THE MODULAR GROUP FOR THE TOTAL ANCESTOR POTENTIAL OF A SIMPLE ELLIPTIC SINGULARITY.

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ABSTRACT. In a series of papers Krawitz, Milanov, Ruan, and Shen have verified the so called Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence for simple elliptic singularities $E_N^{(1,1)}$ ($N = 6, 7, 8$). As a byproduct it was also proved that the orbifold Gromov–Witten invariants of the orbifold projective lines $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{2,4,4}$, and $\mathbb{P}^1_{2,3,6}$ are quasi-modular forms on an appropriate modular group. While the modular group for the first orbifold is easy to find, namely it is $\Gamma(3)$, the modular groups for the other two were left unknown. The goal of this paper is to prove that the modular groups in the remaining two cases are respectively $\Gamma(4)$ and $\Gamma(6)$.

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1. INTRODUCTION

Let $W(x) = x_1^{a_1}+x_2^{a_2}+x_3^{a_3}$ be a Fermat polynomial whose exponents (a_1, a_2, a_3) are given by one of the following triples $(3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$. Since such a W defines a hypersurface in $\mathbb{C}^3$ that has a simple-elliptic singularity at $x = 0$, we will sometimes refer to it as *elliptic* Fermat polynomial. We assign weights $q_i = 1/a_i$ to each variable x_i, so that W becomes a quasi-homogeneous polynomial of degree 1.

1.1. **Formulation of the main results.** Let $H = \mathbb{C}[x_1, x_2, x_3]/(W_{x_1}, W_{x_2}, W_{x_3})$ be the Jacobi algebra of W. Given a triple $e = (e_1, e_2, e_3)$ of non-negative integers we put $\phi_e(x) = x_1^{e_1}x_2^{e_2}x_3^{e_3}$. We choose a set E of exponents e, s.t., the monomials $\phi_e(x)$ project to a basis of H. More presicely, put

$$E = \{(e_1, e_2, e_3) \mid 0 \le e_i \le a_i - 2\}. \tag{1}$$

It will be convenient also to decompose $E = \{(0, 0, 0), m\} \sqcup E_{\mathrm{tw}}$, where E_{tw} corresponds to monomials of non-integral degree and $\phi_m(x)$ is the monomial of degree

1. Let us denote by $\Sigma\subset\mathbb{C}$ the set of all *marginal* deformations

$$f(\sigma,x)=W(x)+\sigma\,\phi_m(x),\quad \sigma\in\Sigma,$$

s.t., $f(\sigma,x)$ has only one critical point. The hypersurfaces

$$X_{\sigma,\lambda}=\{x\in\mathbb{C}^3\mid f(\sigma,x)=\lambda\}$$

form a smooth fibration on $\Sigma\times(\mathbb{C}\setminus\{0\})$, while the homology (resp. cohomology) groups $H_2(X_{\sigma,\lambda};\mathbb{C})$ (resp. $H^2(X_{\sigma,\lambda};\mathbb{C})$) form a vector bundle equipped with a flat Gauss–Manin connection. We fix a reference point, say $(0,1)$, and let

$$\mathfrak{h}=H_2(X_{0,1};\mathbb{C}),\quad \mathfrak{h}^\vee=H^2(X_{0,1};\mathbb{C})$$

be the reference fibers. The parallel transport around $\lambda=0$ induces a monodromy transformation $J\in\mathrm{GL}(\mathfrak{h})$, which commutes with the monodromy action of $\pi_1(\Sigma)$ on $\mathfrak{h}$. In other words, we have a monodromy representation

$$\rho:\pi_1(\Sigma)\to\mathrm{GL}(\mathfrak{h}_0)\oplus\mathrm{GL}(\mathfrak{h}_{\neq0}),$$

where $\mathfrak{h}_0$ is the J-invariant subspace and $\mathfrak{h}_{\neq0}$ is the direct sum of all eigenspaces of J with eigenvalue $\neq0$. Put $\rho=(\rho_0,\rho_{\neq0})$ and let

$$\overline{\rho}_{\neq0}:\pi_1(\Sigma)\longrightarrow\mathrm{GL}(\mathfrak{h}_{\neq0})/\langle J\rangle,$$

the map induced from $\rho_{\neq0}$. Using an explicit computation we will check that

$$\mathrm{Ker}\,(\rho_0)\subset\mathrm{Ker}\,(\overline{\rho}_{\neq0}).\tag{2}$$

It will be nice if one can find a conceptual explanation and determine if (2) is satisfied for other normal forms W of the simple elliptic singularities. Using (2) we get an induced homomorphism

$$\rho_W:\ \widetilde{\Gamma}(W)\longrightarrow\mathrm{GL}(\mathfrak{h}_{\neq0})/\langle J\rangle,$$

where $\widetilde{\Gamma}(W)=\mathrm{Im}(\rho_0)$. Put $\Gamma(W):=\mathrm{Ker}(\rho_W)$.

Theorem 1.1. *The total ancestor potential $\mathcal{A}^W_\sigma(\hbar;\mathbf{q})$ of the simple elliptic singularity W transforms as a quasi-modular form on $\Gamma(W)$.*

The definition of the total ancestor potential in singularity theory as well as the precise meaning of the quasi-modularity will be recalled later on. Our second result can be stated this way.

Theorem 1.2. *If W is an elliptic Fermat polynomial of type $E_6^{(1,1)}$, $E_7^{(1,1)}$, or $E_8^{(1,1)}$, then $\Gamma(W)$ is respectively $\Gamma(3),\Gamma(4)$, or $\Gamma(6)$.*

1.2. **Applications to Gromov–Witten theory.** The LG/CY correspondence was proposed by Chiodo and Ruan [9]. In our settings it can be stated this way. A triplet of non-zero complex numbers $(g_1, g_2, g_3) \in (\mathbb{C}^*)^3$ is called a *diagonal symmetry* of W if

$$W(g_1x_1, g_2x_2, g_3x_3) = W(x_1, x_2, x_3).$$

The diagonal symmetries form a group G_W. It contains the element

$$J_W = \mathrm{diag}(e^{2\pi\sqrt{-1}q_1}, e^{2\pi\sqrt{-1}q_2}, e^{2\pi\sqrt{-1}q_3}), \quad q_i = 1/a_i.$$

The equation $W = 0$ defines an elliptic curve $\widetilde{X}_W$ in the weighted projective plane $\mathbb{CP}^2(c_1, c_2, c_3)$, where $q_i = c_i/d$ for a common denominator d. The action of the group $\widetilde{G}_W := G_W/\langle J_W\rangle$ on $\widetilde{X}_W$ is faithful and the quotient $X_W := \widetilde{X}_W/\widetilde{G}_W$ is an orbifold projective line $\mathbb{P}^1_{a_1,a_2,a_3}$. The LG/CY correspondence predicts that the GW invariants of X_W can be obtained from the so called Fan–Jarvis–Ruan–Witten (FJRW) invariants of the pair (W, G_W) via analytic continuation and a certain quantizatied symplectic transformation (c.f. [9]).

The orbifold GW invariants of $\mathcal{X} = \mathbb{P}^1_{a_1,a_2,a_3}$ are defined as follows. Let $\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,d}$ be the moduli space of degree-d stable maps from a genus-g orbi-curve, equipped with n marked points, to $\mathcal{X}$. Let us denote by π the forgetful map, and by ev_i the evaluation at the i-th marked point

$$\overline{\mathcal{M}}_{g,n} \xleftarrow{\pi} \overline{\mathcal{M}}^{\mathcal{X}}_{g,n,d} \xrightarrow{\mathrm{ev_i}} I\mathcal{X}\,,$$

where $I\mathcal{X}$ is the inertia orbifold of $\mathcal{X}$. Let $\mathrm{Eff}(\mathcal{X}) \subset H_2(\mathcal{X};\mathbb{Z})$ be the cone of effective curve classes. By definition the Novikov ring is the completed group algebra of $\mathrm{Eff}(\mathcal{X})$. In our case, since $H_2(\mathcal{X};\mathbb{Z}) = \mathbb{Z}\cdot[\mathcal{X}]$, where $[\mathcal{X}]$ is the fundamental class of $\mathcal{X}$, we may identify the Novikov ring with the space of formal power series $\mathbb{C}[\![q]\!]$. The moduli space is equipped with a virtual fundamental cycle $[\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,d}]$, s.t., the maps

$$\Lambda^{\mathcal{X}}_{g,n} : H^*_{CR}(\mathcal{X};\mathbb{C}[\![q]\!])^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n};\mathbb{C})$$

defined by

$$\Lambda^{\mathcal{X}}_{g,n} = \sum_{d=0}^{\infty} q^d\, \Lambda^{\mathcal{X}}_{g,n,d}, \quad \Lambda^{\mathcal{X}}_{g,n,d}(\alpha_1,\ldots,\alpha_n) := \pi_*\Big([\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,d}] \cap \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i)\Big)$$

form a CohFT with state space the Chen-Ruan cohomology $H^*_{CR}(\mathcal{X};\mathbb{C}[\![q]\!])$. The *ancestor* GW invariants of $\mathcal{X}$ are by definition the following formal series:

$$\Big\langle \tau_{k_1}(\alpha_1),\ldots,\tau_{k_n}(\alpha_n)\Big\rangle_{g,n} = \sum_{d=0}^{\infty} q^d \int_{\overline{\mathcal{M}}_{g,n}} \Lambda^{\mathcal{X}}_{g,n,d}(\alpha_1,\ldots,\alpha_n)\, \psi_1^{k_1}\cdots\psi_n^{k_n}\,, \tag{3}$$

where ψ_i is the i-th psi class on $\overline{\mathcal{M}}_{g,n}$, $\alpha_i \in H$, and $k_i \in \mathbb{Z}_{\geq 0}$. For more details on orbifold Gromov–Witten theory we refer to [7]. The total ancestor potential of $\mathcal{X}$ is by definition the following generating series

$$\mathcal{A}_q^{\mathcal{X}}(\hbar;\mathbf{q}) = \exp\Big(\sum_{g,n=0}^{\infty} \hbar^{g-1}\Big\langle \mathbf{q}(\psi_1)+\psi_1,\ldots,\mathbf{q}(\psi_n)+\psi_n\Big\rangle_{g,n}\Big),$$

where $\mathbf{q}(z) = \sum_{k=0}^{\infty} q_k z^k$ and $\{q_k\}_{k=0}^{\infty}$ is a sequence of formal vector variables with values in $H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$. The generating function is a formal series in $q_0, q_1+1, q_2, \ldots$.

Following the ideas of Krawitz–Shen [25] and Milanov–Ruan [27], one can obtain a very precise correspondence between the total ancestor potentials $\mathcal{A}_\sigma^W$ and $\mathcal{A}_q^{\mathcal{X}}$ (see [29]). Let us briefly explain this correspondence. Recall that the curve

$$E_\sigma = \{f(\sigma,x)=0\} \subset \mathbb{P}^2(c_1,c_2,c_3)$$

is called *elliptic curve at infinity*. Let us think of Σ as a punctured $\mathbb{P}^1$ and let us select a puncture p, s.t., the j-invariant $j(E_\sigma)\to\infty$ as $\sigma\to p$. For example, if W is the Fermat polynomial of type $E_7^{(1,1)}$, then $p=-2,2$, or ∞. If W is the Fermat polynomial of type $E_8^{(1,1)}$, then p is a solution to $4p^3+27=0$. The main statement is that there exists a function $f(\sigma)$ on $\mathbb{P}^1$ holomorphic near $\sigma=p$, s.t., under the substitution $q=f(\sigma)$, the total ancestor potential $\mathcal{A}_q^{\mathcal{X}}$ coincides with $\mathcal{A}_\sigma^W$. Let us point out that the definition of $\mathcal{A}_\sigma^W$ requires a choice of a *primitive form* in the sense of K. Saito [33]. Part of the statement is that there exists a primitive form, s.t., the identification holds. Combining the mirror symmetry theorem of [29] with Theorem 1.1 and Theorem 1.2 we get

Corollary 1.3. *If $\mathcal{X}$ is one of the orbifolds $\mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{2,4,4}$, or $\mathbb{P}^1_{2,3,6}$, then the Gromov–Witten invariants* (3) *are quasi-modular forms respectively on* $\Gamma(3), \Gamma(4)$, *or* $\Gamma(6)$.

2. The total ancestor potential in singularity theory

Let $\mathcal{S}=\Sigma\times\mathbb{C}^{\mu-1}$. We fix a coordinate system on $\mathcal{S}$, such that the coordinates of $s\in\mathcal{S}$ are indexed by the exponents E (cf. (1)) in such a way that $s_m=\sigma\in\Sigma$ and $s_e\in\mathbb{C}$ for $e\neq m$. The miniversal deformation of W can be given by the following function

$$F(s,x) = W(x) + \sum_{e\in E} s_e\,\phi_e(x), \tag{4}$$

where the domain of F is $X=\mathcal{S}\times\mathbb{C}^3$. The marginal deformations $f(\sigma,x)$ are obtained from $F(s,x)$ by restricting $s_m=\sigma$ and $s_e=0$ for $e\neq m$.

It is well known (see [20, 35]) that Saito's theory of primitive forms (cf. [33]) gives rise to a Frobenius structure (cf. [12]) on $\mathcal{S}$. In this section the goal is to recall the key points in the construction of this Frobenius structure and then use the higher-genus reconstruction formalism of Givental to define the total ancestor potential of W.

2.1. **Saito's theory.** Let $C \subset X$ be the critical variety of F, i.e., the support of the sheaf

$$\mathcal{O}_C := \mathcal{O}_X / \langle \partial_{x_1} F, \partial_{x_2} F, \partial_{x_3} F \rangle.$$

Let $q : X \to \mathcal{S}$ be the projection on the first factor. The Kodair–Spencer map ($\mathcal{T}_{\mathcal{S}}$ is the sheaf of holomorphic vector fields on $\mathcal{S}$)

$$\mathcal{T}_{\mathcal{S}} \longrightarrow q_* \mathcal{O}_C, \quad \partial/\partial s_i \mapsto \partial F/\partial s_i \bmod (F_{x_1}, F_{x_2}, F_{x_3})$$

is an isomorphism, which implies that for any $s \in \mathcal{S}$, the tangent space $T_s\mathcal{S}$ is equipped with an associative commutative multiplication $\bullet_s$ depending holomorphically on $s \in \mathcal{S}$. If in addition we have a volume form $\omega = g(s,x) d^3x$, where $d^3x = dx_1 \wedge dx_2 \wedge dx_3$ is the standard volume form; then $q_*\mathcal{O}_C$ (hence $\mathcal{T}_{\mathcal{S}}$ as well) is equipped with the *residue pairing*:

$$(\psi_1, \psi_2) = \frac{1}{(2\pi i)^3} \int_{\Gamma_\epsilon} \frac{\psi_1(s,y)\psi_2(s,y)}{F_{y_1} F_{y_2} F_{y_3}}\, \omega, \tag{5}$$

where $y = (y_1, y_2, y_3)$ is an unimodular coordinate system for the volume form, i.e., $\omega = d^3y$, and Γ_ϵ is a real 3-dimensional cycle supported on $|F_{x_i}| = \epsilon$ for $1 \le i \le 3$.

Given a semi-infinite cycle

$$\mathcal{A} \in \varprojlim H_3(\mathbb{C}^3, (\mathbb{C}^3)_{-m}; \mathbb{C}) \cong \mathbb{C}^\mu, \tag{6}$$

where

$$(\mathbb{C}^3)_m = \{x \in \mathbb{C}^3 \mid \mathrm{Re}(F(s,x)/z) \le m\}. \tag{7}$$

Put

$$J_{\mathcal{A}}(s,z) = (-2\pi z)^{-3/2}\, z d_{\mathcal{S}} \int_{\mathcal{A}} e^{F(s,x)/z} \omega, \tag{8}$$

where $d_{\mathcal{S}}$ is the de Rham differential on $\mathcal{S}$. The oscillatory integrals $J_{\mathcal{A}}$ are by definition sections of the cotangent sheaf $\mathcal{T}^*_{\mathcal{S}}$.

According to Saito's theory of primitive forms [33, 36], there exists a volume form ω such that the residue pairing is flat and the oscillatory integrals satisfy a system of differential equations, which in flat-homogeneous coordinates $t = (t_e)_{e \in E}$ have the form

$$z \partial_e J_{\mathcal{A}}(t,z) = \partial_e \bullet_t J_{\mathcal{A}}(t,z), \tag{9}$$

where $\partial_e := \partial/\partial t_e$ and the multiplication is defined by identifying vectors and covectors via the residue pairing. Using the residue pairing, the flat structure, and the Kodaira–Spencer isomorphism we have the following isomorphisms:

$$T^*\mathcal{S} \cong T\mathcal{S} \cong \mathcal{S} \times T_0\mathcal{S} \cong \mathcal{S} \times H.$$

Due to homogeneity the integrals satisfy a differential equation with respect to the parameter $z \in \mathbb{C}^*$

$$(z\partial_z + E) J_{\mathcal{A}}(t,z) = \Theta\, J_{\mathcal{A}}(t,z), \tag{10}$$

where

$$E = \sum_{i=-1}^{\mu-2} d_e t_e \partial_e, \quad (d_e := \deg t_e = \deg s_e),$$

is the *Euler vector field* and Θ is the so-called *Hodge grading operator*

$$\Theta : \mathcal{T}_S^* \to \mathcal{T}_S^*, \quad \Theta(dt_e) = \left(\frac{1}{2} - d_e\right) dt_e.$$

The compatibility of the system (9)–(10) implies that the residue pairing, the multiplication, and the Euler vector field give rise to a *conformal Frobenius structure* of conformal dimension 1. We refer to B. Dubrovin [12] for the definition and more details on Frobenius structures.

2.2. **Primitive forms for simple elliptic singularities.** The classification of primitive forms in general is a very difficult problem. In the case of simple elliptic singularities however, all primitive forms are known (see [33]). They are given by $\omega = d^3x/\pi_A(\sigma)$, where $\pi_A(\sigma)$ is a period of the elliptic curve at infinity. Let us recall also that $\pi_A(\sigma)$ can be expressed in terms of a period of the Gelfand-Lerey form d^3x/dF as follows. We embed $\mathbb{C}^3$ in the weighted projective space $\mathbb{P}^3(1, c_1, c_2, c_3)$ via $x_i = X_i/X_0^{c_i}$, $i = 1, 2, 3$. The Zariski closure of the Milnor fiber $X_{\sigma,1}$ is $\overline{X}_{\sigma,1} = X_{\sigma,1} \cup E_\sigma$, therefore we have a *tube* map

$$L : H_1(E_\sigma; \mathbb{Z}) \to H_2(X_{\sigma,1}; \mathbb{Z})$$

which allows us to write

$$\pi_A(\sigma) := \int_{L(A)} \frac{d^3x}{df} = 2\pi\sqrt{-1} \int_A \mathrm{Res}_{E_\sigma} \frac{d^3x}{df}.$$

Let us point out that when $\sigma = 0$ the image of the tube map L is precisely $\mathfrak{h}_0$ and the monodromy representation ρ_0 coincides with the monodromy representation of the elliptic pencil E_σ, $\sigma \in \Sigma$.

2.3. **Givental's higher-genus reconstruction formalism.** Following Givental we introduce the vector space $\mathcal{H} = H((z))$ of formal Laurent series in z^{-1} with coefficients in H, equipped with the symplectic structure

$$\Omega(f(z), g(z)) = \mathrm{res}_{z=0}(f(-z), g(z))dz.$$

Using the polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = H[[z^{-1}]]z^{-1}$ we identify $\mathcal{H}$ with the cotangent bundle $T^*\mathcal{H}_+$. The goal in this subsection is to define the total ancestor potential of W.

2.3.1. *The stationary phase asymptotics.* We fix a primitive form $\omega = d^3x/\pi_A(\sigma)$ and let $t = (t_e)$ be flat coordinates, defined near $s = 0$, s.t., under the Kodaira–Spencer isomorphism $T_0\mathcal{S} \cong H$ we have $\partial_e = \phi_e(x)$. Since, π_A is a multi-valued analytic function on $\mathcal{S}$, the flat coordinates t_e are also multi-valued analytic functions on $\mathcal{S}$.

Let $s \in \mathcal{S}$ be a semi-simple point, i.e., the critical values u_i of F ($1 \le i \le \mu$) form locally near s a coordinate system. Let us also fix a path from $0 \in S$ to s, so that we have a fixed branch of the flat coordinates. Then we have an isomorphism

$$\Psi_s : \mathbb{C}^\mu \to H, \quad e_i \mapsto \sqrt{\Delta_i}\,\partial_{u_i} = \sqrt{\Delta_i}\sum_{e\in E}\frac{\partial u_i}{\partial t_e}\,\phi_e,$$

where Δ_i is determined by $(\partial/\partial u_i, \partial/\partial u_j) = \delta_{ij}/\Delta_i$. It is well known that Ψ_s diagonalizes the Frobenius multiplication and the residue pairing, i.e.,

$$e_i \bullet e_j = \sqrt{\Delta_i} e_i \delta_{i,j}, \quad (e_i, e_j) = \delta_{ij}.$$

Let $\mathcal{S}_{\rm ss}$ be the set of all semi-simple points. The complement $\mathcal{K} = \mathcal{S} \setminus \mathcal{S}_{\rm ss}$ is an analytic hypersurface also known as the *caustic*. It corresponds to deformations, s.t., F has at least one non-Morse critical point. By definition

$$\mathcal{S}_{\rm ss} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^\mu, H), \quad s \mapsto \Psi_s$$

is a multi-valued analytic map.

The system of differential equations (9) and (10) admits a unique formal asymptotical solution of the type

$$\Psi_s R_s(z) e^{U_s/z}, \quad R_s(z) = 1 + R_{s,1}z + R_{s,2}z^2 + \cdots$$

where U_s is a diagonal matrix with entries $u_1(s), \ldots, u_\mu(s)$ on the diagonal and $R_{s,k} \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^\mu, \mathbb{C}^\mu)$. Alternatively this formal solution coincides with the stationary phase asymptotics of the following integrals. Let $\mathcal{B}_i$ be the semi-infinite cycle of the type (6) consisting of all points $x \in \mathbb{C}^3$ such that the gradient trajectories of $-\mathrm{Re}(F/z)$ flow into the critical value u_i. Then

$$(-2\pi z)^{-3/2}\, z d_S \int_{\mathcal{B}_i} e^{F(s,x)/z}\omega \;\sim\; e^{u_i(s)/z}\Psi_s R_s(z) e_i \quad \text{as } z \to 0.$$

We refer to [18] for more details and proofs.

2.3.2. *The quantization formalism.* Let us fix a Darboux coordinate system on $\mathcal{H}$ given by the linear functions q_k^e, $p_{k,e}$ defined as follows:

$$\mathbf{f}(z) = \sum_{k=0}^{\infty}\sum_{e\in E}(q_k^e\,\phi_e\,z^k + p_{k,e}\,\phi^e\,(-z)^{-k-1}) \quad \in \quad \mathcal{H},$$

where $\{\phi^e\}_{e\in E}$ is a basis of H dual to $\{\phi_e\}_{e\in E}$ with respect to the residue pairing.

If $R = e^{A(z)}$, where $A(z)$ is an infinitesimal symplectic transformation, then we define $\widehat{R}$ as follows. Since $A(z)$ is infinitesimal symplectic, the map $\mathbf{f} \in \mathcal{H} \mapsto$

$A\mathbf{f} \in \mathcal{H}$ defines a Hamiltonian vector field with Hamiltonian given by the quadratic function $h_A(\mathbf{f}) = \frac{1}{2}\Omega(A\mathbf{f}, \mathbf{f})$. By definition, the quantization of e^A is given by the differential operator $e^{\widehat{h}_A}$, where the quadratic Hamiltonians are quantized according to the following rules:

$$(p_{k',e'}p_{k'',e''})\widehat{\ } = \hbar\frac{\partial^2}{\partial q_{k'}^{e'}\partial q_{k''}^{e''}}, \quad (p_{k',e'}q_{k''}^{e''})\widehat{\ } = (q_{k''}^{e''}p_{k',e'})\widehat{\ } = q_{k''}^{e''}\frac{\partial}{\partial q_{k'}^{e'}}, \quad (q_{k'}^{e'}q_{k''}^{e''})\widehat{\ } = q_{k'}^{e'}q_{k''}^{e''}/\hbar.$$

Note that the quantization defines a projective representation of the Poisson Lie algebra of quadratic Hamiltonians:

$$[\widehat{F}, \widehat{G}] = \{F, G\}\widehat{\ } + C(F, G),$$

where F and G are quadratic Hamiltonians and the values of the cocycle C on a pair of Darboux monomials is non-zero only in the following cases:

$$C(p_{k',e'}p_{k'',e''}, q_{k'}^{e'}q_{k''}^{e''}) = \begin{cases} 1 & \text{if } (k', e') \neq (k'', e''), \\ 2 & \text{if } (k', e') = (k'', e''). \end{cases} \tag{11}$$

2.3.3. *The total ancestor potential.* By definition, the Kontsevich-Witten tau-function is the following generating series:

$$\mathcal{D}_{\mathrm{pt}}(\hbar; q(z)) = \exp\Big(\sum_{g,n}\frac{1}{n!}\hbar^{g-1}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^{n}(q(\psi_i) + \psi_i)\Big), \tag{12}$$

where $q(z) = \sum_k q_k z^k$, $(q_0, q_1, \ldots)$ are formal variables, ψ_i $(1 \leq i \leq n)$ are the first Chern classes of the cotangent line bundles on $\overline{\mathcal{M}}_{g,n}$. The function is interpreted as a formal series in $q_0, q_1 + \mathbf{1}, q_2, \ldots$ whose coefficients are Laurent series in $\hbar$..

Let $s \in \mathcal{S}_{\mathrm{ss}}$ be *a semi-simple* point. Motivated by Gromov–Witten theory Givental introduced the notion of the *total ancestor potential* of a semi-simple Frobenius structure (see [18]). In our settings the definition takes the form

$$\mathcal{A}_s(\hbar; \mathbf{q}) := \widehat{\Psi_s}\,\widehat{R}_s\, e^{\widehat{U_s/z}}\prod_{i=1}^{\mu}\mathcal{D}_{\mathrm{pt}}(\hbar\,\Delta_i(s); {}^i\mathbf{q}(z)\sqrt{\Delta_i(s)}) \tag{13}$$

where

$$\mathbf{q}(z) = \sum_{k=0}^{\infty}\sum_{e\in E} q_k^e z^k \phi_e, \quad {}^i\mathbf{q}(z) = \sum_{k=0}^{\infty}{}^i q_k\, z^k.$$

The quantization $\widehat{\Psi_s}$ is interpreted as the change of variables

$$\sum_{i=1}^{\mu}{}^i\mathbf{q}(z)e_i = \Psi_s^{-1}\mathbf{q}(z) \quad \text{i.e.} \quad {}^iq_k\sqrt{\Delta_i} = \sum_{e\in E}(\partial_e u^i)\, q_k^e. \tag{14}$$

3. Modularity and monodromy

The flat coordinates are multi-valued analytic functions on $\mathcal{S}$. In this section we will compute their monodromy under analytic continuation. Once this task is completed the proof of Theorem 1.1 will be easy.

3.1. **Picard–Fuchs equations.** We make use of the so called *geometric sections* (see [3])

$$\Phi_e(\sigma,\lambda) := \int x^e \, \frac{d^3x}{df} \quad \in H^2(X_{\sigma,\lambda};\mathbb{C}), \tag{15}$$

where $e = (e_1, e_2, e_3)$, $e_i \in \mathbb{Z}_{\geq 0}$, and $x^e := x_1^{e_1} x_2^{e_2} x_3^{e_3}$. The geometric sections (15) with $e \in E$ give rise to a trivialization of the vanishing cohomology bundle. The Gauss–Manin connection corresponds to a system of Fuchsian differential equations known as *Picard–Fuchs* equations. It is enough to solve this system at $\lambda = 1$, because the homogeneity of $f(\sigma, x)$ yields the following simple relation

$$\Phi_e(\sigma,\lambda) = \lambda^{\deg(e)}\, \Phi_e(\sigma, 1),$$

where $\deg(e) := \deg(x^e) = \sum_i e_i\, q_i$. The Picard–Fuchs equations have the form

$$\partial_\sigma\, \Phi_e(\sigma, 1) = \sum_{e'\in E} G_{e,e'}(\sigma)\, \Phi_{e'}(\sigma, 1), \tag{16}$$

where $G = (G_{e,e'})$ is a square matrix of size $\mu = |E|$, whose entries are holomorphic functions on Σ. Let us denote by $F = (F_{e',e''}(\sigma))$ a fundamental solution to the above system, i.e., F is a non-degenerate matrix satisfying $\partial_\sigma F = G\cdot F$. We have

$$\Phi_e(\sigma,\lambda) = \lambda^{\deg(e)} \sum_{r\in E} F_{e,r}(\sigma)\, A_r, \tag{17}$$

where A_r, $r \in E$ are multi-valued *flat* sections.

The system (16) is block-diagonal in the following sense

$$G_{e',e''} \neq 0 \quad \Rightarrow \quad \deg(e') - \deg(e'') \in \mathbb{Z}.$$

Therefore the matrix F is also block-diagonal. Since analytic continuation around $\lambda = 0$ corresponds to the classical monodromy transformation J, we get that the vectors $\{A_r\}_{r\in E}$ give an eigenbasis of $\mathfrak{h}^\vee$, i.e., $J(A_r) = e^{-2\pi\sqrt{-1}\,\deg(r)} A_r$.

3.2. **Flat coordinates.** We follow the idea of [27], except that we will avoid the use of explicit formulas. It is convenient to introduce the following multi-index notation. We will be interested in sequences $\kappa = (\kappa_e)_{e\in E\setminus\{m\}}$, where κ_e are non-negative integers. Put

$$x^\kappa := \prod_{e\in E\setminus\{m\}} (x_1^{e_1} x_2^{e_2} x_3^{e_3})^{\kappa_e}, \qquad \frac{s^\kappa}{\kappa!} := \prod_{e\in E\setminus\{m\}} \frac{s_e^{\kappa_e}}{\kappa_e!}, \qquad d_\kappa := \deg(s^\kappa) = \sum_e \kappa_e\, d_e,$$

where recall that $d_e = \deg s_e = 1 - \deg(x^e)$.

Let us define a block-diagonal matrix $C = (C_{e,r}(s))_{e,r\in E}$, whose entries are holomorphic functions on $\mathcal{S}$

$$C_{e,r}(s) = \sum_{\kappa: d_\kappa = d_r} c_{e,\kappa}(\sigma)\, \frac{s^\kappa}{\kappa!}, \tag{18}$$

where the functions $c_{e,\kappa}(s)$ are defined from the identity

$$(-2\pi)^{-\frac{3}{2}} \left(\int_0^{-\infty} e^\lambda \lambda^{\deg(x^\kappa)+1} d\lambda \right) \int x^\kappa \, \frac{d^3x}{df} = \sum_{e\in E} c_{e,\kappa}(\sigma)\, \Phi_e(\sigma, 1), \tag{19}$$

where the integration path in the first integral is the negative real axis and the 2-nd one is interpreted as a cohomology class in $H^2(X_{\sigma,1}; \mathbb{C})$. The identity is obtained by performing successively integration by parts until the degree of the monomial x^κ is reduced to some number in the interval $[0, 1]$. In particular, since each integration by parts decreases the degree by an integer number, the sum on the RHS is over all $e \in E$, s.t., $d_e - d_\kappa \in \mathbb{Z}$. It follows that the matrix C is block-diagonal. Given cycles $\alpha_r \in \mathfrak{h}^\vee$, $r \in E$, we define the following multi-valued analytic functions on $\mathcal{S}$:

$$\begin{aligned}
t_m(s) &= \Big(C_{0,m}(s) \Big\langle \Phi_0(\sigma, 1), \alpha_m \Big\rangle + C_{m,m}(s) \Big\langle \Phi_m(\sigma, 1), \alpha_m \Big\rangle \Big) \pi_A^{-1}, \\
t_0(s) &= \Big(C_{m,0}(s) \Big\langle \Phi_m(\sigma, 1), \alpha_0 \Big\rangle + C_{0,0}(s) \Big\langle \Phi_0(\sigma, 1), \alpha_0 \Big\rangle \Big) \pi_A^{-1}, \\
t_r(s) &= \Big(\sum_{e\in E: d_e = d_r} C_{e,r}(s) \Big\langle \Phi_e(\sigma, 1), \alpha_r \Big\rangle \Big) \pi_A^{-1}, \quad r \in E_{\mathrm{tw}}.
\end{aligned}$$

Note that by definition $C_{m,m}(s) = 0$ and $C_{0,m}(s)$ is a constant independent of s.

Proposition 3.1. *There are cycles $\{\alpha_r\}_{r\in E}$ that form an eigenbasis for the classical monodromy J, s.t.,*

(i) *The functions $t_e(s)$, $e \in E$ form a flat coordinate system on $\mathcal{S}$.*
(ii) *We have $\partial_0 = 1$ and $(\partial_0, \partial_m) = 1$, where $\partial_e := \partial/\partial t_e$.*
(iii) *The following identity holds (compare with the definition of $t_0(s)$):*

$$\frac{1}{2} \sum_{e',e''\in E} (\partial_{e'}, \partial_{e''}) t_{e'} t_{e''} = \Big(C_{m,0}(s) \Big\langle \Phi_m(\sigma, 1), \alpha_m \Big\rangle + C_{0,0}(s) \Big\langle \Phi_0(\sigma, 1), \alpha_m \Big\rangle \Big) \pi_A^{-1}.$$

Proof. Let $\sigma \in \Sigma$ be an arbitrary point. We fix a path in Σ from 0 to σ. Our goal is to construct flat coordinates in a neighborhood of σ. Given a basis of cycles $\{\alpha_r\}_{r\in E}$ we denote by $\alpha_r^{\sigma,1} \in H_2(X_{\sigma,1}; \mathbb{C})$ the parallel transport of α_r. The polynomial $f(\sigma, x)$ is weighted homogeneous, so there is a natural $\mathbb{C}^*$-action on $\mathbb{C}^3$, s.t., $f(\sigma, c\cdot x) = c f(\sigma, x)$ for every $c \in \mathbb{C}^*$. Using this action we define

$$\mathcal{A}_r = \{(\lambda z)\cdot y \mid \lambda \in (-\infty, 0], \quad y \in \alpha_r^{\sigma,1}\}.$$

Note that $\mathcal{A}_r^{\sigma,z}$ is a semi-infinite cycle of the type (6), so the corresponding oscillatory integral is convergent. Using the Fubini's theorem we get

$$(-2\pi z)^{-3/2}\int_{\mathcal{A}_r^{\sigma,z}} e^{F(s,x)/z}\omega = (-2\pi z)^{-3/2}\int_0^{-\infty} e^{\lambda}\,(\lambda z)\int_{\alpha_r^{\sigma,1}} e^{\sum_{e\in E\setminus\{m\}} s_e x^e \lambda^{(1-d_e)} z^{-d_e}}\,\frac{\omega}{df}\,d\lambda$$

The exponential in the 2-nd integral on the RHS is

$$\sum_{\kappa}\frac{s^\kappa}{\kappa!}\,x^\kappa\,\lambda^{\deg(x^\kappa)}\,z^{-d_\kappa},$$

where the sum is over all sequences $\kappa = (\kappa_e)_{e\in E\setminus\{m\}}$ of non-negative integers. Substituting the above expansion we get

$$(-2\pi)^{-3/2}z^{-1/2}\sum_{\kappa}\left(\int_0^{-\infty} e^{\lambda}\,\lambda^{1+\deg(x^\kappa)}\,d\lambda\right)\frac{s^\kappa}{\kappa!}\,z^{-d_\kappa}\int_{\alpha_r^{\sigma,1}} x^\kappa\,\frac{\omega}{df}\,.$$

Comparing with formula (19) we get the following formula for the oscillatory integral

$$J_{\mathcal{A}_r}(s,z) = z^{\frac{1}{2}}\,d\left(\sum_{\kappa} z^{-d_\kappa}\frac{s^\kappa}{\kappa!}\sum_{e\in E} c_{e,\kappa}(\sigma)\left\langle \Phi_e(\sigma,1),\alpha_r\right\rangle \pi_A^{-1}\right). \tag{20}$$

The oscillatory integrals $J_{\mathcal{A}_r}(s,z)$ are solutions to the differential equations (9) and (10). On the other hand near $z=\infty$ these equations have a fundamental solution of the type $S_t(z)z^{\Theta}$, where $S_t(z) = 1 + S_{t,1}z^{-1}+\cdots$ and $S_{t,k}\in \mathrm{Hom}_{\mathbb{C}}(H,H)$. Therefore, we can choose the cycles $\{\alpha_r\}$ in such a way that

$$J_{\mathcal{A}_r}(s,z) = S_t(z)z^{\Theta}\,dt_r = z^{\frac{1}{2}-d_r}\Big(dt_r + z^{-1}S_{t,1}dt_r+\cdots\Big), \tag{21}$$

where $t=(t_r)$ is a flat coordinate system. Note that $d_r>0$ for $r\neq m$ and $d_m=0$. Therefore, we have

$$J_{\mathcal{A}_r}(s,z) = z^{\frac{1}{2}-d_r}dt_r + z^{\frac{1}{2}-1}\delta_{r,m}S_{t,1}\,dt_m+\cdots,$$

where the dots stand for terms involving higher order powers of z^{-1}. Let us choose the flat coordinates in such a way that $\partial_0 = 1$ and $(\partial_m,\partial_0)=1$, then we have

$$S_{t,1}\,dt_m = S_{t,1}\,1 = t = \frac{1}{2}d\Big(\sum_{e',e''\in E}(\partial_{e'},\partial_{e''})t_{e'}t_{e''}\Big).$$

Finally we get

$$J_{\mathcal{A}_r}(s,z) = z^{\frac{1}{2}}d\left(z^{-d_r}t_r + z^{-1}\delta_{r,m}\,\frac{1}{2}\sum_{e',e''\in E}(\partial_{e'},\partial_{e''})t_{e'}t_{e''}\right)+\cdots.$$

All statements in the Proposition follow by comparing the above formula with (20).

□

3.3. **The monodromy of the flat coordinates.** Let us choose the fundamental matrix F of the Picard–Fuchs equations (see (17)) in such a way that $\{\alpha_r\}_{r\in E}$ and $\{A_r\}_{r\in E}$ are dual bases. Furthermore, since $C_{0,m}\,\alpha_m$ is a tube cycle we can find $B\in H_1(E_0;\mathbb{C})$ such that $L(B) := C_{0,m}\,\alpha_m$, so we have

$$t_m = \frac{\pi_B(\sigma)}{\pi_A(\sigma)}.$$

The flat coordinate t_0 is such that $\partial/\partial t_0 = 1$. Therefore, the coefficient in front of s_0 in $t_0(s)$ must be 1. On the other hand, using formulas (18) and (19) we get

$$C_{0,m}(s) = c_{0,0},\quad C_{0,0}(s) = c_{0,0}\,s_0 + \cdots,$$

where the dots stand for at least quadratic terms in s. It follows that $L(A) = C_{0,m}\,\alpha_0$.

Let γ be a loop in Σ based at the refernce point $\sigma = 0$. Let us denote by $[\rho_{\neq 0}(\gamma)]_{e,r}$ the matrix of the linear operator $\rho_{\neq 0}(\gamma)$ in the basis $\{\alpha_r\}_{r\in E_{\rm tw}}$, i.e.,

$$\rho_{\neq 0}(\gamma)\alpha_r = \sum_{e\in E_{\rm tw}} [\rho_{\neq 0}(\gamma)]_{e,r}\,\alpha_e.$$

Since the monodromy representation $\rho_{\neq 0}$ commutes with the classical monodromy J and $\{\alpha_r\}$ is an eigenbasis for J, the matrix $[\rho_{\neq 0}(\gamma)]$ is block diagonal

$$[\rho_{\neq 0}(\gamma)]_{e,r} \neq 0 \quad \Rightarrow \quad d_e = d_r.$$

Similarly let us denote by $[\rho_0(\gamma)]$ the matrix of $\rho_0(\gamma)$ in the basis $\{\alpha_m,\alpha_0\}$

$$\begin{aligned} \rho_0(\gamma)\alpha_m &= [\rho_0(\gamma)]_{m,m}\alpha_m + [\rho_0(\gamma)]_{0,m}\alpha_0, \\ \rho_0(\gamma)\alpha_0 &= [\rho_0(\gamma)]_{m,0}\alpha_m + [\rho_0(\gamma)]_{0,0}\alpha_0. \end{aligned}$$

The space $\mathfrak{h}_0 \cong H_1(E_0,\mathbb{C})$ is equipped with a symplectic form that comes from the intersection pairing. By continuity, the linear transformation $\rho_0(\gamma)$ is a symplectic transformation, i.e., the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} := [\rho_0(\gamma)]^T = \begin{bmatrix} [\rho_0(\gamma)]_{m,m} & [\rho_0(\gamma)]_{0,m} \\ [\rho_0(\gamma)]_{m,0} & [\rho_0(\gamma)]_{0,0} \end{bmatrix} \quad \in \mathrm{SL}_2(\mathbb{C}).$$

An immediate corollary of Proposition 3.1 is the transformation rule for the flat coordinates under the analytic continuation along γ.

Corollary 3.2. *The analytic continuation along a loop γ transforms the flat coordinates as follows*

$$\begin{aligned} \widetilde{t_m} &= \frac{a\,t_m + b}{c\,t_m + d}, \\ \widetilde{t_0} &= t_0 + \frac{c}{2(c\,t_m + d)} \sum_{e',e''\in E_{\rm tw}} (\partial_{e'},\partial_{e''})\,t_{e'}t_{e''}, \\ \widetilde{t_r} &= \sum_{e\in E_{\rm tw}: d_e = d_r} [\rho_{\neq 0}(\gamma)]_{e,r}\,t_e, \quad r\in E_{\rm tw}. \end{aligned}$$

3.4. **Monodromy of the asymptotical operator.** Recall the notation from Section 2.3.1. Let us identify the space of linear operators $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^\mu, H)$ with the space of square matrices of size μ by fixing the bases $\phi_e(x) := x^e (e \in E)$ and $e_i (1 \le i \le \mu)$ respectively in H and in $\mathbb{C}^\mu$, i.e.,

$$Ae_i = \sum_{e\in E} [A]_{e,i}\, \phi_e.$$

The asymptotical operator $\Psi_s R_s e^{U_s/z}$ can be viewed as a matrix with entries formal asymptotical series. Let us fix a loop γ in Σ. We would like to find out how the operator changes under the analytic continuation along γ. The answer can be stated in the following way. Let $M(\gamma, t) \in \mathrm{Hom}_{\mathbb{C}}(H, H)$ be the operator whose matrix is

$$[M(\gamma, t)]_{r,e} = (ct_m + d)^{-1} \frac{\partial t_r}{\partial \widetilde{t_e}} - z\, c\, \delta_{e,m}\delta_{0,r},$$

where c, d, and $\widetilde{t_e}$ are determined via γ as it was stated in Corollary 3.2. Analytic continuation along γ transforms the sequence of critical values $(u_1(s), \dots, u_\mu(s))$ via some permutation p. Let us denote by $P(\gamma) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^\mu, \mathbb{C}^\mu)$ the linear operator whose matrix is given by

$$[P(\gamma)]_{i,j} = \delta_{i,p(j)}.$$

Proposition 3.3. *The analytic continuation along the loop γ transforms the asymptotical operator $\Psi_s R_s e^{U_s/z}$ into*

$${}^{\mathrm{T}}M(\gamma, t)\, \Psi_s R_s e^{U_s/z}\, P(\gamma),$$

where for a linear operator $A : H \to H$ we denote by ${}^{\mathrm{T}}A$ the transpose of A with respect to the residue pairing $(\cdot, \cdot)$.

Proof. Let us denote by $I_i(s, z) (1 \le i \le \mu)$ the stationary phase asymptotic of the oscillatory integral

$$(-2\pi z)^{-3/2} \int_{\mathcal{B}_i} e^{F(s,x)/z} d^3x.$$

By definition the asymptotical operator is defined by the following identity:

$$(\phi_e, \Psi_s R_s e^{U_s/z} e_i) = z\partial_e\Big(I_i(s, z)\pi_A^{-1}\Big), \quad e \in E, \quad 1 \le i \le \mu.$$

The analytic continuation along γ transforms the above matrix into

$$z\widetilde{\partial_e}\Big(I_{p(i)}(s, z)\pi_A^{-1}(ct_m + d)^{-1}\Big) = \sum_{r\in E} \frac{\partial t_r}{\partial \widetilde{t_e}}\, z\partial_r \Big(I_{p(i)}(s, z)\pi_A^{-1}(ct_m + d)^{-1}\Big), \qquad (22)$$

where $\widetilde{\partial_e} := \partial/\partial\widetilde{t_e}$. Note that

$$\sum_{r\in E} \frac{\partial t_r}{\partial \widetilde{t_e}}\, z\partial_r (ct_m + d)^{-1} = z\delta_{e,m} \frac{\partial t_m}{\partial \widetilde{t_m}} \Big(- c\, (ct_m + d)^{-2}\Big) = -zc\, \delta_{e,m}$$

and

$$I_{p(i)}(s,z)\pi_A^{-1} = z\partial_0\Big(I_{p(i)}(s,z)\pi_A^{-1}\Big).$$

Hence the RHS of (22) becomes

$$\Big(\sum_{r\in E}(ct_m+d)^{-1}\frac{\partial t_r}{\partial \widetilde{t}_e}\, z\partial_r(I_{p(i)}(s,z)\pi_A^{-1})\Big) - zc\delta_{e,m}\, z\partial_0(I_{p(i)}(s,z)\pi_A^{-1}).$$

It remains only to check that the above expression coincides with

$$(M(\gamma,t)\phi_e, \Psi_s R_s e^{U_s/z} P\, e_i) = \sum_{r\in E}[M(\gamma,t)]_{r,e}\, z\partial_r(I_{p(i)}(s,z)\pi_A^{-1}). \quad \square$$

Let denote by $J(\gamma,t)\in \mathrm{Hom}_{\mathbb{C}}(H,H)$ the linear operator whose matrix is the Jacobian of the transformation $t\mapsto \widetilde{t}$ (see Corollary 3.1)

$$[J(\gamma,t)]_{r,e} = \frac{\partial t_r}{\partial \widetilde{t}_e}.$$

Let us introduce also the linear operator

$$X(\gamma,t) = 1 - \frac{cz}{ct_m+d}\,\phi_m\bullet_{s=0},$$

where $\phi_m\bullet_{s=0} : H\to H$ is the operator of multiplication by ϕ_m in the Jacobi algebra H. Note that $X(\gamma,t)$ is a symplectic transformation (of $\mathcal{H}$).

Proposition 3.4. *Analytic continuation transforms*

$$\mathcal{A}_t(\hbar;\mathbf{q}) \mapsto (\widehat{X}(\gamma,t)\,\mathcal{A}_t)(\hbar(ct_m+d)^2; J(\gamma,t)\mathbf{q}),$$

where we first apply the operator $\widehat{X}(\gamma,t)$ and then we rescale $\hbar$ and $\mathbf{q}$.

We may assume that $P(\gamma,t)=1$ because P is a permutation matrix, so its quantization $\widehat{P}$ will leave the product of Kontsevich–Witten tau functions invariant. Put $M = M_0 + M_1 z$. Then we have

$${}^{\mathrm{T}}M\Psi R e^{U/z} = \widetilde{\Psi}\widetilde{R}e^{U/z}, \quad \text{where} \quad \widetilde{\Psi} = M_0^{-1}\Psi, \quad \widetilde{R} = \Psi^{-1}M_0{}^{\mathrm{T}}M\Psi R.$$

The quantization is in general only a projective representation. However, the quantization of the operators $\Psi^{-1}M_0{}^{\mathrm{T}}M\Psi$ and R involves quantizing only p^2 and $p\,q$-terms. Since the cocycle (11) on such terms vanishes we get

$$(\widetilde{R})\widehat{\ } = (\Psi^{-1}M_0{}^{\mathrm{T}}M\Psi)\widehat{\ }\widehat{R}.$$

The operators M_0 and Ψ are independent of z and their quantizations by definition are just changes of variables. Hence

$$(\widetilde{\Psi}\widetilde{R})\widehat{\ } = \widehat{M_0}^{-1}(M_0{}^{\mathrm{T}}M)\widehat{\ }(\Psi R)\widehat{\ }.$$

By definition Δ_i^{-1} is $(\partial_{u_i}, \partial_{u_i})$, which gains a factor of $(ct_m + d))^{-2}$ under analytic continuation. The ancestor potential (13) is transformed into

$$\widehat{M_0}^{-1}(M_0{}^{\mathrm{T}}M)\widehat{\ }\Big(\mathcal{A}_t((ct_m + d)^2\hbar; (ct_m + d)\mathbf{q})\Big). \tag{23}$$

Note that

$$M_0^{-1} = (ct_m + d)\ J(\gamma, t)^{-1}, \quad M_0({}^{\mathrm{T}}M) = X(\gamma, t).$$

It remains only to notice that the rescaling

$$(\hbar, \mathbf{q}) \mapsto ((ct_m + d)^2\hbar, (ct_m + d)\mathbf{q})$$

commutes with the action of any quantized operator. □

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