THE PHASE FACTORS IN SINGULARITY THEORY

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ABSTRACT. The paper [2] proposed a construction of a twisted representation of the lattice vertex algebra corresponding to the Milnor lattice of a simple singularity. The main difficulty in extending the above construction to an arbitrary isolated singularity is in the so called *phase factors* – the scalar functions produced by composing two vertex operators. They are certain family of multivalued analytic functions on the space of miniversal deformations. The first result in this paper is an explicit formula for the unperturbed phase factors in terms of the classical monodromy operator and the polylogorithm functions. Our second result is that with respect to the deformation parameters the phase factors are analytic functions on the monodromy covering space.

CONTENTS

1. INTRODUCTION

Lattice Vertex Operator Algebras (VOAs) were introduced by R. Borcherds in his work on the Moonshine conjecture (see [3, 4]). After Borcherds' work, lattice VOAs were used as a tool to generalize the notion of a Kac-Moody Lie algebra. In particular, it is very natural to apply Borcherds' construction to singularity theory, because we have all the necessary ingredients: a Milnor lattice equipped with the intersection pairing. What is even more exciting is that there is a natural candidate for a root system, namely the set of vanishing cycles. It is very tempting to investigate to what extend the classical theory of simple Lie algebras extends to singularity theory. We refer to [17, 20] for further details on Lie algebras in singularity theory. The applications of this class of Lie algebras is still not quite clear. In particular, fundamental questions, such as finding the root multiplicities and classifying the irreducible representations are wide open. Nevertheless, Gromov–Witten theory provides a new motivation to develop further Borcherds' ideas in the settings of singularity theory.

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Especially, Givental's higher genus reconstruction formalism [8, 10, 21] and the integrable hierarchies introduced by Dubrovin and Zhang [6] suggest to investigate whether the Lie algebras in singularity theory are a source of integrability in the same way as the ADE affine Lie algebras are a source of integrability for the Kac–Wakimoto hierarchies [14].

The relevance of period integrals in singularity theory for describing integrable hierarchies was first observed by Givental in [10] and developed further in [7]. Partially motivated by these works, it was suggested in [2] that the period integrals associated with a given singularity can be used to construct a σ-twisted representation of the lattice VOA associated with the Milnor lattice, where σ is the classical monodromy. The main issue in confirming the prediction of [2] arises when we compose two vertex operators. We get a set of phase factors that we can not study only in the category of formal Laurent series, due to the fact that exponentiating a formal Laurent series does not make sense in general. The goal of this paper is to establish the analytic properties of the phase factors that are necessary in order to extend the construction of [2] in general.

Our first result is that the vertex operators are mutually local in the sense of VOA. The proof is based on an explicit computation of the unperturbed phase factors. Our main tool is a formula due to Hertling that expresses Saito's higher residue pairing in terms of the *Seifert form*. The final answer is given in terms of the polylogorithm functions. Furthermore, it is not hard to see that the phase factors are convergent and we can express them as integrals along the path of the so called *phase form* (see (2.12)) – quadratic form on the vanishing homology with values 1-forms on the space of deformation parameters. Using the locality and the fact that the Gauss–Manin connection has regular singularities, we prove that the phase factors are analytic functions on the monodromy covering space. This fact was very important in both [2] and [10] for constructing W-constraints and Hirota quadratic equations of the total descendant potential. The compatibility of the phase factors with the monodromy representation is a very non-trivial and somewhat surprising result. It is equivalent to the following question of Givental [10]: given a path C and a pair of integral cycles invariant along C, then is it true that the integral along C of the corresponding phase form is an integer multiple of $2\pi\sqrt{-1}$? Givental answered the above question positively in the case of simple singularities. Unfortunately, his argument is hard to generalize, because it relies on understanding the fundamental group of the complement of the discriminant.

2. Statement of the main results

2.1. **Singularity theory.** Let $W : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated singularity at $x = 0$. We denote by $x = (x_0, x_1, \ldots, x_n)$ the standard coordinate system on $\mathbb{C}^{n+1}$ and by

$$H := \mathcal{O}_{\mathbb{C}^{n+1},0}/(W_{x_0}, W_{x_1}, \ldots, W_{x_n}), \quad W_{x_i} = \partial W/\partial x_i,$$

the Jacobi algebra of W. Let $\{\phi_j(x)\}_{j=1}^N$ be a set of holomorphic germs that represent a basis of H. We may assume that $\phi_1 = 1$. We are interested in the following deformation of W

$$F(x,t) = W(x) + \sum_{j=1}^N t_j\phi_j(x), \quad t = (t_1, \ldots, t_N) \in \mathbb{C}^N.$$

Let us cut a Stein domain $X \subset \mathbb{C}^{n+1} \times \mathbb{C}^N$ around $(0,0)$, s.t., the map

$$\varphi : X \to B \times \mathbb{C}_\epsilon, \quad (x,t) \mapsto (t, F(x,t))$$

is well defined and surjective, where $B \subset \mathbb{C}^N$ is a sufficiently small ball around 0 and $\mathbb{C}_\epsilon = \{\lambda \in \mathbb{C} : |\lambda| < \epsilon\}$, and the fibers $X_{t,\lambda} := \varphi^{-1}(t,\lambda)$ satisfy an appropriate transversality condition (see [1]). Put $S := B \times \mathbb{C}_\epsilon$ to avoid cumbersome notation.

Let C be the critical locus $\{F_{x_0} = \cdots = F_{x_n} = 0\} \subset X$ of F relative to B. The map

$$\mathcal{T}_B \to \mathrm{pr}_* \mathcal{O}_C, \quad \partial/\partial t_i \mapsto \partial F/\partial t_i \ (\mathrm{mod}\ \ F_{x_0}, \ldots, F_{x_n})$$

is called the *Kodaira–Spencer* map. Decreasing X, B, and ϵ if necessary we may arrange that the Kodaira–Spencer map is an isomorphism. In particular, each tangent space $T_t B$ has an associative algebra structure. Let us fix a *primitive* form $\omega \in \Omega^{n+1}_{X/B}(X)$, so that B becomes a Frobenius manifold (see [11, 18]). The flat Frobenius pairing is given by the following residue pairing:

$$(\phi_1(x), \phi_2(x))_t := \frac{1}{(2\pi\sqrt{-1})^{n+1}} \int_\Gamma \frac{\phi_1(y)\phi_2(y)}{F_{y_0}(t,y)\cdots F_{y_n}(t,x)} dy_0 \ldots dy_N, \tag{2.1}$$

where the cycle Γ is a disjoint union of sufficiently small tori around the critical points of F defined by equations of the type $|F_{x_0}| = \cdots = |F_{x_n}| = \epsilon$ and $y = (y_0, \ldots, y_n)$ is a local coordinate system near each critical point, s.t., $\omega = dy_0 \wedge \cdots \wedge dy_n$. We have the following identifications:

$$T^*B \cong TB \cong B \times T_0 B \cong B \times H, \tag{2.2}$$

where the first isomorphism is given by the residue pairing, the second by the Levi–Civita connection of the flat residue pairing, and the last one is the Kodaira–Spencer isomorphism. Let us denote by ∂_i the flat vector fields corresponding to the basis $\{\phi_i\}_{i=1}^N \subset H$.

2.2. **Period vectors.** Removing the singular fibers from X we obtain a smooth fibration $X' \to S'$ known as the Milnor fibration, where S' is the subset parametrizing non-singular fibers. Its complement is an irreducible analytic hypersurface known as the *discriminant*. The homology $H_n(X_{t,\lambda}; \mathbb{C})$ and the cohomology groups $H^n(X_{t,\lambda}; \mathbb{C})$ form vector bundles on S' equipped with a flat Gauss–Manin connection. Let us fix a reference point $(0, \epsilon_0) \in S'$ and denote by

$$\mathfrak{h} := H^n(X_{0,\epsilon_0}; \mathbb{C}), \quad \mathfrak{h}^* := H_n(X_{0,\epsilon_0}; \mathbb{C})$$

the reference fibers. The vector space $\mathfrak{h}^*$ contains the so called *Milnor lattice* $Q := H_n(X_{0,\epsilon_0}; \mathbb{Z})$ and if we assume that n is even then the intersection pairing gives a symmetric bi-linear pairing on Q (which however might be degenerate). We normalize the intersection pairing by the sign $(-1)^{n/2}$ in order to obtain a pairing $(\,|\,)$, s.t., $(\alpha|\alpha) = 2$ for every *vanishing cycle* $\alpha \in Q$. We refer again to [1] for some more details on the vanishing homology.

Using the parallel transport with respect to the Gauss–Manin connection we get the so called monodromy representation

$$\pi_1(S') \to \mathrm{O}(\mathfrak{h}^*), \tag{2.3}$$

where the RHS denotes the group of linear transformations that preserve the intersection pairing. The image of the monodromy representation is a reflection group W generated by the reflections

$$s_\alpha(x) = x - (\alpha|x)\alpha,$$

where α is a vanishing cycle and s_α is the monodromy transformation representing a simple loop corresponding to a path from the reference point to a generic point on the discriminant along which α vanishes.

Let us introduce the notation d_x, where $x = (x_1, \dots, x_m)$ is a coordinate system on some manifold, for the de Rham differential in the coordinates x. This notation is especially useful when we have to apply d_x to functions that might depend on other variables as well. The main object of our interest are the following period integrals (see [10])

$$I^{(k)}_\alpha(t,\lambda) = -d_t\ (2\pi)^{-1}\,\partial_\lambda^{k+1} \int_{\alpha_{t,\lambda}} d_x^{-1}\omega\ \in T_t^*B \cong H, \tag{2.4}$$

where $\alpha \in \mathfrak{h}^*$ is a cycle from the vanishing homology, $\alpha_{t,\lambda} \in H_n(X_{t,\lambda}, \mathbb{C})$ is the parallel transport of α along a reference path, and $d_x^{-1}\omega$ is a holomorphic n-form $\eta \in \Omega^n_{X/B}$ defined in a neighborhood of the fiber $X_{t,\lambda}$, s.t., $d_x\eta = \omega$. The periods are multivalued analytic functions in $(t,\lambda) \in S$ with poles along the discriminant. In other words they are analytic functions $\widetilde{S'} \to H$, where $\widetilde{S'}$ is the monodromy covering space of S'.

The periods satisfy the following system of differential equations

$$\partial_i I^{(k)}_\alpha(t,\lambda) = -\phi_i \bullet_t I^{(k+1)}_\alpha(t,\lambda) \tag{2.5}$$

$$\partial_\lambda I^{(k)}_\alpha(t,\lambda) = I^{(k+1)}_\alpha(t,\lambda) \tag{2.6}$$

$$(\lambda - E\bullet_t)\partial_\lambda I^{(k)}_\alpha(t,\lambda) = \Big(\theta - k - \frac{1}{2}\Big) I^{(k)}_\alpha(t,\lambda), \tag{2.7}$$

where the notation in (2.7) is as follows. The vector field $E \in \mathcal{T}_B$ is the *Euler* vector field, which by definition corresponds to F via the Kodaira spencer isomorphism. Changing the basis $\{\phi_i\}_{i=1}^N \subset H$ if necessary we may arrange that the Euler vector field takes the form

$$E = \sum_{i=1}^N (1-d_i)\tau_i\partial_i + \rho, \tag{2.8}$$

where ρ is a flat vector field and the degree spectrum $1 = d_1 < d_2 \leq \cdots \leq d_{N-1} < d_N =: D$ satisfies $d_i + d_{N+1-i} = D$. The number D is the *conformal dimension* of the Frobenius structure. The linear operator θ is defined by

$$\theta : H \to H, \quad \phi_i \mapsto (D/2 - d_i)\phi_i.$$

It is sometimes called the *Hodge-grading operator*.

2.3. **Extending the domain of the period vectors.** Using the differential equations (2.5)–(2.7) we would like to extend the domain S in such a way that the period vectors have a translation symmetry and that there exists a point $u^{(0)} \in B$, such that the singularities of the period vectors $I^{(m)}_\alpha(u^{(0)},\lambda)$ are at points $\lambda = u_i$ $(1 \leq i \leq N)$ that coincide with the vertices of a regular N-gon with center 0.

The differential equation (2.7) allows us to extend the definition of $I^{(m)}_\alpha(t,\lambda)$ for all $(t,\lambda) \in B\times\mathbb{C}$. Let us denote by $t = (t_1, \dots, t_N)$ the standard coordinate system on $B \subset \mathbb{C}^N$ induced from the linear coordinates on $\mathbb{C}^N$. We denote by $t - \lambda\mathbf{1} = (t_1 - \lambda, t_2, \dots, t_N)$, i.e., this is the time $-\lambda$-flow of t with respect to the flat identity. Note that $X_{t,\lambda} = X_{t-\lambda\mathbf{1},0}$, so the period vectors have the following translation symmetry

$$I^{(k)}_\alpha(t,\lambda) = I^{(k)}_\alpha(t-\lambda\mathbf{1},0), \tag{2.9}$$

where $|\lambda| \ll 1$. Note that the RHS is a multivalued function on $B' := B \times \{0\} \cap S'$, while the LHS is a multivalued function on S'. We choose $-\epsilon_0 \mathbf{1}$ as a reference point in B' and the reference path for the RHS is obtained from the reference path for the LHS via the translation map

$$\tau : S' \to B', \quad (t, \lambda) \mapsto t - \lambda \mathbf{1}.$$

Using (2.9), we extend the definition of the periods also for all t that belong to a domain in $\mathbb{C}^N$ obtained from B by the flow of the flat vector field ∂_1. Slightly abusing the notation we denote by B the extended domain and redefine S to be $B \times \mathbb{C}$. Note that the Frobenius structure also extends on B and now the translation symmetry (2.9) makes sense for all $(t, \lambda) \in S$.

Let us fix a point $t_0 \in B$, s.t., the critical values of $F(x, t_0)$ are pairwise distinct. In particular, there exists an open neighborhood $\mathcal{U} \subset B$ of t_0 on which the critical values $\{u_i(t)\}_{i=1}^N$, $t \in \mathcal{U}$, form a *canonical* coordinate system, i.e., if we put $1/\Delta_i := (\partial/\partial u_i, \partial/\partial u_i)$ $(1 \leq i \leq N)$, then the map

$$\Psi_t : \mathbb{C}^N \to T_t U, \quad e_i \mapsto \sqrt{\Delta_i} \partial/\partial u_i$$

gives a trivialization of the tangent bundle $T\mathcal{U}$ in which the Frobenius multiplication and the residue pairing have a diagonal form

$$e_i \bullet e_j = \delta_{i,j} \sqrt{\Delta_j} e_j, \quad (e_i, e_j) = \delta_{i,j}.$$

If we write the differential equations (2.5)–(2.7) in canonical coordinates, then we obtain a system of linear ODEs whose coefficients depend polynomially on the entries of the matrices

$$V := -\Psi^{-1} \theta \Psi, \quad (\lambda - U)^{-1}, \quad \mathrm{ad}_U^{-1}(V),$$

where U is the diagonal matrix $\mathrm{diag}(u_1, \dots, u_N)$. Here we are using that the matrix V is skew-symmetric and we define $\mathrm{ad}_U^{-1}(V)$ to be the unique matrix X with zero diagonal entries, s.t., $[U, X] = V$. The dependence of the matrix V on the canonical coordinates $\{u_i\}_{i=1}^N$ is quite remarkable. It was proved by Dubrovin that $V = V(u)$, $u = (u_1, \dots, u_N)$, is a solution to an integrable system (see equation (3.74) in [5]) and that V extends analytically along any path in the domain $\mathfrak{D} = \{u \in \mathbb{C}^N \mid u_i \neq u_j, \forall i \neq j\}$ (see Corollary 3.4 in [5]). Using the canonical coordinates, we embed $\mathcal{U} \subset \mathbb{C}^N$. Decreasing $\mathcal{U}$ if necessary we may assume that $\mathcal{U} \subset \mathfrak{D}$. Let us pick a point $u^{(0)} \in \mathbb{C}^N$ whose coordinates are vertices of a regular polygon with center at 0 and fix a path in $\mathfrak{D}$ that connects $\mathcal{U}$ and $u^{(0)}$. We can extend $V(u)$ analytically along the path and obtain a function analytic in a slightly larger domain $\widetilde{\mathcal{U}}$. In particular, we can extend the system (2.5)–(2.7) on $\widetilde{\mathcal{U}} \times \mathbb{C}$, which gives us the required extension of the period vectors. Again, we redenote by B the extended domain obtained by gluing B and $\widetilde{\mathcal{U}}$ along the maximal open subset $\mathcal{U}'$, s.t., $\mathcal{U} \subset \mathcal{U}'$ and the analytic embedding $\mathcal{U} \to \widetilde{\mathcal{U}}$ extends to an analytic embedding $\mathcal{U}' \to \widetilde{\mathcal{U}}$.

2.4. **The phase factors.** Our main goal is to investigate the analytic properties of the following infinite series:

$$\Omega_{\alpha,\beta}(t, \lambda, \mu) = \sum_{n=0}^{\infty} (-1)^{n+1} (I_\alpha^{(n)}(t, \lambda), I_\beta^{(-n-1)}(t, \mu)) \quad \in \quad \mathbb{C}(\!(\lambda^{-1/|\sigma|})\!), \tag{2.10}$$

where σ is the classical monodromy operator and $|\sigma|$ is the order of its semi-simple part. To begin with, it is not hard to determine the radius of convergence. Namely, for fixed

$(t,\mu)\in S'$, the series (2.10) is convergent for

$$|\lambda| > \max_i\{|\mu|, |u_i(t)|\},$$

where $u_i(t)$ is the set of critical values of $F(t,x)$. The main motivation to study the above series comes from the so called *phase factors* $B_{\alpha,\beta}(t,\lambda,\mu) := e^{\Omega_{\alpha,\beta}(t,\lambda,\mu)}$, which are produced naturally when we compose two vertex operators (see Section 5).

Recall the variation operator isomorphism

$$\mathrm{Var} : \mathfrak{h} := H^n(X_{0,\epsilon_0};\mathbb{Z}) \to \mathfrak{h}^* := H_n(X_{0,\epsilon_0};\mathbb{Z})$$

defined by composing the Lefschetz duality $H^n(X_{0,\epsilon_0};\mathbb{Z}) \cong H_n(X_{0,\epsilon_0},\partial X_{0,\epsilon_0};\mathbb{Z})$ and the operator h_*-1, where $h:(X_{0,\epsilon_0},\partial X_{0,\epsilon_0}) \to (X_{0,\epsilon_0},\partial X_{0,\epsilon_0})$ is the geometric monodromy. Let us recall also the *Seifert forms* (see [1]) in respectively cohomology and homology:

$$\begin{aligned}
\mathrm{SF}(A,B) &:= (-1)^{n/2+1}\langle A, \mathrm{Var}(B)\rangle, \quad A,B\in\mathfrak{h},\\
\mathrm{SF}(\alpha,\beta) &:= (-1)^{n/2+1}\langle \mathrm{Var}^{-1}(\alpha), \beta\rangle, \quad \alpha,\beta\in\mathfrak{h}^*,
\end{aligned}$$

where recall that we required n to be even. Both forms are non-degenerate, integer valued, and σ-invariant. The sign here is chosen in such a way that the symmetrization of the Seifert form yields the sign-normalized intersection form introduced above

$$\mathrm{SF}(\alpha,\beta)+\mathrm{SF}(\beta,\alpha) = (\alpha|\beta).$$

Using the variation isomorphism we introduce intersection form in cohomology $(A|B) := (\mathrm{Var}(A)|\,\mathrm{Var}(B))$. Note that the cohomological intersection form is also the symmetrization of the cohomological Seifert form.

Finally, let us introduce the following polylogorithm operator series depending on a linear operator σ whose eigenvalues are roots of unity:

$$\mathrm{Li}_\sigma(x) = \sum_{k=1}^\infty \frac{x^{k+\mathcal{N}}}{k+\mathcal{N}}\,,$$

where $\mathcal{N} := -\frac{1}{2\pi\sqrt{-1}}\log\sigma$, s.t., the eigenvalues of $\mathcal{N}$ belong to the set $(-1,0]\cap\mathbb{Q}$. Note that the series is convergent for $|x|<1$. Moreover, it can be expressed in terms of the standard polylogorithm functions

$$\mathrm{Li}_\sigma(x) = x^{\mathcal{N}_n}\sum_{p=1}^\infty\sum_{r=1}^{|\sigma|}(-\mathcal{N}_n|\sigma|)^{p-1}\,\mathrm{Li}_p(\eta^r x^{1/|\sigma|})\sigma_s^r, \tag{2.11}$$

where we decomposed $\mathcal{N} = \mathcal{N}_s + \mathcal{N}_n$ into a diagonal and nilpotent operator, $\sigma_s := e^{-2\pi\sqrt{-1}\mathcal{N}_s}$, $|\sigma|$ is the order of σ_s, and $\eta = e^{2\pi\sqrt{-1}/|\sigma|}$. In particular, the series $\mathrm{Li}_\sigma(x)$ extends analytically along any path avoiding $x=0$ and $x=1$.

Theorem 2.1. *The following formula holds*

$$\Omega_{\alpha,\beta}(0,\lambda,\mu) = -(\mathrm{Li}_\sigma(\mu/\lambda)\alpha|\beta) + P_{\alpha,\beta}(\lambda,\mu),$$

where $P_{\alpha,\beta}(\lambda,\mu)$ is a polynomial in $\lambda^{\pm1/|\sigma|}$, $\mu^{\pm1/|\sigma|}$, $\log\lambda$, and $\log\mu$ satisfying

$$P_{\alpha,\beta}(\lambda,\mu) - P_{\beta,\alpha}(\mu,\lambda) = \mathrm{SF}\Big(\frac{e^{-2\pi\sqrt{-1}\mathcal{N}}-1}{\mathcal{N}}\,(\mu/\lambda)^{\mathcal{N}}\alpha_1,\beta\Big),$$

where α_1 is the projection of α on the generalized eigen subspace $\mathfrak{h}_1^* := \mathrm{Ker}(\mathcal{N}_s)$.

As a corollary of Theorem 2.1 we obtain the following result (for the proof see Lemma 4.1).

Corollary 2.2. *Let $(\lambda,\mu)\in\mathbb{C}^2$ be a fixed point, s.t., $|\lambda|>|\mu|>0$ and $|\lambda-\mu|\ll 1$. Let $C\subset\mathbb{C}^2$ be a path from (λ,μ) to (μ,λ) avoiding the diagonal and contained in a sufficiently small neighborhood of the line segment $[(\lambda,\mu),(\mu,\lambda)]$. The following symmetry holds $B_{\alpha,\beta}(0,\lambda,\mu)=B_{\beta,\alpha}(0,\mu,\lambda)$, where the second phase factor is obtained from $B_{\beta,\alpha}(0,\lambda,\mu)$ via analytic continuation along C.*

Corollary 2.2 is important for the applications to VOA representations, because it essentially means that the vertex operators are mutually local. In particular, we can construct a representation of the lattice VOA corresponding to the Milnor lattice. In Section 5 we give more details on how to construct such a representation, although a more systematic investigation will be presented elsewhere.

2.5. **The phase form and analytic extension.** The phase 1-form is by definition the following 1-form on B' depending on a parameter ξ:

$$\mathcal{W}_{\alpha,\beta}(t,\xi)=I^{(0)}_\alpha(t,\xi)\bullet_t I^{(0)}_\beta(t,0),\quad \alpha,\beta\in\mathfrak{h}^*, \tag{2.12}$$

where the RHS is interpreted as a cotangent vector in T^*_tB via the identifications (2.2) and the parameter ξ is assumed to be sufficiently small. In other words, we expand the RHS into a Taylor series at $\xi=0$

$$\sum_{m=0}^\infty \mathcal{W}^{(m)}_{\alpha,\beta}(t)\frac{\xi^m}{m!},\quad \mathcal{W}^{(m)}_{\alpha,\beta}(t)=I^{(m)}_\alpha(t,0)\bullet I^{(0)}_\beta(t,0)\in T^*_tB.$$

where for each $t\in B'$ the radius of convergence of the above series is non-zero.

The phase form can be used to describe the dependence of the phase factors in the deformation parameters

$$\Omega_{\alpha,\beta}(t,\lambda,\mu)=\Omega_{\alpha,\beta}(0,\lambda,\mu)+\int_{-\lambda\mathbf{1}}^{t-\lambda\mathbf{1}}\mathcal{W}_{\alpha,\beta}(t',\mu-\lambda), \tag{2.13}$$

where (t,λ,μ) is in the domain of convergence of (2.10), $|\mu-\lambda|\ll 1$, and the integration path should be chosen appropriately. The main result of this paper can be stated as follows.

Theorem 2.3. *Let (t,λ,μ) be a point in the domain of convergence of (2.10), s.t., $|\mu-\lambda|\ll 1$ and $C\subset B'$ be a closed loop based at $t-\lambda\mathbf{1}$, then*

$$\oint_{t'\in C}\mathcal{W}_{\alpha,\beta}(t',\mu-\lambda)-\Omega_{w(\alpha),w(\beta)}(t,\lambda,\mu)+\Omega_{\alpha,\beta}(t,\lambda,\mu)\quad\in\quad 2\pi\sqrt{-1}\mathbb{Z},$$

where w is the monodromy transformation along C.

The integral (2.13) provides analytic extension along any path, so the phase factors can be interpreted as Laurent series in $(\mu-\lambda)$. Theorem 2.3 implies that the coefficients of the Laurent series expansion are analytic on the monodromy covering space, i.e, if a closed loop C is in the kernel of the monodromy representation, then the coefficients are invariant with respect to the analytic continuation along C. Finally, when the cycles α and β are invariant along C, we get an affirmative answer to Givental's question from [10].

Corollary 2.4. *Let C be a closed loop in B' and $\alpha,\beta\in Q$ be cycles invariant with respect to the monodromy transformation along C, then $\oint_{t\in C}\mathcal{W}_{\alpha,\beta}(t,\xi)\in 2\pi\sqrt{-1}\mathbb{Z}$.*

3. The phase factors at $t = 0$

Let $\mathcal{H} = H(\!(z^{-1})\!)$ be Givental's symplectic loop space, where the symplectic form is

$$\Omega(f(z), g(z)) = \mathrm{Res}_{z=0}(f(-z), g(z))dz, \quad f, g \in \mathcal{H}.$$

There is a natural polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\mathcal{H}_+ := H[z]$ and $\mathcal{H}_- := H[\![z^{-1}]\!]z^{-1}$ which allows us to identify $\mathcal{H} \cong T^*\mathcal{H}_+$. Let us introduce the generating series

$$\mathbf{f}_a(t, \lambda; z) = \sum_{n\in\mathbb{Z}} I_a^{(n)}(t, \lambda)\,(-z)^n, \quad a \in \mathfrak{h}^*$$

and note that $\Omega_{\alpha,\beta}(t, \lambda, \mu) = \Omega(\mathbf{f}_\alpha(t, \lambda; z)_+, \mathbf{f}_\beta(t, \mu; z))$, where the index $+$ denotes the projection on $\mathcal{H}_+$ along $\mathcal{H}_-$.

3.1. **The Virasoro grading operator.** The symplectic vector space $\mathcal{H}$ has the following Virasoro grading operator:

$$\ell_0(z) = z\partial_z + \frac{1}{2} - \theta + \rho/z,$$

where $\rho \in \mathrm{End}(H)$ is the operator of multiplication by W. Note that ρ/z is a nilpotent operator, commuting with $\ell_0(z)$, while $\ell_0(z) - \rho/z$ is diagonalizable. Let us decompose, the symplectic vector space

$$\mathcal{H} = \mathcal{H}_{<0} \oplus \mathcal{H}_0 \oplus \mathcal{H}_{>0}, \tag{3.1}$$

according to the sign of the eigenvalues of $\ell_0(z)$, i.e., $\mathcal{H}_{<0}$ is the sum of all generalized eigen subspaces of ℓ_0 with eigenvalue < 0, $\mathcal{H}_0$ – with eigenvalue 0, and $\mathcal{H}_{>0}$ – with eigenvalue > 0.

We are going to compute the symplectic pairing

$$\Omega(\mathbf{f}_\alpha(0, \lambda; z)_{>0}, \mathbf{f}_\beta(0, \mu; z)) = \Omega(\mathbf{f}_\alpha(0, \lambda; z), \mathbf{f}_\beta(0, \mu; z)_{<0}), \quad \alpha, \beta \in \mathfrak{h}^*,$$

where the index > 0 (resp. < 0) corresponds to projection with respect to the spectral decomposition of the operator $\ell_0(z)$ onto the subspace spanned by all generalized eigen subspaces with eigenvalue > 0 (resp. < 0). It is not hard to check that the difference between the phase factor $\Omega_{\alpha,\beta}(0, \lambda, \mu)$ and the above symplectic pairing is a function $P_{\alpha,\beta}$ that has the form described in Theorem 2.1. So the main difficulty is in finding an explicit formula for the above symplectic pairing.

3.2. **Fundamental solution.** The differential equation (2.7) can be solved explicitly when $t = 0$. Assume that $k = -m$ with $m \gg 0$, then the fundamental solution is given by the following operator-valued function

$$\Phi_m(\lambda) = e^{\rho\partial_\lambda\partial_m}\Big(\frac{\lambda^{\theta+m-1/2}}{\Gamma(\theta + m + 1/2)}\Big),$$

where the RHS is defined first for m a complex number, so that ∂_m makes sense and the Gamma function is defined through its Taylor's series expansion at $\theta = 0$, so it makes sense to substitute a linear operator for θ. Note that if we assume that $1/\Gamma(m) = 0$ when m is a negative integer or 0, then the above formula makes sense for all $m \in \mathbb{Z}$.

Using the commutation relations

$$[\rho, \theta] = \rho, \quad \rho\partial_\lambda\Phi_m(\lambda) = \Phi_m(\lambda)\rho, \tag{3.2}$$

it is easy to prove that the analytic continuation around $\lambda = 0$ transforms $\Phi_m(\lambda)$ into

$$\Phi_m(\lambda)\, e^{2\pi\sqrt{-1}\rho}\, e^{2\pi\sqrt{-1}(\theta-1/2)}.$$

3.3. **Geometric sections.** If $\omega \in \Omega^{n+1}_{X/B}(X)$ is a holomorphic form then $s(\omega) := \nabla_{\partial_\lambda} \int d^{-1}\omega$, where ∇ is the Gauss–Manin connection, defines a holomorphic section of the vanishing cohomology bundle known as *geometric section*. Let us denote by $s(\omega, \lambda)$ the value of the section at the point $(0, \lambda) \in S'$.

Put $\ell = n/2$ (we assume that n is even). Note that by definition the period vector

$$I^{(-\ell)}(0, \lambda) = (2\pi)^{-\ell} \sum_{i=1}^{N} \phi^i \otimes s(\omega_i, \lambda),$$

where $\{\phi^i\} \subset H$ is a basis of H dual to $\{\phi_i\}$ with respect to the residue pairing and $\omega_i \in \Omega^{n+1}_{X/B}(X)$ is defined by

$$-\nabla_{\partial_i} \int d^{-1}\omega = \nabla_{\partial_\lambda} \int d^{-1}\omega_i.$$

On the other hand, since $\Phi_\ell(\lambda)$ is a fundamental solution for the differential equation (2.7) with $k = -\ell$, we get that

$$I^{(-\ell)}(0, \lambda) = \sum_{i=1}^{N} \Phi_\ell(\lambda)\phi^i \otimes A_i,$$

where $\{A_i\}_{i=1}^N \subset \mathfrak{h}$ is a basis with some very special properties that guarantee that ω is a primitive form (see [11] for more details). Let us introduce the linear map

$$^{\#} : \mathrm{End}(H) \to \mathrm{End}(\mathfrak{h}), \quad \text{s.t.,} \quad \sum_{i=1}^{N} T\phi^i \otimes A_i = \sum_{i=1}^{N} \phi^i \otimes T^{\#} A_i.$$

Lemma 3.1. *The map $^{\#}$ is anti-homomorphism of algebras, i.e., $(T_1T_2)^{\#} = T_2^{\#}T_1^{\#}$.*

Proof. We just have to use that $T^{\#}A_i = \sum_{j=1}^{N} (T\phi^j, \phi_i)A_j$. □

The numbers

$$s_i := d_i + \ell - \frac{1}{2} - \frac{D}{2}, \quad 1 \le i \le N,$$

are known as the *Steenbrink spectrum* of the singularity. The key result that reveals the Hodge-theoretic origin of the period integrals is the following Lemma.

Proposition 3.2. *The following formula holds*

$$s(\omega_i, \lambda) = (2\pi)^{\ell}\, \lambda^{s_i + \rho^{\#}} \Gamma(s_i + \rho^{\#} + 1)^{-1} A_i,$$

where the value of the Gamma function is defined through its Taylor's series at $\rho^{\#} = 0$.

Proof. We have to compute $\Phi_\ell(\lambda)^{\#} A_i$. Using the commutation relations (3.2) we get

$$\Phi_\ell(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \rho^k \partial_\lambda^k \partial_\ell^k \Big(\lambda^{\theta + \ell - 1/2} \Gamma(\theta + \ell + 1/2)^{-1} \Big) = \sum_{k=0}^{\infty} \frac{1}{k!} \partial_\ell^k \Big(\lambda^{\theta + \ell - 1/2} \Gamma(\theta + \ell + 1/2)^{-1} \Big) \rho^k.$$

Note that

$$\theta^{\#} A_i = \sum_{j=1}^{N} (\theta\phi^j, \phi_i) A_j = (d_i - D/2) A_i.$$

To finish the proof we just need to recall Lemma 3.1 and use Taylor's formula. □

Since the geometric sections are single valued, Proposition 3.2 allows us to conclude that the classical monodromy

$$\sigma = e^{-2\pi\sqrt{-1}\rho^{\#}}\, e^{-2\pi\sqrt{-1}(\theta^{\#}-1/2)}.$$

Let us write $s_i = p_i + \alpha_i$, where $-1 < \alpha_i \leq 0$, then our choice of logarithm $\mathcal{N} = -\frac{1}{2\pi\sqrt{-1}}\log\sigma$ acts on the basis $\{A_i\} \subset \mathfrak{h}$ as follows

$$\mathcal{N}(A_i) = (\alpha_i + \rho^{\#})A_i, \quad 1 \leq i \leq N.$$

Let us point out that since

$$\nabla_{\partial_\lambda}^{p_i} s(\omega_i, \lambda) = \lambda^{\alpha_i+\rho^{\#}}\Gamma(\alpha_i + \rho^{\#} + 1)^{-1}A_i$$

the vectors $\Gamma(\alpha_i + \rho^{\#} + 1)^{-1}A_i \in F_{p_i}\mathfrak{h}$, where $\{F_p\mathfrak{h}\}_{p=0}^n$ is the Steenbrink's Hodge filtration.

3.4. **The higher residue pairing.** Let us denote by $\mathbb{H} \to B$ the vector bundle whose fiber over $t \in B$ is

$$\mathbb{H}_t = \Omega^{n+1}_{X_t}[\![z]\!]/(zd_x + d_xF(t,x)\wedge)\Omega^n_{X_t}[\![z]\!],$$

where $X_t = X \cap \mathbb{C}^{n+1} \times \{t\}$. By definition, the domain X is chosen so small that $\mathbb{H}_t$ is a $\mathbb{C}[\![z]\!]$-free module of rank N. We can think of sections ω of $\mathbb{H}$ as formal oscillatory integrals $\int e^{F/z}\omega$, which allows us to see that $\mathbb{H}$ is equipped with a flat connection, called *Gauss–Manin connection*, corresponding to differentiating the integral formally with respect to the deformation parameters.

Recall, also K. Saito's higher residue pairing

$$K_t : \mathbb{H}_t \otimes \mathbb{H}_t \to \mathbb{C}[\![z]\!]z^{n+1}.$$

It is uniquely determined up to a constant by the following properties

(K1) If $\omega_1, \omega_2 \in \mathbb{H}_t$, then $K_t(\omega_1,\omega_2) = (-1)^{n+1}K_t(\omega_2,\omega_1)^*$, where $*$ is the involution $z \mapsto -z$.

(K2) If $p(z) \in \mathbb{C}[z]$, then

$$p(z)K_t(\omega_1,\omega_2) = K_t(p(z)\omega_1,\omega_2) = K_t(\omega_1, p(-z)\omega_2).$$

(K3) The pairing $K_t^{(p)}(\omega_1,\omega_2)$ defined by the coefficient in front of z^{n+1+p} in $K_t(\omega_1,\omega_2)$ depends analytically on t and the Leibnitz rule holds

$$\xi K_t(\omega_1,\omega_2) = K_t(\nabla_\xi\omega_1,\omega_2) + K_t(\omega_1,\nabla_\xi\omega_2),$$

where $\xi = \partial/\partial t_i$ or $z\partial_z$ and ∇ is the Gauss–Manin connection on $\mathbb{H}$.

(K4) If $\omega_i = \phi_i(x)dx_0\cdots dx_n$ $(i = 1,2)$, then $K_t^{(0)}(\omega_1,\omega_2) = (\phi_1,\phi_2)$, where the residue pairing (see (2.1)) is with respect to the volume form $dx_0\cdots dx_n$.

Given a holomorphic form $\omega \in \Omega^{n+1}_{X_0}(X_0)$, let us denote by

$$\widehat{s}(\omega, z) = (-2\pi z)^{-\ell-1/2}\int_0^\infty e^{\lambda/z}s(\omega,\lambda)d\lambda,$$

where the integration path is $\lambda = -tz$, $t \in [0,+\infty)$, the Laplace transform of the corresponding geometric section. We will make use also of the automorphism $\widehat{s}(\omega,z)^* := \widehat{s}(\omega, e^{\pi\sqrt{-1}}z)$. Note that since we might have $\log z$ dependence, * is no longer an involution.

Let us introduce also the so called *elementary sections*. They are defined as follows. Given a vector $A \in \mathfrak{h}$, we can construct a section of the vanishing cohomology bundle over $S' \cap \{t = 0\}$ as follows

$$s(A, \lambda) := \sum_{-1<\alpha\leq 0} \lambda^{\alpha+\rho^\#} A_\alpha,$$

where A_α is the projection of A on the generalized eigen space of σ corresponding to the eigenvalue $e^{-2\pi\sqrt{-1}\alpha}$. Put

$$\widehat{s}(A, z) = (-2\pi z)^{-\ell-1/2} \int_0^\infty e^{\lambda/z} s(A, \lambda) d\lambda,$$

where the integration path is the same as above. The next result is a reformulation of Hertling's formula for Saito's higher residue pairing in terms of the Seifert form.

Lemma 3.3. *The pairing*

$$K_W(\omega_1, \omega_2) := \mathrm{SF}(\widehat{s}(\omega_1, z)^*, \widehat{s}(\omega_2, z))^*\, z^{n+1}$$

coincides with Saito's higher residue pairing $K_0(\omega_1, \omega_2)$.

Proof. We will prove that the Lemma is equivalent to Hertling's formula for Saito's higher residue pairings. In order to compare our formula to Hertling's one, we have to introduce the vector bundle $\mathbb{H}'' \to B$ whose fiber over a point $t \in B$ is $\mathbb{H}''_t := \Omega^{n+1}_{X_t}/d_xF(t,x) \wedge d_x\Omega^{n-1}_{X_t}$. If $\omega \in \Omega^{n+1}_{X_t}$ is a holomorphic form, then we denote by $s(\omega) \in \mathbb{H}''_t$ the projection of ω . It is known that $\mathbb{H}''_t$ is a $\mathbb{C}[\partial_\lambda^{-1}]$-module and that if we identify $\mathbb{C}[\![z]\!] \cong \mathbb{C}[\![\partial_\lambda^{-1}]\!]$ via $z \mapsto -\partial_\lambda^{-1}$, then the natural map

$$\mathbb{H}_t \to \mathbb{H}''_t \otimes_{\mathbb{C}[\partial_\lambda^{-1}]} \mathbb{C}[\![\partial_\lambda^{-1}]\!], \quad \omega \mapsto s(\omega),$$

is an isomorphism of $\mathbb{C}[\![z]\!]$-modules. Let us define

$$\widetilde{K}_t(s(\omega_1), s(\omega_2)) := K_t(\omega_2, \omega_1) = (-1)^{n+1} K_t(\omega_1, \omega_2)^*.$$

It is easy to check that $\widetilde{K}$ satisfy properties (K1)–(K4) of the higher residue pairing except that we have to replace z by ∂_λ^{-1}. Since the higher residue pairing is uniquely determined by its properties, $\widetilde{K}$ coincides with the higher residue pairing used in [11]. We have to prove that

$$\widetilde{K}_t(s(\omega_1), s(\omega_2)) = \mathrm{SF}(\widehat{s}(\omega_1, z)^*, \widehat{s}(\omega_2, z)) z^{n+1}.$$

Note that using the above formula we can uniquely define a pairing on the space of all elementary sections, s.t., property (K2) holds. We just need to check that the RHS agrees with Hertling's formula for a pair of elementary sections (see formulas (10.81)–(10.83) in [11]).

Let us assume that B_i $(i = 1, 2)$ are eigenvectors with eigenvalues $e^{-2\pi\sqrt{-1}\beta_i}$, $-1 < \beta_i \leq 0$. Then

$$\widehat{s}(B_i, z) = (2\pi)^{-\ell-1/2}(-z)^{\beta_i+\rho^\#-\ell+1/2}\Gamma(\beta_i + \rho^\# + 1)B_i, \quad i = 1, 2.$$

Using that the Seifert form is monodromy invariant and infinitesimally $\rho^\#$-invariant, i.e., $\mathrm{SF}(\rho^\# A, B) + \mathrm{SF}(A, \rho^\# B) = 0$ we get that

$$\mathrm{SF}(\widehat{s}(B_1, z)^*, \widehat{s}(B_2, z))\, z^{n+1} \tag{3.3}$$

is given by the following formula

$$\frac{z^{\beta_1+\beta_2+2}}{(2\pi\sqrt{-1})^{n+1}}\,\mathrm{SF}(B_1, e^{\pi\sqrt{-1}(\beta_2+\rho^{\#}+1)}\Gamma(\beta_1-\rho^{\#}+1)\Gamma(\beta_2+\rho^{\#}+1)B_2).$$

Since the Seifert form is monodromy invariant, the above pairing vanishes unless $\beta_1+\beta_2\in\mathbb{Z}$, i.e., $\beta_1+\beta_2=-1$, or $\beta_1=\beta_2=0$. Recall, the operator $\mathcal{N}=-\frac{1}{2\pi\sqrt{-1}}\log\sigma$ and note that $\mathcal{N}B_i=-(\beta_i+\rho^{\#})B_i$.

$$e^{\pi\sqrt{-1}(\beta_2+\rho^{\#}+1)}\Gamma(\beta_1-\rho^{\#}+1)\Gamma(\beta_2+\rho^{\#}+1)B_2=\begin{cases}-2\pi\sqrt{-1}(\sigma-1)^{-1}B_2 & \text{if } \beta_1+\beta_2=-1,\\ 2\pi\sqrt{-1}\mathcal{N}(\sigma-1)^{-1}B_2 & \text{if } \beta_1=\beta_2=0.\end{cases}$$

Hence the pairing (3.3) takes the form

$$\frac{1}{(2\pi\sqrt{-1})^n}\,(-1)^\ell\langle B_1,\mathrm{Var}\,(\sigma-1)^{-1}B_2\rangle\, z,\qquad \text{for } \beta_1+\beta_2=-1,$$

$$\frac{-1}{(2\pi\sqrt{-1})^{n+1}}\,(-1)^\ell\langle B_1,\mathrm{Var}\;2\pi\sqrt{-1}\mathcal{N}(\sigma-1)^{-1}B_2\rangle\, z^2,\qquad \text{for } \beta_1=\beta_2=0,$$

and it vanishes in all other cases. □

The identity in Lemma 3.3 is really remarkable. It is a relation between two completely different quantities. The LHS is defined via residues of differential forms, while the RHS is purely topological. Let us outline a different way to prove Lemma 3.3, which in particular generalizes the identity for an arbitrary deformation. Given $\omega\in\Omega^{n+1}_{X/B}(X)$ and a critical value $u=u(t)$ of $F(t,x)$ we define the following formal asymptotic series

$$\widehat{s}_u(\omega,z):=(-2\pi z)^{-\ell-1/2}\int_{u(t)}^{\infty}e^{\lambda/z}s(\omega)d\lambda.$$

Note that this is the stationary phase asymptotic as $z\to 0$ of an appropriate oscillatory integral. Let us define

$$K_F(\omega_1,\omega_2)=\sum_u \mathrm{SF}(\widehat{s}_u(\omega_1,z)^*,\widehat{s}_u(\omega_1,z))^*\,z^{n+1},$$

where the sum is over all critical values of F. When $F=W$, this formula reduces to Lemma 3.3. One has to check that the above formula satisfies all properties (K1)–(K4) of the higher residue pairing. The verification is straightforward except for property (K4). In the latter case we take a generic deformation and then we just have to see that the computation on the RHS reduces to proving Lemma 3.3 for A_1-singularity, which is straightforward. It remains only to recall the result of M. Saito [19] that the higher residue pairing is uniquely determined by its properties.

3.5. **Proof of Theorem 2.1.** Using the explicit formula for the geometric sections in Proposition 3.2 we get that

$$\widehat{s}(\omega_i,z)=(2\pi)^{-1/2}(-z)^{p_i+\mathcal{N}-\ell+1/2}A_i,$$

where $p_i=\lfloor s_i\rfloor$ is the floor of the Steenbrink number s_i. It is convenient, to introduce the operator $p\in\mathrm{End}(\mathfrak{h})$, s.t., $p(A_i)=p_iA_i$. Recalling Lemma 3.3 we get

$$K_W(\omega_i,\omega_j)=(\phi_i,\phi_j)z^{n+1}=\frac{1}{2\pi\sqrt{-1}}\langle A_i,e^{\pi\sqrt{-1}\mathcal{N}}e^{\pi\sqrt{-1}p}A_j\rangle\, z^{s_i+s_j-2\ell+1+n+1},\tag{3.4}$$

where $\langle\, ,\, \rangle$ is the Seifert form (without the sign normalization) and the 1st identity holds, because ω is a primitive form. Let us define the *residue pairing* $(\, ,\,)$ on $\mathfrak{h}$, s.t., the $(\phi_i,\phi_j) = (A_i,A_j)$, i.e.,

$$(A,B) = \frac{1}{2\pi\sqrt{-1}}\langle A, e^{\pi\sqrt{-1}\mathcal{N}} e^{\pi\sqrt{-1}p} B\rangle, \quad A,B\in\mathfrak{h}.$$

Finally, if $R\in\operatorname{End}(\mathfrak{h})$, then we denote by R^T the transpose with respect to the residue pairing. Let us point out the following relations

$$\mathcal{N} = \mathcal{N}_s + \rho^{\#},\quad p+\mathcal{N}_s = (\theta+\ell-1/2)^{\#},\quad [\mathcal{N},\rho^{\#}]=0,\quad [p,\rho^{\#}]=\rho^{\#},$$

where $\mathcal{N}_s$ is the semi-simple part of $\mathcal{N}$. Comparing the powers of z in (3.4) we get

$$p+\mathcal{N}_s+p^T+\mathcal{N}_s^T = 2\ell-1.$$

By definition

$$\mathbf{f}_\alpha(0,\lambda;z) = \sum_{m\in\mathbb{Z}} \partial_\lambda^{m+\ell} I_\alpha^{(-\ell)}(0,\lambda)\,(-z)^m = \sum_{i=1}^N\sum_{m\in\mathbb{Z}} (-z)^m \partial_\lambda^{m+\ell-p_i}\langle \lambda^{\mathcal{N}}\Gamma(\mathcal{N}+1)^{-1}A_i,\alpha\rangle\phi^i$$

Let us introduce also the transpose R^{SF} with respect to the Seifert form

$$\langle RA,B\rangle =: \langle A, R^{SF}B\rangle,\quad \forall A,B\in\mathfrak{h}.$$

Note that since the Seifert form is not symmetric, in general $\langle A,RB\rangle\neq\langle R^{SF}A,B\rangle$. Since

$$\Big(z\partial_z+\frac{1}{2}-\theta\Big)(-z)^m\phi^i = (m+\ell-s_i)(-z)^m\phi^i,$$

after shifting $m\mapsto m-\ell+p_i$ we get that the projection

$$\mathbf{f}_\beta(0,\mu;z)_{<0} = \sum_{j=1}^N\sum_{m=-\infty}^{-1}\Big\langle A_j, (-z)^{m-\ell+p^{SF}}\mu^{\mathcal{N}^{SF}-m}\Gamma(\mathcal{N}^{SF}-m+1)^{-1}\beta\Big\rangle\,\phi^j. \tag{3.5}$$

Similarly,

$$\mathbf{f}_\alpha(0,\lambda;z) = \sum_{i=1}^N\sum_{k\in\mathbb{Z}}\Big\langle A_i, (-z)^{k-\ell+p^{SF}}\lambda^{\mathcal{N}^{SF}-k}\Gamma(\mathcal{N}^{SF}-k+1)^{-1}\alpha\Big\rangle\,\phi^i.$$

Let us denote by $\{A^i\}_{i=1}^N$ the basis of $\mathfrak{h}$ dual to $\{A_i\}_{i=1}^N$ with respect to the residue pairing. We need to compute a pairing of the following type:

$$\sum_{i,j=1}^N\langle A_i,x\rangle(\phi^i,\phi^j)\langle A_j,y\rangle = (2\pi\sqrt{-1})^2\sum_{i,j=1}^N (A_i,M^{-1}x)(\phi^i,\phi^j)(A_j,M^{-1}y),$$

where $M = e^{\pi\sqrt{-1}\mathcal{N}}e^{\pi\sqrt{-1}p}$ and the equality follows from the definition of the residue pairing. Using that $(\phi^i,\phi^j) = (A^i,A^j)$ and the standard properties of dual bases we get

$$(2\pi\sqrt{-1})^2((M^{-1})x,(M^{-1})y) = 2\pi\sqrt{-1}\langle M^{-1}x,y\rangle.$$

The symplectic pairing

$$(3.6)\quad \Omega(\mathbf{f}_\alpha(0,\lambda;z)_{>0},\mathbf{f}_\beta(0,\mu;z)) = \Omega(\mathbf{f}_\alpha(0,\lambda;z),\mathbf{f}_\beta(0,\mu;z)_{<0}) = 2\pi\sqrt{-1}\sum_{m=-\infty}^{-1}\sum_{k\in\mathbb{Z}}\mathrm{Res}_{z=0}$$

$$\Big\langle M^{-1}z^{k-\ell+p^{SF}}\lambda^{\mathcal{N}^{SF}-k}\Gamma(\mathcal{N}^{SF}-k+1)^{-1}\alpha, (-z)^{m-\ell+p^{SF}}\mu^{\mathcal{N}^{SF}-m}\Gamma(\mathcal{N}^{SF}-m+1)^{-1}\beta\Big\rangle.$$

Using that by definition $R^{SF}=MR^TM^{-1}$ we get the following formulas

$$\mathcal{N}_s^T=\mathcal{N}_s^{SF},\quad (\rho^\#)^{SF}=-\rho^\#,\quad (\rho^\#)^T=\rho^\#,\quad p^{SF}=p^T+\pi\sqrt{-1}\rho^\#.$$

Let us point out that the two conjugations do not commute in general and also that $(R^{SF})^{SF}\neq R$ in general. Since $M^{-1}p^{SF}=p^TM^{-1}$ and $M^{-1}\mathcal{N}^{SF}=\mathcal{N}^TM^{-1}$, the summand in the sum (3.6) can be written in the following form

$$\Big\langle z^{m+k-2\ell+p+p^T}e^{\pi\sqrt{-1}(m-\ell+p)}\lambda^{\mathcal{N}^T-k}\Gamma(\mathcal{N}^T-k+1)^{-1}M^{-1}\alpha, \mu^{\mathcal{N}^{SF}-m}\Gamma(\mathcal{N}^{SF}-m+1)^{-1}\beta\Big\rangle.$$

Using that $e^{\pi\sqrt{-1}p}\mathcal{N}^T=\mathcal{N}^{SF}e^{\pi\sqrt{-1}p}$ and $e^{\pi\sqrt{-1}p}M^{-1}=e^{-\pi\sqrt{-1}\mathcal{N}}$, we get

$$\Big\langle z^{m+k-2\ell+p+p^T}\lambda^{\mathcal{N}^{SF}-k}\Gamma(\mathcal{N}^{SF}-k+1)^{-1}e^{-\pi\sqrt{-1}(\mathcal{N}-m+\ell)}\alpha, \mu^{\mathcal{N}^{SF}-m}\Gamma(\mathcal{N}^{SF}-m+1)^{-1}\beta\Big\rangle.$$

The terms that contribute to the residue must satisfy

$$m+k+p+p^T=2\ell-1=p+p^T+\mathcal{N}_s+\mathcal{N}_s^T=p+p^T+\mathcal{N}+\mathcal{N}^{SF},$$

so we may substitute $p+p^T=2\ell-1-m-k$ and $\mathcal{N}^{SF}-k=-\mathcal{N}+m$

$$(-1)^\ell z^{-1}\Big\langle(\mu/\lambda)^{\mathcal{N}-m}\Gamma(\mathcal{N}-m+1)^{-1}\Gamma(-\mathcal{N}+m+1)^{-1}e^{\pi\sqrt{-1}(-\mathcal{N}+m)}\alpha,\beta\Big\rangle.$$

Using the well known identity

$$\frac{e^{\pi\sqrt{-1}x}}{\Gamma(1-x)\Gamma(1+x)}=\frac{e^{2\pi\sqrt{-1}x}-1}{2\pi\sqrt{-1}x}$$

with $x=-\mathcal{N}+m$ we get

$$\frac{(-1)^{\ell+1}}{2\pi\sqrt{-1}}z^{-1}\Big\langle\frac{(\mu/\lambda)^{\mathcal{N}-m}}{\mathcal{N}-m}(e^{-2\pi\sqrt{-1}\mathcal{N}}-1)\alpha,\beta\Big\rangle.$$

It remains only to recall that the intersection form can be expressed in terms of the Seifert form

$$(\alpha|\beta)=(-1)^\ell\langle(\sigma-1)\alpha,\beta\rangle,\quad \alpha,\beta\in\mathfrak{h}^*.$$

We get

$$\Omega(\mathbf{f}_\alpha(0,\lambda;z)_{>0},\mathbf{f}_\beta(0,\mu;z))=-(\mathrm{Li}_\sigma(\mu/\lambda)\alpha|\beta).$$

Put

$$P_{\alpha,\beta}(\lambda,\mu)=\Omega(\mathbf{f}_\alpha(0,\lambda;z)_+,\mathbf{f}_\beta(0,\mu;z))-\Omega(\mathbf{f}_\alpha(0,\lambda;z)_{>0},\mathbf{f}_\beta(0,\mu;z)).$$

Using that Ω is a symplectic pairing and that $\mathcal{H}_{>0}$ is symplectic orthogonal to $\mathcal{H}_{\geq 0}$ we get

$$P_{\beta,\alpha}(\mu,\lambda)=\Omega(\mathbf{f}_\beta(0,\mu;z),\mathbf{f}_\alpha(0,\lambda;z)_-)-\Omega(\mathbf{f}_\beta(0,\mu;z),\mathbf{f}_\alpha(0,\lambda;z)_{<0}).$$

Hence

$$P_{\alpha,\beta}(\lambda,\mu)-P_{\beta,\alpha}(\mu,\lambda)=\Omega(\mathbf{f}_\alpha(0,\lambda;z)_0,\mathbf{f}_\beta(0,\mu;z)_0).$$

The above symplectic pairing can be computed in the same way as above. The only difference is that the summation over m in (3.5) collapses to $m=0$ and the summation over j reduces only to j, s.t., $s_j = p_j$, i.e., $\mathcal{N}_s A_j = 0$. This implies that in (3.6) we have to put $m=0$ and replace α (and β) by α_1 (and β_1). The rest of the computation is the same. We get

$$\Omega(\mathbf{f}_\alpha(0,\lambda;z)_0, \mathbf{f}_\beta(0,\mu;z)_0) = (-1)^{\ell+1} \left\langle \frac{(\mu/\lambda)^{\mathcal{N}}}{\mathcal{N}} (e^{-2\pi\sqrt{-1}\mathcal{N}} - 1)\alpha_1, \beta_1 \right\rangle .$$

Finally, the polynomiality statement about $P_{\alpha,\beta}(\lambda,\mu)$ follows from the fact that the vector space $\mathcal{H}_+ \cap \mathcal{H}_{<0}$ is finite dimensional. $\square$

4. Analytic continuation

We will be interested in the phase factors as multivalued analytic functions. Our starting point is the infinite series (2.10), which by definition is interpreted as a formal Laurent series of the type

$$\sum_{r=0}^{|\sigma|-1} \sum_{\ell=0}^{d-1} \lambda^{m+r/|\sigma|} (\log\lambda)^\ell \sum_{k=0}^{\infty} \Omega^{r,\ell,k}_{\alpha,\beta}(t,\mu)\lambda^{-k}, \tag{4.1}$$

where $d \geq 1$ is the smallest integer number, s.t., $\mathcal{N}_n^d = 0$. If $R > 0$ is the smallest real number, s.t., for every r and ℓ the infinite series in k is convergent for all $|\lambda| > R$, then R^{-1} is called the *radius of convergence*. The radius of convergence does not change if we differentiate the series with respect to λ. On the other hand, the derivative of (2.10) with respect to λ can be computed in terms of the period integrals (see [15] Proposition 2.3)

$$\partial_\lambda \Omega(\mathbf{f}_\alpha(t,\lambda;z)_+, \mathbf{f}_\beta(t,\mu;z)) = \frac{1}{\lambda-\mu}\Big(I^{(0)}_\alpha(t,\lambda), (\theta+1/2) I^{(-1)}_\beta(t,\mu)\Big). \tag{4.2}$$

Since the period $I^{(0)}_\alpha(t,\lambda)$ is a solution to a Fuchsian differential equation in λ, whose singularities are at $\lambda=\infty$ and the critical values $u_i(t)$ of $F(x,t)$, we get that the series (4.1) is convergent for

$$|\lambda| > \max\{|\mu|, r(t)\},$$

where $r(t) = \max_i\{|u_i(t)|\}$.

4.1. **The domains at infinity.** Put $S_\infty = \{(t,\lambda) \in S \,|\, |\lambda| > r(t)\}$. Let us fix a real constant $c \in (0,1)$, say $c = |1 - e^{2\pi\sqrt{-1}/|\sigma|}|/3$, s.t., the distance between any two distinct $|\sigma|$-roots of 1 is bigger than c. Let $\epsilon : S_\infty \to \mathbb{R}_{>0}$ be the function

$$\epsilon(t,\lambda) = \sup_C \Big(\min_{(t',\lambda')\in C} (|\lambda'| - r(t'))\Big),$$

where the sup is over all paths $C \subset S_\infty$ from $(0,\lambda)$ to (t,λ). We introduce the domain

$$D_\infty := \Big\{ (t,\lambda,\mu) \in S_\infty \times \mathbb{C} \mid |\lambda - \mu| < c \min\{\epsilon(t,\lambda), \epsilon(t,\mu)\}\Big\}$$

and its subdomain $D_\infty^+ := \{|\lambda| > |\mu|\} \subset D_\infty$.

Let us denote by $\widetilde{S}_\infty$ the universal covering of S_∞. The domains D^+_∞ and D_∞ are (trivial) smooth disk fibrations over S_∞. The phase factors $\Omega_{\alpha,\beta}$ are holomorphic functions on the pullback $\widetilde{D}^+_\infty$ of D^+_∞ via the covering map $\widetilde{S}_\infty \to S_\infty$. In more explicit terms, in order to define the phase factor $\Omega_{\alpha,\beta}(t,\lambda,\mu)$ at some point $(t,\lambda,\mu) \in D^+_\infty$ we have to select a reference path in S_∞ from $(0,\epsilon_0)$ to (t,λ). This choice determines the value of the period vectors $I^{(n)}_\alpha(t,\lambda)$ and using the line segment $[(t,\lambda),(t,\mu)]$ we can specify also the values of $I^{(-n-1)}_\beta(t,\mu)$, so the summands in (2.10) are uniquely defined. Note that if $(t,\lambda,\mu)\in D_\infty$, then $|\lambda-\mu|<|\lambda|-r(t)$, so using the triangle inequality it is easy to verify that the line segment $[(t,\lambda),(t,\mu)]$ does not intersect the discriminant.

4.2. **Symmetry of the phase factors at $t=0$.** According to Theorem 2.1, the phase factor $\Omega_{\alpha,\beta}(0,\lambda,\mu)$ extends analytically from D^+_∞ to $D_\infty-\{\lambda=\mu\}$. Note that the polylogorithms that enter in our formula have singularities for (λ,μ), s.t., $\mu=0$ or $\lambda^{|\sigma|}=\mu^{|\sigma|}$. However, if $(0,\lambda,\mu)\in D_\infty$, then $\mu\neq 0$ and thanks to our choice of the constant c the equality $\lambda^{|\sigma|}=\mu^{|\sigma|}$ implies that $\lambda=\mu$. Therefore, our explicit formula provides the analytic extension along any path in $D_\infty-\{\lambda=\mu\}$. The analytic continuation has the following crucial symmetry. Fix $(0,\lambda,\mu)\in D^+_\infty$ and a path $C\subset D_\infty-\{\lambda=\mu\}$ connecting $(0,\lambda,\mu)$ with $(0,\mu,\lambda)$. The coordinate projections of C along the last two coordinates determine paths C_1 from $(0,\lambda)$ to $(0,\mu)$ and C_2 from $(0,\mu)$ to $(0,\lambda)$ in S_∞.

Lemma 4.1. *Let $C\subset D_\infty-\{\lambda=\mu\}$ be a path from $(0,\lambda,\mu)\in D^+_\infty$ to $(0,\mu,\lambda)$, s.t., the projections C_1 and C_2 of C are homotopic (in S_∞) respectively to the line segments $[(0,\lambda),(0,\mu)]$ and $[(0,\mu),(0,\lambda)]$, then*

$$\Omega_{\alpha,\beta}(0,\lambda,\mu)-\Omega_{\beta,\alpha}(0,\mu,\lambda)=-2\pi\sqrt{-1}\Big(\mathrm{SF}(\alpha,\beta)+k(\alpha|\beta)\Big),\quad k\in\mathbb{Z},$$

where the 2nd phase factor is obtained from $\Omega_{\beta,\alpha}(0,\lambda,\mu)$ via analytic continuation along the path C and the integer k depends on the choice of C.

Proof. Let us first recall the so called Jonquière's inversion formula [12] (see also Appendix A), which provides a description of the analytic continuation of the polylogorithm functions in terms of Bernoulli polynomials

$$\mathrm{Li}_p(1/x)=(-1)^{p+1}\,\mathrm{Li}_p(x)+(-1)^{p+1}\frac{(2\pi\sqrt{-1})^p}{p!}\,B_p\Big(\frac{1}{2\pi\sqrt{-1}}\log x\Big),\quad 0<|x|<1, \tag{4.3}$$

where $B_p(x)$ $(p\geq 0)$ are the Bernoulli polynomials defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{p=0}^\infty B_p(x)\frac{t^p}{p!}. \tag{4.4}$$

The value of $\mathrm{Li}_p(1/x)$ is specified via analytic continuation along a path $C'\subset\mathbb{C}-\{1\}$ from x to $1/x$, which *does not* wind around $x=1$. The choice of the branch of $\log x$ in (4.3) is such that the formula holds for $p=1$. Let us assume first that C' intersects the real axis once and that $\mathrm{Im}(x)\neq 0$. In order to determine the branch of $\log x$, we have to consider 4 cases depending on whether $\mathrm{Im}(x)>0$ or <0 and whether C' intersects the real interval $(1,\infty)$ or not. However, after analyzing the 4 cases we find that there are two possibilities only. Namely, if we walk along C' from x to $1/x$, then when crossing the real axis, 1 is either on our left or on our right. In the first case $\log x:=\mathrm{Log}\,x$, otherwise $\log x:=\mathrm{Log}\,x+2\pi\sqrt{-1}$, where

$$\mathrm{Log}\,x:=\ln|x|+\sqrt{-1}\,\mathrm{Arg}(x),\quad -\pi<\mathrm{Arg}(x)\leq\pi$$

is the principal branch of the logarithm. For general C', the choice of the branch can be deduced easily from the above two cases.

Put $x=\mu/\lambda$. Recalling formula (2.11) and using (4.3) we get

$$\mathrm{Li}_\sigma(1/x)=\mathrm{Li}_\sigma(x)^T+\sum_{r=1}^{|\sigma|}\sum_{p=1}^{\infty}\frac{x^{-\mathcal{N}_n}}{\mathcal{N}_n|\sigma|}\,\frac{(2\pi\sqrt{-1}\mathcal{N}_n|\sigma|)^p}{p!}\,B_p\Big(\frac{1}{2\pi\sqrt{-1}}\log x_r\Big)\sigma_s^{-r},$$

where $x_r=\eta^r x^{1/|\sigma|}$ and the paths C'_r from x_r to x_r^{-1} can be described as follows. Let $C':t\mapsto C_2(t)/C_1(t)$ be the path from x to $1/x$ and C'_0 be the induced path from $x^{1/|\sigma|}:=e^{\mathrm{Log}(x)/|\sigma|}$ to $x^{-1/|\sigma|}:=e^{-\mathrm{Log}(x)|/\sigma|}$. Then C'_r is a composition of a path inside the unit disk from x_r to x_{-r} and $\eta^{-r}C'_0$ (clockwise rotation of C'_0 with angle $2\pi r/|\sigma|$).

The infinite sum over p can be computed via (4.4)

$$\mathrm{Li}_\sigma(1/x)=\mathrm{Li}_\sigma(x)^T+\sum_{r=1}^{|\sigma|}\frac{x^{-\mathcal{N}_n}}{\mathcal{N}_n|\sigma|}\Big(2\pi\sqrt{-1}\mathcal{N}_n|\sigma|\frac{e^{\mathcal{N}_n|\sigma\log x_r}}{e^{2\pi\sqrt{-1}\mathcal{N}_n|\sigma|}-1}-1\Big)\sigma_s^{-r}.$$

Since $(0,\lambda,\mu)\in D^+_\infty$ we have $|x-1|<c<1$ and $|x|<1$. Moreover, since the branch of the period vectors depending on μ is determined from the branch of the period vectors depending on λ via the straight segment from $[\lambda,\mu]$, we get that the multi-valued functions

$$x^{-\mathcal{N}_n}=e^{-\mathcal{N}_n\,\mathrm{Log}\,x},\quad x^{1/|\sigma|}=e^{\frac{1}{|\sigma|}\,\mathrm{Log}\,x}$$

are defined via the principal branch of the logarithm. Hence

$$x^{-\mathcal{N}_n}e^{\mathcal{N}_n|\sigma|\log x_r}=\exp\Big(\mathcal{N}_n|\sigma|\sqrt{-1}\Big(\,\mathrm{Arg}(\eta^r x^{1/|\sigma|})-\mathrm{Arg}(x^{1/|\sigma|})+2\pi\chi_r\Big)\Big),$$

where $\chi_r=1$ or 0, depending whether 1 is on the left or on the right of C'_r. Since C'_0 is in a small neighborhood of 1 we have

$$\mathrm{Arg}(\eta^r x^{1/|\sigma|})-\mathrm{Arg}(x^{1/|\sigma|})+2\pi\chi_r=2\pi r/|\sigma|,\quad 1\le r\le|\sigma|-1.$$

The sum over r becomes

$$\frac{2\pi\sqrt{-1}}{\sigma^{-|\sigma|}-1}\Big(\sum_{r=1}^{|\sigma|-1}\sigma^{-r}+\sigma^{-|\sigma|\chi_0}\Big)+\frac{x^{-\mathcal{N}_n}}{-\mathcal{N}_n}\Big(\frac{1}{|\sigma|}\sum_{r=1}^{|\sigma|}\sigma_s^r\Big).$$

Finally, we get

$$\mathrm{Li}_\sigma(1/x)=\mathrm{Li}_\sigma(x)^T-2\pi\sqrt{-1}\,\frac{\sigma^{1-\chi_0}}{\sigma-1}+\frac{x^{-\mathcal{N}_n}}{-\mathcal{N}_n}\Big(\frac{1}{|\sigma|}\sum_{r=1}^{|\sigma|}\sigma_s^r\Big).\tag{4.5}$$

Using the formula for the phase factors in Theorem 2.1 we compute

$$\Omega_{\alpha,\beta}(0,\lambda,\mu)-\Omega_{\beta,\alpha}(0,\mu,\lambda)=\Big((\mathrm{Li}_\sigma(1/x)-\mathrm{Li}_\sigma(x)^T)\beta|\alpha\Big)+P_{\alpha,\beta}(\lambda,\mu)-P_{\beta,\alpha}(\mu,\lambda).$$

Recalling again Theorem 2.1 and using (4.5) we get

$$\Omega_{\alpha,\beta}(0,\lambda,\mu)-\Omega_{\beta,\alpha}(0,\mu,\lambda)=2\pi\sqrt{-1}\,\mathrm{SF}(\sigma^{1-\chi_0}\beta,\alpha).$$

If $\chi_0=0$, then $\mathrm{SF}(\sigma\beta,\alpha)=-\,\mathrm{SF}(\alpha,\beta)$ otherwise, if $\chi_0=1$, then $\mathrm{SF}(\beta,\alpha)=-\,\mathrm{SF}(\alpha,\beta)+(\alpha|\beta)$. $\square$

4.3. **Analytic extension of the phase factors.** Recall that the series (2.10) is convergent in the domain

$$D^+ := \{(t,\lambda,\mu) \mid |\lambda-\mu| < |\mu| - r(t) < |\lambda| - r(t)\}$$

It is not hard to see that the RHS of (4.2) has a singularity at $\lambda=\mu$ of the form $(\alpha|\beta)(\lambda-\mu)^{-1}$. Hence the Laurent series expansion at $\lambda=\infty$ of the series

$$\widetilde{\Omega}_{\alpha,\beta}(t,\lambda,\mu) := \Omega_{\alpha,\beta}(t,\lambda,\mu) - (\alpha|\beta)\log(\lambda-\mu)$$

is convergent in the domain

$$D = \{(t,\lambda,\mu) \mid |\lambda-\mu| < \min\{|\mu|-r(t), |\lambda|-r(t)\}\}.$$

In particular, we get that $\Omega_{\alpha,\beta}(t,\lambda,\mu)$ extends analytically along any path in the domain $D_\infty - \{\lambda=\mu\}$ and that $e^{\Omega_{\alpha,\beta}(t,\lambda,\mu)}$ has a pole of order $-(\alpha|\beta)$ at $\lambda=\mu$.

Lemma 4.2. *If $(t,\lambda,\mu)\in D^+_\infty$ and $C\subset D_\infty-\{\lambda=\mu\}$ is a path from (t,λ,μ) to (t,μ,λ), s.t., the projections of C along the last two coordinates are homotopic to the line segments $[(t,\lambda),(t,\mu)]$ and $[(t,\mu),(t,\lambda)]$, then*

$$\Omega_{\alpha,\beta}(t,\lambda,\mu) - \Omega_{\beta,\alpha}(t,\mu,\lambda) = -2\pi\sqrt{-1}\Big(\,\mathrm{SF}(\alpha,\beta) + k(\alpha|\beta)\Big), \quad k\in\mathbb{Z},$$

where the 2nd phase factor on the LHS is obtained from $\Omega_{\beta,\alpha}(t,\lambda,\mu)$ via analytic continuation along the path C.

Proof. The main idea is to obtain an integral representation of the infinite series $\Omega_{\alpha,\beta}(t,\lambda,\mu)$. If $(t,\lambda,\mu)\in D_\infty$, then by definition we can find a path $C\subset S_\infty$ from $(0,\lambda)$ to (t,λ), s.t., $|\mu-\lambda| < |\lambda'| - r(t')$ for all $(t',\lambda')\in C$. Put $\mu' := \lambda'+\mu-\lambda$. Using the triangle inequality we get $|\mu'| \geq |\lambda'| - |\mu-\lambda| > r(t')$. Therefore, $(t',\lambda',\mu')\in D$ and it makes sense to think of $\widetilde{\Omega}_{\alpha,\beta}(t',\lambda',\mu')$ as a holomorphic function on the curve C. Using the differential equations (2.5)–(2.7) we get

$$d\widetilde{\Omega}_{\alpha,\beta}(t',\lambda',\mu') = \tau^*\mathcal{W}_{\alpha,\beta}\big|_C\,,$$

where $\tau: S\to B$, $(t,\lambda)\mapsto t-\lambda\mathbf{1}$. Integrating the above identity along C we get

$$\Omega_{\alpha,\beta}(t,\lambda,\mu) = \Omega_{\alpha,\beta}(0,\lambda,\mu) + \int_{-\lambda\mathbf{1}}^{t-\lambda\mathbf{1}} \mathcal{W}_{\alpha,\beta}(t'-\lambda'\mathbf{1},\mu-\lambda). \tag{4.6}$$

The translation $C+(0,\mu-\lambda)$ is a path $\widetilde{C}$ in S_∞ from $(0,\mu)$ to (t,μ). Note that $\widetilde{C}$ is obtained from C also by the map $(t',\lambda')\mapsto(t',\mu')$. Using that by definition

$$\mathcal{W}_{\alpha,\beta}(t'-\lambda'\,\mathbf{1},\mu-\lambda) = I^{(0)}_\alpha(t',\lambda')\bullet_{t'} I^{(0)}_\beta(t',\mu'),$$

and translating the integration variables $(t',\lambda')\mapsto(t',\mu')$ we get

$$\int_{-\lambda\mathbf{1}}^{t-\lambda\mathbf{1}} \mathcal{W}_{\alpha,\beta}(t'-\lambda'\mathbf{1},\mu-\lambda) = \int_{-\mu\mathbf{1}}^{t-\mu\mathbf{1}} \mathcal{W}_{\beta,\alpha}(t'-\mu'\mathbf{1},\lambda-\mu),$$

i.e., the integral in (4.6) is symmetric with respect to exchanging the pairs (α,λ) and (β,μ). To finish the proof it remains only to recall Lemma 4.1. □

4.4. **Proof of Theorem 2.3.** Let us choose $t \in B$, s.t., the canonical coordinates $u_i := u_i(t)$ are vertices of a regular N-gon with center at 0. In particular, $r(t) = |u_i|$ for all i. Let us fix a reference point (t, λ) in $\{t\} \times \mathbb{C} \subset S_\infty$ and take a closed loop C obtained by approaching one of the points u_i along a path in S_∞ and going around u_i. Since the fundamental group of S' is generated by loops of the above type, it suffices to prove that

$$\oint_C \tau^* \mathcal{W}_{\alpha,\beta}(\,\cdot\,, \mu - \lambda) - \Omega_{w(\alpha),w(\beta)}(t, \lambda, \mu) + \Omega_{\alpha,\beta}(t, \lambda, \mu) \quad \in \quad 2\pi\sqrt{-1}\mathbb{Z}, \tag{4.7}$$

where w is the monodromy transformation along C and μ is sufficiently close to λ. The integral in (4.7) can be written also as follows: if $(t, x) \in C$, then $\tau(t, x) = t - x\mathbf{1} \in \tau(C)$ and we have

$$\oint_C \tau^* \mathcal{W}_{\alpha,\beta}(\,\cdot\,, \mu - \lambda) = \oint_{\tau(C)} \mathcal{W}_{\alpha,\beta}(t - x\mathbf{1}, \mu - \lambda).$$

In our argument we always keep t fixed, all paths in S that we use will be identified with paths in $\mathbb{C} \cong \{t\} \times \mathbb{C} \subset S$, and we drop the argument of the phase form.

Lemma 4.3. *If φ is the cycle vanishing over u_i, then $\oint_C \tau^* \mathcal{W}_{\varphi,\varphi} = -4\pi\sqrt{-1}$.*

Proof. The differential equations of the periods imply that the integral is independent of λ and μ. Therefore, we may set $\mu = \lambda$. The computation in this case can be found in [10]. □

Lemma 4.4. *Assume that φ is the cycle vanishing over u_i and α is a cycle invariant along C, then*

$$\oint_C \tau^* \mathcal{W}_{\alpha,\varphi} = -2\Omega_{\alpha,\varphi}(t, \lambda, \mu).$$

Proof. Let us choose $\lambda' \in C$ sufficiently close to u_i with $|\lambda'| > r(t)$ and put $\mu' = \lambda' + \mu - \lambda$. The infinite series (2.10) at $(t, \lambda, \mu) = (t, \lambda', \mu')$ can be interpreted in two different ways: as a convergent Laurent series in $(\lambda')^{-1}$ or as a convergent Laurent series in $(\mu' - u_i)^{1/2}$. The 2nd interpretation allows us to extend the equality $d\Omega_{\alpha,\varphi}(t, \lambda', \mu') = \tau^* \mathcal{W}_{\alpha,\varphi}$ as λ' varies along the loop around u_i. The Lemma follows by the Stoke's theorem. □

If α and β are arbitrary cycles, then we can decompose them as

$$\alpha = \alpha' + (\alpha|\varphi)\varphi/2, \quad \beta = \beta' + (\beta|\varphi)\varphi/2,$$

where α' and β' are invariant along C. We have to compute

$$\oint_C \tau^* \mathcal{W}_{\alpha,\beta} = \frac{(\beta|\varphi)}{2} \oint_C \tau^* \mathcal{W}_{\alpha',\varphi} + \frac{(\alpha|\varphi)}{2} \oint_C \tau^* \mathcal{W}_{\varphi,\beta'} + \frac{(\alpha|\varphi)(\beta|\varphi)}{4} \oint_C \tau^* \mathcal{W}_{\varphi,\varphi}$$

where we used that $\oint_C \tau^* \mathcal{W}_{\alpha',\beta'} = 0$. The last and the first integrals can be computed by respectively Lemma 4.3 and Lemma 4.4. For the middle one, note that changing the variables $(t, x) \to (t, x + \mu - \lambda)$ gives an integral along a loop $\widetilde{C}$ based at (t, μ) which can be computed again by Lemma 4.4

$$\oint_C \tau^* \mathcal{W}_{\varphi,\beta'}(\cdot, \mu - \lambda) = \oint_{\widetilde{C}} \tau^* \mathcal{W}_{\beta',\varphi}(\cdot, \lambda - \mu) = -2\Omega_{\beta',\varphi}(t, \mu, \lambda).$$

In other words, we have

$$\oint_C \tau^* \mathcal{W}_{\alpha,\beta} = -(\beta|\varphi)\Omega_{\alpha',\varphi}(t, \lambda, \mu) - (\alpha|\varphi)\Omega_{\beta',\varphi}(t, \mu, \lambda) - \pi\sqrt{-1}(\alpha|\varphi)(\beta|\varphi).$$

Therefore, (4.7) takes the form

$$(\alpha|\varphi)\Big(\Omega_{\varphi,\beta'}(t,\lambda,\mu)-\Omega_{\beta',\varphi}(t,\mu,\lambda)\Big)-\pi\sqrt{-1}(\alpha|\varphi)(\beta|\varphi).$$

Recalling Lemma 4.2 we get

$$\begin{aligned}&-2\pi\sqrt{-1}(\alpha|\varphi)\,\mathrm{SF}(\varphi,\beta')-\pi\sqrt{-1}(\alpha|\varphi)(\beta|\varphi)=\\&=-2\pi\sqrt{-1}(\alpha|\varphi)\,\mathrm{SF}(\varphi,\beta)+\pi\sqrt{-1}(\alpha|\varphi)(\beta|\varphi)\Big(\mathrm{SF}(\varphi,\varphi)-1\Big).\end{aligned}$$

Since $\mathrm{SF}(\varphi,\varphi)=1$, the above number is an integer multiple of $2\pi\sqrt{-1}$. □

5. VOA representations

In this section we would like to explain the origin of the phase factors and the importance of Theorem 2.3 from the point of view of the representation theory of lattice VOAs.

5.1. **The tame Fock space.** Let us denote by

$$\mathfrak{W}:=\Big(\bigoplus_{m=0}^{\infty}\mathcal{H}^{\otimes m}\otimes\mathbb{C}(\!(\hbar)\!)\Big)/I$$

the Weyl algebra of $\mathcal{H}$, where I is the two-sided ideal generated by

$$v\otimes w-w\otimes v-\Omega(v,w)\hbar,\quad v,w\in\mathcal{H}.$$

Furthermore we introduce the *Fock space* $\mathbb{V}=\mathfrak{W}/\mathfrak{W}\mathcal{H}_+$. For any $\mathbb{C}$-algebra A, let us denote by $\widehat{\mathbb{V}}_A$ the completion of $\mathbb{V}\otimes A$ corresponding to the filtration $\{\mathcal{H}_-^m\mathbb{V}\otimes A\}_{m=0}^\infty$. When $A=\mathbb{C}$ then the index A will be omitted.

As a vector space the Weyl algebra $\mathfrak{W}$ is isomorphic to

$$\bigoplus_{m',m''=0}^{\infty}\mathcal{H}_-^{\otimes m'}\otimes\mathcal{H}_+^{\otimes m''}\otimes\mathbb{C}(\!(\hbar)\!).$$

We introduce the *tame Weyl algebra*[1]

$$\mathfrak{W}_{\rm tame}\subset\widehat{\mathfrak{W}}:=\prod_{m',m''=0}^{\infty}\Big(\underbrace{\mathcal{H}_-\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{H}_-}_{m'}\widehat{\otimes}\underbrace{\mathcal{H}_+\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{H}_+}_{m''}\widehat{\otimes}\mathbb{C}(\!(\hbar)\!)\Big),$$

where $\widehat{\otimes}$ is the completion of the tensor product induced by the filtrations $\{z^{-m}\mathcal{H}_-\}_{m=0}^\infty$, $\{z^m\mathcal{H}_+\}_{m=0}^\infty$, and $\{\hbar^m\mathbb{C}(\!(\hbar)\!)\}_{m=0}^\infty$ of respectively $\mathcal{H}_-$, $\mathcal{H}_+$, and $\mathbb{C}(\!(\hbar)\!)$. Every element of $\widehat{\mathfrak{W}}$ is an infinite series of monomials of the type $\hbar^{g-1}v\otimes w$, where $g\in\mathbb{Z}$, v and w have the form respectively $v_1z^{-k_1-1}\otimes\cdots\otimes v_{m'}z^{-k_{m'}-1}$ and $w=w_1z^{\ell_1}\otimes\cdots\otimes w_{m''}z^{\ell_{m''}}$. The tame Weyl algebra $\mathfrak{W}_{\rm tame}$ consists of those series for which the non-zero monomials satisfy the inequality

$$k_1+\cdots+k_{m'}-m'\le 3(g-1+m''/2).$$

Let us point out that if $m''=0$, then this is the notion of a tame function introduced in [10]. One can check that $\mathfrak{W}_{\rm tame}$ is in fact an algebra with the product induced from $\mathfrak{W}$. Finally, let us introduce also the *tame* Fock space $\mathbb{V}_{\rm tame}:=\mathfrak{W}_{\rm tame}/\mathfrak{W}_{\rm tame}\mathcal{H}_+$.

[1]This notion was invented by H. Iritani

5.2. **Heisenberg fields and Vertex operators.** Let us denote by $\mathcal{O}$ the algebra of analytic functions on the monodromy covering space $\widetilde{S}'$ with at most polynomial growth at $\lambda=\infty$. By definition (and since the Gauss–Manin connection has regular singularities) all period vectors $I^{(m)}_\alpha \in \mathcal{O}\otimes H$. Let us define the following set of linear operators $\mathbb{V}_{\rm tame}\to \widehat{\mathbb{V}}_{\mathcal{O}}$:

$$\phi_a(t,\lambda)=\hbar^{-1/2}\partial_\lambda \mathbf{f}_a(t,\lambda;z),\quad a\in\mathfrak{h}^* \tag{5.1}$$

and

$$\Gamma^\alpha(t,\lambda)=e^{\hbar^{-1/2}\mathbf{f}_\alpha(t,\lambda;z)_-}e^{\hbar^{-1/2}\mathbf{f}_\alpha(t,\lambda;z)_+},\quad \alpha\in Q, \tag{5.2}$$

where $\pm$ denotes the projection on $\mathcal{H}_\pm$. If we want to define a VOA-representation using the above operators we need to define a composition of vertex operators. Let us first compose two vertex operators ignoring the possible divergence issues. We get

$$\Gamma^\alpha(t,\lambda)\Gamma^\beta(t,\mu)=e^{\Omega_{\alpha,\beta}(t,\lambda,\mu)} : \Gamma^\alpha(t,\lambda)\Gamma^\beta(t,\mu) :, \tag{5.3}$$

where : : is the normally ordered product for which the action of all elements of $\mathcal{H}_+$ precedes the action of $\mathcal{H}_-$. The problem is that in general we can not exponentiate elements of the ring $\mathbb{C}(\!(\mu^{-1/|\sigma|})\!)(\!(\lambda^{-1/|\sigma|})\!)$. To offset this difficulty we recall that the series (2.10) is convergent for all $(t,\lambda,\mu)\in D_\infty-\{\lambda=\mu\}$. Moreover, using the integral presentation (2.13) we can extend analytically $\Omega_{\alpha,\beta}(t,\lambda,\mu)$ with respect to $(t,\mu)\in S'$ provided λ is sufficiently close to μ. Therefore, the *phase factor*

$$B_{\alpha,\beta}(t,\lambda,\mu):=\exp\left(\Omega_{\alpha,\beta}(t,\lambda,\mu)\right) \tag{5.4}$$

has a convergent Laurent series expansion in $(\mu-\lambda)$ with coefficients multi-valued analytic functions on S', i.e., the coefficients are analytic on the universal cover of S'. According to Theorem 2.3 the coefficients are analytic on the *monodromy* covering space $\widetilde{S}'$. In other words we have the following proposition.

Proposition 5.1. *The phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ has a Laurent series expansion in $\mu-\lambda$, whose coefficients are elements of $\mathcal{O}$.*

5.3. **Applications to VOA.** Let us denote by

$$V_Q:=\mathrm{Sym}(\mathfrak{h}^*[s^{-1}]s^{-1})\otimes\mathbb{C}[Q]$$

the vector space underlying the lattice vertex algebra corresponding to the Milnor lattice Q. Note that we did not put a VOA structure on V_Q yet. Following [2], we can introduce a family of products $a_{(k)}b$, $a,b\in V_Q$, $k\in\mathbb{Z}$ and a linear map

$$X: V_Q\to \mathrm{Hom}_{\mathbb{C}}(\mathbb{V}_{\rm tame},\widehat{\mathbb{V}}_{\mathcal{O}}),$$

defined recursively by the initial conditions

$$X(as^{-1}\otimes 1):=\phi_a,\quad X(1\otimes e^\alpha):=\Gamma^\alpha, \tag{5.5}$$

and the *operator product expansion* (OPE) formula

$$X_t(a_{(M-k-1)}b,\lambda)\mathcal{A}=\frac{1}{k!}\partial_\mu^k\left((\mu-\lambda)^M X_t(a,\mu)X_t(b,\lambda)\mathcal{A}\right)\Big|_{\mu=\lambda}, \tag{5.6}$$

where $a,b\in V_Q$ and $\mathcal{A}\in\mathbb{V}_{\rm tame}$ are arbitrary, the number M is sufficiently large, $k\geq 0$, and we denote by $X_t(a,\lambda)$ the value of the operator $X(a)$ at a point $(t,\lambda)\in S'$. One has to check that the above definition is correct and that all the fields $X_t(a,\lambda)$, $a\in V_Q$, are mutually local. Let us point out that the products $a_{(k)}b$ will depend on the choice of a point on $\widetilde{S}'$, i.e., a point (t,λ) with a reference path. Our expectation is that V_Q has a unique structure of a VOA, s.t., the products defined via the OPE formula coincides with the Borcherds' products. Moreover, the VOA structure on V_Q is isomorphic to the lattice VOA structure (see [13] for some background). The details will be presented elsewhere.

Appendix A. Jonquière's inversion formula

We would like to give a proof of formula (4.3), because as it was pointed out by the Wikipedia article about polylogorithms, some authors are not very careful about the choice of the branch of $\log x$, while for our purposes choosing the correct branch is essential.

Let us assume that C' intersects the real axis once and that $\mathrm{Im}(x)\neq 0$. We can always achieve this by replacing C' with a homotopy equivalent path and by deforming slightly x. Put

$$f_p(x) := \mathrm{Li}_p(x) + (-1)^p\,\mathrm{Li}_p(1/x).$$

Since $x\partial_x f_p(x) = f_{p-1}(x)$, we get that $f_p(x)$ is a polynomial in $\log x$. Let us write it as

$$f_p(x) =: -\frac{(2\pi\sqrt{-1})^p}{p!} h_p\Big(\frac{\log x}{2\pi\sqrt{-1}}\Big).$$

In order to determine the polynomials h_p, let us see what happens when we analytically continue x along a closed loop inside the unit disk going counterclockwise around 0. There are 4 cases to be considered, depending on whether C' intersects $(1,\infty)$ or not and whether $\mathrm{Im}(x)>0$ or <0. Let us analyze the case: C' does not intersect $(1,\infty)$ and $\mathrm{Im}(x)<0$. The remaining cases are similar. Note that in this case $\log x = \mathrm{Log}\, x + 2\pi\sqrt{-1}$. We have the following integral representation

$$\mathrm{Li}_p(x^{-1}) = \frac{x^{-1}}{\Gamma(p)}\int_1^\infty \frac{(\mathrm{Log}\ z)^{p-1}}{z-x^{-1}}\frac{dz}{z}.$$

When x is varying around 0, the point x^{-1} will vary clockwise along a closed loop C around the unit disk. Let y be a point on C which we reach slightly after we cross the interval $(1,\infty)$. Using the Cauchy residue theorem, we get

$$\mathrm{Li}_p(y) = \frac{y}{\Gamma(p)}\int_1^\infty \frac{(\mathrm{Log}\ z)^{p-1}}{z-y}\frac{dz}{z} + \frac{2\pi\sqrt{-1}}{\Gamma(p)}(\mathrm{Log}\ y)^{p-1}.$$

Extending analytically the above identity as y varies along the remaining part of C, we get the following formula

$$\mathrm{a.c.} f_p(x) - f_p(x) = (-1)^p\Big(\mathrm{a.c.}\,\mathrm{Li}_p(x^{-1}) - \mathrm{Li}_p(x^{-1})\Big) = -\frac{2\pi\sqrt{-1}}{\Gamma(p)}\Big(-\mathrm{Log}\ x^{-1} + 2\pi\sqrt{-1}\Big)^{p-1}.$$

Expressing f_p in terms of h_p we arrive at the following difference equation

$$h_p(L+1) - h_p(L) = pL^{p-1},\quad L := \log x/2\pi\sqrt{-1},\quad p\geq 1.$$

On the other hand, since $x\partial_x f_p(x) = f_{p-1}(x)$ we also have $\partial_L h_p(L) = p h_{p-1}(L)$. The solution to these relations is unique and it is given by the Bernoulli polynomials.

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