On the geometry of supersymmetric AdS solutions

Nakwoo Kim

Physics Department
Kyung Hee University, Seoul Korea

IPMU

Based on works with
J. Gauntlett, D. Waldram, A. Donos, J.D. Park

Table of Contents

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Introduction and Motivation

AdS/CFT correspondence :
Weakly coupled gravity provides a holographic dual description to strongly coupled gauge field theory.
More concretely,

$$\text{IIB strings in } AdS_5 \times SE_5 \longleftrightarrow D = 4, N = 1, \text{ Super YM}$$

and it is the geometry of the Sasaki-Einstein space which determines the matter content and interactions of the dual field theory.

Sasaki-Einstein space $\sim$
Vacuum moduli space of SUSY gauge theory

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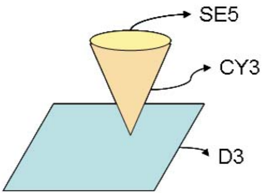
D-brane description

We put D3-branes at the tip of singular CY3, and take the near-horizon limit. The backreacted geometry is $AdS_5 \times SE_5$, which is dual to the gauge theory living on D3.



Sasaki-Einstein space

- Einstein space : $R_{ab} = \lambda g_{ab}$
- Sasakian : The metric cone is Kahler.
- When combined, the metric cone of SE is Ricci-flat and Kahler (Singular Calabi-Yau)
- Thus allows nontrivial Killing spinor, satisfying $\nabla_a \eta = i\gamma_a \eta$
- Examples : $S^5, T^{1,1}, Y^{p,q}, L^{p,q,r}$ ... (with explicitly known metric)

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KE, SE and Singular CY

- SE by definition is the base manifold of Calabi-Yau with conical singularity.

$$ds^2 = dr^2 + r^2(\text{Sasaki-Einstein})$$

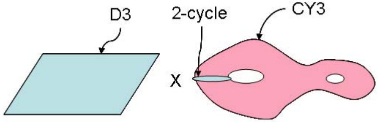
- SE is odd-dimensional, and $(r\frac{\partial}{\partial r})_b J^b{}_a$ provides a Killing vector of SE.
- Locally, SE is always written as a Hopf-fibration over Kahler-Einstein space.

$$ds^2_{SE} = (d\psi + A)^2 + ds^2_{KE}, \quad dA = R_{KE}$$

- The most well-known nontrivial example: $T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$ or U(1) fibration over $S^2 \times S^2$

Wrapped D3

But more generally, we can consider wrapped branes which are also supersymmetric. (They are useful when we want product gauge groups with different ranks.)



Then the worldvolume is $1+1$ dimensional and we expect to get AdS_3 in the near horizon limit.

Wrapped D3 and M2

In the same way one can consider wrapped branes of different kinds. But here we are particularly interested in $D3$ and $M2$ wrapping 2-cycles in CY. And their near-horizon geometry, we expect

- D3 wrapped on 2-cycle : $AdS_3 \times_w M_7$
- M2 wrapped on 2-cycle : $AdS_2 \times_w M_9$

Preserving SUSY: in general 1/8-BPS.

Main Result

For both D3 and M2 wrapped on 2-cycles, or AdS3 from D3, and AdS2 from D2, the geometry is built on Kahler space satisfying the following eq:

$$\nabla^2 R - \frac{1}{2}R^2 + R_{ij}R^{ij} = 0$$

and it will be explained:

- How this result is obtaind.
- What kind of ansatz can provide explicit solutions to it.
- Extension/generalization.

AdS_3 in IIB

Ansatz for the IIB solution :

$$ds_{10}^2 = e^{2A}(ds^2(AdS_3) + ds^2(M_7))$$

$$F^{(5)} = (1 + *)Vol(AdS_3) \wedge F$$

and turn off all the remaining fields. Then the Killing spinor equation for M_7 is :

$$[\gamma^a \partial_a A - i + \frac{1}{2} e^{-4A} F_{ab} \gamma^{ab}]\eta = 0$$

$$[\nabla_a + \frac{i}{2}\gamma_a - \frac{1}{2} e^{-4A} F_{ab} {\gamma_c}^{ab}]\eta = 0$$

Killing spinor analysis

- The next step is to consider all spinor bilinears, $\eta^\dagger\gamma_{\dots}\eta$, and study the algebraic and differential relations between them. One makes use of Fierz identities and the Killing spinor equation.
- We find, for instance, $\eta^\dagger\eta$ is constant, $\eta^\dagger\gamma^a\eta$ is a Killing vector with constant norm etc.
- Then the M_7 metric can be written

$$ds^2(M_7) = \frac{1}{4}(dz + P)^2 + e^{-4A} ds^2(K_6)$$

BPS relations

The two-form $J_{ab} = \eta^\dagger \gamma_{ab} \eta$ provides a complex structure to K_6, which is in fact Kahler. The remaining field equations are summarised as

$$e^{-4A} = \frac{1}{8} R$$

$$F = \frac{1}{2} J - \frac{1}{8} d(e^{4A}(dz + P))$$

$$\nabla^2 R - \frac{1}{2} R^2 + R_{ij} R^{ij} = 0(*)$$

once we find a solution to (*), we can construct the whole 10d solution.

M2 brane analysis

One can perform a similar analysis for M2-branes.

$$ds_{11}^2 = e^{2A}[ds^2(AdS_2) + ds^2(M_9)]$$

$$G^{(4)} = Vol(AdS_2) \wedge F$$

After a similar procedure with 9d Killing spinor equation for M_9, constant norm Killing vector in M_9 etc.

$$ds^2(M_9) = (dz + P)^2 + e^{-3A} ds^2(K_8)$$

$$e^{-3A} = \frac{1}{2}R, \quad F = -J + d[e^{3A}(dz + P)]$$

$$\nabla^2 R - \frac{1}{2}R^2 + R_{ij}R^{ij} = 0$$

Summary so far

- For D3 (or M2) wrapped on 2-cycles, in the AdS limit the internal geometry takes locally a form of U(1)-fibration over 6(8) dimensional Kahler geometry, satisfying the higher-order differential equation (*).
- This is very similar to SE, which in canonical form is a U(1) fibration over Kahler-Einstein space.
- Through analytic continuation, one gets IIB(M) theory with $S^3(S^2)$ parts as well. The same master equation.

Questions

- Can we find a nice interpretation of (*)?

$$\nabla^i \mathcal{J}_i = 0, \text{ with } \mathcal{J} = dR + 2 * P \wedge R \wedge J$$

- Is it plausible to solve (*)?
- Generalization to higher dimensions?
- What is the cone geometry? Other constructions like GLSM?

Kahler base as products of Kahler-Einstein

- For simplicity, let us first consider the Kahler base as (product of) Kahler-Einstein space. Like $S^2 \times H^2 \times T^2$ etc. (*) becomes algebraic.
- For a single KE, (*) holds only if $d = 2$. We can thus take $K_6 = S^2 \times T^4, K_8 = S^2 \times T^6$. They lead to
  - IIB theory : $AdS_3 \times S^3 \times T^4$ (1/2-BPS)
  - M theory : $AdS_2 \times S^3 \times T^6$ (1/4-BPS)
- More generally, if $ds^2(K) = \sum ds^2(KE_2^{(i)})$, with $R = \sum \ell_i J_i$,

$$\sum \ell_i^2 = (\sum \ell_i)^2$$

Kahler base as products of Kahler-Einstein

- For K_6, $(\ell_1, \ell_2, \ell_3) = (\ell, -\ell/(\ell+1), 1)$.
  - Special cases are $(0, 0, 1)$ and $(-1/2, 1, 1)$.
  - The latter, for $K_6 = H_2 \times CP^2$, has been known, from the gauged sugra solution of wrapped D3-brane (M. Naka)
- For K_8, $(\ell_1, \ell_2, \ell_3, \ell_4) = (\ell_1, \ell_2, -(\ell_1\ell_2 + \ell_1 + \ell_2)/(\ell_1 + \ell_2 + 1), 1)$.
  - special cases are $(0, 0, 0, 1)$ and $(-1, 1, 1, 1)$.
  - The latter, for $K_8 = H_2 \times CP^3$, also found from gauged sugra (Gauntlett et. al)

Calabi-ansatz

- It is well-known that the following metric is always Kahler ($U = U(\rho)$).

$$ds^2_{2n+2} = \frac{d\rho^2}{U} + U\rho^2(D\phi)^2 + \rho^2 ds^2(KE_{2n})$$

- For our purpose KE is positively curved, and $D\phi = d\phi + B$ with $dB = 2J_{KE}$.
- Now one can compute the Ricci tensor and then the master equation becomes nonlinear ODE.

Solution of the Calabi-ansatz

- One can find a polynomial solution for general n.
- $U = 1 - \alpha x^{n-1}(x - \beta)^2, \quad x = 1/\rho^2$
- The issue is whether this solution can be made into a compact $M_7(M_9)$, regular and smooth.
- It turns out, the IIB solution is equivalent to the AdS_3 solution found by Gauntlett, Mac Conamhna, Mateos and Waldram (hep-th/0606221, PRL)
- K_6 is not smooth. But upon including $(Dz)^2$, M_7 can be made regular if we impose periodic b.c. for $3\phi + z$ and z.

Geometry in $2n + 2$ dimensions

- We consider dimensional reduction of IIB sugra on $R^{1,1}$, or 11d sugra on R. From the effective action and Killing spinor equation, in 8d and 10d, we can find the generalization to arbitrary higher dimensions ($d = 2n + 2$).
- We have a system consisting of metric, scalar ϕ, 2-form gauge field b.
- One can construct: bosonic action, and the associated Killing spinor system.

Action and the Killing spinor equations

- Action : $(f = db)$

$$L = e^{2(n-1)\phi}[R + 2n(2n-3)(\nabla\phi)^2 + \frac{1}{2}e^{-4\phi}f^2]$$

- Killing spinor equations

$$[\gamma^a\nabla_a\phi + \frac{i}{12}e^{-2\phi}f_{abc}\gamma^{abc}]\epsilon = 0$$

$$[\nabla_a - \frac{i}{24}e^{-2\phi}f_{bcd}\gamma_a{}^{bcd}]\epsilon = 0$$

- They are consistent, in the sense that any susy configuration satisfying the gauge field equation and Bianchi identity, solves the field equation.

G-structure

Since we have $2n+2$-dimensional space, susy solution should come with $SU(n+1)$ structure. For $(1,1)$-form J and $(n+1,0)$-form Ω,

$$\begin{aligned} d[e^{n\phi}\Omega] &= 0 \\ d[e^{2(n-1)\phi}J^n] &= 0 \\ d[e^{2\phi}J] &= f \end{aligned}$$

From above, we see the space is complex, but not Kahler.

Geometry in (2n+1) dimensions

Now consider reducing on the *radial* direction.

$$e^{-2\phi} = r^{\frac{2(n-1)}{n-2}} e^{B}, \quad f = r^{\frac{n}{2-n}} dr \wedge F$$

We again find action and the associated Killing spinor system, for $(2n+1)d$ metric, scalar B, and 2-form f.s. F.

$$L = e^{(1-n)B}[R + \frac{n(2n-3)}{2}(\nabla B)^2 + \frac{1}{4}e^{2B}F^2 - \frac{2n}{(n-2)^2}]$$

again, any susy configuration, combined with Bianchi identity and the equation of motion for F, satisfy the entire field equations.

SUSY in $(2n+1)$ dimensions

Following the usual Killing spinor analysis, one can show that, any susy solution of the above $(2n+1)$-dim system, can be written as follows. $(c = (n-2)/2)$

$$ds^2_{2n+1} = c^2(dz+P)^2 + e^B ds^2_{2n}$$

$$e^B = c^2 R/2, \quad F = -J_{2n}/c + cd[e^{-B}(dz+P)]$$

where the base space ds^2_{2n} is Kahler, and satisfies

$$\nabla^2 R - \frac{1}{2}R^2 + R_{ij}R^{ij} = 0$$

Calabi-ansatz

- For general n, we use the solution $U = 1 - \alpha x^{n-2}(x-1)^2$
- For large enough α, we have two positive roots $0 < x_1 < 1 < x_2$.
- Upon coordinate transformation $\phi = (\psi - z)/n$,

$$\begin{aligned} \frac{1}{c^2} ds^2_{2n+1} &= (dz+P)^2 + \frac{R}{2} ds^2_{2n} \\ &= wDz^2 + \frac{RU}{2n^2 wx} D\psi^2 + \frac{R}{8x^3 U} dx^2 + \frac{R}{2x} ds^2 (KE^+_{2n-2}) \end{aligned}$$

  with $R = 8\alpha x^{n-1}$.

- At x_1, x_2, we have potential conical singularities. But in the form given above, giving 2π periodicity to ψ can make the base regular at both ends. Finally α should take discrete values to make Dz good U(1) fibration. (One demands $d(Dz)$ integrated over 2-cycles be integral.)

Solutions from the Calabi ansatz

$$f = n(1-U) + x\frac{DU}{dx}$$

$$R = 4(n-1)xf - 4x^2\frac{df}{dx}$$

$$w = (1-f/n)^2 + \frac{RU}{2n^2x}$$

$$Dz = dz + gD\psi, \quad g = \frac{1}{n^2w}(nf - f^2 - \frac{RU}{2x})$$

LLM inspired ansatz

- The 1/2-BPS fluctuations of max susy AdS backgrounds have been studied by Lin, Lunin and Maldacena.
- IIB with $S^3 \times S^3$, or M-theory with $S^5 \times S^2$. The reduced 4d systems have been studied.
- Being 1/2-BPS, they are special cases of our 1/8-BPS geometries.
- We try to find similar reduction of (*) equation to 4 dimensions.

Metric ansatz and BPS relation

- Metric ansatz

$$ds^2 = dy^2/U + y^2 U(D\psi)^2 + f/U(dx_1^2 + dx_2^2) + y^2 ds^2(KE_{2n-4})$$

with $D\psi = d\psi + \sigma + V$.

- Choose the Kahler form

$$J = ydy \wedge D\psi + (f/U)dx_1 \wedge dx_2 + y^2 J_{KE}$$

- $dJ = 0$, provided

$$d\sigma = 2J_{KE}, \quad d_2 V = 1/y\partial_y(f/U)dx_1 \wedge dx_2$$

- Kahler in fact, if we impose $\partial_y V = 1/y *_2 d_2(1/U)$

Reduction to 3d

- Integrability condition leads to ($\Delta = \partial_1^2 + \partial_2^2$)

$$\Delta \frac{1}{U} + y\partial_y[\frac{1}{y}\partial_y(\frac{f}{U})] = 0$$

- Now assuming a BPS relation through a new field D, as $\frac{1}{U} = \frac{y}{2}\partial_y D, \quad f = y^{2p}e^{qD}$ the (*) equation is satisfied, if (k is scalar curvature of KE)

$$p = 3 - n, \quad (q - k)[(q(n-1) - k(n-3)] = 0$$

- When $n = 3$, $p = q = 0$ we recover $\Delta D + \frac{1}{y}\partial_y(y\partial_y D) = 0$.
- For $n > 3$, after some coordinate change, $\Delta D + x^{\frac{n-4}{n-3}}\partial_x^2 e^D = 0$.

With 3-form flux in IIB

One can in fact turn on 3-forms, for instance to IIB construction.

$$\nabla^2 R - \frac{1}{2}R^2 + R_{ij}R^{ij} + \frac{2}{3}G^{*ijk}G_{ijk} = 0$$

where G is the complexified 3-form, imaginary self-dual in 6d. $G = i\tilde{G}$.

conclusion

- We have found an interesting new class of complex geometry related to wrapped brane solutions. (Or, giant gravitons, BPS fluctuations of gauge theory etc.)
- Quite similar to the hierarchy of singular CY, SE, KE. Extraordinary higher-order equation in place of Einstein condition, for Kahler base.
- Presented several classes of explicit solutions. More systematic construction? (e.g. Toric data?)