C}}) \times (\mathbb{Z}_2)^{N_{\mathrm{C}}-1}} \\ &\supset \mathbb{R}^{N_{\mathrm{C}}}_{>0} \times \frac{U(N_{\mathrm{F}})}{U(N_{\mathrm{F}}-N_{\mathrm{C}}) \times (\mathbb{Z}_2)^{N_{\mathrm{C}}-1}} \ , \\ \mathcal{M}^{\mathrm{HK}}_{USp(2M_{\mathrm{C}})} &\simeq \mathbb{R}^{M_{\mathrm{C}}}_{>0} \times \frac{SO(2N_{\mathrm{F}})}{SO(2N_{\mathrm{F}}-4M_{\mathrm{C}}) \times USp(2)^{M_{\mathrm{C}}}} \\ &\supset \mathbb{R}^{M_{\mathrm{C}}}_{>0} \times \frac{U(N_{\mathrm{F}})}{U(N_{\mathrm{F}}-2M_{\mathrm{C}}) \times USp(2)^{M_{\mathrm{C}}}} \ ,\end{aligned}$$

These spaces are HK spaces of cohomogeneity N_{C} and M_{C}, respectively.

Non-linear σ model lumps

Lumps are stringy topological textures which are supported by a **non-trivial** $\pi_2(\mathcal{M})$ associated with a **holomorphic map** from the $\mathbb{C}$-plane (spatial) to a 2-cycle of the target space of the non-linear σ model.

The lumps we want to consider here are 1/2 BPS configurations.

Lumps in $U(1) \times G'$ Kähler quotients

Considering a non-linear σ model of a $U(1) \times G'$ Kähler quotient. Let $\phi^\alpha \in \{I^i\} /\!\!/ U(1)^{\mathbb{C}}$ be **inhomogeneous coordinates** on the manifold then a **static lump solution** is obtained by

$$\phi^\alpha(t, z, \bar{z}, x^3) \to \phi^\alpha(z; \varphi^i) \ ,$$

with φ^i being complex constants which indeed are the **moduli parameters** of the lump and the tension is

$$T = 2 \int_{\mathbb{C}} K_{\alpha\bar{\beta}} \partial\phi^\alpha \bar{\partial}\bar{\phi}^{\bar{\beta}} \Big|_{\phi\to\phi(z)} = 2 \int_{\mathbb{C}} \bar{\partial}\partial K \Big|_{\phi\to\phi(z)} \ .$$

Description of lumps

It will prove convenient to use the holomorphic G' invariants I^i satisfying constraints as **homogeneous coordinates**

$$I^i(z) = I^i_{\rm vev} z^{n_i \nu} + \mathcal{O}(z^{n_i\nu - 1}) \ ,$$

with n_i the $U(1)$ charges of the invariants I^i.

ν is some number

$$\nu = \frac{k}{n_0} \ , \quad k \in \mathbb{Z}_+ \ , \quad n_0 = \gcd(\{n_i\} \mid I^i_{\rm vev} \neq 0) \ ,$$

such the invariants are holomorphic

Finally, the inhomogeneous coordinates $\{\phi^\alpha\}$ can be found from the ratios of these G' invariants, namely being $U(1)^{\mathbb{C}}$ invariants, which is analogous to the **rational maps** in the Abelian case

The lump condition - small lump singularity

Lump condition

All points in the base manifold $\mathbb{C}$ must be mapped to the full Higgs phase by the holomorphic map

Common zeros

Common zeros in the G' invariants give rise to a small lump singularity - **zero size** $\sim$ local vortex

- for $U(N)$ the two conditions above are **in fact identical**
- the lump condition is **stronger** than the other condition
- the existence of the difference implies that there exists a type of singularity of non-vanishing size
- this singularity is a typical property of lumps in NLσMs with a **singular submanifold**

Taking the strong coupling limit

gauge theory	$\xrightarrow{g \to \infty}$	NLσM
semi-local vortex		lump

- even for **finite** gauge coupling there the two are closely related
- in fact the dimensions of their moduli spaces have relations

Dimensions of the moduli spaces : $U(N)$

Grassmann sigma model:

$$\dim_{\mathbb{C}} \mathcal{M}^{k-\text{vortex}}_{U(N),\, N_{\text{F}}} = \dim_{\mathbb{C}} \mathcal{M}^{k-\text{lump}}_{U(N),\, N_{\text{F}}} = kN_{\text{F}} \ .$$

[A.Hanany & D.Tong, JHEP **0307**, 037 (2003) [arXiv:hep-th/0306150]]

[Eto et.al. J.Phys.A **39**, R315 (2006) [arXiv:hep-th/0602170]]

In fact the moduli space $\mathcal{M}^{k-\text{vortex}}_{U(N),\, N_{\text{F}}}$ is **identical** to $\mathcal{M}^{k-\text{lump}}_{U(N),\, N_{\text{F}}}$ when the lump condition has been applied, that is

$$\mathcal{M}^{k-\text{vortex}}_{U(N),\, N_{\text{F}}} \subset \mathcal{M}^{k-\text{lump}}_{U(N),\, N_{\text{F}}} \ .$$

Dimensions of the moduli spaces : $SO(N)$

The dimension of the moduli space for k vortices in a $U(1) \times G'$ gauge theory with $N_{\mathrm{F}} = N$ is

$$\dim_{\mathbb{C}} \mathcal{M}^{k-\text{vortex}}_{U(1)\times G'} = \frac{kN^2}{n_0} ,$$

where n_0 is the **greatest common divisor** of the Abelian charges of the G' invariants.

[Eto et.al., arXiv:0802.1020 [hep-th]]

In fact we find for both odd and even N (and $N_{\mathrm{F}} = N$) that

$$\dim_{\mathbb{C}} \mathcal{M}^{k-\text{vortex}}_{SO(N)} = \dim_{\mathbb{C}} \mathcal{M}^{k-\text{lump}}_{SO(N)} = \frac{kN^2}{n_0} ,$$

where $n_0 = 2$ for $SO(N = 2M)$ and $n_0 = 1$ for $SO(N = 2M + 1)$.

There are **internal moduli** in the lump solutions and the moduli of the lump solutions are sufficient to describe the vortex moduli space. This is quite different from the Grassmann lump.

Dimensions of the moduli spaces : $USp(2M)$

The number of moduli for $M_{\rm F} = M$ is

$$\#\,\text{moduli in } M(z) = \dim_{\mathbb{C}} \mathcal{M}^{k-\text{vortex}}_{USp(2M)} - kM \ ,$$

Note, there exists **no regular solution** in this NLσM.
The difference is due to the **surviving color flavor symmetry** $USp(2)^M$ even at a generic point in the vacuum.
We would guess that

$$\mathcal{M}^{k-\text{vortex}}_{USp(2M)} \sim \mathcal{M}^{k-\text{singular lump}}_{USp(2M)} \times \left(\mathbb{C}P^1\right)^{kM} \ .$$

To cure this singular configuration, we need **more flavors** $M_{\rm F} > M$.

Moduli matrix formalism

Both in the case of $U(1) \times SO(N)$ and $U(1) \times USp(2M)$, **additional NG modes** can emerge as moduli at **special points** of a vortex $M_{\rm vev} = J$.
To study the vortex moduli, it proves convenient to consider the **moduli matrix** Q(z) which is redundant up to V-equivalence

$$Q(z) \sim V(z) Q(z) \ .$$

[Eto et.al. J.Phys.A **39**, R315 (2006) [arXiv:hep-th/0602170]]

The boundary conditions for the moduli matrix are

$$SO(2M),\ USp(2M) : Q^{\rm T}(z) J Q(z) = M_{\rm vev} z^k + \mathcal{O}(z^{k-1}) \ ,$$
$$SO(2M+1) : Q^{\rm T}(z) J Q(z) = M_{\rm vev} z^{2k} + \mathcal{O}(z^{2k-1}) \ .$$

[Eto et.al., arXiv:0802.1020 [hep-th]]

Relation between the vortex moduli space and the lump moduli space

We expect on the grounds of the former results the following relation

$$\mathcal{M}^{k-\text{lump}} \simeq \left\{ a | a \in \mathcal{M}^{k-\text{vortex}},\ \text{the lump condition} \right\}.$$

Effective action of lumps

By promoting the moduli parameters to superfields on the **lump world volume**

$$\phi^\alpha(t, z, \bar{z}, x^3) \to \phi^\alpha(z; \varphi^i(t, x^3)) \ .$$

we can write the effective action

$$\mathcal{K}_{\text{lump}} = \int dz d\bar{z}\, K\left(\phi(z, \varphi^i(t, x^3)),\ \phi^\dagger(\bar{z}, \bar{\varphi}^i(t, x^3))\right) \ .$$

Identifying non-normalizable modes

We can identify non-normalizable modes by finding a **divergence** in the Kähler potential which **cannot be removed by Kähler transformations**.

The **only normalizable modulus in a single lump** in $U(1) \times SO(2M)$ and $U(1) \times USp(2M)$ is the **center of mass**.

Constructing vortices from the new lumps

We can use the our technology from the lumps to construct the vortices or similarly consider the lumps configurations and turning on a finite gauge coupling. The BPS equations in the case of $SO(N)$ and $USp(2M)$ are

$$\bar{D}H = 0,$$

$$F^0_{12} - \frac{e^2}{\sqrt{2N}}\left(\mathrm{Tr}\,(HH^\dagger) - v^2\right) = 0 \ ,$$

$$F^a_{12}t^a - \frac{g^2}{4}\left(HH^\dagger - J^\dagger (HH^\dagger)^T J\right) = 0 \ ,$$

and we can apply the same ansatz as for the $U(N)$ case

$$H = S_e^{-1}(z,\bar{z}) S'^{-1}(z,\bar{z}) H_0(z) \ ,$$

with $S_e \in U(1)^{\mathbb{C}}$ and $S' \in G'^{\mathbb{C}} = \{SO, USp\}^{\mathbb{C}}$.

Constructing the vortices from the new lumps

This leaves us with the **master equations** for $SO(N)$ and $USp(2M)$

$$\bar{\partial}\partial\psi = -\frac{e^2}{4N}\big(\mathrm{Tr}\,(\Omega_0\Omega'^{-1})e^{-\psi} - v^2\big), \qquad (2)$$

$$\bar{\partial}(\Omega'\partial\Omega'^{-1}) = \frac{g^2}{8}\big(\Omega_0\Omega'^{-1} - J^\dagger(\Omega_0\Omega'^{-1})^T J\big)e^{-\psi},$$

where $e^{\psi} \equiv S_e S_e^\dagger$, $\Omega' \equiv S'S'^\dagger$ and $\Omega_0 \equiv H_0 H_0^\dagger$.
Now, inserting the moduli matrices found for the lump **we have obtained the vortices** of the $SO(N)$ and $USp(2M)$ gauge theories.

[Eto et.al., arXiv:0802.1020 [hep-th]]

Local vortices

Having the construction of the vortices for a general gauge group and explicitly for $SO(N)$ and $USp(2M)$, we can **restrict** them to **local/ANO-like vortices**. This corresponds to having **common zeros** in the holomorphic invariants

$$I^i_{G'}(H_{0,\text{local}}) = \left[\prod_{l=1}^{k}(z - z_l)\right]^{\frac{n_i}{n_0}} I^i_{\text{vev}} \ ,$$

which in terms of the moduli matrix is

$$H^{\text{T}}_{0,\text{local}}(z) J H_{0,\text{local}}(z) = \prod_{l=1}^{k}(z - z_l) J \ .$$

Local single vortex in $SO(2M)$ or $USp(2M)$

A single local vortex in $SO(2M)$ or $USp(2M)$ can be written as

$$H_{0,\text{local}}(z) = \begin{pmatrix} (z - z_0)\mathbb{1}_M & 0 \\ \mathbf{B}_{A/S} & \mathbb{1}_M \end{pmatrix} ,$$

with $\mathbf{B}_{A/S}$ being anti-symmetric for $SO(2M)$ and symmetric for $USp(2M)$.

A simple example : $U(1) \times SO(2)$ lumps with $N_{\mathrm{F}} = 2$

The moduli space is

$$\mathcal{M} = \mathbb{C}P^1 \times \mathbb{C}P^1 \ ,$$

of which the second homotopy group is

$$\pi_2(\mathcal{M}) = \mathbb{Z}_+ \otimes \mathbb{Z}_- \ .$$

The quark fields i.e. the moduli matrix can be written as

$$Q(z) = \begin{pmatrix} Q_1^+(z) & Q_2^+(z) \\ Q_1^-(z) & Q_2^-(z) \end{pmatrix} \ ,$$

which are holomorphic functions of degree $k_\pm$, respectively. The tension reads

$$T = \int_{\mathbb{C}} 2\partial\bar{\partial} K_{U(1)\times SO(2)} = \pi\xi(k_+ + k_-) \equiv \pi\xi k \ .$$

The minimal $U(1) \times SO(2)$ lump solution

Taking a moduli matrix for a $k = 1$ solution

$$Q(z) = \begin{pmatrix} z - z_1 & c_1 \\ 0 & 1 \end{pmatrix} ,$$

which has the meson field

$$M(z) = \begin{pmatrix} 0 & z - z_1 \\ z - z_1 & 2c_1 \end{pmatrix} ,$$

and the Kähler potential

$$K = \xi \log \left(2\sqrt{|z - z_1|^2 + |c_1|^2} \right) .$$

where z_1 is the position and c_1 is a size/regularizer.

The minimal $U(1) \times SO(2M)$ lump solution

Extending this solution for $k = 1$

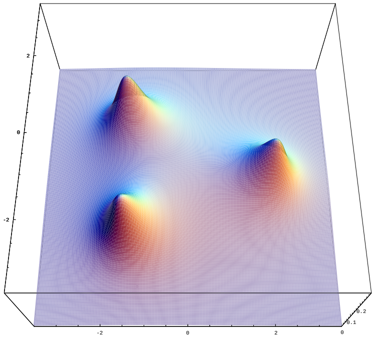
$$Q_{k=1} = \begin{pmatrix} z\mathbf{1}_M - \mathbf{A} & \mathbf{C} \\ 0 & \mathbf{1}_M \end{pmatrix} , \begin{cases} \mathbf{A} = \mathrm{diag}(z_1, z_2, \cdots, z_M) , \\ \mathbf{C} = \mathrm{diag}(c_1, c_2, \cdots, c_M) . \end{cases}$$

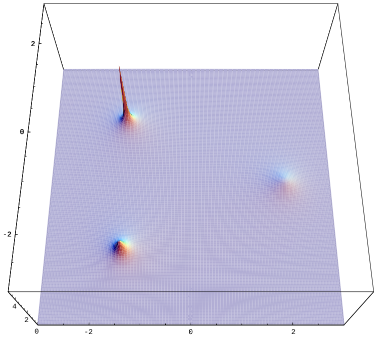
and the Kähler potential

$$K = \xi \log \left(2 \sum_{i=1}^{M} \sqrt{|z - z_i|^2 + |c_i|^2} \right) .$$

where z_i are M positions and c_i are sizes/regularizers.

Multi-center solutions





Conclusion

- We have obtained explicitly the Kähler metric and potential for $SO(N)$, $USp(2M)$, $U(1) \times SO(2M)$ and $U(1) \times USp(2M)$ theories
- Key point : **bigger algebra** with **constraints** in the form of **Lagrange multipliers**
- Expansion of the Kähler potential around the VEV, from which we have obtained the **scalar curvatures**
- We have obtained explicitly the Hyper-Kähler metric and potential for $SO(N)$ and $USp(2M)$ theories
- We have studied the normalizability in the new lump solutions

Further developments

- Metrics and potentials for other representations, especially the adjoint representations
- Hyper-Kähler quotients of exceptional groups, $E_{6,7}$ etc.
- Admission of a Ricci-flat non-compact Calabi-Yau metric
- Construction of a massive deformed theory and domain wall solutions
- Q-lumps in the $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ Kähler quotients
- Dynamics of lumps
- Cosmic lump strings and their reconnection
- Composite lumps like triple lump-string interactions
- Lump-strings stretched between domain walls