THE EYNARD–ORANTIN RECURSION FOR SIMPLE SINGULARITIES

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Abstract. According to [9] and [19], the ancestor correlators of any semi-simple cohomological field theory satisfy *local* Eynard–Orantin recursion. In this paper, we prove that for simple singularities, the local recursion can be extended to a global one. The spectral curve of the global recursion is an interesting family of Riemann surfaces defined by the invariant polynomials of the corresponding Weyl group. We also prove that for genus 0 and 1, the free energies introduced in [10] coincide up to some constant factors with respectively the genus 0 and 1 primary potentials of the simple singularity.

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1. Introduction

1.1. **Motivation.** According to [9] and [19], the ancestor correlators of any semi-simple cohomological field theory satisfy *local* Eynard–Orantin recursion. The term local refers to the fact that the spectral curve is just a disjoint union of several discs. If we are interested in computing specific ancestor Gromov–Witten (GW) invariants in terms of Givental's R-matrix, then the local recursion is all that we need. However, if we want to understand the nature of the generating function from the point of view of representations of vertex algebras (see [2]) and integrable systems (see [15]), then it is important to extend the local recursion to a global one, i.e., extend the spectral curve and the recursion kernel to global objects (see [3]). The appropriate spectral curve however, looks quite complicated in general, since

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it is parametrized by period integrals. In particular, finding whether an appropriate generalization of the global Eynard–Orantin recursion [10, 3] exists in the settings of semi-simple cohomological field theories is a very challenging and important problem.

In this paper we would like to solve the above problem for simple singularities. In this case, the spectral curve turns out to be a classical Riemann surface defined by the invariant polynomials of the monodromy group of non-maximal degree, while the invariant polynomial of maximal degree defines a branched covering of $\mathbb{P}^1$. This brunched covering was also studied by K. Saito (unfortunately he did not write a text), because it is a covering of what he called a *primitive direction* in the space of miniversal deformations of the singularity.

I think that the spectral curve for simple singularities is important also in the representation theory of the corresponding simple Lie algebras. For example, one can obtain a simple proof of the well known fact that the order of the Weyl group is the product of the degrees of the invariant polynomials (see Appendix A).

Finally, after a small modification our argument should work also for all finite reflection groups. The spectral curve is a certain family of Hurwitz covers of $\mathbb{P}^1$ parametrized by an open subset in the space of orbits of the corresponding reflection group. It would be interesting to obtain the Frobenius structure on the space of orbits of the reflection group (see [8, 22]) via the construction of a Frobenius structure on the moduli space of Hurwitz covers (see [8], Lecture 5).

1.2. **Singularity theory.** Let $f \in \mathbb{C}[x_1, x_2, x_3]$ be a weigthed-homogeneous polynomial that has an isolated critical point at 0 of ADE type. Such polynomials correspond to the ADE Dynkin diagrams and are listed in Table 1, where we have included also the Coxeter number h and the Coxeter exponents of the corresponding simple Lie algebra.

TABLE 1. Simple singularities

Type	$\boldsymbol{f(x)}$	**Exponents**	$\boldsymbol{h}$
A_N	$x_0^{N+1}+x_1^2+x_2^2$	$1,2,\dots,N$	$N+1$
D_N	$x_0^{N-1}+x_0x_1^2+x_2^2$	$1,3,\dots,2N-3,N-1$	$2N-2$
E_6	$x_0^4+x_1^3+x_2^2$	$1,4,5,7,8,11$	12
E_7	$x_0^3x_1+x_1^3+x_2^2$	$1,5,7,9,11,13,17$	18
E_8	$x_0^5+x_1^3+x_2^2$	$1,7,11,13,17,19,23,29$	30

We fix a miniversal deformation

$$F(t,x) = f(x) + \sum_{i=1}^{N} t_i v_i(x), \quad t = (t_1,\dots,t_N) \in B := \mathbb{C}^N, \tag{1}$$

where $\{v_i(x)\}_{i=1}^N$ is a set of weighted-homogeneous polynomials that represent a basis of the Jacobi algebra

$$H = \mathbb{C}[x_1, x_2, x_3]/(f_{x_1}, f_{x_2}, f_{x_3}).$$

The form $\omega := dx_1dx_2dx_3$ is primitive in the sense of K. Saito [21, 24] and the space B inherits a Frobenius structure (see [16, 23]). For some background on Frobenius structures we refer to [8]. The Frobenius multiplication on T_tB is obtained from the multiplication in the Jacobi algebra of $F(t,\cdot)$ via the Kodaira-Spencer isomorphism

$$T_tB \cong C[x_1, x_2, x_3]/(F_{x_1}(t,x), \ldots, F_{x_3}(t,x)), \quad \partial/\partial t_i \mapsto \partial F/\partial t_i.$$

While the Frobenius pairing $(\ ,\ )$ on TB is the residue pairing

$$(\phi_1(x), \phi_2(x))_t := \frac{1}{(2\pi\sqrt{-1})^3}\int_\Gamma \frac{\phi_1(x)\phi_2(x)}{F_{x_1}(t,x)\cdots F_{x_3}(t,x)}dx_1\ldots dx_N,$$

where the cycle Γ is a disjoint union of sufficiently small tori around the critical points of F defined by equations of the type $|F_{x_1}| = \cdots = |F_{x_3}| = \epsilon$. In particular, we have the following identifications:

$$T^*B \cong TB \cong B \times T_0B \cong B \times H,$$

where the first isomorphism is given by the residue pairing, the second by the Levi–Civita connection of the flat residue pairing, and the last one is the Kodaira–Spencer isomorphism

$$T_0B \cong H, \quad \partial/\partial t_i \mapsto \partial_{t_i}F\Big|_{t=0} \text{ mod } (f_{x_1}, \ldots, f_{x_3}). \tag{2}$$

Let $B_{ss} \subset B$ be the subset of semi-simple points, i.e., points $t \in B$ such that the critical values of $F(t,\cdot)$ form a coordinate system in a neighborhood of t. For every $t \in B_{ss}$, using Givental's higher-genus reconstruction formalism [12, 13], we define ancestor correlation functions of the following form (c.f. [19])

$$\langle a_1\psi_1^{k_1}, \ldots, a_n\psi_n^{k_n}\rangle_{g,n}(t), \quad a_i \in H, \quad k_i \in \mathbb{Z}_{\geq 0}(1 \leq i \leq n). \tag{3}$$

A priory, each correlator depends analytically on $t \in B_{ss}$, but it might have poles along the divisor $B \setminus B_{ss}$. According to [20] the correlation functions (3) extend analytically to the entire domain B.

1.3. **The period vectors.** Put $X = B \times \mathbb{C}^3$ and $S = B \times \mathbb{C}$. Let $\Sigma \subset S$ be the *discriminant* of the map

$$\varphi : X \to S, \quad \varphi(t,x) := (t, F(t,x)).$$

Removing the singular fibers $X' = X \setminus \varphi^{-1}(\Sigma)$ we obtain a smooth fibration $X' \to S'$, where $S' = S \setminus \Sigma$, known as the Milnor fibration. Let us denote by $X_{t,\lambda} = \varphi^{-1}(t,\lambda)$ the fiber over $(t,\lambda) \in S'$. The vector spaces $H^2(X_{t,\lambda};\mathbb{C})$ and $H_2(X_{t,\lambda},\mathbb{C})$ form the so called vanishing cohomology and homology bundles. They are equipped with flat Gauss–Manin connections.

We fix $(0,1)\in S$ as a reference point and denote by $\mathfrak{h} := H^2(X_{0,1},\mathbb{C})$. The dual space $\mathfrak{h}^* = H_2(X_{0,1},\mathbb{C})$ is equipped with a non-degenerate intersection pairing and we denote by $(\ |\ )$ the negative of the intersection pairing, so that $(\alpha|\alpha)=2$ for every vanishing cycle α. The set R of all vanishing cycles together with the pairing $(\ |\ )$ is a root system of type ADE. Moreover, according to the Picard–Lefschetz theory (see [1]) the image of the monodromy representation

$$\pi_1(S')\to \mathrm{GL}(\mathfrak{h}^*) \tag{4}$$

is the Weyl group of R, i.e., the monodromy transformation s_α along a simple loop around the discriminant corresponding to a path along which the cycle α vanishes is the following reflection

$$s_\alpha(x) = x-(\alpha|x)\alpha,\quad \alpha\in R,\quad x\in\mathfrak{h}^*.$$

Let us introduce the notation d_x, where $x=(x_1,\dots,x_m)$ is a coordinate system on some manifold, for the de Rham differential in the coordinates x. This notation is especially useful when we have to apply d_x to functions that might depend on other variables as well. The main object of our interest are the following period integrals

$$I^{(k)}_\alpha(t,\lambda) = -d_t\,(2\pi)^{-1}\,\partial_\lambda^{k+1}\int_{\alpha_{t,\lambda}} d_x^{-1}\omega\ \in T_t^*B\cong H, \tag{5}$$

where $\alpha\in\mathfrak{h}$ is a cycle from the vanishing homology, $\alpha_{t,\lambda}\in H_2(X_{t,\lambda},\mathbb{C})$ is the parallel transport of α along a reference path, and $d_x^{-1}\omega$ is any 2-form $\eta\in\Omega^2_{\mathbb{C}^3}$ such that $d_x\eta=\omega$. The periods are multivalued analytic functions in $(t,\lambda)\in B\times\mathbb{C}$ with poles along the discriminant Σ.

1.4. **The period isomorphism.** Let us fix a coordinate system $t=(t_1,\dots,t_N)$ on B defined by a miniversal unfolding of f of the type (1). We may assume that $v_N(x)=1$ and denote by $t-\lambda\mathbf{1}$ the point with coordinates $(t_1,t_2,\dots,t_N-\lambda)$. Note that $X_{t,\lambda}=X_{t-\lambda\mathbf{1},0}$, so the period vectors have the following translation symmetry

$$I^{(k)}_\alpha(t,\lambda) = I^{(k)}_\alpha(t-\lambda\mathbf{1},0). \tag{6}$$

Sometimes we restrict the period integrals to $\lambda=0$ and it will be convenient to use as a reference point $-\mathbf{1}\in B$. Note that this choice is compatible with the choice of the other reference point $(0,1)\in B\times\mathbb{C}$ in a sense that the values of the period vectors at these two points are identified via the translation symmetry (6).

Now we can state the following result that goes back to Looijenga [18] and Saito [21]. The monodromy covering space of $B' := S'\cap B$ is the covering $\widetilde{B'}$ of B' corresponding to the kernel of the monodromy representation (4). It can be constructed as the set of equivalence classes of pairs (t,C), where $t\in B'$ and C is a path in B' from the reference point $-\mathbf{1}$ to t and the equivalence relation $(t_1,C_1)\sim(t_2,C_2)$ is $t_1=t_2$ and $C_1\circ C_2^{-1}$ is in the kernel of the monodromy representation (4). Note that the period integrals are by definition functions on $\widetilde{B'}$. In particular we have a well defined *period map*

$$\widetilde{\Phi} : \widetilde{B'} \to \mathfrak{h}', \quad \langle \widetilde{\Phi}(C,t), \alpha \rangle := (I^{(-1)}_{\alpha}(t,0), 1),$$

where $\mathfrak{h}'$ is the complement in $\mathfrak{h}$ of the reflection hyperplanes of the roots R, i.e., $\mathfrak{h}' = \{x \in \mathfrak{h} \mid \langle \alpha, x\rangle \neq 0 \ \forall \alpha \in R\}$. The first statement is that $\widetilde{\Phi}$ is an analytic isomorphism. In particular, there is an induced isomorphism $\Phi : B' \to \mathfrak{h}'/W$. The 2nd statement is that Φ extends analytically across the discriminant and the extension provides an analytic isomorphism $B \cong \mathfrak{h}/W := \mathrm{Spec}(S(\mathfrak{h}^*)^W)$. Using the isomorphism $\widetilde{\Phi}$ and the natural projection $\widetilde{B'} \to B'$ we can think of the coordinates t_i as W-invariant holomorphic functions on $\mathfrak{h}'$. The 2nd statement is equivalent to saying that each coordinate t_i extends holomorphically through the reflection mirrors, the extension is in fact a W-invariant polynomial in $\mathfrak{h}$, and the ring of all W-invariant polynomials is $S(\mathfrak{h}^*)^W = \mathbb{C}[t_1, \dots, t_N]$. We refer to [18, 21] for the proof of all these statements.

1.5. **The spectral curve.** Let us fix a set of simple roots $\{\alpha_i\}_{i=1}^N \subset \mathfrak{h}^*$ and denote by $x = (x_1, \dots, x_N)$ the coordinate system in $\mathfrak{h}$ corresponding to the basis of fundamental weights $\{\omega_i\}_{i=1}^N \subset \mathfrak{h}$, i.e.,

$$x = \sum_{i=1}^N x_i \omega_i, \quad x_i = \langle \alpha_i, x\rangle.$$

As explained above $t_i \in \mathbb{C}[x_1, \dots, x_N]^W$ are invariant polynomials and since the period mapping is weighted-homogeneous, t_i are homogeneous polynomials of certain degrees d_i. Let us assume that the degrees are in an increasing order, then the numbers $1 = d_1 - 1 \leq d_2 - 1 \leq \dots d_N - 1 =: h - 1$ are known as the Coxeter exponents (see Table 1). Given $s \in \mathbb{C}^{N-1}$ we define the algebraic curve $V_s \subset \mathbb{P}^N$

$$t_i(X_1, \dots, X_N) = s_i X_0^{d_i}, \quad 1 \leq i \leq N-1.$$

As we will see later on if $s \in B_{ss}$, then V_s is non-singular. In fact, the points s for which V_s has singularities are precisely the caustic $B - B_{ss}$. I am not aware if the family of algebraic curves V_s, $s \in B_{ss}$ has an official name attached, but since it will be the spectral curve for the EO recursion, we will refer to it as the *spectral curve of the singularity* or just the *spectral curve* when the singularity is understood from the context.

There is a natural projection

$$\lambda : V_s \to \mathbb{P}^1, \quad [X_0, X_1, \dots, X_N] \mapsto [X_0^h, t_N(X_1, \dots, X_N)], \tag{7}$$

which is a branched covering of degree $|W|$, where $|A|$ denotes the number of elements of the set A. The branching points are $\lambda = u_1, \dots, u_N, \infty$, where u_i are the critical values of $F(s,x)$. By definition, the period integral $(I^{(-1)}(s,\lambda), 1)$ defines locally near a non-branching point $\lambda \in \mathbb{P}^1$ a section of the branched covering (7). It follows that the set of ramification points

$$\lambda^{-1}(u_i), \quad 1 \leq i \leq N$$

is precisely the intersections of V_s and the reflection mirrors

$$\langle \alpha, X \rangle = \sum_{i=1}^{N} \langle \alpha, \omega_i \rangle X_i = 0, \quad \alpha \in R_+,$$

where R_+ is the set of positive roots. The remaining ramification points are $\lambda^{-1}(\infty)$. They correspond to eigenvectors of the Coxeter transformations with eigenvalue $\eta := e^{2\pi\sqrt{-1}/h}$:

$$[X_0, X_1, \ldots, X_N] \in \lambda^{-1}(\infty)$$

if and only if $X_0 = 0$ and $\sum_{i=1}^{N} X_i \omega_i \in \mathfrak{h}$ is an eigenvector with eigenvalue η for a Coxeter transformation. It is easy to see that the ramification index of any point in $\lambda^{-1}(u_i)$ is 2, while the ramification index of a point in $\lambda^{-1}(\infty)$ is h.

1.6. **The Eynard–Orantin recursion.** We make use of the following formal series

$$\mathbf{f}^{\alpha}(t, \lambda; z) = \sum_{k \in \mathbb{Z}} I^{(k)}_{\alpha}(t, \lambda)\,(-z)^k, \quad \phi^{\alpha}(t, \lambda; z) = \sum_{k \in \mathbb{Z}} I^{(k+1)}_{\alpha}(t, \lambda)(-z)^k\, d\lambda.$$

Note that $\phi^{\alpha}(t, \lambda; z) = d_{\lambda}\mathbf{f}^{\alpha}(t, \lambda; z)$. Given n cycles $\alpha_1, \ldots, \alpha_n$ and a semi-simple point $s \in B_{ss}$ we define the following n-point symmetric forms

$$\omega^{\alpha_1,\ldots,\alpha_n}_{g,n}(s; \lambda_1, \ldots, \lambda_n) = \Big\langle \phi^{\alpha_1}_{+}(s, \lambda_1; \psi_1), \ldots, \phi^{\alpha_n}_{+}(s, \lambda_n; \psi_n) \Big\rangle_{g,n}(s), \tag{8}$$

where the + means truncation of the terms in the series with negative powers of z. The functions (8) will be called *n-point series* of genus g or simply *correlator forms*. The ancestor correlators (3) are known to be *tame* (see [14]), which by definition means that they vanish if $k_1 + \cdots + k_n > 3g - 3 + n$. Hence the correlator (8) is a polynomial expression of the components of the period vectors (5). Thanks to the translation symmetry, we may assume that $s_N = 0$, then (8) is a meromorphic function on the spectral curve $V_s \times \cdots \times V_s$ with possible poles at the ramification points of the covering (7).

Let us fix $s = (s_1, \ldots, s_{N-1}) \in \mathbb{C}^{N-1}$ and denote by $\gamma \in \mathfrak{h}^*$ an arbitrary cycle, s.t., $(\gamma|\alpha) \neq 0$ for all $\alpha \in R$. We define a set of symmetric meromorphic differentials on V_s^n with poles along the ramification points of V_s

$$\omega_{g,n}(s; p_1, \ldots, p_n) := \omega^{\gamma,\ldots,\gamma}_{g,n}(s; \lambda_1, \ldots, \lambda_n), \tag{9}$$

where the RHS is defined by fixing a reference path for each $(s, \lambda_i) \in S'$, s.t., $p_i = (I^{(-1)}(s, \lambda_i), 1)$. Our main result can be stated as follows.

Theorem 1.1. *If $s \in B_{ss}$, then the forms $\omega_{g,n}$, $2g-2+n>0$, satisfy the Eynard–Orantin recursion associated with the branched covering* (7) *and the meromorphic function $f_\gamma : V_s \to \mathbb{P}^1$, $f_\gamma(x) := \langle \gamma, x\rangle$.*

The recursion will be recalled later on (see Section 3.3). We would like however to emphasize that the Eynard–Orantin recursion in our case differs from the standard one by the initial condition:

$$\omega_{0,2}(x,y) = \sum_{w\in W} (\gamma|w\gamma) B(x,wy), \tag{10}$$

where $B(x,y)$ is the Bergman kernel of V_s and for $x = wy$ one has to regularize the RHS by removing an appropriate singular term (see Section 3.3). Let us point out that while the Bergman kernel depends on the choice of a Torelli marking of V_s, i.e., a symplectic basis of $H_1(V_s;\mathbb{Z})$, our initial condition is independent of the Torelli marking (see Corollary 2.5). This fact could be proved also directly by using the explicit formula (24) for the holomorphic 1-forms and some standard facts for W-invariant polynomials. Finally, let us point out that the set of correlators (9) determines the set (8), because by definition

$$\omega_{g,n}^{w_1\gamma,\dots,w_N\gamma}(s;\lambda_1,\dots,\lambda_N) = \omega_{g,n}(s; w_1^{-1}p_1,\dots,w_N^{-1}p_N),$$

where $w_1,\dots,w_N \in W$ are arbitrary.

The branched covering (7) and the meromorphic function f_γ determine a birational model of V_s in $\mathbb{C}^2$. Following [10] we can introduce the tau-function $Z(\hbar,s) := Z_\hbar(V_s,\lambda,f_\gamma)$ of the birational model of the spectral curve. It has the form

$$Z(\hbar,s) = \exp\Big(\sum_{g=0}^{\infty} \hbar^{g-1}\underline{F}^{(g)}(s)\Big),$$

where $\underline{F}^{(g)}(s)$ is called the genus-g free energy of V_s. It is very natural to compare $\underline{F}^{(g)}(s)$ with Givental's primary genus-g potentials $F^{(g)}(s)$. Unfortunately we could not solve this problem in general, but only for $g=0$ and $g=1$

$$\underline{F}^{(0)}(s) = -(\gamma|\gamma)\frac{|W|}{N}F^{(0)}(s), \quad \underline{F}^{(1)}(s) = \frac{|W|}{2}F^{(1)}(s) = 0,$$

where the first identity is valid up to quadratic terms in the Frobenius flat coordinates of $s\in B$, while the second one up to a constant independent of s. It is known that the genus-1 potential of the Frobenius structure is homogeneous of degree 0, so it must vanish in the case of a simple singularity. According to [10], $Z(\hbar,s)$ satisfies Hirota bilinear equations. According to [15], the total ancestor potential $\mathcal{A}_s(\hbar;\mathbf{q})$ of an ADE singularity satisfies the Hirota bilinear equations of the corresponding generalized KdV hierarchy. It will be interesting to clarify the relation between $\underline{F}^{(g)}(s)$ and $F^{(g)}(s)$ for $g\geq 2$, as well as to determine whether Givental's *primary* ancestor potential $\mathcal{A}_s(\hbar;0)$ also satisfies Hirota bi-linear equations.

2. Analytic extension of the kernel of the local Eynard–Orantin recursion

It was proved in [19] (see also [9]) that the correlator forms (8) satisfy a local Eynard–Orantin (EO) recursion, whose kernel is defined by the symplectic pairing of certain period series. In this section, we will prove that these symplectic pairings are convergent and can be extended to the entire spectral curve V_s. Moreover, the corresponding extensions can be expressed in an elegant way via the so called *Bergman kernel* of V_s.

2.1. **The kernel of the local recursion.** Recall the symplectic pairing

$$\Omega(f(z), g(z)) = \mathrm{Res}_{z=0}(f(-z), g(z))dz, \quad f, g \in H(\!(z^{-1})\!).$$

The local recursion is defined in terms of the symplectic pairings

$$\Omega(\phi_+^\alpha(s,\lambda;z), \mathbf{f}_-^\beta(s,\mu;z)) = d\lambda \sum_{k=0}^\infty (-1)^{k+1}(I_\alpha^{(k+1)}(s,\lambda), I_\beta^{(-k-1)}(s,\mu)),$$

where the infinite series is interpreted formally in a neighborhood of a point $\mu = u_i(s)$, s.t., the cycle $\beta_{s,\mu}$ vanishes over $\mu = u_i(s)$. We are going to prove that this infinite series expansion is convergent to a meromorphic function on $V_s \times V_s$.

To begin with, let us recall that the periods satisfy the following system of differential equations

$$\begin{aligned} \partial_a I^{(n)}(s,\lambda) &= -v_a \bullet_s I^{(n+1)}(s,\lambda) \\ \partial_\lambda I^{(n)}(s,\lambda) &= I^{(n+1)}(s,\lambda) \\ (\lambda - E\bullet_s)\partial_\lambda I^{(n)}(s,\lambda) &= \Big(\theta - n - \frac{1}{2}\Big) I^{(n)}(s,\lambda), \end{aligned} \tag{11}$$

where $\partial_a := \partial/\partial s_a$, E is the Euler vector field $E = \sum_{i=1}^N \deg(t_i) t_i \partial_{t_i}$, and θ is the Hodge-grading operator

$$\theta : H \to H, \quad v_a \mapsto (D/2 - \deg(v_a))v_a,$$

where $D = \deg(\mathrm{Hess}(f)) = 1 - 2/h$ is the conformal dimension of the Frobenius structure. The key to proving the convergence is the so called *phase 1-form* (see [14, 2])

$$\mathcal{W}_{\alpha,\beta}(s,\xi) = I_\alpha^{(0)}(s,\xi) \bullet I_\beta^{(0)}(s,0) \quad \in \quad T_s^*S,$$

where the period vectors are interpreted as elements in T_s^*S and the multiplication in T_s^*S is induced by the Frobenius multiplication via the natural identification $T_s^*S \cong T_sS$. The dependence on the parameter ξ is in the sense of a germ at $\xi = 0$, i.e., Taylor's series expansion about $\xi = 0$. The phase form is a power series in ξ whose coefficients are multivalued 1-forms on B'.

Lemma 2.1. *We have*

$$(\alpha|\beta) = -\iota_E \mathcal{W}_{\alpha,\beta}(s,0) = -(I^{(0)}_\alpha(s,0), E\bullet I^{(0)}_\beta(s,0)).$$

This is a well known fact due originally to K. Saito [21].

Lemma 2.2. *The phase form is weighted-homogeneous of weight* 0*, i.e.,*

$$(\xi\partial_\xi + L_E)\mathcal{W}_{\alpha,\beta}(s,\xi) = 0,$$

where L_E is the Lie derivative with respect to the vector field E.

Proof. Note that

$$\mathcal{W}_{\alpha,\beta}(s,\xi) = (I^{(0)}_\alpha(s,\xi), dI^{(-1)}_\beta(s,0)).$$

It is easy to check that $\mathcal{W}_{\alpha,\beta}$ is a closed 1-form, so using the Cartan's magic formula $L_E = d_s\iota_E + \iota_E d_s$, where ι_E is the contraction by the vector field E, we get

$$L_E\mathcal{W}_{\alpha,\beta} = d_s(I^{(0)}_\alpha(s,\xi),(\theta+1/2)I^{(-1)}_\beta(s,0)) = -d_s((\theta-1/2)I^{(0)}_\alpha(s,\xi), I^{(-1)}_\beta(s,0)).$$

We used that θ is skew-symmetric with respect to the residue pairing and that

$$\iota_E d_s I^{(-1)}_\beta(s,0) = EI^{(-1)}_\beta(s,0)) = (\theta+1/2)I^{(-1)}_\beta(s,0),$$

where the last equality comes from the differential equation (11) with $n=-1$ and $\lambda = 0$. Furthermore, using the Leibnitz rule we get

$$-((\theta-1/2)d_sI^{(0)}_\alpha(s,\xi), I^{(-1)}_\beta(s,0)) - ((\theta-1/2)I^{(0)}_\alpha(s,\xi), d_sI^{(-1)}_\beta(s,0)).$$

The first residue pairing is

$$(AI^{(1)}_\alpha(s,\xi),(\theta+1/2)I^{(-1)}_\beta(s,0)) = -(AI^{(1)}_\alpha(s,\xi), E\bullet I^{(0)}_\beta(s,0)), \tag{12}$$

where we used that θ is skew-symmetric and that $d_sI^{(0)}_\alpha = -AI^{(1)}_\alpha$ with $A = \sum_{a=1}^N(\partial/\partial s_a\bullet)ds_a$. Similarly, the 2nd residue pairing becomes

$$((\xi\partial_\xi + E)I^{(0)}_\alpha(s,\xi), d_sI^{(-1)}_\beta(s,0)) = \xi\partial_\xi\mathcal{W}_{\alpha,\beta}(s,\xi) + (E\bullet I^{(1)}_\alpha(s,\xi), AI^{(0)}_\beta(s,0)). \tag{13}$$

On the other hand, since the Frobenius multiplication is commutative, $[A, E\bullet] = 0$, so the terms (12) and (13) add up to $\xi\partial_\xi\mathcal{W}_{\alpha,\beta}(s,\xi)$, which completes the proof. □

Let us define the following meromorphic 1-forms on $V_s\times V_s$. Given $(x,y)\in V_s\times V_s$, s.t., x, y are not ramification points, there are unique pairs (C,λ) and (C',μ), s.t., $x = \widetilde{\Phi}(C, s-\lambda\mathbf{1})$ and $y = \widetilde{\Phi}(C', s-\mu\mathbf{1})$, where C and C' are paths in B' connecting respectively $s-\lambda\mathbf{1}$ and $s-\mu\mathbf{1}$ with the reference point. Put

$$K_{\alpha,\beta}(s,x,y) := \frac{d\lambda}{\lambda-\mu}\,(I^{(0)}_\alpha(s,\lambda),(\theta+1/2)I^{(-1)}_\beta(s,\mu)),$$

where the branches of $I^{(0)}_\alpha$ and $I^{(-1)}_\beta$ are determined respectively by the paths C and C'. This definition extends analytically across the ramification points and the form has a pole only along the divisor

$$\{(x,y)\in V_s\times V_s\ :\ \lambda(x)=\lambda(y)\} = \cup_{w\in W}\{x=wy\},$$

where the action of W on $\mathfrak{h}$ is induced from the action on $\mathfrak{h}^*$, i.e.,

$$\langle \alpha, wx\rangle = \langle w^{-1}\alpha, x\rangle,\quad \alpha\in\mathfrak{h}^*,\quad x\in\mathfrak{h}.$$

Proposition 2.3. *The symplectic pairing*

$$\Omega(\phi^\alpha_+(s,\lambda;z),\mathbf{f}^\beta_-(s,\mu;z)) = K_{\alpha,\beta}(s,x,y),$$

where $x=\widetilde{\Phi}(C, s-\lambda\mathbf{1})$*,* $y=\widetilde{\Phi}(C,s-\mu\mathbf{1})$ *and* C *is the path that specifies the value of the symplectic pairing.*

Proof. Using the differential equations for the periods, it is easy to verify that

$$d_s\Omega(\phi^\alpha_+(s,\lambda;z),\mathbf{f}^\beta_-(s,\mu;z)) = d\lambda I^{(1)}_\alpha(s,\lambda)\bullet_s I^{(0)}_\beta(s,\mu) = d_\lambda\mathcal{W}_{\alpha,\beta}(s-\mu\mathbf{1},\lambda-\mu).$$

According to Lemma 2.2 we have

$$\partial_\lambda\mathcal{W}_{\alpha,\beta}(s',\lambda-\mu) = -d_{s'}\Big(\frac{1}{\lambda-\mu}\,\iota_E\mathcal{W}_{\alpha,\beta}(s',\lambda-\mu)\Big), \tag{14}$$

which by definition is

$$d_{s'}\Big(\frac{1}{\lambda-\mu}\,(I^{(0)}_\alpha(s',\lambda-\mu),(\theta+1/2)I^{(-1)}_\beta(s',0)\Big).$$

Integrating (14) with respect to s' along a short path from $s_0:=s-u_i(s)\mathbf{1}$ to $s-\mu\mathbf{1}$ and using that $I^{(-1)}_\beta(s',0)$ vanishes as $s'\to s_0$, we get

$$\Omega(\phi^\alpha_+(s,\lambda;z),\mathbf{f}^\beta_-(s,\mu;z)) = \frac{d\lambda}{\lambda-\mu}\,(I^{(0)}_\alpha(s,\lambda),(\theta+1/2)I^{(-1)}_\beta(s,\mu)).\quad\square \tag{15}$$

2.2. **The local kernel and the Bergman kernel.** Now we are in a position to prove the key result in this paper. Let us fix a symplectic basis $\{\mathcal{A}_i,\mathcal{B}_i\}_{i=1}^g$ of $H_1(V_s;\mathbb{Z})$, s.t., $\mathcal{A}_i\circ\mathcal{B}_| = \delta_{i,j}$. There is a unique symmetric differential $B(x,y)\in\Omega^1_{V_s}\boxtimes\Omega^1_{V_s}(2\Delta)$ which is holomorphic on $V_s\times V_s$ except for a pole of order 2 with no residue along the diagonal $\Delta\subset V_s\times V_s$, normalized by

$$\oint_{y\in\mathcal{A}_i} B(x,y)=0,\quad 1\le i\le N$$

and

$$B(x,y) = \frac{d\lambda(x)d\lambda(y)}{(\lambda(x)-\lambda(y))^2}+\cdots$$

for any local coordinate $\lambda: U\to\mathbb{C}$ $(U\subset V_s)$ and for all $x,y\in U\times U$. The differential $B(x,y)$ is called the *Bergman kernel*. We refer to [10] for more details and references.

Proposition 2.4. *The following identity holds*

$$d_y K_{\alpha,\beta}(s,x,y) = \sum_{w\in W} (\alpha|w\beta) B(x,wy), \quad \forall \alpha,\beta \in \mathfrak{h}^*.$$

Proof. Put

$$x = \widetilde{\Phi}(C, s-\lambda\mathbf{1}), \quad y = \widetilde{\Phi}(C', s-\mu\mathbf{1}).$$

By definition $wy = \widetilde{\Phi}(C'\circ w^{-1}, s-\mu\mathbf{1})$. Therefore, if x and wy are near by, then w must be the monodromy along the loop $C^{-1}\circ C'$. Using Saito's formula (2.1) we get that the leading order term of $K_{\alpha,\beta}(s,x,y)$ near the w-diagonal $x = wy$ is

$$\frac{d\lambda}{(\lambda-\mu)}\,(I^{(0)}_{\alpha}(s,\mu), (\mu - E\bullet) I^{(0)}_{w\beta}(s,\mu)) = (\alpha|w\beta)\frac{d\lambda}{(\lambda-\mu)},$$

where we used that

$$I^{(0)}_{\beta_{C'}}(s,\mu) = I^{(0)}_{(w\beta)_C}(s,\mu),$$

where the index in the cycle denotes the path along which the cycle has to be transported in order to define the period.

We get that the difference of the two sides of the identity that we want prove is a holomorphic symmetric 2-form $D(x,y)$ on $V_s\times V_s$. To prove that such a form vanishes it is enough to prove that

$$\oint_{x\in\mathcal{A}_i} D(x,y) = 0, \quad \forall i = 1,2,\dots,g.$$

This is true for the Bergman kernel by definition, while for $d_y K_{\alpha,\beta}(s,x,y)$, since it is an exact form, the corresponding integral vanishes for all cycles $\mathcal{A}\in H_1(V_s,\mathbb{Z})$ not only $\mathcal{A}_i$. □

Corollary 2.5. *The 2-form* (10) *is independent of the choice of Torelli marking.*

3. From local to global

In this section we prove Theorem 1.1.

3.1. The unstable range.

By definition, the ancestor potential does not have non-zero correlators in the unstable range $(g,n) = (0,0), (0,1), (0,2)$ and $(1,0)$. However, in order to formulate the EO recursion, it is convenient to extend the definition of the correlators in the unstable range as well in the following two cases:

$$\omega^{\alpha_1,\alpha_2}_{0,2}(s;\lambda_1,\lambda_2) := \Omega(\phi^{\alpha_1}_+(s,\lambda_1;z), \phi^{\alpha_2}_+(s,\lambda_2;z)_-), \tag{16}$$

$$\omega^{\alpha_1,\alpha_2}_{0,2}(s;\lambda,\lambda) := P^{(0)}_{\alpha_1,\alpha_2}(s,\lambda), \tag{17}$$

where $P^{(0)}_{\alpha_1,\alpha_2}(s,\lambda)$ is defined as the limit $\mu\to\lambda$ of

$$\Omega(\phi^{\alpha_1}_+(s,\lambda;z), \phi^{\alpha_2}_+(s,\mu;z)_-) - (\alpha_1|\alpha_2)\frac{d\lambda d\mu}{(\lambda-\mu)^2}.$$

The limit exists, because the above difference is analytic near $\mu = \lambda$ (see [19] for more details). Let us point out that in the definition of the correlator form (17) we assume that there is a fixed path C from the reference point to $s - \lambda\mathbf{1}$. It is more natural however to assume that there are two such paths C_1 and C_2: one for the 1st and one for the 2nd slot of the correlator form. Since

$$P_{C_2}\alpha_2 = P_{C_1}(P_{C_1^{-1}\circ C_2})\alpha_2,$$

where $P_{C_i} : H_2(X_{-\mathbf{1},0};\mathbb{C}) \to H_2(X_{s-\lambda\mathbf{1},0};\mathbb{C})$ is the parallel transport with respect to the Gauss–Manin connection, we get that if we want to allow two different paths in the definition (17) and still have compatibility with the monodromy representation, then we should define

$$\omega_{0,2}^{\alpha_1,\alpha_2}(s;\lambda,\lambda) := P^{(0)}_{\alpha_1,w\alpha_2}(s,\lambda),$$

where $w = P_{C_1^{-1}\circ C_2}$ and the branch on the RHS is determined by C_1.

3.2. **The local EO recursion.** According to [19], the ancestor correlators satisfy the following recursion

$$\omega_{g,n+1}^{\alpha_0,\alpha_1,\dots,\alpha_n}(s;\lambda_0,\lambda_1,\dots,\lambda_n) = -\frac{1}{4}\sum_{j=1}^N \mathrm{Res}_{\lambda=u_j}\frac{\Omega(\phi_+^{\alpha_0}(s,\lambda_0;z)\,,\mathbf{f}_-^{\beta_j}(s,\lambda;z))}{(I^{(-1)}_{\beta_j}(s,\lambda),\mathbf{1})d\lambda}\times$$
$$\left(\omega_{g-1,n+2}^{\beta_j,-\beta_j,\alpha_1,\dots,\alpha_n}(s;\lambda,\lambda,\lambda_1,\dots,\lambda_n) + \sum_{\substack{g'+g''=g\\ I'\subset\{1,\dots,n\}}} \omega_{g',n'+1}^{\beta_j,\alpha_{I'}}(s;\lambda,\lambda_{I'})\,\omega_{g'',n''+1}^{-\beta_j,\alpha_{I''}}(s;\lambda,\lambda_{I''})\right), \tag{18}$$

for all stable pairs $(g, n+1)$, i.e., $2g-2+n \geq 0$, where the notation is as follows. All unstable correlators on the RHS are set to 0, except for the ones of the type (16) and (17). The summation is over all subsets $I' \subseteq \{1,2,\dots,n\}$ and for each subset $I' = \{i_1 < \cdots < i_{n'}\}$ we put

$$I'' = \{1,2,\dots,n\} - I' =: \{j_1 < \cdots < j_{n''}\}.$$

In particular, $n' = |I'|$ and $n'' = |I''|$. If $x = (x_1,\dots,x_n)$ is a sequence of n elements, then we define

$$x_{I'} = (x_{i_1},\dots,x_{i_{n'}}),\quad x_{I''} = (x_{j_1},\dots,x_{j_{n''}}).$$

Finally, β_j ($1 \leq j \leq N$) is a vanishing cycle vanishing over $\lambda = u_j$.

3.3. **The global EO recursion.** Let us write down the recursion from Theorem 1.1. Let us denote by

$$\{y_{j,a} \ :\ 1 \leq a \leq |W|/2\,\} := \lambda^{-1}(u_j),\quad 1 \leq j \leq N,$$

the ramification points on V_s with ramification index 2. There is a unique root $\beta_{j,a} \in R_+$, s.t., $\langle \beta_{j,a}, y_{j,a}\rangle = 0$ and the reflection $s_{\beta_{j,a}}$ induces a deck transformation

$\theta_{j,a} : V_s \to V_s$ which is a generator for the Galois group of a neighborhood of $y_{j,a}$ viewed as a 2-sheeted covering of a neighborhood of u_j.

$$\omega_{g,n+1}(s; x_0, x_1, \ldots, x_n) = \sum_{j=1}^{N} \sum_{a=1}^{|W|/2} \mathrm{Res}_{y=y_{j,a}} \frac{\frac{1}{2}\int_y^{\theta_{j,a}(y)} B(x_0, y)}{(f_\gamma(y) - f_\gamma(\theta_{j,a}(y)))d\lambda(y)}$$
$$\Big(\omega_{g-1,n+1}(s; y, \theta_{j,a}(y), x_1, \ldots, x_n) + \sum_{\substack{g'+g''=g \\ I'\subset\{1,\ldots,n\}}} \omega_{g',n'+1}(s; y, x_{I'})\, \omega_{g'',n''+1}(s; \theta_{j,a}(y), x_{I''})\Big),$$

where the summation is the same as in the local recursion (see Section 3.2). Let us also point out that in the above recursion all unstable correlators are set to 0, except for

$$\omega_{0,2}(x_1, x_2) = \omega_{0,2}^{\gamma,\gamma}(s; \lambda_1, \lambda_2), \quad x_1, x_2 \in V_s,$$

where in order to define the RHS we choose $\lambda_i = \lambda(x_i)$ and paths C_i in B' from $-\mathbf{1}$ to $s - \lambda_i\mathbf{1}$, s.t., $\widetilde{\Phi}(C_i, s - \lambda_i\mathbf{1}) = x_i$. Using Proposition 2.4 we can express the form $\omega_{0,2}(x, y)$ in terms of the Bergman kernel. Namely, if $\lambda(x) \neq \lambda(y)$, then $\omega_{0,2}(x, y)$ is given by formula (10). If $\lambda(x) = \lambda(y)$, then $x = w_0 y$ for some $w_0 \in W$ and we have

$$\omega_{0,2}(x, y) = \lim_{y'\to y} \left(\sum_{w\in W} (\gamma|w\gamma) B(x, wy') - (\gamma|w_0\gamma) \frac{d\lambda(x) d\lambda(y')}{(\lambda(x) - \lambda(y'))^2} \right),$$

where y' is sufficiently close to y. Note that the set of poles of $\omega_{0,2}(x, y)$ is the following set of points in $V_s \times V_s$:

$$\bigcup_{j=1}^{N} \lambda^{-1}(u_j) \times \lambda^{-1}(u_j).$$

3.4. **Proof of Theorem 1.1.** We are going to prove that the global recursion reduces to the local one. To begin with let us simplify the kernel of the global recursion. Put

$$S_{y_1,y_2}(x) = \int_{y_2}^{y_1} B(x, y'), \quad x, y_1, y_2 \in V_s.$$

This is the unique form on V_s with vanishing $\mathcal{A}_i$-periods, with poles of order 1 at y_1 and y_2 with residues respectively $+1$ and -1. The kernel of the local recursion has the following symmetry

$$K_{\alpha,\beta}(s, x, wy) = K_{\alpha,w^{-1}\beta}(s, x, y).$$

In particular, using this symmetry when $w = \theta_{j,a}$ and Proposition 2.4 we get

$$K_{\alpha,\beta_{j,a}}(s, x, y) = -\frac{1}{2} \sum_{w\in W} (\alpha|w\beta_{j,a}) S_{w\theta_{j,a}(y),wy}(x). \tag{19}$$

Furthermore, we have

$$f_\gamma(y)-f_\gamma(\theta_{j,a}(y))=\langle\gamma,y-s_{\beta_{j,a}}(y)\rangle=\langle\beta_{j,a},y\rangle\,(\gamma|\beta_{j,a}).$$

For fixed j, let us fix the local coordinate $y_a:=\widetilde{\Phi}(C_a,s-\lambda\mathbf{1})$ near the ramification point $y_{j,a}$ (C_a is a path along which $\beta_{j,a}$ vanishes over $\lambda=u_j$). There is a unique element $w_a\in W$, s.t., $\beta_{j,a}=w_a\beta_{j,1}$ and $y_{j,a}=w_ay_{j,1}$ (recall that we chose $\beta_{j,a}\in R_+$). We express the residue at a given ramification point $y_{j,a}$ in terms of the residue at $y_{j,1}$. Let us denote for brevity $\beta_j:=\beta_{j,1}$, $\theta_j:=\theta_{j,1}$, and $y:=y_1$. The contribution on the RHS of the global recursion corresponding to the jth term in the outer sum is

$$\frac12\sum_{a=1}^{|W|/2}\mathrm{Res}_{y_a=y_{j,a}}\frac{S_{\theta_{j,a}(y_a),y_a}(x_0)}{(\gamma|\beta_{j,a})\langle\beta_{j,a},y_a\rangle\,d\lambda}\times\Big(\omega_{g-1,n+2}(s;y_a,\theta_{j,a}(y_a),x_1,\dots,x_n)+\cdots\Big),$$

where the omitted term differs from the corresponding term on the RHS of the global recursion via the substitution $y\mapsto y_a$. After changing the variables $y_a=w_ay$, the residue turns into a residue at $y_{j,1}$, i.e.,

$$\frac12\sum_{a=1}^{|W|/2}\mathrm{Res}_{y=y_{j,1}}\frac{S_{w_a\theta_j(y),w_ay}(x_0)}{(\gamma|\beta_{j,a})\langle\beta_j,y\rangle\,d\lambda}\times\Big(\omega_{g-1,n+2}(s;w_ay,w_a\theta_j(y),x_1,\dots,x_n)+\cdots\Big).$$

The term in the bracket is by definition

$$\omega_{g-1,n+2}^{w_a^{-1}\gamma,s_{\beta_j}w_a^{-1}\gamma,\gamma,\dots,\gamma}(s;\lambda,\lambda,\lambda_1,\dots,\lambda_n)+\sum\omega_{g',n'+1}^{w_a^{-1}\gamma,\gamma_{I'}}(\lambda,\lambda_{I'})\,\omega_{g'',n''+1}^{s_{\beta_j}w_a^{-1}\gamma,\gamma_{I''}}(\lambda,\lambda_{I''})\qquad(20)$$

where λ and $\lambda_1,\dots,\lambda_n$ are the projections of y and $x_1,\dots,x_n$ on the base of the branched covering (7). We have the following decomposition

$$w_a^{-1}\gamma=\gamma'+(w_a^{-1}\gamma|\beta_j)\beta_j/2=\gamma'+(\gamma|\beta_{j,a})\beta_j/2,\qquad s_{\beta_j}(w_a^{-1}\gamma)=\gamma'-(\gamma|\beta_{j,a})\beta_j/2,$$

where γ' is a cycle invariant with respect to the local monodromy around $\lambda=u_j$. The period vectors $\phi_{\gamma'}(s,\lambda;z)$ are analytic near $\lambda=u_j$, so up to terms that are analytic at $y=y_{j,1}$ we get that (20) coincides with

$$\frac14\,(\gamma|\beta_{j,a})^2\Big(\omega_{g-1,n+2}^{\beta_j,-\beta_j,\gamma,\dots,\gamma}(s;\lambda,\lambda,\lambda_1,\dots,\lambda_n)+\sum\omega_{g',n'+1}^{\beta_j,\gamma_{I'}}(\lambda,\lambda_{I'})\,\omega_{g'',n''+1}^{-\beta_j,\gamma_{I''}}(\lambda,\lambda_{I''})\Big).$$

Note that $(1/2)\,\mathrm{Res}_{y=y_{j,1}}=\mathrm{Res}_{\lambda=u_j}$. To finish the proof we just need to compute the sum

$$\sum_{a=1}^{|W|/2}(\gamma|\beta_{j,a})S_{w_a\theta_j(y),w_ay}(x_0)=\frac12\sum_{w\in W}(\gamma|w\beta_j)S_{w\theta_j(y),wy}(x_0)=-K_{\gamma,\beta_j}(s,x_0,y).$$

It remains only to recall Proposition 2.3

$$K_{\gamma,\beta_j}(s,x_0,y)=\Omega(\phi_+^\gamma(s,\lambda_0;z),\mathbf{f}_-^{\beta_j}(s,\lambda;z))$$

and to recall that by definition

$$\langle\beta_j,y\rangle=(I^{(-1)}_{\beta_j}(s,\lambda),1).\qquad\square$$

4. The free energies and the primary potentials in genus 0 and 1

The main goal of this section is to compute the genus-0 and genus-1 free energies. However, let us first prove that the spectral curve is non-singular.

4.1. **Smoothness of the spectral curve.** The spectral curve is a branched covering (7) of a smooth curves, so the only singularities could be at the ramification points. The ramification points of index 2 are easy to analyze, because locally the covering near such a point is equivalent to a covering defined by the period map of an A_1-singularity. Therefore, we can reduce the proof of the general case to the case of an A_1-singularity. The latter case is straightforward, so we omit the details. It is more interesting to prove the regularity at the ramification points $\lambda^{-1}(\infty)$.

Let us first recall several properties of the Coxeter transformations. Given a Coxeter transformation σ, all other Coxeter transformations have the form $w\sigma w^{-1}$, $w \in W$ and the set of all Coxeter transformations consist of $|W|/h$ elements. Note that the number of ramification points above $\lambda = \infty$ is also $|W|/h$. By definition, the ramification points are the solutions of the following equations in $\mathbb{P}^{N-1}$:

$$t_a(X_1, \dots, X_N) = 0, \quad 1 \le a \le N-1.$$

We assign a ramification point $\xi = [\xi_1, \dots, \xi_N]$ to each Coxeter transformation σ, by letting $\sum_{i=1}^N \xi_i \omega_i \in \mathfrak{h}$ be an eigenvector of σ with eigenvalue $\eta := e^{2\pi\sqrt{-1}/h}$. Recall the so called Coleman lemma [5]: $\langle \alpha, \xi \rangle \neq 0$ for all $\alpha \in R$, i.e., each eigenvector ξ is inside some Weyl chamber.

Proposition 4.1. *The map that associates a ramification point to a Coxeter transformation is a bijection.*

Proof. Let us assume that $\sigma_1 \xi = \eta \xi = \sigma_2 \xi$ for two Coxeter transformations σ_1 and σ_2. Since the Weyl group acts faithfully on the set of Weyl chambers and $\sigma_1^{-1}\sigma_2$ fixes the Weyl chamber to which ξ belongs, we must have $\sigma_1 = \sigma_2$. Since both sets have the same number of elements, the map must be onto. □

Assume now that $\xi = [0, \xi_1, \dots, \xi_N] \in V_s$ is a ramification point. We may assume that $\xi_N = 1$, so the ramification point is in the affine chart $U_N := \{X_N \neq 0\} \subset \mathbb{P}^N$. Let $u_i = X_i/X_N$, $0 \le i \le N-1$ be the affine coordinates of U_N. The equation of $V_s \cap U_N$ can be written as

$$t_a(u_1, \dots, u_{N-1}, 1) = s_a u_0^{d_a}, \quad 1 \le a \le N-1.$$

Using the Jacobian criterion, we get that we have to prove that the determinant

$$\det \begin{bmatrix} \frac{\partial t_1}{\partial x_1} & \cdots & \frac{\partial t_1}{\partial x_{N-1}} \\ \vdots & & \vdots \\ \frac{\partial t_{N-1}}{\partial x_1} & \cdots & \frac{\partial t_{N-1}}{\partial x_{N-1}} \end{bmatrix}$$

is non-zero at $x = (\xi_1, \dots, \xi_N)$. Let us look at the larger determinant

$$\det \begin{bmatrix} \frac{\partial t_1}{\partial x_1} & \cdots & \frac{\partial t_1}{\partial x_{N-1}} & \frac{\partial t_1}{\partial x_N} \\ \vdots & & \vdots & \vdots \\ \frac{\partial t_{N-1}}{\partial x_1} & \cdots & \frac{\partial t_{N-1}}{\partial x_{N-1}} & \frac{\partial t_{N-1}}{\partial x_N} \\ \frac{\partial t_N}{\partial x_1} & \cdots & \frac{\partial t_N}{\partial x_{N-1}} & \frac{\partial t_N}{\partial x_N} \end{bmatrix} = \prod_{\alpha\in R_+} \langle \alpha, x\rangle.$$

Since the invariant polynomials are weighted homogeneous, we have

$$\sum_{i=1}^N x_i \frac{\partial t_a}{\partial x_i} = d_a t_a.$$

Therefore, when we evaluate the bigger determinant at $x = (\xi_1, \dots, \xi_N)$ we may replace the last column by $(0, \dots, 0, d_N t_N(\xi))^t$, where we used that $t_a(\xi) = 0$ for $1 \le a \le N-1$. Again, the Coleman's lemma implies that the big determinant is non-zero, so both $t_N(\xi)$ and the determinant that we are interested in must be non-zero.

Finally, let us point out that our argument proves that the ramification points $\lambda^{-1}(\infty)$ are smooth for all $s \in \mathbb{C}^{N-1}$.

4.2. **Genus-0 free energy.** The genus-0 free energy is defined through the meromorphic differential

$$f_\gamma(x)d\lambda = (I^{(-1)}_\gamma(s,\lambda), 1)d\lambda = d_\lambda(I^{(-2)}_\gamma(s,\lambda), 1),$$

where $x = \widetilde{\Phi}(C, s-\lambda\mathbf{1})$ and C is the path to the reference point that determines the value of the period. The poles of this differential are only at the ramification points $\{x_a\}_{a=1}^{|W|/h} := \lambda^{-1}(\infty)$ and the integrals along any closed path in V_s is 0, so the definition from [10] takes the form

$$\underline{F}^{(0)} = \frac{1}{2}\sum_{a=1}^{|W|/h} \mathrm{Res}_{x=x_a} V_a(x) f_\gamma(x) d\lambda(x),$$

where

$$V_a(x) = \mathrm{Res}_{y=x_a} \log(1-\zeta(x)/\zeta(y))\, f_\gamma(y)\, d\lambda(y),$$

where $\zeta : U_a - \{x_a\} \to \mathbb{C}$ is a local coordinate in a neighborhood U_a of x_a, s.t., $\lambda(y) = \zeta(y)^h\ \forall y \in U_a$.

We have $\mathbf{f}^\alpha(s,\lambda;z) = S_s(z)\mathbf{f}^\alpha(0,\lambda;z)$, where $S_s = 1 + S_1 z^{-1} + \cdots$ is a fundamental solution for the Dubrovin's connection

$$z\partial_{t_a} S_t(z) = v_a \bullet S_t(z), \quad S_0(z) = 1.$$

Let us denote by $\sigma : \mathfrak{h} \to \mathfrak{h}$ the Coxeter transformation corresponding to the monodromy along a big loop around the discriminant (in counterclockwise direction), then using the homogeneity of v_i, we get

$$(I^{(0)}_\alpha(0,\lambda), v_i) = \lambda^{-m_i/h}\langle H_i, \alpha\rangle,$$

where $m_i = d_i - 1$ are the Coxeter exponents and $H_i \in \mathfrak{h}$ is an eigenvector of σ with eigenvalue η^{m_i}. Note that $m_1 = 1$, $m_N = h-1$, $m_i + m_{N+1-i} = h$. Using Saito's formula (2.1) we get that the eigenbasis $\{H_i\}_{i=1}^N$ satisfy

$$(H_i|H_j) = \delta_{i+j,N+1}, \quad 1 \le i, j \le N.$$

It follows that we can express the free genus-0 energy in terms of the eigenbasis and the matrices S_k. After a direct computation we get the following formula for $(I^{(-2)}_\gamma(s,\lambda), 1)$

$$\sum_{i=1}^N \frac{\lambda^{-m_i/h+2}}{(-m_i/h+1)(-m_i/h+2)}\langle\gamma, H_i\rangle(v^i,1) - \sum_{i=1}^N \frac{\lambda^{-m_i/h+1}}{-m_i/h+1}\langle\gamma, H_i\rangle(S_1 v^i, 1) +$$
$$+\sum_{k=0}^\infty\sum_{i=1}^N (m_i/h)(m_i/h+1)\cdots(m_i/h+k-1)\lambda^{-m_i/h-k}\langle\gamma, H_i\rangle(S_{k+2}v^i, 1),$$

where $\{v^i\}$ is a basis of H dual to $\{v_i\}$ with respect to the residue pairing.

Let us assume first that x_a is the ramification point corresponding to the classical monodromy, then

$$V_a(x) = \zeta^{h+1}\frac{h^2}{h+1}\langle\gamma, H_N\rangle - \sum_{i=1}^N \frac{\zeta^{m_i}}{m_i}\langle\gamma, H_{N+1-i}\rangle(S_1 v^{N+1-i}, 1),$$

where $\zeta = \zeta(x) = \lambda(x)^{1/h}$ is the local coordinate near x_a. From this formula we get that

$$\mathrm{Res}_{x=x_a} V_a(x) f_\gamma(x) d\lambda(x) = -\mathrm{Res}_{\zeta=\infty}(I^{(-2)}_\gamma(s,\zeta^h), 1)\, d_\zeta V_a(x)$$

is

$$h\Big(\langle\gamma, H_1\rangle\,\langle\gamma, H_N\rangle(S_3 1, 1) - \sum_{i=1}^N \langle\gamma, H_i\rangle\,\langle\gamma, H_{N+1-i}\rangle(S_2 v^i, 1)(S_1 v^{N+1-i}, 1)\Big). \qquad (21)$$

The above formula can be simplified as follows. Note that

$$h\langle\gamma, H_i\rangle\, H_{N+1-i} = \sum_{k=1}^h \eta^{m_i k}\sigma^k\gamma,$$

so (21) takes the form

$$\sum_{k=1}^h \eta^k\langle\sigma^k\gamma|\gamma)(S_3 1, 1) - \sum_{k=1}^h\sum_{i=1}^N \eta^{m_i k}\langle\sigma^k\gamma|\gamma)(S_2 v^i, 1)(S_1 v^{N+1-i}, 1). \qquad (22)$$

The expression (22) is the contribution to $2\underline{F}^{(0)}$ coming from the residue at the ramification point x_a. Note that after adding the remaining contributions we get that $2\underline{F}^{(0)}$ is the sum of (22) over all Coxeter transformations σ.

Lemma 4.2. *The following identity holds*

$$\sum_{k=1}^{h} \eta^{m_i k} \sum_{\sigma} \sigma^k = |W|/N,$$

where the sum is over all Coxeter transformations σ.

Proof. The operator $\sum_\sigma \sigma^k$ commutes with the action of W, so By Schur's lemma, it must act by some constant c_k. After taking trace we get

$$c_k N = \mathrm{Tr}(\sigma^k)|W|/h,$$

where we used that there are $|W|/h$ Coxeter transformations and that the trace of σ^k is the same for all Coxeter transformations. On the other hand,

$$\sum_{k=1}^{h} \eta^{m_i k}\, \mathrm{Tr}(\sigma^k) = \sum_{j=1}^{N} \sum_{k=1}^{h} \eta^{(m_i - m_j)k} = h. \quad \square$$

Applying the above Lemma and using that $S_t(z)S_t(-z)^T = 1$, we get

$$\underline{F}^{(0)} = \frac{1}{2}((S_3 - S_2 S_1)1, 1)\,|W|(\gamma|\gamma)/N.$$

Using that $S_t(z)$ is a solution for the Dubrovin's connection, it is easy to verify that

$$F^{(0)} = \frac{1}{2}((S_2 S_1 - S_3)1, 1)$$

is a potential of the Frobenius structure, so up to quadratic terms in t we have

$$\underline{F}^{(0)}(t) = -(\gamma|\gamma)\,\frac{|W|}{N} F^{(0)}(t).$$

4.3. **Genus-1 free energy.** Let us denote by $x_{j,a}$ ($1 \le j \le N$, $1 \le a \le |W|/2$) the double ramification points and by $u_j := \lambda(x_{j,a})$ the corresponding branching points. The genus-1 free energy is by definition

$$\underline{F}^{(1)}(s) = -\frac{1}{2}\log \tau_B(s) - \frac{1}{24}\sum_{i=1}^{N}\sum_{a=1}^{|W|/2} \log f_\gamma'(x_{j,a}),$$

where f'_γ is the derivative with respect to the local parameter $\sqrt{\lambda(x) - u_j}$ near $x = x_{j,a}$ and τ_B is the Bergman tau-function of V_s.

Recall that the period mapping has the following Laurent series expansion near $\lambda = u_j$ (see [14, 19]):

$$I^{(0)}_{\beta_j}(s,\lambda) = \pm \frac{2}{\sqrt{2(\lambda - u_j)\Delta_j}}\Big(du_j + \cdots\Big),$$

where the dots represents higher order terms in $\lambda - u_j$, $\Delta_j := (du_j, du_j)$, and β_j is a cycle vanishing over $\lambda = u_j$. The points $x = \widetilde{\Phi}(C, s - \lambda\mathbf{1})$ in a neighborhood of $x_{j,a}$ correspond to λ in a neighborhood of u_j, so we get

$$f'_\gamma(x_{j,a}) = \lim_{\lambda\to u_j} \frac{d_\lambda(I^{(-1)}_\gamma(s,\lambda), 1)}{d_\lambda \sqrt{\lambda - u_j}} = \lim_{\lambda\to u_j} 2\sqrt{\lambda - u_j}\,(I^{(0)}_\gamma(s,\lambda), 1).$$

Decomposing the cycle $\gamma = \gamma' + (\gamma|\beta_j)\beta_j/2$ into invariant and anti-invariant parts with respect to the local monodromy we get

$$f'_\gamma(x_{j,a}) = \lim_{\lambda\to u_j} (\gamma|\beta_j)\sqrt{\lambda - u_j}\,\frac{\pm 2}{\sqrt{2(\lambda - u_j)\Delta_j}} = \pm\sqrt{2}\,(\gamma|\beta_j)\,\Delta_j^{-1/2}.$$

4.3.1. *The Bergman τ-function.* To define the Bergman tau-function we have to think of the pair (V_s, λ) as a point in an appropriate moduli space $\mathcal{M}$ of Hurwitz covers of $\mathbb{P}^1$ whose genus and ramification profile is the same as of V_s. The critical values $u_{j,a} = \lambda(x_{j,a})$ provide local coordinates on $\mathcal{M}$ and the differential of τ_B at (V_s, λ) is defined via

$$d\log\tau_B = \sum_{j=1}^{N}\sum_{a=1}^{|W|/2} du_{j,a}\ \mathrm{Res}_{x=x_{j,a}} \frac{B(x, \theta_{j,a}(x))}{d\lambda},$$

where B is the Bergman kernel of V_s.

Let $u = (\lambda(x) - u_j)^{1/2}$ and $v = (\lambda(y) - u_j)^{1/2}$ be the local coordinates of two points $x, y \in V_s$ near $x_{j,a}$. The Bergman kernel has the form

$$B(x, y) = \frac{du dv}{(u - v)^2} + f_{j,a}(u, v) du dv,$$

where $f_{j,a} \in \mathbb{C}\{u, v\}$ is a convergent power series in u and v. If $y = \theta_{j,a}(x)$, then $v = -u$ and we get that

$$\mathrm{Res}_{x=x_{j,a}} \frac{B(x, \theta_{j,a}(x))}{d\lambda} = -\frac{1}{2} f_{j,a}(0, 0).$$

4.3.2. *The Bergman τ-function and the R-matrix.* Following the notation in [19] we recall the following formula for the correlator

$$\omega^{\beta_j,\beta_j}_{0,2}(s; \lambda, \lambda) = P^{jj}_0(s, \lambda) d\lambda \cdot d\lambda,$$

where β_j is the vanishing cycle vanishing over $\lambda = u_j$:

$$P^{jj}_0(s, \lambda) = \frac{1}{4}(\lambda - u_j)^{-2} + 2(e_j, V_{00}(s)e_j)(\lambda - u_j)^{-1},$$

where $e_j = du_j/\sqrt{\Delta_j} \in T_s^*B \cong H$ and $V_{k\ell}(s)$ are linear operators of H defined via Givental's R-matrix $\mathcal{R}(s,z) = 1 + R_1(s)z + R_2(s)z^2 + \cdots$

$$\sum_{k,\ell=0}^{\infty} V_{k\ell}(s) w^k z^\ell = \frac{1 - {}^T\mathcal{R}(s,-w)\mathcal{R}(s,-z)}{z+w}.$$

Note that $V_{00} = R_1$ and since $\{e_j\}$ is an orthonormal basis of H, we get that $(e_j, V_{00}(s)e_j) = R_1^{jj}(s)$ is the jth diagonal entry of $R_1(s)$.

Recalling the definition of the 2-point genus-0 correlators we get that

$$\mathrm{Res}_{x=x_{j,a}} \frac{1}{d\lambda} \Big(\Omega(\phi_+^\gamma(s,\lambda;z), \phi_+^\gamma(s,\mu;z) - (\gamma|\gamma)\frac{d\lambda\cdot d\mu}{(\lambda-\mu)^2}\Big)\Big|_{\mu=\lambda} \tag{23}$$

is

$$\mathrm{Res}_{x=x_{j,a}} P^{(0)}_{\gamma,\gamma}(s,\lambda)d\lambda = \mathrm{Res}_{\lambda=u_j}\Big(P^{(0)}_{\gamma,\gamma}(s,\lambda) + P^{(0)}_{\theta_{j,a}\gamma,\theta_{j,a}\gamma}(s,\lambda)\Big)d\lambda =$$
$$\mathrm{Res}_{\lambda=u_j}\Big(2P^{(0)}_{\gamma',\gamma'}(s,\lambda) + \frac{1}{2}(\gamma|\beta_j)^2\, P_0^{jj}(s,\lambda)\Big)d\lambda = (\gamma|\beta_j)^2 R_1^{jj}(s),$$

where $x_{j,a}$ is the ramification point corresponding to the reference path that defines the residue (23) and γ' is the invariant part of γ with respect to the local monodromy around $\lambda = u_j$. Note that $P^{(0)}_{\gamma',\gamma'}$ is holomorphic near $\lambda = u_j$, so it does not contribute to the residue.

On the other hand we can compute the residue (23) in terms of the Bergman kernel. Recalling Proposition 2.3 and Proposition 2.4 we transofrm the residue (23) into

$$\mathrm{Res}_{x=x_{j,a}}\Big(\sum_{w\in W}(\gamma|w\gamma)\,\frac{B(x,wy)}{d\lambda(x)} - (\gamma|\gamma)\frac{d\lambda(y)}{(\lambda(x)-\lambda(y))^2}\Big)\Big|_{y=x}.$$

The only terms in the above sum that contribute to the residue are the ones for which $w = 1$ or $w = \theta_{j,a}$. Using again the local coordinates $u = \sqrt{\lambda(x) - u_j}$ and $v = \sqrt{\lambda(y) - u_j}$ we get that the term with $w = 1$ and the term outside of the sum add up to

$$(\gamma|\gamma)\Big(\frac{1}{8}u^{-3} + \frac{1}{2}f_{j,a}(u,u)u^{-1}\Big)du$$

and the contribution to the residue is $(\gamma|\gamma)f_{j,a}(0,0)/2$. The term with $w = \theta_{j,a}$ contributes to the residue

$$-\frac{1}{2}(\gamma|\theta_{j,a}\gamma)f_{j,a}(0,0).$$

Comparing the two computations of the residue (23) we get

$$(\gamma|\beta_j)^2 R_1^{jj}(s) = \frac{1}{2}(\gamma|\gamma - \theta_{j,a}(\gamma))f_{j,a}(0,0) = \frac{1}{2}(\gamma|\beta_j)^2 f_{j,a}(0,0),$$

i.e., $\frac{1}{2}f_{j,a}(0,0) = R_1^{jj}$. Therefore the differential of the Bergman τ-function is the following

$$d\log\tau_B = -\frac{|W|}{2}\sum_{j=1}^N R_1^{jj}(s)du_j.$$

Finally, the genus-1 free energy becomes

$$\underline{F}^{(1)}(s) = \frac{|W|}{2}\left(\frac{1}{2}\int\sum_{j=1}^N R_1^{jj}(s)du_j + \frac{1}{48}\sum_{j=1}^N \log\Delta_j\right).$$

It remains only to recall that the term in the brackets is the genus-1 primary potential of the Frobenius structure also known as the G-function (see [13] for more details and references).

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Appendix A. The space of holomorphic 1-forms

In this appendix we would like to give a description of the space of holomorphic 1-forms and to prove that our initial condition (10) is independent of the choice of a Torelli marking. Unfortunately, our argument does not work in the exceptional cases. Of course, we can use Corollary 2.5, but it would be nice to find a direct algebraic proof. We also give an amusing proof that the order of the Weyl group is the product of the degrees of the invariant polynomials.

A.1. **The space of holomorphic 1-forms.** Using the Riemann–Hurwitz formula we can compute the genus of V_s as

$$g = 1 + d(V)|W|/(2h),$$

where

$$d(V) = Nh/2 - h - 1 = d_1 + d_2 + \cdots + d_{N-1} - N - 1.$$

The genus can be computed also using that V_s is a complete intersection, which allows us to compute the canonical bundle via the adjunction formula (see [6])

$$g = 1 + d_2\cdots d_{N-1}(1 + d_2 + \cdots + d_{N-1} - N).$$

In particular, we get a uniform proof that $|W| = d_1 d_2 \ldots d_N$.

The space of holomorphic 1-forms on V_s can be described as follows (see [7]). In the affine chart $X_0 \neq 0$ and the affine coordinates $x_i = X_i/X_0$ it is easy to see that

$$\phi(x_1,\ldots,x_N)\frac{dx_1\wedge\cdots\wedge dx_N}{dt_1\wedge\cdots\wedge dt_{N-1}} \tag{24}$$

extends to a holomorphic form on V_s if and only if $\phi \in \mathbb{C}[x_1, \dots, x_N]$ is a polynomial of degree at most $d(V)$. Note that the form (24) is identically 0 on V_s if and only if $h \in (t_1(x) - s_1, \dots, t_{N-1}(x) - s_{N-1})$. It remains only to check that the number of elements in the ring

$$\mathbb{C}[x_1, \dots, x_N]/(t_1(x) - s_1, \dots, t_{N-1}(x) - s_{N-1}).$$

of degree at most $d(V)$ is g.

Proposition A.1. *If $\phi \in \mathrm{Sym}(\mathfrak{h}^*)$ is a polynomial of degree at most $d(V)$ and $\gamma \in \mathfrak{h}^*$ is a linear function, then*

$$\sum_{w\in W} \det(w)\,(w\gamma \otimes w^{-1}\phi) = 0.$$

Proof. Our argument works in the A and D cases only. The identities in the exceptional cases, can be verified with a computer.

The LHS will be viewed as a function f on $\mathfrak{h}\times\mathfrak{h}$

$$f(x,y) = \sum_{w\in W} \det(w)\langle w\gamma, x\rangle\, \phi(wy), \quad (x,y) \in \mathfrak{h}\times\mathfrak{h}.$$

Since the function depends linearly on γ, it is enough to prove the identity for a set of γ's that form a basis of $\mathfrak{h}^*$. Similarly, we may assume that ϕ is a monomial in y.

Let us take γ to be a fundamental weight corresponding to a node of the Dynkin diagram, s.t., if we remove that node, then we get a Dynkin diagram of the same type but with rank one less. Note that the number of positive roots orthogonal to γ is $\frac{1}{2}(N-1)N$ for A_N and $(N-2)(N-1)$ for D_N. In both cases, the number is greater than

$$d(V) = \begin{cases} \frac{1}{2}N(N-1) - 2 & \text{for } A_N \\ (N-2)(N-1) - 1 & \text{for } D_N. \end{cases}$$

In particular, the polynomial

$$\Delta_\gamma(y) = \prod_{\alpha\in R_+:(\alpha|\gamma)=0} \langle \alpha, y\rangle$$

has degree at least $d(V)+1$. The zero locus of Δ_γ is contained in the zero locus of f: if $\langle \alpha, y_0\rangle = 0$, then in the definition of f let us shift the summation by replacing $w \mapsto ws_\alpha$, we get $f(x,y_0) = -f(x,y_0)$. The ideal generated by Δ_γ is a radical ideal, so using Hilbert's Nullstellensatz, we get that $f(x,y) = g(x,y)\Delta_\gamma(y)$ for some polynomial g. If we assume that $f \neq 0$, then we get a contradiction by comparing the degrees of the monomials in y on both sides: on the left they all have degree $\deg(\phi) \leq d(V)$, while on the right, they all have degree at least $\deg(\Delta_\gamma) > d(V)$.

To finish the proof, we just need to use that the above argument applies to the entire orbit $W\gamma$ and that this orbit contains a basis of $\mathfrak{h}^*$. □.

The above proposition implies that our initial condition is independent of the Torelli marking. Indeed, changing the Torelli marking will modify the Bergman kernel via a quadratic expression of holomorphic differentials on V_s. Using the explicit description of the holomorphic differentials from above we see that if we replace $B(x, y)$ in (10) by a product $\theta_1(x)\theta_2(y)$ of holomorphic differentials, then we get precisely the identity in Proposition A.1.

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