ON UNIRATIONAL QUARTIC HYPERSURFACES

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Abstract. We study the unirationality property of an algebraic variety X (over $\mathbb{C}$) versus the so-called *stable birational infinite transitivity* of X. We show that in the case when X is a smooth quartic hypersurface these two notions do not coincide.

1. Introduction

1.1. Consider a smooth projective variety X over an algebraically closed field $\mathbf{k} \subseteq \mathbb{C}$. Recall that X is called *unirational* if there exists a rational dominant map $\mathbb{P}^N \dashrightarrow X$ from some projective space. In dimensions ≤ 2, unirationality implies rationality, whereas in higher dimensions this is no longer true (examples are due to Iskovskikh - Manin, Clemens - Griffiths, Artin - Mumford, Bogomolov, Kollár, and others). At the same time not a single example of rationally connected X that is not unirational has been found yet (see [11] for an overview of the current state of art). In this respect the paper [5] suggested a possible criterion for projective manifold to be unirational (cf. [6]).

Namely, it was proposed in [5] that unirationality of X is equivalent to the *stable birational infinite transitivity* (or *stable b-inf. trans.* for short), with respect to the group **SAut** (see **2.8** below for some recollections). The latter means that the product $X \times \mathbb{P}^k$, $k \gg 1$, carries a Zariski open subset U such that the group $\mathbf{SAut}(U)$ acts infinitely transitively on it. One of the goals, given the stated unirationality criterion, was to find non-unirational Fano hypersurfaces among those treated in [10].

The aim of the present paper is to show that the above-mentioned criterion for unirationality does not hold as stated. We prove the following:

Theorem 1.2. *There exists a unirational X which is* not *stably b-inf. trans.*

Let us briefly outline the strategy of the proof of Theorem 1.2.

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1.3. Recall that in [5, Section 3] smooth cubic (resp. some singular quartic) hypersurfaces were shown to be stably b - inf. trans. This makes it reasonable to test *smooth* quartic hypersurfaces on having the same property (compare with [8]). In Section 3, we prove that the latter does not hold for some smooth quartic X of a given dimension ≥ 3, and since every such X of sufficiently large dimension is unirational (see e. g. [7, Corollary 3.7]), Theorem 1.2 will follow.

Originally, we wanted to test unirationality criterion from [5] over a field $\mathbf{k}_p$ of positive characteristic p, where one finds a very interesting example of unirational variety, that is a *supersingular* K3 surface S. In Section 2, we employ further the analogy between such S and *singular* K3 surfaces, defined over $\mathbf{k}$, namely the "torus - like" structure of their *Hodge groups* Hg.

More precisely, the idea is that once S is stably b - inf. trans., $\mathrm{Hg}(S)$ must be isomorphic (as an algebraic group) to the additive group $(\mathbf{k}_p, +)$, which is absurd (see **3.8** below). However, while realizing this idea we heavily relied on another interplay between the supersingular and singular cases, which is the deep structural result proved in [13] (cf. **2.4**). It then remains to choose X in such a way that its mod p reduction will be a cone over S and deduce the stable b - inf. trans. for S from that for X to get the needed contradiction.

In turn, the " stable b - inf. trans. conservation " step is carried in **3.1**, **3.4** (see also **2.8**) and is based on the observation made in Lemma 3.3, together with a standard technique of deforming rational curves in families and the fact that S (when lifted to $\mathbf{k}$) actually contains a rational curve.

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2. Preliminaries

2.1. Hodge group. Let X be an Abelian variety and $V := H^1(X, \mathbb{Q}) \otimes \mathbb{R}$. The complex structure on X corresponds to a homomorphism $\varphi : S^1 \longrightarrow \mathrm{Sp}(V)$, the symplectic group of V (w. r. t. the symplectic form coming from an ample divisor on X), so that the usual Riemann positivity conditions are satisfied.

The *Hodge group* $\mathrm{Hg}(X)$ is the smallest algebraic subgroup of $\mathrm{Sp}(V)$ defined over $\mathbb{Q}$ and such that $\varphi(S^1) \subseteq \mathrm{Hg}(X)$. There is the following criterion for X to have a complex multiplication (i. e. for the algebra $\mathrm{End}(X) \otimes \mathbb{Q}$ to be a product of fields):

Proposition 2.2 (see [15, §2]). *X is of CM - type iff $Hg(X)$ is a torus algebraic group.*

More generally, if X is any smooth projective variety, then the Hodge group of X is the largest subgroup $\mathrm{Hg}(X)$ of automorphisms of $H^*(X, \mathbb{Q})$ that acts trivially on all the spaces $H^{k,k} \cap H^{2k}(X^n, \mathbb{Q})$ (for the natural Hodge filtration on $H^*(X^n, \mathbb{C})$), where k, n are arbitrary and $X^n := X \times \underbrace{\ldots}_{\text{n times}} \times X$. (Note that this definition of $\mathrm{Hg}(X)$ coincides with the previous one in the case when X is an Abelian variety.) Similarly, X is said to be of *CM - type*, if the group $\mathrm{Hg}(X)$ is commutative.

Finally, given a K3 surface X together with the corresponding Kuga – Satake Abelian variety A_X, the group $\mathrm{Hg}(A_X)$ is an extension of $\mathrm{Hg}(X)$ by $\mathbb{Z}/2\mathbb{Z}$. In particular, since the group $\mathrm{Hg}(A_X)$ is reductive, we obtain the following (cf. Proposition 2.2):

Corollary 2.3 (see e. g. [17, §3]). *In the previous setting, X is of CM - type iff A_X is, iff $Hg(X)$ is a torus algebraic group.*

2.4. Supersingular K3 surfaces. Let us now briefly recall a few properties of the K3 surfaces in question (see [18], [2], [21], [20] and [12] for an extensive treatment of the subject).

Fix an algebraically closed field $\mathbf{k}_p$ of characteristic $p \geq 5$ and consider a supersingular K3 surface S over $\mathbf{k}_p$ (i. e. $\mathrm{rk}\,\mathrm{NS}(S) = 22 = b_2(S)$) with its Artin invariant σ_0 (for the discriminant of Néron – Severi lattice $\mathrm{NS}(S)$ being equal $p^{-2\sigma_0}$). Then, given a supersingular elliptic curve E with the associated Kummer surface $\mathrm{Km}(E^2)$, there exist rational dominant maps

$$\mathrm{Km}(E^2) \dashrightarrow S \dashrightarrow \mathrm{Km}(E^2),$$

which are purely inseparable and generically finite of degree $p^{2\sigma_0 - 2}$ (see [13]). This and unirationality of supersingular Kummer surfaces established in [20] immediately yield the following:

Theorem 2.5 (C. Liedtke)**.** *Any supersingular* K3 *surface S is unirational.*[1)]

Example 2.6. In [19], Fermat quartic surface $S := (\sum x_i^4 = 0) \subset \mathbb{P}^3$ was shown to be supersingular, provided $p \equiv 3 \mod 4$.

Further, through the rest of the paper $\mathbf{k}_p$ will be the residual field $R/\mathfrak{m}$ of a local algebra $R \subset \mathbf{k}$ with respect to the maximal ideal $\mathfrak{m} \subset R$, so that $\mathbf{k}$ is the field of fractions of R (cf. [11, Ch. II, **5.10**]). Similarly, given any variety X over R, replacing $\mathbf{k}$-scalars by the $\mathbf{k}_p$-ones will be referred to (as usual) as the *mod p reduction of* X and denoted X_p (variety X in turn will be called the *lift of X_p (to* $\mathbf{k}$*)*).

Remark 2.7. Note that for the above supersingular surface S, when identified with its model (if any) over $R \subset \mathbf{k}$, the action of the group $\mathrm{Hg}(S)$ on $H^*_{\text{ét}}(S, \mathbb{Q}_\ell)$ (here $\ell \neq p$ is another prime number) is induced by the Frobenius action on S. In particular, we have $\mathrm{Hg}(S) \simeq \mathbf{k}_p^*$ (as algebraic groups), which suggests further analogy between supersingular K3 and the singular ones (cf. the discussion in [13, Section 2]). Indeed, the latter can be lifted to $\mathbf{k}$ (see [13, Theorem 2.6]), and so Corollary 2.3 applies.

2.8. The group SAut. Fix a smooth affine $\mathbf{k}$-variety U and consider some derivation D (identified with a vector field on U) of the algebra $\mathbf{k}[U]$. Recall that there is a 1-to-1 correspondence between the locally nilpotent D and regular $(\mathbf{k}, +)$-actions on U (see e.g. [3, 1.19]). Let us form the subgroup $\mathbf{SAut}(U) \subseteq \mathbf{Aut}(U)$ generated by various additive groups $(\mathbf{k}, +) =: \mathbb{G}_a$ parameterized by all D.

It is not difficult to show that once $D_1, \ldots, D_N$ are non-trivial locally nilpotent derivations of $\mathbf{k}[U]$ spanning the tangent space $T_{U,\zeta}$ at some point $\zeta \in U$, then $\mathbf{SAut}(U)$ acts on U with the dense orbit $\mathbf{SAut}(U) \cdot \zeta$ (compare with [5, Theorem 2.2]).

Let further U be defined over R (cf. the end of **2.4**). In this case, after multiplying by $\mathbf{k}$-scalars, one may assume D and all D_i to be derivations of the algebra $R[U]$. Then D (resp. D_i) descends again to a locally nilpotent derivation for U

[1)] This theorem is stated here only for the completeness of our present exposition and will not be used anywhere in the text.

replaced by U_p . However, even though the $\mathbf{k}$-group $\mathbb{G}_a$ corresponding to D coincides with $\exp(tD)$, $t \in \mathbf{k}$, we have failed to establish a similar description for the mod p reduction of $\mathbb{G}_a$ acting on U_p.

In particular, it is not at all clear a priori whether the preceding D_i define at least *non-trivial* $(\mathbf{k}_p, +)$-actions on U_p, not to mention the dense orbit property (for different D_i may become linearly dependent modulo p). The latter obstacle will be partially circumvented below by an additional argument (see **3.4**) and the former one is dealt with by the next

Lemma 2.9. *In the previous setting, after possibly replacing U with a smaller affine subset, there exists an R-scheme $\mathbb{A}^1 \times Z$ for some quasi-projective variety Z such that $U \simeq \mathbb{A}^1 \times Z$ over $\mathbf{k}$. In particular, if U is flat over R, there is a* faithful *$(\mathbf{k}_p, +)$-action on U_p, which is the mod p reduction of the $\mathbb{G}_a$-action on U. The same holds for all the groups* $\exp(tD_i)$*, $t \in \mathbf{k}$, $1 \le i \le N$.*

Proof. It follows from [9, Proposition 3.5] that there is a $\mathbf{k}$-isomorphism $U \simeq \mathbb{A}^1 \times Z$ for some quasi-projective variety Z.

More precisely, the natural projection $U \longrightarrow Z$ is actually an R-*morphism*, corresponding to the inclusion of algebras $\operatorname{Ker} D \subset R[U]$ (recall that both U and D are defined over R). Furthermore, there exists an element $g \in R[U]$ with $Dg \ne 0$ and $D^2 g = 0$, which yields $s := g/Dg \in \mathbf{k}[U]$ such that $Ds = 1$.[2)] Multiplying by $\mathbf{k}$-scalars, one may assume that $s \in R[U]$, satisfying $Ds \in R \setminus 0$. This shows that

$$\mathbb{A}^1 \times Z = \operatorname{Spec} \operatorname{Ker} D[s]$$

(via Slice Theorem) and the inclusion $\operatorname{Ker} D[s] \subseteq R[U]$ induces a $\mathbf{k}$-isomorphism $U \simeq \mathbb{A}^1 \times Z$.

Suppose now that U is flat over R. Replacing Z with another affine model one may assume the R-flatness for Z as well. Then it follows from [16, Ch. III, §10, Proposition 1] that $U = \mathbb{A}^1 \times Z$ over R. The same continues to hold for U_p also, i.e. $U_p = \mathbb{A}^1 \times Z_p$ with fiberwise $(\mathbf{k}_p, +)$-action, and lemma follows. $\square$

[2)] Here one possibly needs to replace U with a smaller affine subset. Note however that D remains nilpotent due to $D^2 g = 0$.

3. Proof of Theorem 1.2

3.1. Let us now return to the strategy outlined in **1.3**. Namely, we consider a smooth unirational quartic hypersurface $X \subset \mathbb{P}^N$, $N \gg 1$, whose mod p reduction is a cone X_0 over supersingular (quartic) K3 surface S, with defining equation of X being

$$p \cdot Q + \sum_{i=0}^{3} x_i^4 = 0,$$

say, for some generic quartic form $Q \in R[x_0, \ldots, x_N]$ (cf. Example 2.6). One easily sees via Bertini theorem that such X is smooth.

Remark 3.2. Note that X is flat over R because all of its (scheme) fibers are reduced.

Suppose that X *is* actually stably b-inf. trans. In this case, for $X_k := X \times \mathbb{P}^k$, $k \gg 1$, and various subgroups $G_i := (\mathbf{k}, +) \subset \mathbf{SAut}(U)$, $i \in I$, one defines the rational curves $C_i \subset X_k$ as Zariski closures in X_k of generic G_i-orbits on an inf. trans. open subset $U \subset X_k$ (cf. **2.8**).

Lemma 3.3. *The group $H_2(X_k, \mathbb{Q})$ is generated by the classes of C_i.*

Proof. Let C_1, C_2 be the two curves for G_1, G_2 as above, together with induced rational fibrations $f_i : X_k \dashrightarrow B_i$ having C_i as generic fibers (one for each i). Suppose the classes of C_i are proportional in $H_2(X_k, \mathbb{Q})$.

Choose some common resolution $f : W \longrightarrow X_k$ for both f_1, f_2, so that the maps $g_i := f_i \circ f : W \longrightarrow B_i$ become regular. Note that according to our assumption $f^*C_1 \equiv af^*C_2$ (numerically on W) for some $a \in \mathbb{Q}$. At the same time, we have $f^*C_i \cdot g_i^*\mathcal{O}_{B_i}(1) = 0$ by construction, for both $i \in \{1, 2\}$ and some very ample line bundles $\mathcal{O}_{B_i}(1)$. Altogether this gives $f^*C_1 \cdot g_1^*\mathcal{O}_{B_1}(1) = f^*C_1 \cdot g_2^*\mathcal{O}_{B_2}(1) = 0$, i. e. $g_1 = g_2$ (hence $f_1 = f_2$ as well), a contradiction.

Thus the classes of C_1, C_2 are not proportional in $H_2(X_k, \mathbb{Q})$. Furthermore, since X_k is a Fano manifold, we have $H_2(X_k, \mathbb{Q}) \simeq \mathrm{Pic}(X_k) \otimes \mathbb{Q} = \mathbb{Q}^2$, which implies that $H_2(X_k, \mathbb{Q}) = \mathbb{Q}^2$ by construction. In particular, the classes of C_i generate $H_2(X_k, \mathbb{Q})$, as wanted. □

3.4. Next we employ the standard technique (see [11], [1], [4], [14]) of lifting rational curves from $S = X_0 \cap (x_4 = \ldots = x_N = 0)$ to X_k.

Consider an immersion $\phi : \mathbb{P}^1 \longrightarrow S \subset X_0$ whose image $C := \phi(\mathbb{P}^1)$ is a rational curve from an arbitrary ample class on S (see e. g. [4, Proposition 17]). Regard X_k and $X_{0,k} := X_0 \times \mathbb{P}^k$ as fibers — generic and the special one, respectively, — of an algebraic family $\mathcal{X}$ over $\operatorname{Spec} R$. Let $\mathrm{pr} : X_{0,k} \longrightarrow X_0$ (resp. $\pi : X_{0,k} \dashrightarrow S$) be the natural projection (resp. the composition of pr with projection $X_0 \dashrightarrow S$ to the base of the cone). Choose also a point $o \in \mathbb{P}^k$ and denote the composition $\phi : \mathbb{P}^1 \longrightarrow S \simeq S \times o \subset X_{0,k}$ again by ϕ.

Lemma 3.5. *The images of deformations of $\phi : \mathbb{P}^1 \longrightarrow X_{0,k}$ do not form a* dominant *family of rational curves on $X_{0,k}$ (resp. are not contained in the fibers of* pr*).*

Proof. Suppose C varies in a family $\{C_t\} \subset X_{0,k}$. Then, since $\phi \circ \pi = \phi$ and C does not intersect the indeterminacy locus of π, we obtain a family $\{\pi(C_t)\}$ on S of *immersed* rational curves. Thus $\{C_t\}$ can not be dominant because S is not *separably* uniruled. Finally, since $\mathrm{pr}_* C = C \neq 0$ by construction, we get $\mathrm{pr}_* C_t \neq 0$ as well. This proves the claim. □

Recall that $C \subset X_{0,k} \setminus \operatorname{Sing} X_{0,k}$ by construction. Then Lemma 3.5 implies that the curve C deforms in the preceding family $\mathcal{X}$ to a curve $\widetilde{C} \subset X_k$ as long as

$$\begin{aligned}\chi\big(C, T_{X_{0,k}}\big|_C\big) &\geq -K_{X_{0,k}} \cdot C + \dim X + k \geq \\ &\geq \dim X_{0,k} + \dim \mathbf{PGL}(2, \mathbf{k}) = \dim X_{0,k} + 3\end{aligned}$$

(cf. [1, Theorem 15]). But the needed estimate is evident because $-K_{X_{0,k}} \cdot C \geq N-3$ and so

$$\chi\big(C, T_{X_{0,k}}\big|_C\big) \geq 2N - 4 + k \geq N + k + 2 = \dim X_{0,k} + 3.^{3)}$$

Let C_i be as in Lemma 3.3. Then for the mod p reductions $(C_i)_p \subset X_{0,k}$ we get the following:

Lemma 3.6. *There exists $j \in I$ such that the cycle* $\mathrm{pr}_*(C_j)_p \neq 0$.

Proof. Suppose the contrary. Thus we have $\mathrm{pr}_*(C_i)_p = 0$ for all i.

Let $C \subset S$ be as above. We have seen that there exists a lift $\widetilde{C}$ of C to X_k. Choose a prime $\ell \neq p$ and express the class of $\widetilde{C}$ in $H^{2(N+k-1)-2}_{\text{ét}}(X_k, \mathbb{Q}_\ell)$ via C_i (cf. Lemma 3.3). Taking the mod p reduction (a. k. a. specializing to $X_{0,k}$) we find an expression for the class of C in $H^{2(N+k-1)-2}_{\text{ét}}(X_{0,k}, \mathbb{Q}_\ell)$ via $(C_i)_p$. But then $\mathrm{pr}_* C = \mathrm{pr}_* C_i = 0$ by assumption. This contradiction concludes the proof. □

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3) Alternatively, one may set $Q \in R[x_4, \ldots, x_N]$ in the equation for X, so that *any* curve from the class $\mathcal{O}_{\mathbb{P}^N}(1)\big|_S$ deforms to X_k (this fact can again be applied to prove Lemmas 3.6 and 3.7 below). However, the presented technique with C, $\widetilde{C}$, etc. covers more general situation.

It follows from the construction (cf. **3.1**) that there exist infinitely many curves $(C_j)_p$ as in Lemma 3.6 which are not contained in the indeterminacy locus of $\pi : X_{0,k} \dashrightarrow S$. Then the following holds:

Lemma 3.7. $\dim \pi((C_j)_p) = 1$.

Proof. Suppose $\dim \pi((C_j)_p) = 0$ for all C_j. Let the notation be as in the proof of Lemma 3.6. One can express the class of $C = \mathrm{pr}(C) \subset X_0$ in $H^{2(N-1)-2}_{\text{ét}}(X_0, \mathbb{Q}_\ell)$ via $\mathrm{pr}_*(C_j)_p$. Then resolving π if necessary, we find that $\pi(C) = C$ must be a point, which is absurd. $\square$

3.8. We work over the field $\mathbf{k}_p = R/\mathfrak{m}$ in what follows. In particular, $U_p \subset X_{0,k}$ is an affine open subset, carrying a non-trivial (but possibly not inf. trans.) action of the group $\mathbf{SAut}(U_p)$ (see Lemma 2.9 and Remark 3.2). One may obviously assume that $S \cap U_p \neq \emptyset$.

Now, given any $g \in \mathbb{G}_a \subset \mathbf{SAut}(U_p)$, composing the induced map $S \simeq S \times o \dashrightarrow g_*(S \times o)$ (onto the proper birational transform of $S \times o \subset X_{0,k}$ under g) with $\pi\big|_{g_*(S\times o)} : g_*(S \times o) \dashrightarrow S$, yields a rational dominant endomorphism g_S of S (here $o \in \mathbb{P}^k$ is as in **3.4**).

Lemma 3.9. *g_S has degree > 1 for a generic choice of $\mathbb{G}_a \subset \mathbf{SAut}(U_p)$.*

Proof. Suppose the contrary. Then we have $g_S = \mathrm{id}$ because S is not ruled and hence $\mathbb{G}_a \not\subset \mathbf{Aut}(S)$.

Note that by construction (cf. **3.1**) a general $\mathbb{G}_a$-orbit on U_p coincides with one of the curves $(C_j)_p$ as in Lemma 3.6. Thus, for $g_S = \mathrm{id}$, we obtain that $\pi((C_j)_p)$ is a point. But this contradicts Lemma 3.7. $\square$

By the discussion in **2.4** one may assume that $S := \mathrm{Km}(E^2)$. Let $\psi : E^2 \dashrightarrow S$ be the natural $2:1$-map.

Lemma 3.10. *There exists a lift of g_S to a rational self-map g_E of E^2 such that $g_S \circ \psi = \psi \circ g_E$.*

Proof. Note that ψ ramifies precisely at the locus of 2-torsion points on E^2. This implies that the field extension $\mathbf{k}_p(E^2) \supset \psi^*(\mathbf{k}_p(S))$ is generated by an element θ whose minimal polynomial has coefficients only in $\mathbf{k}_p$. Then we can set g_E to act on $\mathbf{k}_p(E^2) = \psi^*(\mathbf{k}_p(S))[\theta]$ as follows:

- $g_E\big|_{\psi^*(\mathbf{k}_p(S))} = g_S$;

- $g_E(\theta) = \theta$.

$\square$

Note that g_E constructed in Lemma 3.10 is *regular* because E^2 does not contain rational curves. Further, by varying g_S in $\mathbb{G}_a$ we obtain an algebraic family of graphs of the corresponding self-maps of E^2, all isomorphic to E^2.

In particular, the corresponding family (over $\mathbb{G}_a$) of groups $H^2_{\text{ét}}(\mathbb{Q}_\ell)$ is a trivial fibration, which yields a 1-parameter family of automorphisms $(g_S)_*$ of $H^2_{\text{ét}}(E^2, \mathbb{Q}_\ell)$, mapping the lattice $\text{Pic}(E^2)$ to itself. This is only possible when all g_S are Frobenius twists (cf. Remark 2.7). Moreover, $g_S \neq \text{id}$ according to Lemma 3.9, and hence we get $\mathbf{k}_p^* \simeq \mathbb{G}_a$ as affine curves, a contradiction.

Theorem 1.2 is proved.

Remark 3.11. In general, algebraic family of rational endomorphisms of a manifold X need not lead to (again algebraic) family of automorphisms of $H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$, for the corresponding family of graphs need not admit a natural trivialization.

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