Growth functions for Artin monoids

Kyoji Saito [*]

Abstract

In [S1], we showed that the growth function $P_M(t)$ for an Artin monoid of finite type M is a rational function of the form $1/(1-t)N_M(t)$, and formulated three conjectures on the zeros of the denominator polynomial $N_M(t)$ of a finite type.[1] In the present note, we observe that the same formula holds for arbitraly Artin monoids, and formulate slightly modified conjectures for the denominator polynomials of the growth function of affine type. The new conjectures are verified for affine types $\tilde{A}_2,\cdots,\tilde{A}_8,\tilde{C}_2,\cdots,\tilde{C}_8,\tilde{D}_4,\tilde{E}_7,\tilde{E}_8,\tilde{F}_4,\tilde{G}_2$ among others. In Appendix, we formally define the denominator polynomials for elliptic root systems ([S2]). Then, the new conjectures are verified also for elliptic types $A_2^{(1,1)},\cdots,A_7^{(1,1)},D_4^{(1,1)},E_6^{(1,1)},E_7^{(1,1)},E_8^{(1,1)}$ and $G_2^{(1,1)}$.

1 Growth function for an Artin monoid

We recall the definition (1.3) of a *growth function* for an Artin monoid, and introduce the *denominator polynomial* (1.6) in the following *Lemma-Definition* 1 (see [2]).

Let $M=(m_{ij})_{i,j\in I}$ be a Coxeter matrix ([B]). The Artin monoid G_M^+ ([B-S, §1.2]) associated with M (or, of type M) is a monoid generated by the letters a_i, $i\in I$ which are subordinate to the relation generated by

(1.1) $$a_ia_ja_i\cdots \;=\; a_ja_ia_j\cdots \qquad i,j\in I,$$

where both hand sides of (1.1) are words of alternating sequences of letters a_i and a_j of the same length $m_{ij}=m_{ji}$ with the initials a_i and a_j, respectively. More precisely, G_M^+ is the quotient of the free monoid generated by the letters a_i $(i\in I)$ by the equivalence relation: two words U and V in the letters are equivalent, if there exists a sequence $U_0:=U,U_1,\cdots,U_m:=V$ such that the word U_k $(k=1,\cdots,m)$ is obtained by replacing a phrase in U_{k-1} of the form on LHS of (1.1) by RHS of (1.1) for some $i,j\in I$. We write by $U \underset{\bullet}{=} V$ if U and V are equivalent. The equivalence class (i.e. an element of G_M^+) of a word W is denoted by the same notation W. By the definition, equivalent words have the same length. Hence, we define the degree homomorphism:

(1.2) $$\deg\ :\ G_M^+ \longrightarrow \mathbb{Z}_{\geq 0}$$

by assigning to each equivalence class of words the length of the words.

*Institute for the Physics and Mathematics of the Universe, the University of Tokyo

[1] Here M is a Coxeter matrix [B]. We shall refer to M as the type of the Coxeter group $\bar{G}_M$, Artin group G_M, Artin monoid G_M^+, growth function $P_M(t)$, etc. associated with M. In the present note, we shall say a Coxeter matrix M is of finite (resp. affine) type if the associated Coxeter group ([B, Ch.IV §1]) is finite (resp. affine), or equivalently, if the associated bilinear form B_M [B] is positive definite (resp. semi-positive with rank 1 kernel).

[2] The definition is a slight generalization of that for finite type in [S1, (2.2),(2.3)]. A similar generalization is made by Albenque and Nadeau ([A-N, (1.2)]) for cancellative monoids such that if a subset of atomic generators has a common multiple then it admits a least common multiple. This class covers the Artin monoids, which we discuss in the present paper.

The growth function $P_{G^+_M,I}(t)$ for the Artin monoid G^+_M is defined by

$$(1.3)\qquad P_{G^+_M,I}(t) \;:=\; \sum_{n\in\mathbb{Z}_{\geq 0}} \#\{W\in G^+_M \mid \deg(W)\leq n\}\; t^n.$$

The *spherical growth function of the monoid* G^+_M of type M is defined by

$$(1.4)\qquad \dot{P}_{G^+_M,I}(t) \;:=\; \sum_{n\in\mathbb{Z}_{\geq 0}} \#(\deg^{-1}(n))\; t^n,$$

so that one has the obvious relation: $P_{G^+_M,I}(t) = \dot{P}_{G^+_M,I}(t)/(1-t)$.

Lemma-Definition 1. *Let G^+_M be the Artin monoid of any type M. Then the spherical growth function of the monoid is given by the Taylor expansion of the rational function of the form*

$$(1.5)\qquad \dot{P}_{G^+_M,I}(t) \;=\; \frac{1}{N_M(t)}.$$

Here, $N_M(t)$ is called the denominator polynomial *and is given by*

$$(1.6)\qquad N_M(t) \;:=\; \sum_{J\subset I} (-1)^{\#(J)}\; t^{\deg(\Delta_J)},$$

where the summation index J runs over subsets of I such that the restricted Coxeter matrix $M|_J$ is of finite type,[3] *and Δ_J is the fundamental element in G^+_M associated with J (*[B-S, §5 Definition]. *See also* Lemma-Definition 2 *and* Remark 1.2 *of the present note).*

Proof. The proof is achieved by a recursion formula (1.12) on the coefficients of the growth function. For the proof of the formula, we use the method used to solve the word problem for the Artin monoid ([B-S, §6.1]), which we recall below. We first recall the fact that an Artin monoid satisfies the cancellation condition in the following sense ([B-S, Prop.2.3]).

Lemma 1.1. *Let $A,B,X,Y\in G^+_M$. If $AXB \underset{\bullet}{=} AYB$. Then $X\underset{\bullet}{=}Y$.*

A word U is said to be divisible (from the left) by a word V, and denoted by $V|U$, if there exists a word W such that $U\underset{\bullet}{=}VW$. Since $V\underset{\bullet}{=}V'$, $U\underset{\bullet}{=}U'$ and $V|U$ implies $V'|U'$, we use the notation "$|$" of divisibility also between elements of the monoid G^+_M. We have the following basic concepts ([B-S, §5 Definition and §6.1]).

Lemma-Definition 2. *Let M be a Coxeter matrix of any type, and let $J\subset I$ be a subset of I such that $M|_J$ is of finite type (which may not necessarily be indecomposable). Then, there exists a unique element $\Delta_J\in G^+_M$, called the* fundamental element*, such that i) $a_i|\Delta_J$ for all $i\in J$, and ii) if $W\in G^+_M$ and $a_i|W$ for all $i\in J$, then $\Delta_J|W$.*

3. *To an element $W\in G^+_M$, we associate the subset of I:*

$$(1.7)\qquad I(W) \;:=\; \{i\in I \;\mid\; a_i|W\}.$$

The restricted Coxeter matrix $M|_{I(W)}$ is of finite type for any $W\in G^+_M$.

Proof. **2.** This follows from the fact that the existence of Δ_J is achieved under a weaker assumption than M_J is of finite type, rather that there exists a common multiple of a_j for $j\in J$ in G^+_M (see [B-S, Prop. (4.1)]. □

By the definition (1.7), one has $\Delta_{I(W)}|W$ and if $\Delta_J|W$ then $J\subset I(W)$.

[3]For a Coxeter matrix $M=(m_{ij})_{i,j\in I}$ and a subset J of I, we define the restricted Coxeter matrix by $M|_J := (m_{ij})_{i,j\in J}$, which, obviously, is again a Coxeter matrix.

We return to the proof of Theorem.

For $n \in \mathbb{Z}_{\geq 0}$ and for any subset $J \subset I$, put

$$G_n^+ \ := \ \{W \in G_M^+ \mid \deg(W) = n\} \tag{1.8}$$

$$G_{n,J}^+ \ := \ \{W \in G_n^+ \mid I(W) = J\}. \tag{1.9}$$

We note that $G_{n,J}^+ = \emptyset$ if $M|_J$ is not of finite type. By the definition, we have the disjoint decomposition:

$$G_n^+ \ = \ \amalg_{J \subset I} \ G_{n,J}^+, \tag{1.10}$$

where J runs over all subsets of I. Note that $G_{n,\emptyset}^+ = \emptyset$ if $n > 0$ but $G_{0,\emptyset}^+ = \{\emptyset\} \neq \emptyset$. For any subset J of I, the union $\amalg_{J \subset K \subset I} G_{n,K}^+$, where the index K runs over all subsets of I containing J, is equal to the subset of G_n^+ consisting of elements divisible by a_j for $j \in J$. That is, one has

$$\amalg_{J \subset K \subset I} G_{n,K}^+ \ = \ \begin{cases} \Delta_J \cdot G_{n-\deg(\Delta_J)}^+ & \text{if } M|_J \text{ is of finite type,} \\ \emptyset & \text{if } M|_J \text{ is not of finite type.} \end{cases}$$

Thus, if $M|_J$ is of finite type, due to the cancellation condition Lemma 1.1, the multiplication map of Δ_J is injective and we obtain a bijection: $G_{n-\deg(\Delta_J)}^+ \simeq \amalg_{J \subset K \subset I} G_{n,K}^+$. This implies a numerical relation:

$$\#(G_{n-\deg(\Delta_J)}^+) \ = \ \textstyle\sum_{J \subset K \subset I} \#(G_{n,K}^+). \tag{1.11}$$

If $M|_J$ is not of finite type, still the formula (1.11) holds formally, by putting $\deg(\Delta_J) := \infty$ and $G_{-\infty}^+ := \emptyset$, i.e. $\#(G_{n-\deg(\Delta_J)}^+) := 0$. Then, for $n > 0$, using (1.11), we get the recursion relation:

$$\sum_{J \subset I} (-1)^{\#(J)} \#(G_{n-\deg(\Delta_J)}^+) \ = \ 0, \tag{1.12}$$

where the index J may run either over all subsets of I, or, over only subset J such that the restricted Coxeter matrix $M|_J$ is of finite type. Together with $\#(G_0^+) = 1$ for $n = 0$, this is equivalent to the formula:

$$\dot{P}_{G_M^+, I}(t) N_M(t) \ = \ 1. \tag{1.13}$$

This completes the proof of Lemma-Definition 1. □

Remark 1.2. We have the equality ([B-S, §5.7]):
$\deg(\Delta_J) = \#\{\text{reflections in } \overline{G}_{M|_J}\} = \text{the length of the longest element of } \overline{G}_{M|_J}$.

By the definition (1.6) of the denominator polynomial, one has

$$N_M(1) = \sum_{\substack{J \subset I, M|_J \text{is} \\ \text{of finite type}}} (-1)^{\# J}$$

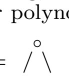
This, in particular, implies

i) $N_M(t)$ has the factor $1 - t$ if M contains a component of finite type, and

ii) $N_M(1) = (-1)^l$ if M is of indecomposable affine type of rank l (i.e. M is indecomposable and affine such that $\#(I) = l + 1$).[4]

We refer to [S1] and http://www.kurims.kyoto-u.ac.jp/~saito/FFST/ for examples of finite type (the author express his gratitude to S. Tsuchioka for making this page). Here, we give a few examples of affine type.

Example. There are three types of indecomposable affine Coxeter matrices of rank 2. In the following, for each type, we associate the Coxeter diagram Γ_M and the denominator polynomial $N_M(t)$.

1. $\tilde{A}_2$ $\quad \Gamma_{\tilde{A}_2} = $  $\quad N_{\tilde{A}_2}(t) = 1 - 3t + 3t^3$
2. $\tilde{C}_2$ $\quad \Gamma_{\tilde{C}_2} = \circ \overset{}{\underset{4}{\text{—}}} \circ \underset{4}{\text{—}} \circ$ $\quad N_{\tilde{C}_2}(t) = 1 - 3t + t^2 + 2t^4$
3. $\tilde{G}_2$ $\quad \Gamma_{\tilde{G}_2} = \circ \text{—} \circ \underset{6}{\text{—}} \circ$ $\quad N_{\tilde{G}_2}(t) = 1 - 3t + t^2 + t^3 + t^6$

[4]The discrepancy between the rank l and the number $\#(I) = l + 1$ for a Coxeter matrix M of indecomposable affine type comes from the fact that the associated affine Coxeter group acts on a semi-positive $\mathbb{R}$-vector space of signature $(l, 1, 0)$.

2 A bound on the zeros of the denominator polynomial $N_M(t)$ of affine type

The following lemma gives a numerical bound on the zeros of the denominator polynomials for indecomposable affine type.

Lemma 2.1. *Let M be a Coxeter matrix of indecomposable affine type of rank l. Then, all the roots of $N_M(t) = 0$ are contained in the open disc of radius r centered at the origin, where r is give by*

$$r := \Big(\frac{2^{l+1} - s - 1}{s}\Big)^{1/(\deg(\Delta_{M|_{I\setminus\{v\}}})-d)}, \tag{2.1}$$

and $\deg(\Delta_{M|_{I\setminus\{v\}}}), d, s$ *are invariants of M explained in the proof.*

Proof. In the affine Coxeter graph Γ_M (whose vertex set is identified with I, and hence $\#(\Gamma_M) = \#(I) = l+1$), there is a vertex v, called *special* [B, p.87] such that $\Gamma_M \setminus \{v\}$ is the Coxeter graph of the finite Coxeter group isomorphic to the radical quotient of the affine Coxeter group $\overline{G}_M$. Let s be the number of special vertices in Γ_M. For types $\tilde{A}_l, \tilde{B}_l, \tilde{C}_l, \tilde{D}_l, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$, the number s is given by $l+1$, 2, 2, 4, 3, 2, 1, 1, 1, respectively.

Noting the fact that the type of $\Gamma_{M\setminus\{v\}}$ (and, hence, $\deg(\Delta_{M|_{I\setminus\{v\}}})$) does not depend on the choice of a special vertex v, we see that the monomial $N(t) := (-1)^l s \cdot t^{\deg(\Delta_{M|_{I\setminus\{v\}}})}$ (v a special vertex) is the leading term of $N_M(t)$, one has $|N_M(t) - N(t)| \le (2^{l+1} - s - 1)|t|^d$ for $t \in \mathbb{C}$ with $|t| > 1$ (strict inequality holds except for the type $\tilde{A}_1$), where we put

$$d := \max\{\deg(\Delta_J) \mid J \subset I \text{ such that } I \setminus J \text{ is not a single special vertex}\}.$$

Hence $|N_M(t) - N(t)|/|N(t)| \le \frac{2^{l+1}-s-1}{s}|t|^{d-\deg(\Delta_{M|_{I\setminus\{v\}}})}$. If $r \in \mathbb{R}_{>1}$ satisfies an inequality $\frac{2^{l+1}-s-1}{s} r^{d-\deg(\Delta_{M|_{I\setminus\{v\}}})} \le 1$, then, due to Rouche's theorem, the number of zeros of $N_M(t) = 0$ in the disc of radius r is equal to that of $N(t) = 0$, which has zeros only at 0 of multiplicity $\deg(N(t)) = \deg(N_W(t))$. That is, all roots of $N_M(t) = 0$ are in the disc $\{|t| < r\}$ for r given in (2.1). □

3 Conjectures on the zeros of the denominator polynomial $N_M(t)$ of affine type

Motivated by a study of the author on certain limit functions associated withfinitely generated monoids or groups (see §11 and 12 of [S] in the reference [S1] of the present note), some discussions and examples at the end of §1 lead us to the following three conjectures on the distribution of the zeros of the denominator polynomial $N_M(t)$ of indecomposable finite or affine type.[5]

Conjecture 1. i) The polynomial $\tilde{N}_M(t) := N_M(t)/(1-t)$ is irreducible over $\mathbb{Z}$ for any indecomposable finite type M. ii) The polynomial $N_M(t)$ is irreducible over $\mathbb{Z}$ for any indecomposable affine type M.

Conjecture 2. There are l mutually distinct real roots of $N_M(t) = 0$ on the interval $(0, 1]$ where l is the rank of positive definite part of the root lattice associated with M.

Conjecture 3. Let r_M be the smallest of the roots on the interval $(0, 1]$. Then, the absolute values of the other roots of $N_M(t) = 0$ are strictly larger than r_W.

[5] As we shall observe in Appendix, these conjectures are (formally) valid also for elliptic root systems [S2]. After a suitable modification, the conjectures seems to be valid also for some Artin monoids of hyperbolic type (see Remark 3.1 and 3.4). It is interesting to clarify the range where the conjecture works, and to develop a unified understanding of them (hopefully, in connection with the original motivation to study the limit functions associated with monoids).

Conjectures on the denominator polynomials of finite type were already stated in [S1] and verified by computer calculations for the types A_l, B_l, C_l, D_l ($l \leq 30$), E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(p)$ ($p \in \mathbb{Z}_{\geq 3}$) by M. Fuchiwaki, S. Tsuchioka and others (see http://www.kurims.kyoto-u.ac.jp/˜ saito/FFST/). Some theoretical approach on the conjectures is in progress by S. Yasuda.

Conjectures on the denominator polynomial of affine type are positively confirmed directly for the three types $\tilde{A}_2$, $\tilde{C}_2$ and $\tilde{G}_2$ of rank 2 from the explicit expressions in §2 Example. Further cases, include $\tilde{A}_3,\cdots,\tilde{A}_8, \tilde{C}_3,\cdots,\tilde{C}_8, \tilde{D}_4$, $\tilde{E}_7$, $\tilde{E}_8$ and $\tilde{F}_4$, are calculated affirmatively by S. Tsuchioka by use of computer.

Remark 3.1. As we observed, a denominator polynomial $N_X(t)$ of finite type has zeros of order 1 at $t = 1$, and that of affine type does not vanish there. In Appendix, we observe that a denominator polynomial of elliptic type does not vanish there either. It is an immediate calculation to show that a denominator polynomial of hyperbolic type (2,3,7), (2,4,5), (3,3,4), (2,3,8), (3,3,5), (2,5,5), (2,3,9), (2,4,7), (2,5,6), (3,4,5) or (4,4,4) has zeros at $t = 1$. It is interesting to find a formula of the order d of zeros at $t = 1$ in general and to ask more precise question than Conjecture 1: whether or when is $N_X(t)/(1-t)^d$ irreducible?

Remark 3.2. The definition of rank in Conjecture 2 is restrictive in the sense that some Coxeter systems (which are not cristalographic such as type H_3, H_4 and $I_2(p)$) do not admit a root system. Therefore, we should read here the word "the rank of positive part of the root lattice" loosely in the sense "the rank of the positive part of the real quadratic form B_M ([B])" associated with M. On the other side, in the following Appedix, we study formally denominator polynomials associated with elliptic root systems, which are not associated with Coxeter systems. However their root systems and root lattices make sense of the definition of the rank in Cojecture 2 (cf. footnote 5).

Remark 3.3. In Conjecture 3, the fact that r_M is less than or equal the absolute values of any other roots of $N_W(t) = 0$ is trivially true, since r_M is equal to the radius of convergence of the series $P_M(t)$ of non-negative real coefficients. Therefore, the true question here is that there are no other roots of $N_W(t) = 0$ whose absolute value is equal to r_W. This question is motivated from a study of the author on certain limit functions associated with the monoid G_M^+ (see [S1, §5]).

Appendix.

Pursuing formal analogy (i.e. without an explicit relation with the growth functions[6]), let us introduce the denominator polynomial $N_X(t)$ of elliptic type: let (R, G) be an irreducible marked elliptic root system of type X and let $\Gamma_X := \Gamma(R, G)$ be the associated Dynkin diagram [S2, I,§8]. Then, similar to (1.6), we define the *elliptic denominator polynomial of type X* by

$$N_X(t) \ := \sum_{J\subset\Gamma_X} (-1)^{\#(J)} t^{\deg(\Delta_J)},$$

where the summation index J runs over all subdiagrams of Γ_X which is of finite type.

For these denominator polynomials of elliptic type, we ask again whether Conjectures 1.ii) (replacing indecomposable affine type by irreducible marked elliptic type), 2. and 3. hold or not. Then, the conjectures are affirmatively verified for the types $A_2^{(1,1)},\cdots, A_7^{(1,1)}, D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$ and $G_2^{(1,1)}$ using computer by S. Tsuchioka.

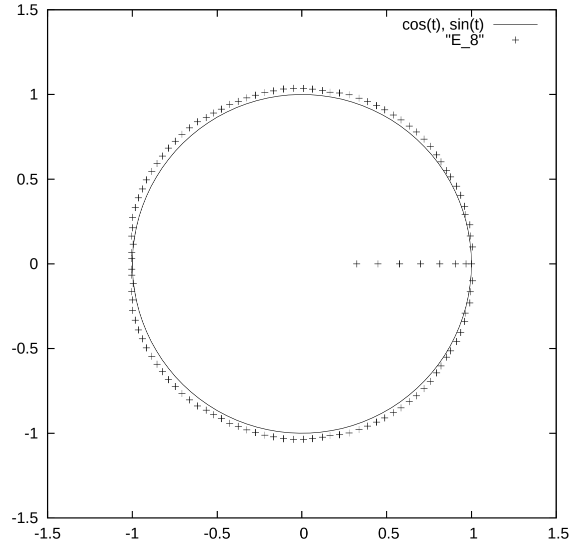
Remark 3.4. The above observation suggests that it seems of interest to define formally the denominator polynomial associated with the Gabrielov diagram of the basis of the lattice of vanishing cycles associated with 14 regular systems of weights with $\varepsilon = -1$ and to compare them with the denominator polynomials of the Artin monoids associated with the Picard lattice of the K3 surfaces obtained by Pinkham compatification of the Milnor fiber (see [S3, §13 and §18]).

[6] Associated with elliptic root systems, there are concepts of elliptic Weyl groups, elliptic Lie groups, elliptic Hecke algebra,... etc. However, at present, there is no clear description of the growth function associated with the elliptic Artin monoid (since they are no longer Coxeter groups or Kac-Moody groups,... etc), which is a subject to be studied yet.

Example (S. Tsuchioka). We illustrate the zero loci of the denominator polynomials of finite type E_8, affine type $\tilde{E}_8$ and elliptic type $E_8^{(1,1)}$. In the following figures, zero-loci are indicated by crosses "+".

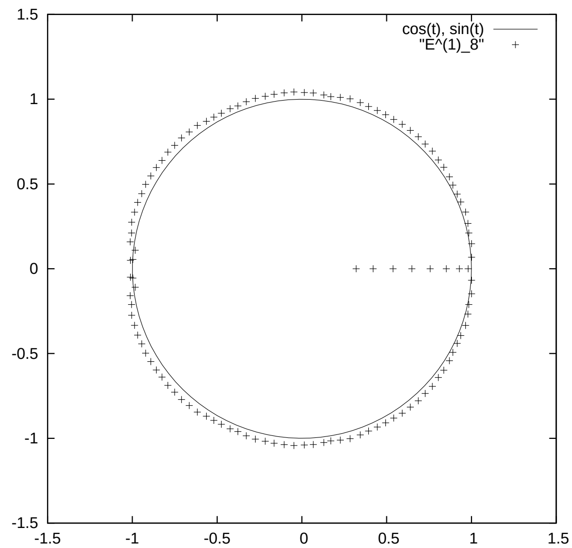
Type E_8

$N_{E_8}(t) = 1-8t+21t^2-14t^3-21t^4+28t^5-7t^6+12t^7-8t^8-10t^9+10t^{1}0-12t^{1}1+7t^{1}2+2t^{13}-t^{14}-3t^{15}+2t^{16}-2t^{20}+6t^{21}-t^{22}-t^{23}-t^{28}+t^{30}+t^{36}-t^{37}-t^{42}-t^{63}+t^{120}$.



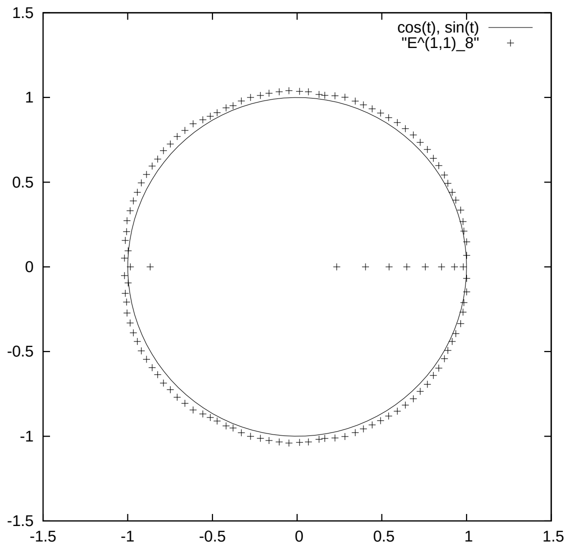
Type $\tilde{E}_8$

$N_{\tilde{E}_8}(t) = 1-9t+28t^2-28t^3-22t^4+54t^5-20t^6+10t^7-17t^8-13t^9+21t^{10}-23t^{11}+19t^{12}+7t^{13}-5t^{14}-3t^{15}+4t^{16}-3t^{17}-3t^{18}+t^{19}-t^{20}+9t^{21}-4t^{22}-3t^{23}+t^{26}-3t^{28}+t^{29}+t^{30}-t^{31}+2t^{36}-2t^{37}+t^{39}-t^{42}+t^{56}-t^{63}+t^{64}+t^{120}$.



Type $E_8^{(1,1)}$

$N_{E_8^{(1,1)}}(t) = 1-10t+33t^2-32t^3-35t^4+73t^5-23t^6+21t^7-30t^8-28t^9+36t^{10}-38t^{11}+34t^{12}+12t^{13}-8t^{14}-5t^{15}+5t^{16}-4t^{17}-5t^{18}+t^{19}-2t^{20}+18t^{21}-8t^{22}-6t^{23}+2t^{26}-6t^{28}+2t^{29}+2t^{30}-2t^{31}+4t^{36}-4t^{37}+2t^{39}-2t^{42}+2t^{56}-2t^{63}+2t^{64}+2t^{120}$.



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