Open strings and noncommutative algebraic geometry

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I would like to thank my colleagues and collaborators P. Aspinwall, C. Beil, and D. Berenstein. This talk will discuss some work in progress, as well as reviewing a number of older papers, including:

- D. Berenstein, *Reverse geometric engineering of singularities*, JHEP **04** (2002) 052, `hep-th/0201093`.
- P. S. Aspinwall, *A point's point of view of stringy geometry*, JHEP **01** (2003) 002, `hep-th/0203111`.

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Also relevant are:

- M. Van den Bergh, *Three-dimensional flops and noncommutative rings*, Duke Math. J. **122** (2004) 423–455, `math.AG/0207170`.
- V. Ginzburg, *Calabi–Yau algebras*, `math.AG/0612139`.

AdS/CFT

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Gauge theories

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Algebraic geometry

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Non-spherical horizons

Early in the study of the AdS/CFT correspondence, it was realized that the same "AdS/CFT" idea could be applied if the branes were located at a singular point of spacetime, in which the transverse space is a (real) cone over a "horizon manifold " H which need not be a sphere. In fact, it had long been known that some spacetime singularities (such as orbifolds) are just as well-behaved in string theory as are nonsingular points, and the AdS/CFT idea gave a new approach to understanding why that might be true. In this approach, on the "AdS" side we need to be concerned with the differential geometry of the horizon manifold H.

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Today, I will discuss a different point of view on this correspondence, in which a careful analysis of the "CFT" side produces some novel (noncommutative) algebraic structures which—at least in many examples—permit one to recover the singular Calabi–Yau manifold used on the "AdS" side (and hence the horizon H).

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To achieve a full understanding of these brane states, we need to introduce a *category* of D-branes. The objects in the category will be all possible (stacks of) D-branes located at P, and the morphisms in the category describe the open string states, where the open strings are stretched between one kind of brane and another.

For stacks of branes located at a single point P, this categorical description is rather elementary: we simply get the category of finite-dimensional complex vector spaces. A vector space of dimension N corresponds to a stack of N D3-branes, and the strings from N branes to M branes are described by linear transformations $T : \mathbb{C}^N \to \mathbb{C}^M$.

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In the more general case with a singularity at P, for many types of singularities we will be able to describe the corresponding D-brane category as the category of (left) $\mathcal{A}$-modules for a particular algebra $\mathcal{A}$ associated to the point P.

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Morita equivalence

One fundamental notion in the theory of noncommutative rings and algebras is the notion of *Morita equivalence*. Two rings $\mathcal{A}$ and $\mathcal{B}$ are said to be Morita equivalent if there is an equivalence of categories between the category of left $\mathcal{A}$-modules and the category of left $\mathcal{B}$-modules.

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A key fact is that if $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then the centers of the rings $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{B})$ are isomorphic.

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A key fact is that if $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then the centers of the rings $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{B})$ are isomorphic. (This is obvious in the case of R and $M_n(R)$.)

The McKay correspondence

An important example of the types of algebras we will encounter is the "twisted group algebra". Let G be a finite subgroup of SU(2) so that G acts on the polynomial algebra $\mathbb{C}[x, y]$. The *twisted group algebra* $\mathbb{C}[x, y] \star G$ consists of pairs $(f(x), g)$ and a multiplication

$$(f(x), g) \cdot (\phi(x), \gamma) = (f(x) \cdot \phi^g(x), g \circ \gamma),$$

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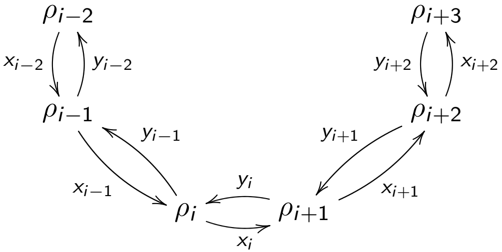
$$(f(x), g) \cdot (\phi(x), \gamma) = (f(x) \cdot \phi^g(x), g \circ \gamma),$$

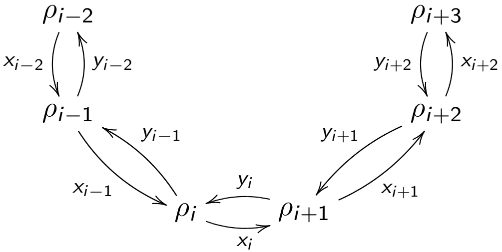
where $\phi^g(x)$ is the function obtained from acting on $\phi(x)$ by g. Kapranov and Vasserot interpreted the McKay correspondence as a statement about the structure of this algebra: we can write

$$\mathbb{C}[x,y] \star G = \bigoplus_{\rho \in \mathrm{Irrep}(G)} M_\rho \otimes \rho$$

and describe the algebra in terms of the modules M_ρ. Its structure is determined by the graph of representations which McKay used:



In the A_n case, for example, the graph forms a cycle and one has relations $x_i y_i = y_{i-1} x_{i-1}$.



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Its not too hard to determine the center of the algebra from this description: it is generated by

$$X = x_0 \cdots x_{n-1} + \text{ cyclic permutations}$$
$$Y = y_0 \cdots y_{n-1} + \text{ cyclic permutations}$$
$$Z = x_i y_i + \text{ cyclic permutations}$$

subject to the relation $Z^n = XY$. Of course, the center can also be identified with $\mathbb{C}[x, y]^G$, so this was not unexpected.

D-brane algebras

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More precisely, one does *not* specify in advance which unitary groups $G_i = U(N_i)$ take part in the theory, but one *does* specify the matter representation (as a collection of adjoint and bifundamental fields for the gauge groups G_i) and one specifies a superpotential W which is the trace of a polynomial in the matter fields.

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From the quiver, we directly get the *path algebra*, which is the algebra of all paths on the quiver (i.e., all ordered monomials in matter fields). However, a universal feature of this family of theories is the relations in the path algebra determined by what are called "F-term constraints" in physics: these are the algebra relations dictated by $\partial W/\partial X_j$.

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So, given a field theory description of the family of D-branes in the form above, the D-brane algebra is

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This is sometimes called a *superpotential algebra*, or a *Calabi–Yau algebra*.

Example 1

As a first example, we consider the case in which P is a smooth point. In physics language, the conformal field theory is the $\mathcal{N} = 4$ super-Yang–Mills theory, here written in $\mathcal{N} = 1$ language. The $\mathcal{N} = 4$ gauge multiplet decomposes as an $\mathcal{N} = 1$ gauge multiplet plus three complex scalar fields X, Y, Z, each transforming in the adjoint representation of the group. The superpotential is

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the (commutative) polynomial algebra in three variables. (So we get the category of $\mathbb{C}[X, Y, Z]$-modules; if the branes are fixed at P, this is just the category of $\mathbb{C}$-vector spaces as remarked earlier.)

Example 2

To realize the McKay quiver $\widehat{A_n}$ in this context, we need to add loops at each of the vertices, represented by fields ϕ_i; then the superpotential

$$W = \mathrm{tr}\,(\phi_i(x_i y_i - y_{i-1}x_{i-1}))$$

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gives the same relations as before. (This is known to be the appropriate field theory description when P lies on a curve of A_{n-1} singularities.) The algebra is the twisted group algebra, tensored with the algebra of the ϕ_i's, which have no relations among them. The center is:

$$\mathcal{Z}(\mathcal{A}) = \mathbb{C}[x,y]^G \otimes \mathbb{C}[\Phi],$$

where $\Phi = \sum \phi_i$.

Algebraic AdS/CFT correspondence

The algebraic version of the AdS/CFT correspondence which we would like to formulate relates a point P of a Calabi–Yau space X on the AdS side to a field theoretic description of the D-branes at P on the CFT side. To recover the standard AdS picture, one needs to know about metrics near P and take a scaling limit; instead, we will focus on the formal completion of the power series ring, we carries the local algebraic information about the Calabi–Yau space.

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Our conjectured correspondence relates the D-brane algebra $\mathcal{A}$ (such that the D-brane category is the category of left $\mathcal{A}$-modules) to the formal completion of the coordinate ring of X at P by asserting that the center $\mathcal{Z}(\mathcal{A})$ and the coordinate ring of X at P have *isomorphic formal completions.*

Remarks:

1. The algebra $\mathcal{A}$ is not uniquely determined by the D-brane category (this is related to Seiberg duality and has been studied extensively); however, Morita equivalence guarantees that the center does not change.

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2. As stated, this provides a way of passing from a Lagrangian description of a family of field theories to the algebro-geometric structure near the AdS dual. It is also possible to go backwards (work of Aspinwall and collaborators): given $P \in X$, one studies the (derived) category of coherent sheaves on X supported at P, and determines a so-called tilting module for the category. That tilting module, and some further computations of *Ext* groups of the sheaves, gives a Lagrangian description for the family of D-branes (including a superpotential).

3. To the best of my knowledge, there is no known algorithm for computing the center of $\mathcal{A}$. There is also no known algorithm for producing a tilting module for the derived category of coherent sheaves supported at X. And of course, there is no proof at present that the formal completion of $\mathcal{Z}(\mathcal{A})$ is isomorphic to the formal completion of the coordinate ring of X at P.

3. To the best of my knowledge, there is no known algorithm for computing the center of $\mathcal{A}$. There is also no known algorithm for producing a tilting module for the derived category of coherent sheaves supported at X. And of course, there is no proof at present that the formal completion of $\mathcal{Z}(\mathcal{A})$ is isomorphic to the formal completion of the coordinate ring of X at P.
4. Van den Bergh has conjectured that a "canonical singularity" $P \in X$ in dimension 3 admits an algebra $\mathcal{A}$ whose category of modules describe the coherent sheaves supported at P if and only if there is a resolution of singularities $\pi : Y \to X$ which is relatively Calabi–Yau (that is, no zeros are introduced into the holomorphic 3-form), and he has proven this conjecture in a number of cases. However, neither his conjecture nor his proof address the question of whether $\mathcal{A}$ can be written as a superpotential algebra.

5. Ginzburg has introduced a more general notion of Calabi–Yau algebras, which suggests that these ideas may be true more globally, but it is fair to say that the study of such algebras is still in its infancy.

Example 3 (Cachazo–Katz–Vafa; Aspinwall–Katz)

Let us modify the theory for $\widehat{A_1}$ by adding another term to the superpotential. That is, we have fields x_i, y_i, ϕ_i for $i = 0, 1$ with superpotential

$$W = \mathrm{tr}\,(\phi_0(x_0y_0 - y_1x_1) + \phi_1(x_1y_1 - y_0x_0) + P(\phi_0) + P(\phi_1))$$

where P is some fixed polynomial.
The F-term relations are:

$$\begin{aligned}
y_0\phi_0 &= \phi_1 y_0\\
\phi_0 y_1 &= y_1\phi_1\\
\phi_0 x_0 &= x_0\phi_1\\
x_1\phi_0 &= \phi_1 x_1\\
x_0 y_0 &= y_1 x_1 - P'(\phi_0)\\
x_1 y_1 &= y_0 x_0 - P'(\phi_1)
\end{aligned}$$

$$y_0\phi_0 = \phi_1 y_0$$
$$\phi_0 y_1 = y_1\phi_1$$
$$\phi_0 x_0 = x_0\phi_1$$
$$x_1\phi_0 = \phi_1 x_1$$
$$x_0 y_0 = y_1 x_1 - P'(\phi_0)$$
$$x_1 y_1 = y_0 x_0 - P'(\phi_1)$$

One can see that the following elements are central:

$$X = x_0 x_1 + x_1 x_0$$
$$Y = y_1 y_0 + y_0 y_1$$
$$Z = x_0 y_0 + x_1 y_1$$
$$\Phi = \phi_0 + \phi_1$$

and satisfy the relation

$$Z^2 = XY + ZP'(\Phi).$$

These singularities are closely related to flops: by an old result of Reid, the blowdown of a flop with normal bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ always has an equation of the above form, where P has a zero at $\Phi = 0$ of order at least 3.

Example 4: the conifold

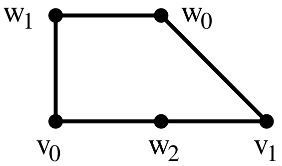
The previous example also describes the conifold, if we take $P(\Phi) = \frac{1}{2}\Phi^2$. The standard physics approach to this is to "integrate out" ϕ_i, using the F-term equations to solve $\phi_0 = y_1x_1 - x_0y_0$, $\phi_i = y_0x_0 - x_1y_1$. The new superpotential is

$$W = \operatorname{tr}\Big((y_1x_1 - x_0y_0)(x_0y_0 - y_1x_1) + (y_0x_0 - x_1y_1)(x_1y_1 - y_0x_0) + \frac{1}{2}(y_1x_1 - x_0y_0)^2 + \frac{1}{2}(y_0x_0 - x_1y_1)^2\Big)$$

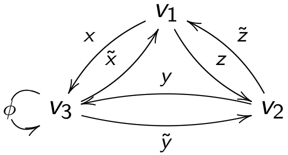
This is the standard superpotential for the conifold, first found by Klebanov and Witten.

Example 5: the suspended pinch point

As one final example, we will compute the superpotential algebra and its centered for the "suspended pinch point" (first considered in *hep-th/9810201*). The method which Plesser and I used to find the superpotential in this case was toric geometry. The suspended pinch point singularity can be described torically as the cone over the following lattice polyhedron:



Plesser and I calculated a field theory dual, which can be expressed in terms of the quiver



and superpotential

$$W = \mathrm{tr}\left(\phi(\tilde{Y}Y - \tilde{X}X) + \lambda(Z\tilde{Z}X\tilde{X} - \tilde{Z}Z\tilde{Y}Y)\right) .$$

The F-term constraints become:

$$\tilde{Y}Y = \tilde{X}X$$
$$\phi\tilde{X} = \lambda Z\tilde{Z}X$$
$$X\phi = \lambda X\tilde{Z}X$$
$$\phi\tilde{Y} = \lambda\tilde{Y}\tilde{Z}Z$$
$$Y\phi = \lambda\tilde{Z}ZY$$
$$\lambda\tilde{Z}X\tilde{Z} = \lambda Y\tilde{Y}\tilde{Z}$$
$$\lambda X\tilde{X}Z = \lambda ZY\tilde{Y}$$

There are central elements $A = \phi + \lambda Z\tilde{Z} + \lambda\tilde{Z}Z$, $B = \tilde{X}X + X\tilde{X} + Y\tilde{Y}$, $C = \tilde{Y}\tilde{Z}X + \tilde{Z}X\tilde{Y} + X\tilde{Y}\tilde{Z}$, $D = \tilde{X}ZY + ZY\tilde{X} + Y\tilde{X}Z$.
The relation is:

$$AB^2 = \lambda CD.$$

Noncommutative algebraic geometry

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There have been many efforts in the past few decades to extend the familiar notions of commutative algebraic geometry to noncommutative settings. This work has met with considerable skepticism from the algebraic geometry community. However, the corner of noncommutative algebraic geometry which I described today seems to be very close to central issues in (commutative) algebraic geometry. One can hope that further exploration of these ideas will lead to both progress in the physics, and also to a deeper understanding of the noncommutative aspects of algebraic geometry *per se*.