Toward
a Proof of Montonen-Olive Duality
via Multiple M2-branes

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based on the work (arXiv:0809.2137)
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1. Introduction

Why String Theory is interesting?

String Theory

- is well defined and understood perturbatively
- includes Quantum Gravity (at least perturbatively)
- is useful for Mathematics (ex. Mirror symmetry)
- is applied to the QCD (ex. AdS/QCD)
- can be applied to Particle Phenomenology

and can be the Theory Of Everything.

But, our understanding of string theory is obviously incomplete.

Further investigations are needed.

Recent exciting progress in string theory:

Low energy actions of

multiple Membranes in M-theory

was found !

Why this is so exciting?

What is M-theory?

M-theory will unify all string theories, but still mysterious

- **What is known for M-theory**
  - **11d SUGRA at low energy**
    - There is an extra dimension
  - **M-theory on S^1 should be 10D type IIA string theory**
  - **M2-brane (Membrane) and M5-brane**

    For a single M2 or an M5 $\rightarrow$ Nambu-Goto action

We do not know much about M-theory

For string theory,
we know much about pertubative aspects.

String perturbation theory is well understood
and
we can compute, for example,
scattering amplitudes of gravitons

But, for M-theory,
we do not have well defined perturbative description,
because quantization of membrane have serious problems,
like presence of continuous spectrum.

D-branes have been very important objects to understand string theory:
For example, AdS/CFT, Matrix Models, etc

Why D-branes are useful?

Because
D-brane can be described by ***open strings***
even though they are non-perturbative objects

→ **Yang-Mills action** as multiple D-brane action!
AdS/CFT, Matrix Models, MQCD, etc

On the other hand, until very recently,
multiple M2-brane action had not been obtained.

Recently,
Bagger and Lambert (BL) proposed
multiple membrane actions,
then
Aharony, Bergman, Jafferis and Maldacena (ABJM)
found different multiple membrane actions.

We will understand many aspects of M-theory
(and string theory) !!

Many possible applications,
ex. AdS4/CFT3

(3+1)d gravity theory $\leftrightarrow$ (2+1)d field theory

Fields in ABJM action:

4 complex scalars (A=1,2,3,4)
bi-fundamental rep. of *U(N) x U(N)*

$$Y^A \, , \; Y^\dagger_A$$

4 (2+1)d Dirac spinors
bi-fundamental rep. of *U(N) x U(N)*

$$\psi_A \, , \; \psi^{A\dagger}$$

(2+1)d *U(N) x U(N)* gauge fields

$$A_\mu \, , \; \hat{A}_\mu$$

ABJM action is following (2+1)d Chern-Simons + matter action:

$$S = \int d^3x \left[\frac{k}{4\pi} \varepsilon^{\mu\nu\lambda} \mathrm{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right.$$
$$\left. - \mathrm{Tr} D_\mu Y_A^\dagger D^\mu Y^A - i \mathrm{Tr}\, \psi^{A\dagger} \gamma^\mu D_\mu \psi_A - V_{\mathrm{bos}} - V_{\mathrm{ferm}} \right]$$

$$V_{bos} = -\frac{4\pi^2}{3k^2} \mathrm{Tr} \left(Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \right.$$
$$\left. + 4 Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6 Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger \right)$$

$$V_{ferm} = -\frac{2i\pi}{k} \mathrm{Tr} \left(Y_A^\dagger Y^A \psi^{B\dagger} \psi_B - \psi^{B\dagger} Y^A Y_A^\dagger \psi_B - 2 Y_A^\dagger Y^B \psi^{A\dagger} \psi_B + 2 \psi^{B\dagger} Y^A Y_B^\dagger \psi_A \right.$$
$$\left. + \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D - \epsilon_{ABCD} Y^A \psi^{B\dagger} Y^C \psi^{D\dagger} \right),$$

No F^2 term ! (not like D-brane)

ABJM action has

12 SUSY and SU(4)xU(1) global symmetry

and

Conformal symmetry

(1) This action describes

N M2-branes on $\mathbf{C}^4/\mathbf{Z}_k$

$$(y^1, y^2, y^3, y^4) \to (e^{\frac{2\pi i}{k}} y^1, e^{\frac{2\pi i}{k}} y^2, e^{\frac{2\pi i}{k}} y^3, e^{\frac{2\pi i}{k}} y^4)$$

(2) ABJM derived this action

as a limit of a D-brane configuration

An interesting application:

SL(2,Z) duality of (3+1)d Maximally Supersymmetric Yang-Mills theory

(SL(2,Z) duality is also called
Montonen-Olive duality)

Why is this related to M2-brane action (ABJM action)?

Consider M2-branes in M-theory compactified on S^1

M-theory on S^1 = IIA string in 10d

(Radius of S^1 ~ string coupling)

Thus, M-theory is the strong coupling limit of IIA string, and

M2-brane at a point in S^1 = D2-brane in IIA

(M2-brane extending in S^1 = fund. string in IIA)

Consider M2-branes in M-theory compactified on **Torus**

M-theory on T^2 = IIB string in 10d

complex moduli τ of T^2 ~ τ of string coupling + RR 0-form where area of T^2 is taken to be very small

SL(2,Z) of τ = SL(2,Z) duality of IIB string

Thus, SL(2,Z) duality of IIB is manifest in M-theory!

M2-brane at a point in T^2 = D3-brane in IIB

SL(2,Z) of τ = SL(2,Z) duality of D3-branes
= SL(2,Z) duality of SYM

Thus we could prove
SL(2,Z) duality of (3+1)d SYM!

Indeed, we will show that

(3+1)d SUSY Yang-Mills with θ term
can be constructed from
(2+1)d Chern-Simons-Matter theory
(which is an orbifold of ABJM action).

Starting from one single (2+1)d CSM theory,
we will find infinitely many equivalent (3+1)d SYM theories
differing up to SL(2,Z) of the gauge coupling

However, S-transformation can be thought of
as a parity transformation because our τ is not generic.

2. M2-branes

Consider M2-branes in M-theory compactified on S^1

M-theory on S^1 = IIA string in 10d

(Radius of S^1 ~ string coupling)

Thus, M-theory is the strong coupling limit of IIA string, and

M2-brane at a point in S^1 = D2-brane in IIA

(M2-brane extending in S^1 = fund. string in IIA)

D2-brane effective action is (2+1)d N=8 Yang-Mills theory which have

7 scalars = location of D2-brane

16 SUSY and SO(7) global symmetry

Not Conformal (Yang-Mills coupling is not dimensionless)

M2-brane effective action should have

8 scalars = location of M2-brane

16 SUSY and SO(8) global symmetry

Conformal symmetry (=not Yang-Mills theory)

For (2+1)d Yang-Mills theory,
Strong coupling limit = low energy limit

M2-brane action = low energy limit of D2-brane action.
Thus, we should solve the strong coupling dynamics.
$\rightarrow$ very difficult.

We want to find a conformal action for M2-brane

3. BLG action of multiple M2-branes

Fields in BLG action:

8 scalar fields (I=1,2,,,8) $$X_a^I$$

16 component spinor
(~ a (10+1)d majorana spinor) $$\Psi_a$$

(2+1)d gauge fields $$A_{\mu ab}$$

a and *b* are indices related to the number of M2-brnaes
(like Chan-Paton indices for D2-branes)

Instead of Lie algebra, BLG action is based on Lie 3-algebra!

Structure constant: f^{abcd}
which satisfy (i) and (ii)

(i) fundamental identities

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d .$$

(ii) total anti-symmetry

$$f^{abcd} = f^{[abcd]}$$

Ex. (called A4 algebra) $f^{abcd} \propto \varepsilon^{abcd}$

Bagger and Lambert proposed the following Lagrangian as a multiple membrane action (motivated by Basu-Harvey):

c.f. Gustavsson

Lagrangian:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(D_\mu X^{aI})(D^\mu X^I_a) + \frac{i}{2}\bar{\Psi}^a\Gamma^\mu D_\mu \Psi_a + \frac{i}{4}\bar{\Psi}_b\Gamma_{IJ}X^I_c X^J_d \Psi_a f^{abcd} \\ &\quad -V + \frac{1}{2}\varepsilon^{\mu\nu\lambda}(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}).\end{aligned}$$

$$V = \frac{1}{12}f^{abcd}f^{efg}{}_d X^I_a X^J_b X^K_c X^I_e X^J_f X^K_g$$

Gauge symmetry:

$$\delta X_a = \Lambda_{cd} f^{cdb}{}_a X_b \equiv \tilde{\Lambda}^b{}_a X_b$$

$$\delta \tilde{A}_\mu{}^b{}_a = \partial_\mu \tilde{\Lambda}^b{}_a - \tilde{\Lambda}^b{}_c \tilde{A}_\mu{}^c{}_a + \tilde{A}_\mu{}^b{}_c \tilde{\Lambda}^c{}_a$$

$$\tilde{A}_\mu{}^b{}_a \equiv f^{cdb}{}_a A_{\mu cd}$$

((2+1)d N=8) SUSY transformation:

$$\begin{aligned} \delta X_a^I &= i\bar{\epsilon}\Gamma^I\Psi_a \\ \delta\Psi_a &= D_\mu X_a^I \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X_b^I X_c^J X_d^K f^{bcd}{}_a \Gamma^{IJK}\epsilon \\ \delta\tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X_c^I \Psi_d f^{cdb}{}_a . \end{aligned}$$

This Lagrangian has

16 SUSY and SO(8) global symmetry

and

Conformal symmetry

(No such action had been known.)

However, there are problems in this action as a M2-action

The problems of BLG action:

(1) Only one 3-Lie algebra exists,

i.e. A4 algebra $f^{abcd} \propto \varepsilon^{abcd}$ which would describe 2 M2-branes

(assuming finite dimensional, positive definite)

(2) No derivation, just a proposal

(16 SUSY and conformal symmetry will constrain the action so much, but not unique.)

4. ABJM action
of multiple M2-branes

Prelude:

BLG action with A4 algebra is equivalent to Chern-Simons action with gauge group SU(2) x SU(2)

van Raamsdonk

SO(4) ~ SU(2) x SU(2)

$$A_{\mu ab} = -\frac{1}{2f}(A^{+}_{\mu ab} + A^{-}_{\mu ab}) \qquad A_\mu = A^{+}_{\mu 4i}\sigma_i \qquad \hat{A}_\mu = A^{-}_{\mu 4i}\sigma_i$$

vector rep of SO(4) = bi-fundamental of SU(2)xSU(2) $\quad X^I \quad \Psi$

$$\begin{aligned}\mathcal{L} \;=\; & \mathrm{Tr}(-(D^\mu X^I)^\dagger D_\mu X^I + i\bar{\Psi}^\dagger\Gamma^\mu D_\mu \Psi) \\ & +\mathrm{Tr}(-\frac{2}{3}if\bar{\Psi}^\dagger\Gamma_{IJ}(X^I X^{J\dagger}\Psi + X^J\Psi^\dagger X^I + \Psi X^{I\dagger}X^J) - \frac{8}{3}f^2 X^{[I}X^{J\dagger}X^{K]}X^{K\dagger}X^J X^{I\dagger}) \\ & +\frac{1}{2f}\epsilon^{\mu\nu\lambda}\mathrm{Tr}(A_\mu\partial_\nu A_\lambda + \frac{2}{3}iA_\mu A_\nu A_\lambda) - \frac{1}{2f}\epsilon^{\mu\nu\lambda}\mathrm{Tr}(\hat{A}_\mu\partial_\nu\hat{A}_\lambda + \frac{2}{3}i\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda)\end{aligned}$$

different sign!

where

$$D_\mu X^I = \partial_\mu X^I + iA_\mu X^I - iX^I\hat{A}_\mu$$

a generalization to

U(N) x U(N)

(or SU(N) x SU(N))

ABJM action

12 SUSY (N=6) instead of 16 SUSY

SU(4) x U(1) global symmetry

Fields in ABJM action:

4 complex scalars (A=1,2,3,4)
bi-fundamental rep. of *U(N) x U(N)* $Y^A \, , \; Y_A^\dagger$

4 (2+1)d Dirac spinors
bi-fundamental rep. of *U(N) x U(N)* $\psi_A \, , \; \psi^{A\dagger}$

(2+1)d *U(N) x U(N)* gauge fields $A_\mu \, , \; \hat{A}_\mu$

ABJM action:

$$S = \int d^3x \left[\frac{k}{4\pi} \varepsilon^{\mu\nu\lambda} \mathrm{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right.$$
$$\left. - \mathrm{Tr} D_\mu Y_A^\dagger D^\mu Y^A - i \mathrm{Tr}\, \psi^{A\dagger} \gamma^\mu D_\mu \psi_A - V_{\mathrm{bos}} - V_{\mathrm{ferm}} \right]$$

$$V_{bos} = -\frac{4\pi^2}{3k^2} \mathrm{Tr} \left(Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \right.$$
$$\left. + 4 Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6 Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger \right)$$

$$V_{ferm} = -\frac{2i\pi}{k} \mathrm{Tr} \left(Y_A^\dagger Y^A \psi^{B\dagger} \psi_B - \psi^{B\dagger} Y^A Y_A^\dagger \psi_B - 2 Y_A^\dagger Y^B \psi^{A\dagger} \psi_B + 2 \psi^{B\dagger} Y^A Y_B^\dagger \psi_A \right.$$
$$\left. + \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D - \epsilon_{ABCD} Y^A \psi^{B\dagger} Y^C \psi^{D\dagger} \right),$$

((2+1)d N=6) SUSY transformation:

$$\begin{aligned}
\delta Y^A &= i\omega^{AB}\psi_B, \\
\delta Y^\dagger_A &= i\psi^{\dagger B}\omega_{AB}, \\
\delta\psi_A &= -\gamma_\mu\omega_{AB}D_\mu Y^B + \frac{2\pi}{k}\left(-\omega_{AB}(Y^C Y^\dagger_C Y^B - Y^B Y^\dagger_C Y^C) + 2\omega_{CD}Y^C Y^\dagger_A Y^D\right), \\
\delta\psi^{A\dagger} &= D_\mu Y^\dagger_B \gamma_\mu \omega^{AB} + \frac{2\pi}{k}\left(-(Y^\dagger_B Y^C Y^\dagger_C - Y^\dagger_c Y^C Y^\dagger_B)\omega^{AB} + 2Y^\dagger_D Y^A Y^\dagger_C \omega^{CD}\right), \\
\delta A_\mu &= -\frac{2\pi}{k}(Y^A\psi^{B\dagger}\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu\psi_A Y^\dagger_B), \\
\delta\hat{A}_\mu &= \frac{2\pi}{k}(\psi^{A\dagger}Y^B\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu Y^\dagger_A\psi_B), \qquad (
\end{aligned}$$

$$(\omega^{AB})_\alpha = ((\omega_{AB})^*)_\alpha, \quad \omega^{AB} = \frac{1}{2}\epsilon^{ABCD}\omega_{CD}$$

Gaiotto-Giobi-Yin, Hosomichi et.el, Bagger-Lambert, ST, Bandres-Lipstein-Schwarz

ABJM action has

12 SUSY and SU(4)xU(1) global symmetry
and
Conformal symmetry

(1) This action describes
N M2-branes on $\mathbf{C}^4/\mathbf{Z}_k$

$$(y^1, y^2, y^3, y^4) \to (e^{\frac{2\pi i}{k}} y^1, e^{\frac{2\pi i}{k}} y^2, e^{\frac{2\pi i}{k}} y^3, e^{\frac{2\pi i}{k}} y^4)$$

(2) ABJM derived this action
as a limit of a D-brane configuration

Problems of BL action are resolved!

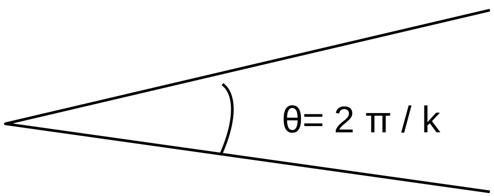
5. ABJM to 3d YM

Orbifold $\mathbf{C}^4/\mathbf{Z}_k$ to R^7 x S¹

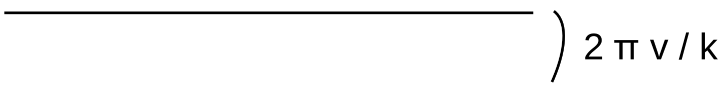
M2-branes probing $\mathbf{C}^4/\mathbf{Z}_k$ $\Rightarrow$ M2-branes probing R^7 x S¹ = D2-branes probing R^7

$\Downarrow$ $\Downarrow$

(2+1)d ABJM theory (Chern-Simon) $\Rightarrow$ (2+1)d SuperYM theory



$\Rightarrow$



Mukhi et.al.

$\mathbf{C}^4/\mathbf{Z}_k$

Scaling limit

$v \longrightarrow \infty, \quad k \longrightarrow \infty, \quad v\,/\,k$: fixed

where v is the distance between the M2 and singularity

Bosonic part of ABJM

$$S = \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \mathrm{tr} \left(A^{(1)}_\mu \partial_\nu A^{(1)}_\lambda + \frac{2i}{3} A^{(1)}_\mu A^{(1)}_\nu A^{(1)}_\lambda - A^{(2)}_\mu \partial_\nu A^{(2)}_\lambda - \frac{2i}{3} A^{(2)}_\mu A^{(2)}_\nu A^{(2)}_\lambda \right) \right.$$
$$\left. - \mathrm{tr} \left((D_\mu Z_A)^\dagger D^\mu Z^A \right) - \mathrm{tr} \left((D_\mu W^A)^\dagger D^\mu W_A \right) - V(Z,W) \right]$$

$$D_\mu Z^A = \partial_\mu Z^A + i A^{(1)}_\mu Z^A - i Z^A A^{(2)}_\mu ,$$
$$D_\mu W^A = \partial_\mu W^A + i A^{(2)}_\mu W^A - i W^A A^{(1)}_\mu$$

where we change the notation: $Y \to \{ Z, W^* \}$

Consider $Z^1 = v \mathbf{1}_{N \times N}$ and take a linear combination

$$A^{(\pm)}_\mu \equiv \frac{1}{2} \left(A^{(1)}_\mu \pm A^{(2)}_\mu \right)$$

$$S_{\mathrm{CS}} = \int d^3x \frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \mathrm{tr} \left[A^{(-)}_\mu F^{(+)}_{\nu\lambda} + \frac{2i}{3} A^{(-)}_\mu A^{(-)}_\nu A^{(-)}_\lambda \right]$$

$$S_{\mathrm{mass}} = - \int d^3x \ \mathrm{tr} \left[\{ A^{(-)}_\mu , v \}^2 \right] = - \int d^3x \ 4v^2 \mathrm{tr} \left[(A^{(-)}_\mu)^2 \right]$$

$A_\mu^{(-)}$ is massive and can be integrated out.
Then we have

$$A_\mu^{(-)} = \frac{k}{16\pi v^2}\epsilon_{\mu\nu\lambda}F^{(+)\nu\lambda}$$

$$S = -\int d^3x \frac{k^2}{32\pi^2 v^2}\mathrm{tr}\left[(F_{\mu\nu}^{(+)})^2 + \frac{k^4}{v^6}\mathcal{O}((F^{(+)})^3)\right]$$

3D YM from CS theory through Higgsing!

6. orbifold of ABJM to 4d YM

To get D3-branes, we need T^2, instead of S^1

⇩

Further orbifolding of ABJM action will be needed

For the D-branes, we know how to obtain orbifold theory a la Douglas-Moore. (Scalars are adjoint of U(N).)

Even though, in ABJM,
scalars are bi-fundamental of U(N) x U(N),
The standard orbifold action of Douglas-Moore can be applied to ABJM by regarding the bi-fundamental as adjoint.

we consider $\mathbb{Z}_n$ orbifold by the action

$$y^A \to e^{2\pi i/n_A} y^A$$

$$(n_1, n_2, n_3, n_4) = (n, n, -n, -n)$$

Then we have a quiver Chern-Simons-Matter theory corresponding to the M2-branes on $\mathbf{C}^4/(\mathbf{Z}_n \times \mathbf{Z}_{nk})$

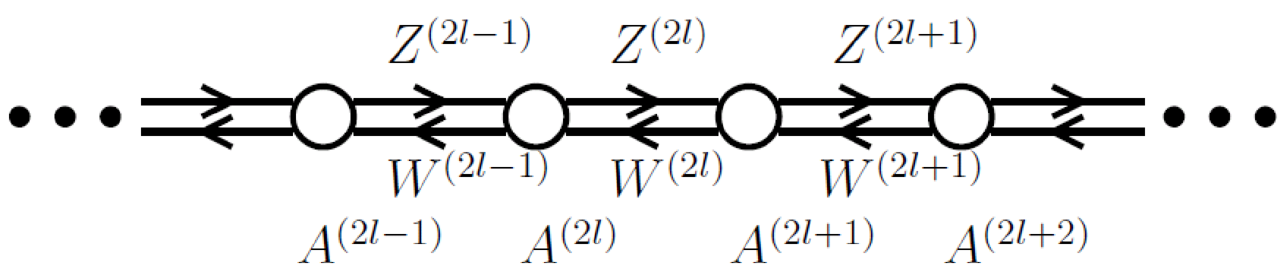
Here $\mathbb{Z}_k$ is replaced by $\mathbf{Z}_{nk}$ because of the over all *1/n* factor of the orbifold action.

Imamura-Kimura
Yagi-Terashima
c.f. Klebanov et.al.

The bosonic action of this orbifolded ABJM action is

$$S=\int d^3x\ \Bigg[\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\sum_{l=1}^{n}\mathrm{tr}\bigg(A_\mu^{(2l-1)}\partial_\nu A_\lambda^{(2l-1)}+\frac{2i}{3}A_\mu^{(2l-1)}A_\nu^{(2l-1)}A_\lambda^{(2l-1)}$$
$$-A_\mu^{(2l)}\partial_\nu A_\lambda^{(2l)}-\frac{2i}{3}A_\mu^{(2l)}A_\nu^{(2l)}A_\lambda^{(2l)}\bigg)$$
$$-\mathrm{tr}\sum_{s=1}^{2n}\left((D_\mu Z^{(s)})^\dagger D^\mu Z^{(s)}+(D_\mu W^{(s)})^\dagger D^\mu W^{(s)}\right)-V(Z,W)\Bigg]$$

$$D_\mu Z^{(2l-1)}=\partial_\mu Z^{(2l-1)}+iA_\mu^{(2l-1)}Z^{(2l-1)}-iZ^{(2l-1)}A_\mu^{(2l)},$$
$$D_\mu Z^{(2l)}=\partial_\mu Z^{(2l)}+iA_\mu^{(2l)}Z^{(2l)}-iZ^{(2l)}A_\mu^{(2l+1)},$$
$$D_\mu W^{(2l-1)}=\partial_\mu W^{(2l-1)}+iA_\mu^{(2l)}W^{(2l-1)}-iW^{(2l-1)}A_\mu^{(2l-1)},$$
$$D_\mu W^{(2l)}=\partial_\mu W^{(2l)}+iA_\mu^{(2l+1)}W^{(2l)}-iW^{(2l)}A_\mu^{(2l)}.$$



Following points in the moduli space is expected to give torus compactification:

$$Z^{(2l-1)} = v, \quad Z^{(2l)} = \tilde{v}, \quad W^{(2l-1)} = W^{(2l)} = 0$$

To get a D3-brane, we will take

$$\tilde{v},\ n \to \infty, \qquad v/n,\ \tilde{v}/n \to 0, \qquad v/\tilde{v} : \text{fixed}, \quad k : \text{fixed}$$

First, we will take linear combinations of gauge fields

$$A_\mu^{(\pm)(2l-1)} \equiv \frac{1}{2}\left(A_\mu^{(2l-1)} \pm A_\mu^{(2l)}\right)$$

Then, the action (relevant in the limit) for gauge fields is CS term and mass term;

$$S_{\rm CS} = \int d^3x \sum_{l=1}^{n} \frac{k}{2\pi}\epsilon^{\mu\nu\lambda} \sum_{l=1}^{n} {\rm tr}\left[A_\mu^{(-)(2l-1)} F_{\nu\lambda}^{(+)(2l-1)} + \frac{2i}{3} A_\mu^{(-)(2l-1)} A_\nu^{(-)(2l-1)} A_\lambda^{(-)(2l-1)}\right]$$

$$S_{\rm mass} = \int d^3x \sum_{l,l'=1}^{n} {\rm tr}\left[A_\mu^{(-)(2l-1)} M_{ll'}^{(-)} A^{(-)(2l'-1)\mu}\right.$$

$$\left. + A_\mu^{(-)(2l-1)} M_{ll'}^{(\rm cross)} A^{(+)(2l'-1)\mu} + A_\mu^{(+)(2l-1)} M_{ll'}^{(+)} A^{(+)(2l'-1)\mu}\right]$$

$$M^{(-)} \equiv -4(v^2+\tilde{v}^2)\mathbf{1}_{n\times n} + 2\tilde{v}^2\Lambda, \quad M^{(\rm cross)} \equiv 2\tilde{v}^2(\Omega - \Omega^{-1}), \quad M^{(+)} \equiv (-\tilde{v}^2)\Lambda,$$

$$\Lambda \equiv 2\mathbf{1}_{n\times n} - (\Omega + \Omega^{-1})$$

$$\Omega_{ij} \equiv \delta_{i+1,j}$$

$A_\mu^{(-)}$ is massive and can be integrated out.
Then we have

$$A_\mu^{(-)(2l-1)} = -\frac{k}{4\pi}\epsilon_{\mu\nu\lambda}((M^{(-)})^{-1})^l{}_{l'}F^{(+)(2l'-1)\nu\lambda} - \frac{1}{2}((M^{(-)})^{-1}(M^{(\mathrm{cross})})^{\mathrm{T}})^l{}_{l'}A_\mu^{(+)(2l'-1)}$$

$$S = \int d^3x \ \mathrm{tr}\left[-\eta^{\mu\mu'}\left(\frac{k}{4\pi}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}^{(+)(2l-1)} + \frac{1}{2}A_\mu^{(+)(2l'-1)}(M^{(\mathrm{cross})})_{l'}{}^l\right)((M^{(-)})^{-1})_{ll''} \right.$$
$$\times\left(\frac{k}{4\pi}\epsilon_{\mu'\nu'\lambda'}F^{(+)(2l''-1)\nu'\lambda'} + \frac{1}{2}((M^{(\mathrm{cross})})^{\mathrm{T}})^{l''}_{l'''}A_{\mu'}^{(+)(2l'''-1)}\right)$$
$$\left. + A_\mu^{(+)(2l-1)}M_{ll'}^{(+)}A^{(+)(2l'-1)\mu}\right]$$

S= *F*^2 + *A*^2 + *A F*
kin mass CS

We will evaluate this action in the limit:

$$\tilde{v},\ n \to \infty, \qquad v/n,\ \tilde{v}/n \to 0, \qquad v/\tilde{v} : \text{fixed}, \quad k : \text{fixed}$$

For kinetic term and mass term, taking this limit is same as “Deconctruction” of Arkani-Hamed-Cohen-Georgi (or Taylor’s T-duality)

We find kinetic term and mass term are written as

$$S = \int d^3x \ \mathrm{tr} \left[\sum_s \frac{-nk^2}{32\pi^2(v^2+\tilde{v}^2)} \hat{\mathcal{L}}_{\mathrm{kin}} - 4\frac{nv^2\tilde{v}^2}{v^2+\tilde{v}^2} \sum_s \left(\frac{s\pi}{n}\right)^2 (\hat{A}_\mu^{(+)(s)})^2 \right]$$

which is equivalent to following 4D YM action compactified on S^1

$$S = \frac{-k^2}{32\pi^2(v^2+\tilde{v}^2)} \frac{1}{2\pi R} \int d^4x \ \mathrm{tr}\left[F_{MN}^2\right] = \frac{-kv\tilde{v}}{8\pi(v^2+\tilde{v}^2)} \int d^4x \ \mathrm{tr}\left[F_{MN}^2\right]$$

where the radious of S^1 is given by $\dfrac{1}{R} = \dfrac{8\pi^2 v\tilde{v}}{kn}$

In the limit, the CS term

$$S_{\text{cross}} = -\int d^3x\ \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\text{tr}\left[A_\mu^{(+)(2l-1)}(M^{(\text{cross})}(M^{(-)})^{-1})_{ll'}F_{\nu\lambda}^{(+)(2l'-1)}\right]$$

becomes

$$S_{\text{cross}} = -\int d^3x\ \frac{ik\tilde{v}^2}{2(v^2+\tilde{v}^2)}\epsilon^{\mu\nu\lambda}\left[\sum_{l'} l'\text{tr}\left(B_\mu^{(l')}(\partial_\nu B_\lambda^{(-l')} - \partial_\lambda B_\nu^{(-l')})\right)\right.$$
$$\left. + \sum_{l'+l'''+l''''=0} il'\text{tr}\left(B_\mu^{(l')}[B_\nu^{(l''')}, B_\lambda^{(l'''')}]\right)\right].$$

where we have used $A_\mu^{(+)(2l-1)} \equiv q^{ll'}B_\mu^{(l')}$ $\quad q \equiv \exp[2\pi i/n]$

This action is equivalent to the KK reduction of 4D θ term!

$$S_{\text{cross}} = \frac{k\tilde{v}^2}{16\pi(v^2+\tilde{v}^2)}\int d^4x\ \text{tr}\left[\epsilon^{MNPQ}F_{MN}F_{PQ}\right]$$

Therefore, from 3D orbifolded ABJM theory we obtain the 4D YM theory with the action:

$$S = \int d^4x \ \mathrm{tr} \left[-\frac{kv\tilde{v}}{8\pi(v^2+\tilde{v}^2)} F_{MN}^2 + \frac{k\tilde{v}^2}{16\pi(v^2+\tilde{v}^2)} \epsilon^{MNPQ} F_{MN} F_{PQ} \right]$$

the complexified gauge couplig is $\tau = \dfrac{-k\tilde{v}^2}{v^2+\tilde{v}^2} + i \dfrac{kv\tilde{v}}{v^2+\tilde{v}^2}$

where τ is defined in the standard notation,

$$S = \frac{-1}{8\pi} \int d^4x \ \mathrm{tr} \left[\mathrm{Im}(\tau) F_{MN} F^{MN} + \mathrm{Re}(\tau) \frac{1}{2} \epsilon^{MNPQ} F_{MN} F_{PQ} \right], \quad \tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2}$$

Instead of $A_\mu^{(\pm)(2l-1)} \equiv \frac{1}{2}\left(A_\mu^{(2l-1)} \pm A_\mu^{(2l)}\right)$

we can take $A_\mu^{(\pm)(2l-1)} \equiv \frac{1}{2}(A_\mu^{(2l-1)} \pm A_\mu^{(2l+2)})$

and then we obtain YM action with

$$\tau' = \frac{-k(v^2 + 2\tilde{v}^2)}{v^2 + \tilde{v}^2} + i\frac{kv\tilde{v}}{v^2 + \tilde{v}^2}$$

Because this is just a change of labeling of gauge fields, this action should be equivalent to previous one.

Indeed, these τs are related by the T-transformation of the SL(2,Z) duality!

$$\tau' = \tau - k = T^{-k}(\tau) \qquad \tau = \frac{-k\tilde{v}^2}{v^2 + \tilde{v}^2} + i\frac{kv\tilde{v}}{v^2 + \tilde{v}^2}$$

we can take $$A_\mu^{(\pm)(2l-1)} \equiv \frac{1}{2}(A_\mu^{(2l-1)} \pm A_\mu^{(2l-2)})$$

and then we obtain YM action with

$$\tau' = \frac{-kv^2}{v^2+\tilde{v}^2} + i\frac{kv\tilde{v}}{v^2+\tilde{v}^2}$$

These τs are related by
the S-transformation of the SL(2,Z) duality for k=1,2

$$\tau' = S(T^2(S(T(\tau)))) \quad (\text{for } k=1)$$
$$\tau' = T^{-1}(S(T(\tau))) \quad (\text{for } k=2)$$

$$\tau = \frac{-k\tilde{v}^2}{v^2+\tilde{v}^2} + i\frac{kv\tilde{v}}{v^2+\tilde{v}^2}$$

This can be regarded as a proof of S-duality, however, for very special τ

Eventually, this can be also understood as parity transformation

ABJM is not enough to prove S-duality, but,,,,

7. Conclusion

- Low energy effective action for multiple M2-branes were found by ABJM (motivated by Bagger-Lambert-Gustavsson) .

- 4D YM action with θterm was obtained from 3D Chern-Simons theory (orbifolded ABJM model).

- Some of SL(2,Z) duality was proven just from the relabeling of the gauge fields.

Many interesting works will be done!

Fin.