# GENERALIZED BOGOMOLOV-GIESEKER TYPE INEQUALITIES ON FANO 3-FOLDS 

DULIP PIYARATNE


#### Abstract

We modify the conjectural Bogomolov-Gieseker type inequality introduced by Bayer, Macrì and Toda to construct a family of geometric Bridgeland stability conditions on smooth projective 3 -folds. We give an equivalent conjecture which needs to check these inequalities for a small class of tilt stable objects. We extend some of the techniques in Li's work for Fano 3-folds of Picard rank one to establish our modified Bogomolov-Gieseker type inequality conjecture for general Fano 3-folds.


## 1. Introduction

The notion of stability conditions on triangulated categories was introduced by Bridgeland (see [Bri]). Such a stability condition on a triangulated category is defined by giving a bounded t-structure together with a stability function on its heart satisfying the Harder-Narasimhan property. This can be interpreted essentially as an abstraction of the usual slope stability for sheaves. Construction of Bridgeland stability conditions on the bounded derived category of a given projective threefold is an important problem. However, unlike for a projective surface, there is no known construction which gives stability conditions for all projective threefolds. See Huy2 for further details.

In [BMT], Bayer, Macrì and Toda introduced a conjectural construction of Bridgeland stability conditions for any projective threefold. Here the problem was reduced to proving so-called Bogomolov-Gieseker type inequality holds for certain tilt stable objects. It has been shown to hold for all Fano 3-folds with Picard rank one (see [BMT, Mac, Sch1, Li]), abelian 3 -folds (see [MP1, MP2, Piy1, Piy2, BMS]) and étale quotients of abelian 3 -folds (see [BMS]). Properties of tilt stable objects are crucial in all of these examples. Recently, Schmidt found a counterexample to the Bogomolov-Gieseker type inequality conjecture when X is the blowup at a point of $\mathbb{P}^{3}$ (see [Sch2]). Therefore, this inequality needs some modifications in general setting and this is one of the main goals of this paper.

In this paper we formulate a conjecture for general 3 -folds modifying the conjectural Bogomolov-Gieseker type inequality in BMT. Let $X$ be a smooth projective 3 -fold, and let $H \in N S(X)$ be an ample divisor class and $B \in N S_{\mathbb{R}}(X)$. For $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$, we define

$$
\mathrm{D}_{\alpha, \boldsymbol{\beta}}^{\mathrm{B}, \boldsymbol{\xi}}(\mathrm{E})=\operatorname{ch}_{3}^{\mathrm{B}+\beta \mathrm{H}^{2}}(\mathrm{E})+\left(\frac{\mathrm{c}_{2}(\mathrm{X})}{12}-\left(\frac{\mathrm{c}_{2}(\mathrm{X}) \cdot \mathrm{H}}{12 \mathrm{H}^{3}}+\xi+\frac{1}{6} \alpha^{2}\right) \mathrm{H}^{2}\right) \operatorname{ch}_{1}^{\mathrm{B}+\beta \mathrm{H}}(\mathrm{E}) .
$$

[^0]Conjecture 1.1 ( $=4.2$ ). There exists a constant $\xi \in \mathbb{R}_{\geqslant 0}$ and a continuous function $A_{B}$ : $\mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}$ such that, for any $\alpha \in \mathbb{R}_{>0}, \beta \in \mathbb{R}$ satisfying $\alpha \geqslant A_{B}(\beta)$, all tilt slope $v_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}^{-}}$ stable objects $\mathrm{E} \in \mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}$ with $\nu_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}(\mathrm{E})=0$ satisfy the inequality $D_{\alpha, \beta}^{\mathrm{B}, \xi}(\mathrm{E}) \leqslant 0$.

This modified conjectural inequality coincides with the inequality in BMT for all the known 3-folds where it holds. Following similar ideas in BMS, Lemma 8.3], if this conjecture holds we get a family of geometric Bridgeland stability conditions. More precisely, if Conjecture 1.1 holds for $X$ with respect to some $\alpha, \beta$ then the pair

$$
\left(\mathcal{A}_{\sqrt{3} \alpha H, B+\beta H}, Z_{\sqrt{3} \alpha H, B+\beta H}^{a, b}\right)
$$

defines a Bridgeland stability condition on the 3 -fold $X$. Here $\mathcal{A}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}$ is the heart of a bounded t-structure as constructed in BMT and

$$
\begin{aligned}
& Z_{\sqrt{3} \alpha H, B+\beta H}^{a, b}=\left(-\operatorname{ch}_{3}^{B+\beta H}+b H \operatorname{ch}_{2}^{B+\beta H}+\left(-\frac{c_{2}(X)}{12}+a H^{2}\right) \operatorname{ch}_{1}^{B+\beta H}\right)+ \\
& \mathfrak{i}\left(\mathrm{Hch}_{2}^{\mathrm{B}+\beta \mathrm{H}}-\frac{\alpha^{2}}{2} \mathrm{H}^{3} \operatorname{ch}_{0}\right)
\end{aligned}
$$

with $a, b \in \mathbb{R}$ satisfying $a \geqslant\left(c_{2}(X) \cdot H\right) /\left(12 H^{3}\right)+\left(\xi+\alpha^{2} / 6+\alpha|b| / 2\right)$.
We extend the notion of $\bar{\beta}$-stability in [Li, BMS, PT] as follows. Let $A_{B}: \mathbb{R} \rightarrow \mathbb{R} \geqslant 0$ be a continuous function. For an object $E$ in the derived category, $\bar{\beta}_{B}(E)$ is defined as a particular solution for $\beta$ in the equation $\mathrm{Hch}_{2}^{B+\beta H}(E)-A(\beta)^{2} H^{3} \operatorname{ch}_{0}(E) / 2=0$; see Definition 4.7 for further details. An object $E$ is called $\bar{\beta}_{B, H, A}$-stable if there is an open neighbourhood $U \subset \mathbb{R}^{2}$ containing $\left(A_{B}\left(\bar{\beta}_{B}(E)\right), \bar{\beta}_{B}(E)\right)$ such that for any $(\alpha, \beta) \in U$ with $\alpha>A_{B}\left(\bar{\beta}_{B}(E)\right)$, $E \in \mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}$ is $v_{\sqrt{3}} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}^{\text {-stable. }}$.

We show that our modified conjectural inequality holds for X if it only holds for this small class of $\bar{\beta}$-stable objects. More precisely, we prove that Conjecture 1.1 is equivalent to the following conjecture. (See Theorem 4.11)

Conjecture 1.2 (=4.10). Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be a $\bar{\beta}_{\mathrm{B}, \mathrm{H}, \mathrm{A}^{-}}$-stable object. Then it satisfies the inequality $D_{A_{B}\left(\bar{\beta}_{B}(E)\right), \bar{\beta}_{B}(E)}^{B, \xi}(E) \leqslant 0$.

In this paper, we see that tilt stability on 3 -folds is preserved under the dualizing of objects. More precisely, we can see that objects in the tilted category behave somewhat similar to coherent sheaves on a projective surface under the dualizing. Therefore, we can further restrict the class of tilt stable objects that we need to check the conjectural BogomolovGieseker type inequalities. Following similar ideas in Li's work for Fano 3-folds of Picard rank one [Li], we obtain certain inequalities involving the Euler characteristic for this smaller class of objects on general Fano 3-folds. Consequently, we show that these inequalities establish the modified Bogomolov-Gieseker type inequality conjecture for those 3 -folds. In particular, we prove the following. See also [BMSZ].

Theorem 1.3 ( $=6.2$, 6.4, 7.4, (7.6). Let X be a Fano 3-fold of index $\mathrm{r} \in\{1,2\}$; so that the canonical divisor class is -rH for an ample divisor class H . Let B be any class proportional to H ; so that $\mathrm{B}=\mathrm{bH}$ for some $\mathrm{b} \in \mathbb{R}$. Then we have the modified Bogomolov-Gieseker type inequality Conjecture 1.2 holds for X with respect to $\xi$ and the function $\mathcal{A}_{\mathrm{B}}: \mathbb{R} \rightarrow \mathbb{R} \geqslant 0$ defined as follows:
(i) when $\mathrm{r}=2$ : if $\mathrm{d} \leqslant 6$ then $\xi=0$ and $A_{\mathrm{B}}: \beta \mapsto 0$; otherwise, that is $\mathrm{d}=7, \xi>0$ is defined by (11) and $A_{B}: \beta \mapsto A_{0}(b+\beta)$, where $A_{0}$ is defined by (13).
(ii) when $\mathrm{r}=1$ : if $\mathrm{d} \leqslant 48$ then $\xi=0$ and $\mathcal{A}_{\mathrm{B}}: \beta \mapsto 0$; otherwise, that is $\mathrm{d}>48, \xi>0$ is defined by (16) and $A_{B}: \beta \mapsto A_{0}(b+\beta)$, where $A_{0}$ is defined by (17).

In a forthcoming paper, we discuss the case when $B$ and $H$ are not necessarily proportional to the anticanonical divisor class.

Notation. Let us collect some of the important notations that we use in this paper as follows:

- When $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, by $\mathrm{H}_{\mathcal{A}}^{i}(-)$ we denote the corresponding $i$-th cohomology functor.
- For a set of objects $\mathcal{S} \subset \mathcal{D}$ in a triangulated category $\mathcal{D}$, by $\langle\mathcal{S}\rangle \subset \mathcal{D}$ we denote its extension closure, that is the smallest extension closed subcategory of $\mathcal{D}$ which contains $\mathcal{S}$.
- Unless otherwise stated, throughout this paper, all the varieties are smooth projective and defined over $\mathbb{C}$. For a variety $X$, by $\operatorname{Coh}(X)$ we denote the category of coherent sheaves on $X$, and by $D^{b}(X)$ we denote the bounded derived category of $\operatorname{Coh}(X)$.
- For a variety $X$, by $\omega_{X}$ we denote its canonical line bundle, and let $K_{X}=c_{1}\left(\omega_{X}\right)$.
- For $E, F \in D^{b}(X)$, denote $\operatorname{hom}_{x}(E, F)=\operatorname{dim} \operatorname{Hom}_{X}(E, F)$, and when $E$ is a sheaf, $h^{i}(E)=$ $\operatorname{dim} H^{i}(E, X)$.
- For the bounded derived category of a variety $X$, we simply write $\mathcal{H}^{i}(-)$ for $H_{C o h(X)}^{i}(-)$.
- For $0 \leqslant i \leqslant \operatorname{dim} X, \operatorname{Coh}_{\leqslant i}(X)=\{E \in \operatorname{Coh}(X): \operatorname{dim} \operatorname{Supp}(E) \leqslant i\}, \operatorname{Coh}_{\geqslant i}(X)=\{E \in \operatorname{Coh}(X)$ : for $0 \neq F \subset E$, $\operatorname{dim} \operatorname{Supp}(F) \geqslant i\}$ and $\operatorname{Coh}_{i}(X)=\operatorname{Coh}_{\leqslant i}(X) \cap \operatorname{Coh}_{\geqslant i}(X)$.
- For $E \in D^{b}(X), E^{\vee}=\mathbf{R} \mathcal{H} \operatorname{com}\left(E, \mathcal{O}_{X}\right)$. When $E$ is a sheaf we write its dual sheaf $\mathcal{H}^{0}\left(E^{\vee}\right)$ by $E^{*}$.
- The skyscraper sheaf of a closed point $x \in X$ is denoted by $\mathcal{O}_{x}$.

Acknowledgements. The author is grateful to Sergey Galkin, Chen Jiang, Ilya Karzhemanov, and Alexander Kuznetsov for some useful discussions on Fano varieties. This work is supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

Note. In BMSZ, Marcello Bernardara, Emanuele Macrì, Benjamin Schmidt, and Xiaolei Zhao independently modified the Bogomolov-Gieseker type inequality in BMT] for Fano 3 -folds.

## 2. Preliminaries

2.1. Tilt stability on $\mathbf{3}$-folds. Let us quickly recall the notions of slope and tilt stabilities for a given smooth projective threefold $X$ as introduced in BMT].

Let $\omega, B$ be in $N S_{\mathbb{R}}(X)$ with $\omega$ an ample class, i.e. $B+i \omega \in \mathrm{NS}_{\mathbb{C}}(X)$ is a complexified ample class. The twisted Chern character with respect to $B$ is defined by $\operatorname{ch}^{B}(-)=e^{-B} \operatorname{ch}(-)$. The twisted slope $\mu_{\omega, B}(E)$ of $E \in \operatorname{Coh}(X)$ is defined by

$$
\mu_{\omega, B}(E)= \begin{cases}+\infty & \text { if } E \text { is a torsion sheaf } \\ \frac{\omega^{2} \operatorname{ch}_{1}^{B}(E)}{\omega^{3} \operatorname{ch}_{0}^{B}(E)} & \text { otherwise } .\end{cases}
$$

We say $E \in \operatorname{Coh}(X)$ is $\mu_{\omega, B}$ (semi)stable, if for any $0 \neq F \nsubseteq E, \mu_{\omega, B}(F)<(\leqslant) \mu_{\omega, B}(E / F)$. The Harder-Narasimhan property holds for $\operatorname{Coh}(X)$, and for a given interval $\mathrm{I} \subset \mathbb{R} \cup\{+\infty\}$,
we define the subcategory $\mathrm{HN}_{\omega, \mathrm{B}}^{\mu}(\mathrm{I}) \subset \operatorname{Coh}(\mathrm{X})$ by

$$
\operatorname{HN}_{\omega, \mathrm{B}}^{\mu}(\mathrm{I})=\left\langle\mathrm{E} \in \operatorname{Coh}(X): E \text { is } \mu_{\omega, \mathrm{B}} \text {-semistable with } \mu_{\omega, \mathrm{B}}(\mathrm{E}) \in \mathrm{I}\right\rangle
$$

The subcategories $\mathcal{T}_{\omega, \mathrm{B}}$ and $\mathcal{F}_{\omega, \mathrm{B}}$ of $\operatorname{Coh}(\mathrm{X})$ are defined by

$$
\mathcal{T}_{\omega, \mathrm{B}}=\operatorname{HN}_{\omega, \mathrm{B}}^{\mu}((0,+\infty]), \quad \mathcal{F}_{\omega, \mathrm{B}}=\operatorname{HN}_{\omega, \mathrm{B}}^{\mu}((-\infty, 0]) .
$$

Now $\left(\mathcal{T}_{\omega, \mathrm{B}}, \mathcal{F}_{\omega, \mathrm{B}}\right)$ forms a torsion pair on $\operatorname{Coh}(\mathrm{X})$ and let the abelian category $\mathcal{B}_{\omega, \mathrm{B}}=$ $\left\langle\mathcal{F}_{\omega, \mathrm{B}}[1], \mathcal{T}_{\omega, \text { В }}\right\rangle \subset \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be the corresponding tilt of $\operatorname{Coh}(\mathrm{X})$.

Define the central charge function $Z_{\omega, B}: K(X) \rightarrow \mathbb{C}$ by $Z_{\omega, B}(E)=-\int_{X} e^{-B-i \omega} \operatorname{ch}(E)$. Following [BMT], the tilt-slope $v_{\omega, B}(E)$ of $E \in \mathcal{B}_{\boldsymbol{\omega}, \mathrm{B}}$ is defined by

$$
v_{\omega, B}(E)= \begin{cases}+\infty & \text { if } \omega^{2} \operatorname{ch}_{1}^{B}(E)=0 \\ \frac{\operatorname{Im}_{\omega, B}(E)}{\omega^{2} \operatorname{ch}_{1}^{B}(E)} & \text { otherwise. }\end{cases}
$$

In BMT the notion of $v_{\omega, B}$-stability for objects in $\mathcal{B}_{\omega, B}$ is introduced in a similar way to $\mu_{\omega, B}$-stability for $\operatorname{Coh}(X)$. Also it is proved that the abelian category $\mathcal{B}_{\omega, B}$ satisfies the Harder-Narasimhan property with respect to $v_{\omega, B}$-stability. Then one can define the subcategory $\mathrm{HN}_{\boldsymbol{\omega}, \mathrm{B}}^{v}(\mathrm{I}) \subset \mathcal{B}_{\omega, \mathrm{B}}$ for an interval $\mathrm{I} \subset \mathbb{R} \cup\{+\infty\}$. The subcategories $\mathcal{T}_{\boldsymbol{\omega}, \mathrm{B}}^{\prime}$ and $\mathcal{F}_{\boldsymbol{\omega}, \mathrm{B}}^{\prime}$ of $\mathcal{B}_{\boldsymbol{\omega}, \mathrm{B}}$ are defined by $\mathcal{T}_{\boldsymbol{\omega}, \mathrm{B}}^{\prime}=\mathrm{HN}_{\boldsymbol{\omega}, \mathrm{B}}^{\boldsymbol{v}}((0,+\infty])$ and $\mathcal{F}_{\boldsymbol{\omega}, \mathrm{B}}^{\prime}=\mathrm{HN}_{\boldsymbol{\omega}, \mathrm{B}}^{v}((-\infty, 0])$. Then the pair $\left(\mathcal{T}_{\omega, B}^{\prime}, \mathcal{F}_{\omega, \mathrm{B}}^{\prime}\right)$ forms a torsion pair on $\mathcal{B}_{\omega, \mathrm{B}}$ and let the abelian category

$$
\begin{equation*}
\mathcal{A}_{\omega, \mathrm{B}}=\left\langle\mathcal{F}_{\omega, \mathrm{B}}^{\prime}[1], \mathcal{T}_{\omega, \mathrm{B}}^{\prime}\right\rangle \subset \mathrm{D}^{\mathrm{b}}(\mathrm{X}) \tag{1}
\end{equation*}
$$

be the corresponding tilt.
2.2. Some homological algebraic results. An object of an abelian category is called minimal when it has no proper subobjects or equivalently no nontrivial quotients in the category. For example skyscraper sheaves of closed points are the only minimal objects of the abelian category of coherent sheaves on a scheme. Moreover, for the abelian category $\mathcal{B}_{\omega, B}$ of a 3 -fold, we have the following:

Proposition 2.1. The objects which are isomorphic to the following types are minimal in $\mathcal{B}_{\omega, \mathrm{B}}$ :
(i) skyscraper sheaves $\mathcal{O}_{x}$ of $x \in X$.
(ii) $\mathrm{E}[1]$, where E is a $\mu_{\omega, \mathrm{B}}$-stable reflexive sheaf with $\mu_{\omega, \mathrm{B}}(\mathrm{E})=0$.

Proof. Similar to the proof of Huy1, Proposition 2.2]. (Also one can see these objects as examples of the class of minimal objects considered abstractly in [PT, Aside 2.12].)

Let $E, F$ be objects in the derived category $D^{b}(X)$ of a smooth projective variety $X$. The Euler characteristic $\chi(E, F)$ is defined by

$$
\chi(E, F)=\sum_{i \in \mathbb{Z}} \operatorname{hom}_{x}(E, F[i])
$$

We write $\chi\left(\mathcal{O}_{X}, E\right)$ by $\chi(E)$, and so $\chi(E, F)=\chi\left(E^{\vee} \otimes F\right)$. The Hirzebruch-Riemann-Roch theorem says,

$$
\begin{equation*}
\chi(E)=\int_{X} \operatorname{ch}(E) \cdot \operatorname{td}(X) \tag{2}
\end{equation*}
$$

Here $\operatorname{td}(X)$ is the Todd class $\operatorname{td}\left(T_{X}\right)$ of the tangent bundle $T_{X}$ of $X$. When $X$ is 3 -dimensional, from [Har, Section 4, Appendix A]

$$
\begin{equation*}
\operatorname{td}(X)=1+\frac{1}{2} c_{1}(X)+\frac{1}{12}\left(c_{1}(X)^{2}+c_{2}(X)\right)+\frac{1}{24} c_{1}(X) c_{2}(X) \tag{3}
\end{equation*}
$$

Here $\boldsymbol{c}_{\mathfrak{i}}(X)$ denotes the $\mathfrak{i}$-th Chern class $\boldsymbol{c}_{\mathfrak{i}}\left(T_{X}\right)$ of the tangent bundle $T_{X}$.
2.3. Some sheaf theory. Let us recall the following useful results for coherent sheaves under the dualizing. See [OSS, HL] for further details.

Proposition 2.2. Let $X$ be an $n$-dimensional smooth projective variety. Then we have the following for $\mathrm{E} \in \operatorname{Coh}(\mathrm{X})$ :
(i) If $\mathrm{E} \in \mathrm{Coh}_{\leqslant \mathrm{d}}(\mathrm{X})$ then it fits into the short exact sequence

$$
0 \rightarrow \mathrm{E}_{\leqslant \mathrm{d}-1} \rightarrow \mathrm{E} \rightarrow \mathrm{E}_{\mathrm{d}} \rightarrow 0
$$

in $\operatorname{Coh}(\mathrm{X})$ for some $\mathrm{E}_{\leqslant \mathrm{d}-1} \in \mathrm{Coh}_{\leqslant \mathrm{d}-1}(\mathrm{X})$ and $\mathrm{E}_{\mathrm{d}} \in \operatorname{Coh}_{\mathrm{d}}(\mathrm{X})$.
(ii) $\mathcal{E} x t^{i}\left(E, \mathcal{O}_{X}\right) \in \operatorname{Coh}_{\leqslant n-i}(X)$.
(iii) If $\mathrm{E} \in \mathrm{Coh}_{\mathrm{d}}(\mathrm{X})$ then it fits into the short exact sequence

$$
0 \rightarrow \mathrm{E} \rightarrow \mathcal{E} x t^{n-\mathrm{d}}\left(\mathcal{E} x t^{\mathrm{n}-\mathrm{d}}(\mathrm{E})\right) \rightarrow \mathrm{Q} \rightarrow 0
$$

in $\operatorname{Coh}(\mathrm{X})$ for some $\mathrm{Q} \in \mathrm{Coh}_{\leqslant \mathrm{d}-2}(\mathrm{X})$.
2.4. Fano 3-folds. Let us recall some important notions associated to Fano varieties. A Fano variety $X$ is a smooth projective variety whose anticanonical divisor $-K_{X}$ is ample. A basic invariant of $X$ is its index, this is the maximal integer $r(X)$ such that $K_{X}$ is divisible by $r(X)$ in $N S(X)$. So $-K_{X}=r(X) \cdot H$ for an ample divisor class $H$ in NS $(X)$. The number $d(X)=H^{\operatorname{dim} X}$ is usually called the degree of $X$.

If $X$ is an $n$-dimensional Fano variety then $r(X) \leqslant n+1$. Moreover, if $r(X)=n+1$ then $X \cong \mathbb{P}^{n}$, and if $r(X)=n$ then $X$ is a quadric. For Fano 3 -folds there is an explicit Iskovskikh-Mori-Mukai classification. See [IP, MM] for further details.

Let us collect some basic properties for Fano 3-folds, that we will need in the proceeding sections.

Lemma 2.3. Let X be a Fano 3-fold of index $\mathrm{r}(\mathrm{X})=\mathrm{r}$ and degree $\mathrm{d}(\mathrm{X})=\mathrm{d}$. Then we have the following:
(i) $h^{i}\left(\Theta_{\mathrm{X}}\right)=0$ for all $i>0$, and $\chi\left(\Theta_{\mathrm{X}}\right)=1$.
(ii) $\mathrm{H} \cdot \mathrm{c}_{2}(\mathrm{X})=24 / \mathrm{r}$.
(iii) $\operatorname{td}(\mathrm{X})=\left(1, \frac{1}{2} \mathrm{rH}, \frac{1}{12}\left(\mathrm{r}^{2} \mathrm{H}^{2}+\mathrm{c}_{2}(\mathrm{X})\right), 1\right)$.

Proof. Since $-K_{X}$ is ample, from the Kodaira's vanishing theorem $H^{i}\left(\mathcal{O}_{X}, X\right)=0$ for all $i>0$. So we have $\chi\left(O_{X}\right)=h^{0}\left(\mathcal{O}_{X}\right)=1$.

Let us compute the Todd class of the tangent sheaf $T_{X}$ of $X$. Since the cotangent bundle is $\Omega_{X} \cong T_{X}^{*}, c_{1}(X)=-c_{1}\left(\Omega_{X}\right)$. Also $\omega_{X}=\operatorname{det}\left(\Omega_{X}\right)$ and so $c_{1}(X)=-c_{1}\left(\omega_{X}\right)=-K_{X}=$ $r H$. From the Hirzebruch-Riemann-Roch theorem (2), $\chi\left(\Theta_{X}\right)=\int_{X} \operatorname{ch}\left(\Theta_{X}\right) \cdot \operatorname{td}(X)$, and so $\frac{1}{24} c_{1}(X) c_{2}(X)=1$. The required expression for Todd class follows from (3).

## 3. Tilt Stability Under the Dualizing

Let $X$ be a smooth projective 3 -fold. We follow the same notations for tilt stability introduced in Section 2.1 for $X$.

By construction, $\mathrm{Coh}_{\leqslant 2}(\mathrm{X}) \subset \mathcal{B}_{\omega, \mathrm{B}}$. Moreover, we have the following for its subcategory $\mathrm{Coh}_{\leqslant 1}(\mathrm{X})$.

Proposition 3.1. We have $\operatorname{Coh}_{\leqslant 1}(\mathrm{X}) \subset \mathrm{HN}_{\boldsymbol{\omega}, \mathrm{B}}^{v}(+\infty)$.
Proof. Let $E \in \operatorname{Coh}_{\leqslant 1}(X)$. Assume the opposite for a contradiction; so that $0 \rightarrow P \rightarrow E \rightarrow$ $\mathrm{Q} \rightarrow 0$ is a short exact sequence on $\mathcal{B}_{\omega, \mathrm{B}}$ with $\boldsymbol{v}_{\omega, \mathrm{B}}(\mathrm{P})<+\infty$. By considering the long exact sequence of $\operatorname{Coh}(X)$ cohomologies we have $\mathcal{H}^{-1}(P)=0$, and since $\operatorname{ch}_{1}(E)=0, \operatorname{ch}_{1}\left(\mathcal{H}^{-1}(Q)\right)=$ $\operatorname{ch}_{1}\left(\mathcal{H}^{0}(P)\right)$. Since $\mathcal{H}^{-1}(Q) \in \operatorname{HN}_{\omega, B}^{\mu}((-\infty, 0])$ and $\mathcal{H}^{0}(P) \in \operatorname{HN}_{\omega, B}^{\mu}((0,+\infty])$, we have $\omega^{2} \operatorname{ch}_{1}^{\mathrm{B}}\left(\mathcal{H}^{0}(\mathrm{P})\right)=0$. So $\mathcal{H}^{0}(\mathrm{P}) \in \mathrm{Coh}_{\leqslant 1}(\mathrm{X})$, and hence, $v_{\omega, \mathrm{B}}(\mathrm{P})=+\infty$. This is the required contradiction.

Let us recall the following slope bounds from [PT] for cohomology sheaves of complexes in the abelian category $\mathcal{B}_{\omega, B}$.
Proposition 3.2 ([РT, Proposition 3.13]). Let $\vartheta$ be any real number and let $\eta=\sqrt{3 \vartheta^{2}+1}$. Let $\mathrm{E} \in \mathcal{B}_{\omega, \mathrm{B}}$ and $\mathrm{E}_{\mathrm{i}}=\mathcal{H}^{\mathrm{i}}(\mathrm{E})$. Then we have the following:
(i) if $\mathrm{E} \in \operatorname{HN}_{\omega, \mathrm{B}}^{v}((-\infty, \vartheta))$, then $\mathrm{E}_{-1} \in \operatorname{HN}_{\omega, \mathrm{B}}^{\mu}((-\infty, \vartheta-\eta / \sqrt{3}))$;
(ii) if $\mathrm{E} \in \mathrm{HN}_{\omega, \mathrm{B}}^{v}((\vartheta,+\infty))$, then $\mathrm{E}_{0} \in \mathrm{HN}_{\omega, \mathrm{B}}^{\mu}((\vartheta+\eta / \sqrt{3},+\infty])$; and
(iii) if E is tilt semistable with $\boldsymbol{v}_{\omega, \mathrm{B}}(\mathrm{E})=\vartheta$, then
(a) $\mathrm{E}_{-1} \in \mathrm{HN}_{\boldsymbol{\omega}, \mathrm{B}}^{\mu}((-\infty, \vartheta-\eta / \sqrt{3}])$ with equality $\mu_{\omega, \mathrm{B}}\left(\mathrm{E}_{-1}\right)=\vartheta-\eta / \sqrt{3}$ holds if and only if $\omega^{2} \operatorname{ch}_{2}^{\mathrm{B}+(\vartheta-\eta / \sqrt{3}) \omega}\left(\mathrm{E}_{-1}\right)=0$, and
(b) when $\mathrm{E}_{0}$ is torsion free $\mathrm{E}_{0} \in \operatorname{HN}_{\omega, \mathrm{B}}^{\mu}([\vartheta+\eta / \sqrt{3},+\infty))$ with equality $\mu_{\omega, B}\left(\mathrm{E}_{0}\right)=$ $\vartheta+\eta / \sqrt{3}$ holds if and only if $\omega^{2} \operatorname{ch}_{2}^{\mathrm{B}+(\vartheta+\eta / \sqrt{3}) \omega}\left(\mathrm{E}_{0}\right)=0$.

Consequently, we have the following:
Corollary 3.3. Let $\mathrm{E} \in \mathcal{B}_{\omega, \mathrm{B}}$ be a tilt stable object with $\boldsymbol{v}_{\omega, \mathrm{B}}(\mathrm{E})=0$. Then we have

$$
\text { (i) } \omega^{2} \operatorname{ch}_{1}^{\mathrm{B}+\omega / \sqrt{3}}(\mathrm{E}) \geqslant 0, \quad \text { and } \text { (ii) } \omega^{2} \operatorname{ch}_{1}^{\mathrm{B}-\omega / \sqrt{3}}(\mathrm{E}) \geqslant 0
$$

Proof. From Proposition 3.2, we have $\omega^{2} \operatorname{ch}_{1}^{\mathrm{B}+\omega / \sqrt{3}}\left(\mathcal{H}^{0}(\mathrm{E})\right) \geqslant 0$ and $\omega^{2} \operatorname{ch}_{1}^{\mathrm{B}-\omega / \sqrt{3}}\left(\mathcal{H}^{-1}(\mathrm{E})\right) \geqslant$ 0 . Hence, we obtain the required inequalities for $E$.

Recall the following result about the walls for tilt stable objects from [PT]:
Proposition 3.4 ([PT, Lemma 3.15]). Let the object $\mathrm{E} \in \mathcal{B}_{\omega, \mathrm{B}}$ be $\boldsymbol{v}_{\omega, \mathrm{B}}$-stable. Then $\mathrm{E} \in$


$$
\alpha^{2}+3\left(\beta-v_{\omega, B}(E)\right)^{2}=3 v_{\omega, B}(E)^{2}+1
$$

Definition 3.5. For $E \in D^{b}(X)$ we define

$$
\bar{\Delta}_{\omega, B}(E)=\left(\omega^{2} \operatorname{ch}_{1}^{B}(E)\right)^{2}-2 \omega^{3} \operatorname{ch}_{0}(E) \omega \operatorname{ch}_{2}^{B}(E)
$$

The following is crucial for us.
Proposition 3.6 ([BMT, Corollary 7.3.2, Proposition 7.4.1]). We have the following:
(i) If $\mathrm{E} \in \mathcal{B}_{\omega, \mathrm{B}}$ is a tilt semistable object then $\bar{\Delta}_{\omega, \mathrm{B}}(\mathrm{E}) \geqslant 0$.
(ii) If E is a $\mu_{\omega, \mathrm{B}}-\left(\right.$ semi) stable object with $\bar{\Delta}_{\omega, \mathrm{B}}(\mathrm{E})=0$, then E or $\mathrm{E}[1]$ in $\mathcal{B}_{\omega, \mathrm{B}}$ is $\nu_{\omega, \mathrm{B}}-($ semi) stable.
We have following result for certain short exact sequences in $\mathcal{B}_{\boldsymbol{\omega}, \mathrm{B}}$.
Proposition 3.7. Let $0 \rightarrow \mathrm{E}_{1} \rightarrow \mathrm{E} \rightarrow \mathrm{E}_{2} \rightarrow 0$ be a short exact sequence in $\mathcal{B}_{\omega, \mathrm{B}}$ such that $\mathrm{E}, \mathrm{E}_{1}, \mathrm{E}_{2}$ are $\mathrm{v}_{\omega, \mathrm{B}}$-semistable with $\mathrm{v}_{\omega, \mathrm{B}}\left(\mathrm{E}_{1}\right)=v_{\omega, \mathrm{B}}\left(\mathrm{E}_{2}\right)<+\infty$. Then

$$
\bar{\Delta}_{\omega, \mathrm{B}}(\mathrm{E}) \geqslant \bar{\Delta}_{\omega, \mathrm{B}}\left(\mathrm{E}_{1}\right)+\bar{\Delta}_{\omega, \mathrm{B}}\left(\mathrm{E}_{2}\right)
$$

where the equality holds when $\bar{\Delta}_{\omega, \mathrm{B}}=0$ for $\mathrm{E}, \mathrm{E}_{1}, \mathrm{E}_{2}$.
Proof. Assume $v_{\omega, B}\left(E_{1}\right)=v_{\omega, B}\left(E_{2}\right)=0$. Let us write, for $\mathfrak{i}=1,2$ :

$$
A_{\mathfrak{i}}=\omega^{2} \operatorname{ch}_{1}^{B}\left(E_{\mathfrak{i}}\right), \quad \text { and } \quad B_{i}=\omega^{3} \operatorname{ch}_{0}\left(E_{\mathfrak{i}}\right) / \sqrt{3}
$$

Since $E_{1}, E_{2}$ are tilt semistable with zero tilt slopes, from Corollary 3.3 we have $A_{i}+B_{i} \geqslant 0$ and $A_{i}-B_{i} \geqslant 0$ for $i=1,2$. Therefore,

$$
2\left(A_{1} A_{2}-B_{1} B_{2}\right)=\left(A_{1}-B_{1}\right)\left(A_{2}+B_{2}\right)+\left(A_{1}+B_{1}\right)\left(A_{2}-B_{2}\right) \geqslant 0
$$

Since $E, E_{1}, E_{2}$ have zero tilt slopes,

$$
\bar{\Delta}_{\omega, B}(E)=\left(A_{1}+A_{2}\right)^{2}-\left(B_{1}+B_{2}\right)^{2}=\sum_{i=1}^{2}\left(A_{1}^{2}-B_{i}^{2}\right)+2\left(A_{1} A_{2}-B_{1} B_{2}\right) \geqslant \sum_{i=1}^{2} \bar{\Delta}_{\omega, B}\left(E_{i}\right)
$$

and the equality holds when $A_{i}=B_{i}$ for $\mathfrak{i}=1$, 2; i.e. $\bar{\Delta}_{\omega, B}=0$ for $E, E_{1}, E_{2}$.
If $v_{\omega, B}\left(E_{1}\right)=v_{\omega, B}\left(E_{2}\right)=\vartheta$ for some $\vartheta<+\infty$ then from Proposition 3.4 there exists $\alpha>0, \beta$ such that $E, E_{1}, E_{2} \in \mathcal{B}_{\alpha \omega, B+\beta \omega}$ are tilt semistable with zero $v_{\alpha \omega, B+\beta \omega}$ slopes. Since $\bar{\Delta}_{\alpha \omega, B+\beta \omega}=\alpha^{4} \bar{\Delta}_{\omega, \mathrm{B}}$ we have the required inequality from the zero tilt slope case.

Notation 3.8. For $E \in \mathcal{B}_{\omega, B}$ we write

$$
E^{i}=H_{\mathcal{B}_{\omega,-B}}^{i}\left(E^{\vee}\right)
$$

So for example $E^{12}=H_{\mathcal{B}_{\omega, B}}^{2}\left(\left(H_{\mathcal{B}_{\omega,-B}}^{1}\left(E^{\vee}\right)\right)^{\vee}\right)$
We have the following.
Proposition 3.9. Let $\mathrm{E} \in \mathrm{HN}_{\boldsymbol{\omega}, \mathrm{B}}^{v}((-\infty,+\infty))$. Then $\mathrm{E}^{\mathfrak{i}}=0$ for $\mathfrak{i} \neq 1,2$ with $\mathrm{E}^{2} \in \operatorname{Coh}_{0}(\mathrm{X})$. Proof. For $E \in \operatorname{HN}_{\omega, B}^{v}((-\infty,+\infty))$, let us denote $E_{j}=\mathcal{H}^{j}(E)$. Object $E$ fits into the short exact sequence

$$
\begin{equation*}
0 \rightarrow \mathrm{E}_{-1}[1] \rightarrow \mathrm{E} \rightarrow \mathrm{E}_{0} \rightarrow 0 \tag{4}
\end{equation*}
$$

in $\mathcal{B}_{\omega, \mathrm{B}}$. Here $\mathrm{E}_{-1}$ is torsion free and so it fits into the short exact sequence $0 \rightarrow E_{-1} \rightarrow$ $\mathrm{E}_{-1}^{* *} \rightarrow \mathrm{Q} \rightarrow 0$ in $\operatorname{Coh}(\mathrm{X})$ for some $\mathrm{Q} \in \operatorname{Coh}_{\leqslant 1}(X)$. Therefore, $0 \rightarrow \mathrm{Q} \rightarrow \mathrm{E}_{-1}[1] \rightarrow \mathrm{E}_{-1}^{* *}[1] \rightarrow 0$ is a short exact sequence in $\mathcal{B}_{\omega, B}$. Hence $Q$ is a subobject of $E \in \operatorname{HN}_{\omega, B}^{v}((-\infty,+\infty))$. By Proposition 3.1, $\mathrm{Q} \in \mathrm{HN}_{\omega, \mathrm{B}}^{v}(+\infty)$, and so $\mathrm{Q}=0$. That is $\mathrm{E}_{-1}$ is reflexive.

By dualizing (4), we have the following distinguished triangle

$$
\begin{equation*}
\mathrm{E}_{0}^{\vee} \rightarrow \mathrm{E}^{\vee} \rightarrow \mathrm{E}_{-1}^{\vee}[-1] \rightarrow \mathrm{E}_{0}^{\vee}[1] \tag{5}
\end{equation*}
$$

By considering $\vartheta \rightarrow+\infty$ in (i) of Proposition 3.2, we have $\mathrm{E}_{-1} \in \operatorname{HN}_{\omega, \mathrm{B}}^{\mu}((-\infty, 0))$. So $\mathrm{E}_{-1}^{*} \in \operatorname{HN}_{\omega, \mathrm{B}}^{\mu}((0,+\infty))$. Since $\mathrm{E}_{-1}$ is reflexive $\mathcal{E x t}{ }^{1}\left(\mathrm{E}_{-1}, \mathcal{O}_{\mathrm{X}}\right)=\mathcal{H}^{1}\left(\mathrm{E}_{-1}^{\vee}\right) \in \operatorname{Coh}_{0}(\mathrm{X})$ and $\mathcal{E x t}{ }^{\mathfrak{i}}\left(E_{-1}, \mathcal{O}_{X}\right)=\mathcal{H}^{\mathfrak{i}}\left(\mathrm{E}_{-1}\right)=0$ for $\mathfrak{i} \geqslant 2$. Therefore, $\left(\mathrm{E}_{-1}[1]\right)^{\mathfrak{i}}=0$ for $\mathfrak{i} \neq 1,2$.

The sheaf $\mathrm{E}_{0} \in \operatorname{HN}_{\omega, \mathrm{B}}^{\mu}((0,+\infty))$ and so $\mathrm{E}_{0}^{*} \in \operatorname{HN}_{\omega,-\mathrm{B}}^{\mu}((-\infty, 0)) \subset \mathcal{B}_{\omega,-\mathrm{B}}[-1]$. Moreover, for $\mathfrak{i} \geqslant 1, \mathcal{E} x t^{i}\left(\mathrm{E}_{0}, \mathcal{O}_{\mathbf{X}}\right)=\mathcal{H}^{\mathfrak{i}}\left(\mathrm{E}_{0}^{\vee}\right) \in \mathrm{Coh}_{\leqslant(3-\mathfrak{i})}(\mathrm{X}) \in \mathcal{B}_{\boldsymbol{\omega},-\mathrm{B}}$. So $\mathrm{E}_{0}^{i}=0$ for $\mathfrak{i} \neq 1,2,3$ with $\mathrm{E}_{0}^{2} \in \mathrm{Coh}_{\leqslant 1}(\mathrm{X})$ and $\mathrm{E}_{0}^{3} \in \mathrm{Coh}_{0}(\mathrm{X})$. Therefore, by considering the long exact sequence of $\mathcal{B}_{\omega,-B}$-cohomologies associated to the triangle (5), we have $E^{i}=0$ for $i \neq 1,2,3$ with $\mathrm{E}^{2} \in \mathrm{Coh}_{\leqslant 1}(\mathrm{X})$ and $\mathrm{E}^{3} \in \mathrm{Coh}_{0}(\mathrm{X})$.

For any $x \in X$,

$$
\begin{aligned}
\operatorname{Hom}_{x}\left(\mathrm{E}^{3}, \mathcal{O}_{x}\right) & \cong \operatorname{Hom}_{x}\left(\mathrm{E}^{\vee}[3], \mathcal{O}_{x}\right) \\
& \cong \operatorname{Hom}_{x}\left(\mathrm{E}^{\vee}, \mathcal{O}_{\chi}[-3]\right) \\
& \cong \operatorname{Hom}_{x}\left(\left(\mathcal{O}_{x}[-3]\right)^{\vee}, \mathrm{E}\right) \\
& \cong \operatorname{Hom}_{x}\left(\mathcal{O}_{\chi}, \mathrm{E}\right)=0,
\end{aligned}
$$

as the skyscraper sheaf $\mathcal{O}_{\chi} \in \operatorname{Coh}_{0}(X) \subset \operatorname{HN}_{\boldsymbol{\omega}, \mathrm{B}}^{v}(+\infty)$ and $\mathrm{E} \in \mathrm{HN}_{\boldsymbol{\omega}, \mathrm{B}}^{v}((-\infty,+\infty))$. Therefore, $\mathrm{E}^{3}=0$.

For any $\mathrm{T} \in \operatorname{Coh}_{1}(\mathrm{X})$,

$$
\begin{aligned}
\operatorname{Hom}_{\times}\left(E^{2}, T\right) & \cong \operatorname{Hom}_{\times}\left(E^{\vee}[2], T\right) \\
& \cong \operatorname{Hom}_{\times}\left(E^{\vee}, T[-2]\right) \\
& \cong \operatorname{Hom}_{x}\left((T[-2])^{\vee}, E\right) \\
& \cong \operatorname{Hom}_{x}\left(\varepsilon x t^{2}\left(\mathrm{~T}, \mathcal{O}_{X}\right), \mathrm{E}\right)=0,
\end{aligned}
$$

as $\mathcal{E x t}{ }^{2}\left(\mathrm{~T}, \mathcal{O}_{\mathrm{X}}\right) \in \operatorname{Coh}_{1}(\mathrm{X}) \subset \operatorname{HN}_{\omega, \mathrm{B}}^{v}(+\infty)$ and $\mathrm{E} \in \operatorname{HN}_{\omega, \mathrm{B}}^{v}((-\infty,+\infty))$. Therefore, $\mathrm{E}^{2} \in$ $\mathrm{Coh}_{0}(\mathrm{X})$. This completes the proof.

Proposition 3.10. We have the following for $\mathrm{E} \in \operatorname{HN}_{\omega,-\mathrm{B}}^{\nu}((-\infty,+\infty))$ :
(i) E fits into the short exact sequence

$$
0 \rightarrow \mathrm{E} \rightarrow \mathrm{E}^{11} \rightarrow \mathrm{E}^{23} \rightarrow 0
$$

in $\mathcal{B}_{\omega, \mathrm{B}}$, where $\mathrm{E}^{23} \in \mathrm{Coh}_{0}(\mathrm{X})$,
(ii) $\mathrm{E}^{1, \mathrm{k}}=0$ for $\mathrm{k} \neq 1$,
(iii) $\operatorname{Hom}_{\mathrm{x}}\left(\mathrm{Coh}_{\leqslant 1}(\mathrm{X}), \mathrm{E}^{1}\right)=0$, and
(iv) $\operatorname{Hom}_{\mathrm{x}}\left(\operatorname{Coh}_{0}(\mathrm{X}), \mathrm{E}^{1}[1]\right)=0$.

Proof. By Proposition [3.9, $\mathrm{E}^{\mathfrak{i}}=0$ for $\mathfrak{i} \neq 1,2$ and $\mathrm{E}^{2} \in \mathrm{Coh}_{0}(\mathrm{X})$. So $\left(\mathrm{E}^{2}\right)^{\vee} \cong \mathrm{E}^{23}[-3]$.
Since $E \vee \vee E$, we have the spectral sequence:

$$
\begin{equation*}
H_{\mathcal{B}_{\omega, B}}^{p}\left(\left(H_{\mathcal{B}_{\omega,-B}}^{-q}\left(E^{\vee}\right)\right)^{\vee}\right) \Longrightarrow H_{\mathcal{B}_{\omega, B}}^{p+q}(E) . \tag{6}
\end{equation*}
$$

Consider the convergence of this spectral sequence for $E \in \operatorname{HN}_{\omega,-B}^{v}((-\infty,+\infty))$ :


From the convergence of the above spectral sequence $E^{1, k}=0$ for $k \neq 1$ and we have the short exact sequence $0 \rightarrow \mathrm{E} \rightarrow \mathrm{E}^{11} \rightarrow \mathrm{E}^{23} \rightarrow 0$ in $\mathcal{B}_{\omega, \mathrm{B}}$.

For $\mathrm{T} \in \mathrm{Coh}_{\leqslant 1}(\mathrm{X}), \mathcal{E x t}\left(\mathrm{T}, \mathcal{O}_{X}\right) \in \operatorname{Coh}_{(3-\mathfrak{i})}(\mathrm{X})$ and so $\mathrm{T}^{\vee} \in\left\langle\mathcal{B}_{\omega,-\mathrm{B}}[-2], \mathcal{B}_{\omega,-\mathrm{B}}[-3]\right\rangle$. On the other hand $\left(E^{1}\right)^{\vee} \in \mathcal{B}_{\omega,-B}[-1]$. Hence,

$$
\operatorname{Hom}_{x}\left(\mathrm{~T}, \mathrm{E}^{1}\right) \cong \operatorname{Hom}_{\times}\left(\left(\mathrm{E}^{1}\right)^{\vee}, \mathrm{T}^{\vee}\right)=0
$$

as required in part (iii).
For any skyscraper sheaf $\mathcal{O}_{\chi}$ of $x \in X$, we have

$$
\operatorname{Hom}_{\chi}\left(\mathcal{O}_{\chi}, \mathrm{E}^{1}[1]\right) \cong \operatorname{Hom}_{\chi}\left(\left(\mathrm{E}^{1}[1]\right)^{\vee}, \mathcal{O}_{\chi}^{\vee}\right) \cong \operatorname{Hom}_{\chi}\left(\mathrm{E}^{11}[-2], \mathcal{O}_{\chi}[-3]\right)=0
$$

as required in part (iv).
Proposition 3.11. Let $\mathrm{E} \in \operatorname{HN}_{\omega, \mathrm{B}}^{\vee}((-\infty,+\infty))$. Then
(i) E is $\boldsymbol{v}_{\omega, \mathrm{B}}$-stable (resp. $\quad \boldsymbol{v}_{\omega, \mathrm{B}}$-semistable) if and only if $\mathrm{E}^{11}$ is $\boldsymbol{v}_{\omega, \mathrm{B}}$-stable (resp. $v_{\omega, \mathrm{B}}$-semistable),
(ii) $v_{\omega,-\mathrm{B}}\left(\mathrm{E}^{1}\right)=-\boldsymbol{v}_{\omega, \mathrm{B}}(\mathrm{E})$,
(iii) E is $\boldsymbol{v}_{\omega, \mathrm{B}}$-stable (resp. $\quad \boldsymbol{v}_{\omega, \mathrm{B}}$-semistable) if and only if $\mathrm{E}^{1}$ is $\boldsymbol{v}_{\omega,-\mathrm{B}}$-stable (resp. $\nu_{\omega, \mathrm{B}}$-semistable), and
(iv) $\mathrm{E}^{1} \in \operatorname{HN}_{\omega,-\mathrm{B}}^{v}((-\infty,+\infty))$.

Proof. From part (2) of Proposition 3.5 in [LM], we have (i).
By Proposition 3.9 and from definition of the twisted Chern character we have

$$
\begin{aligned}
-\operatorname{ch}^{-\mathrm{B}}\left(\mathrm{E}^{1}\right)+\operatorname{ch}^{-\mathrm{B}}\left(\mathrm{E}^{2}\right) & =\operatorname{ch}^{-\mathrm{B}}\left(\mathrm{E}^{\vee}\right)=e^{\mathrm{B}} \operatorname{ch}\left(\mathrm{E}^{\vee}\right)=\left(e^{-\mathrm{B}} \operatorname{ch}(\mathrm{E})\right)^{\vee} \\
& =\left(\operatorname{ch}^{\mathrm{B}}(\mathrm{E})\right)^{\vee}=\left(\operatorname{ch}_{0}^{\mathrm{B}}(\mathrm{E}),-\operatorname{ch}_{1}^{\mathrm{B}}(\mathrm{E}), \operatorname{ch}_{2}^{\mathrm{B}}(\mathrm{E}),-\operatorname{ch}_{3}^{\mathrm{B}}(\mathrm{E})\right) .
\end{aligned}
$$

Since $E^{2} \in \operatorname{Coh}_{0}(X)$, we have $v_{\omega,-B}\left(E^{1}\right)=-v_{\omega, B}(E)$.
Let $E \in \mathcal{B}_{\omega, B}$ be a $v_{\omega, B}$-semistable object. Assume $E^{1} \in \mathcal{B}_{\omega,- \text { B }}$ be $v_{\omega, B}$-unstable. From the Harder-Narasimhan filtration there exists a quotient $E^{1} \rightarrow Q$ in $\mathcal{B}_{\omega,-B}$, where $Q$ is the lowest $v_{\omega,-B}$-semistable Harder-Narasimhan factor. Since $v_{\omega,-B}\left(E^{1}\right)=-v_{\omega, B}(E)$, $v_{\omega,-\mathrm{B}}(\mathrm{Q})<v_{\omega,-\mathrm{B}}\left(\mathrm{E}^{1}\right)<+\infty$. By (ii), $v_{\omega, \mathrm{B}}\left(\mathrm{Q}^{1}\right)>v_{\omega, \mathrm{B}}\left(\mathrm{E}^{11}\right)$ with $\mathrm{Q}^{1} \hookrightarrow \mathrm{E}^{11}$ in $\mathcal{B}_{\omega, B}$; this is not possible as $E^{11}$ is $v_{\omega, B}$-semistable by (i).

Part (iv) is a direct consequence of (iii).
Consequently, we have the following:
Corollary 3.12. We only need to check Bogomolov-Gieseker type inequalities in BMT, BMS, PT], Conjectures 1.1 and 1.2 for tilt stable objects E such that

- $\mathrm{E} \cong \mathrm{E}^{11}$
- $\operatorname{ch}_{0}(E) \geqslant 0\left(\right.$ or $\left.\operatorname{ch}_{0}(E) \leqslant 0\right)$.

Aside 3.13. Let $E$ be an $v_{\omega, B}$-stable object in $\mathcal{B}_{\omega, B}$ with $v_{\omega, B}(E)=0$. By Proposition 3.10, it fits into the short exact sequence $0 \rightarrow E \rightarrow E^{11} \rightarrow E^{23} \rightarrow 0$ in $\mathcal{B} \omega, B$ with $E^{23} \in \operatorname{Coh}_{0}(X)$. Moreover, by Proposition 3.11, $E^{11} \in \mathcal{B}_{\omega, B}$ is $v_{\omega, B}$-stable with $v_{\omega, B}\left(E^{11}\right)=0$. Also by Proposition 3.10, $\operatorname{Hom}_{\mathrm{X}}\left(\operatorname{Coh}_{0}(\mathrm{X}), \mathrm{E}^{11}[1]\right)=0$. Hence by MP1, Lemma 2.3] or [PT, Aside 2.12], $\mathrm{E}^{11}[1] \in \mathcal{A}_{\omega, \mathrm{B}}$ is a minimal object.

## 4. BG Inequality Conjecture for Smooth Projective 3-folds

Let X be a smooth projective 3 -fold, and let H be an ample divisor class on it. Let $B \in \mathrm{NS}_{\mathbb{Q}}(X)$.
4.1. Modified conjectural inequality. Let us state the modified Bogomolov-Gieseker type inequality conjecture for smooth projective 3 -folds. First we introduce the expression of the inequality as follows:
Definition 4.1. For $\xi \in \mathbb{R}_{\geqslant 0}, \alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$, we define

$$
D_{\alpha, \beta}^{\mathrm{B}, \xi}(\mathrm{E})=\operatorname{ch}_{3}^{\mathrm{B}+\beta \mathrm{H}}(\mathrm{E})+\left(\frac{\mathrm{c}_{2}(\mathrm{X})}{12}-\left(\mathrm{k}+\xi+\frac{1}{6} \alpha^{2}\right) \mathrm{H}^{2}\right) \operatorname{ch}_{1}^{\mathrm{B}+\beta \mathrm{H}}(\mathrm{E})
$$

Here $\mathrm{k}=\left(\mathrm{c}_{2}(\mathrm{X}) \cdot \mathrm{H}\right) /\left(12 \mathrm{H}^{3}\right)$.
From the definition of $k$, we have $\left(c_{2}(X) / 12-\mathrm{kH}^{2}\right) \cdot \mathrm{H}=0$. So we can write

$$
\begin{equation*}
D_{\alpha, \beta}^{B, \xi}(E)=\operatorname{ch}_{3}^{B+\beta H}(E)+\left(\frac{c_{2}(X)}{12}-k H^{2}\right) \operatorname{ch}_{1}^{B}(E)-\left(\xi+\frac{1}{6} \alpha^{2}\right) H^{2} \operatorname{ch}_{1}^{B+\beta H}(E) \tag{7}
\end{equation*}
$$

Moreover, $D_{\alpha, \beta}^{B, \xi}=D_{\alpha, 0}^{B+\beta H, \xi}$.
Conjecture 4.2. There exist some constant $\xi \in \mathbb{R}_{\geqslant 0}$ and a continuous function $A_{B}: \mathbb{R} \rightarrow$ $\mathbb{R}_{\geqslant 0}$ such that, for any $\alpha \in \mathbb{R}_{>0}, \beta \in \mathbb{R}$ satisfying $\alpha \geqslant A_{B}(\beta)$, all tilt slope $\nu_{\sqrt{3} \alpha H, B+\beta H^{-}}$ stable objects $\mathrm{E} \in \mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}$ with $v_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}(\mathrm{E})=0$ satisfy the following inequality:

$$
D_{\alpha, \beta}^{B, \xi}(E) \leqslant 0
$$

Remark 4.3. This modified conjectural inequality coincides with Bogomolov-Gieseker type inequality in BMT when

$$
\mathrm{c}_{2}(\mathrm{X}) \text { is proportional to } \mathrm{H}^{2}, \xi=0, \text { and } A_{\mathrm{B}}=0
$$

In particular, all the known 3-folds where Bogomolov-Gieseker type inequality conjecture in BMT holds satisfy this condition. That is, when $X$ is an abelian 3-fold or an étale quotient of an abelian 3 -fold $\left(c_{2}(X)=0\right)$, or a Fano 3 -fold with Picard rank one $\left(c_{2}(X)\right.$ is proportional to $\mathrm{H}^{2}$ ).

This modification of the conjectural inequalities does not affect the corresponding constructions of Bridgeland stability conditions. In particular, similar to [BMS, Lemma 8.3] we have the following:

Theorem 4.4. If Conjecture 4.2 holds for X with respect to some $\alpha, \beta$ then the pair

$$
\left(\mathcal{A}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{~B}+\beta \mathrm{H}}, Z_{\sqrt{3} \alpha \mathrm{H}, \mathrm{~B}+\beta \mathrm{H}}^{\mathrm{a}, \mathrm{~b}}\right)
$$

defines a Bridgeland stability condition on X . Here $\mathcal{A}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}$ is the heart of a bounded $t$-structure as constructed in (1) and

$$
\begin{aligned}
Z_{\sqrt{3} \alpha \mathrm{H}, \mathrm{~B}+\beta \mathrm{H}}^{\mathrm{a}, \mathrm{~b}}=\left(-\mathrm{ch}_{3}^{\mathrm{B}+\beta \mathrm{H}}+\mathrm{bH} \operatorname{ch}_{2}^{\mathrm{B}+\beta \mathrm{H}}+( \right. & \left.\left(-\frac{\mathrm{c}_{2}(\mathrm{X})}{12}+\mathrm{aH}^{2}\right) \mathrm{ch}_{1}^{\mathrm{B}+\beta \mathrm{H}}\right)+ \\
& \mathfrak{i}\left(\mathrm{Hch}_{2}^{\mathrm{B}+\beta \mathrm{H}}-\frac{\alpha^{2}}{2} \mathrm{H}^{3} \mathrm{ch}_{0}\right)
\end{aligned}
$$

with $\mathrm{a}, \mathrm{b} \in \mathbb{R}$ satisfying $\mathrm{a} \geqslant\left(\mathrm{c}_{2}(\mathrm{X}) \cdot \mathrm{H}\right) /\left(12 \mathrm{H}^{3}\right)+\left(\xi+\alpha^{2} / 6+\alpha|\mathfrak{b}| / 2\right)$.
4.2. Equivalent form of the conjecture. In this subsection we formulate an equivalent form of Conjecture 4.2 which only considers the modified Bogomolov-Gieseker type inequalities for a small class of tilt stable objects. This can be considered as a modification of BMS, Conjecture 5.3] and in the next subsection we show that it is equivalent to Conjecture 4.2. We adapt some methods from BMS, Section 5] and Mac.

Let us consider the complexified ample classes $(\mathrm{B}+\beta \mathrm{H})+i \sqrt{3} \alpha \mathrm{H}$ parametrized by $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$. By definition

$$
v_{\sqrt{3} \alpha H, B+\beta H}=\frac{H \operatorname{ch}_{2}^{B+\beta H}-\frac{\alpha^{2}}{2} H^{3} \mathrm{ch}_{0}}{\sqrt{3} \alpha \mathrm{H}^{2} \operatorname{ch}_{1}^{\mathrm{B}+\beta H}} .
$$

Therefore, we can consider

$$
\mathrm{Z}_{\alpha, \beta}^{v}=-\left(\mathrm{Hch}_{2}^{\mathrm{B}+\beta \mathrm{H}}-\frac{\alpha^{2}}{2} \mathrm{H}^{3} \mathrm{ch}_{0}\right)+\mathrm{iH}^{2} \mathrm{ch}_{1}^{\mathrm{B}+\beta \mathrm{H}}
$$

as the associated group homomorphism, more precisely, the weak stability function as introduced in [PT] of the corresponding tilt stability.

In the rest of this section $\alpha_{0} \in \mathbb{R}_{>0}$.
Definition 4.5. Let $E$ be an object in $\mathcal{B}_{\sqrt{3} \alpha_{0} H, B}$ with $v_{\sqrt{3} \alpha_{0} H, B}(E)=0$.

$$
C(E)=\left\{(\alpha, \beta): \operatorname{Re} Z_{\alpha, \beta}^{\nu}(E)=H \operatorname{ch}_{2}^{B+\beta H}(E)-\frac{\alpha^{2}}{2} H^{3} \operatorname{ch}_{0}(E)=0, \text { and } 0 \leqslant \alpha \leqslant \alpha_{0}\right\} .
$$

From the definition $\left(\alpha_{0}, 0\right) \in C(E)$.
Lemma 4.6. Let E be an object in $\mathcal{B}_{\sqrt{3} \alpha_{0} \mathrm{H}, \mathrm{B}}$ with $\vee_{\sqrt{3} \alpha_{0} \mathrm{H}, \mathrm{B}}(\mathrm{E})=0$. Then along $\mathrm{C}(\mathrm{E})$ we have

$$
\frac{\mathrm{d}}{\mathrm{~d} \alpha}\left(\mathrm{D}_{\alpha, \beta}^{\mathrm{B}, \boldsymbol{\xi}}(\mathrm{E})\right)=\frac{-\alpha \bar{\Delta}_{\mathrm{H}, \mathrm{~B}}(\mathrm{E})-3 \xi\left(\mathrm{H}^{3} \mathrm{ch}_{0}(\mathrm{E})\right)^{2}}{3 \mathrm{H}^{2} \mathrm{ch}_{1}^{\mathrm{B}+\beta \mathrm{H}}(\mathrm{E})} .
$$

Proof. For $(\alpha, \beta) \in C(E)$, we have $\mathrm{Hch}_{2}^{\mathrm{B}+\beta \mathrm{H}}(\mathrm{E})-\left(\alpha^{2} / 2\right) \mathrm{H}^{3} \mathrm{ch}_{0}(\mathrm{E})=0$. By differentiating both sides with respect to $\alpha$ we get

$$
\begin{equation*}
\frac{\mathrm{d} \beta}{\mathrm{~d} \alpha}=-\frac{\alpha \mathrm{H}^{3} \mathrm{ch}_{0}(\mathrm{E})}{\mathrm{H}^{2} \operatorname{ch}_{1}^{\mathrm{B}+\beta \mathrm{H}}(\mathrm{E})} . \tag{8}
\end{equation*}
$$

By differentiating the expression of $D_{\alpha, \beta}^{B, \xi}(E)$ in (7) with respect to $\alpha$, we get

$$
\frac{\mathrm{d}}{\mathrm{~d} \alpha}\left(\mathrm{D}_{\alpha, \beta}^{\mathrm{B}, \xi}(\mathrm{E})\right)=-\mathrm{H}_{2}^{\mathrm{B}+\beta \mathrm{H}}(\mathrm{E}) \frac{\mathrm{d} \beta}{\mathrm{~d} \alpha}-\frac{\alpha}{3} H^{2} \operatorname{ch}_{1}^{\mathrm{B}+\beta \mathrm{H}}(\mathrm{E})+\left(\xi+\frac{\alpha^{2}}{6}\right) H^{3} \operatorname{ch}_{0}(E) \frac{\mathrm{d} \beta}{\mathrm{~d} \alpha} .
$$

Since $H \operatorname{ch}_{2}^{B+\beta H}(E)=\left(\alpha^{2} / 2\right) H^{3} \operatorname{ch}_{0}(E)$ and by substituting the expression of $d \beta / d \alpha$, we obtain the required expression.

Let $A_{B}: \mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}$ be a continuous function. For a given object $E$, if we have

$$
\lim _{\alpha \rightarrow A_{B}(\beta)^{+}}-\operatorname{Re} Z_{\alpha, \beta}^{v}(E)=0
$$

when $\beta \rightarrow \bar{\beta}$, then $\bar{\beta}$ satisfies

$$
\begin{equation*}
\mathrm{H} \operatorname{ch}_{2}^{\mathrm{B}+\bar{\beta}} \mathrm{H}(\mathrm{E})-\frac{\mathrm{A}(\bar{\beta})^{2}}{2} \mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})=0 \tag{9}
\end{equation*}
$$

That is, $\bar{\beta}^{2}\left(H^{3} \operatorname{ch}_{0}(E)\right)-2 \bar{\beta}\left(H^{2} \operatorname{ch}_{1}^{B}(E)\right)-A_{B}(\bar{\beta})^{2}\left(H^{3} \operatorname{ch}_{0}(E)\right)+2 H \operatorname{ch}_{2}^{B}=0$.
Definition 4.7. We define $\bar{\beta}_{B}(E)$ to be the root of (9) such that $\left(A_{B}\left(\bar{\beta}_{B}(E)\right), \bar{\beta}_{B}(E)\right) \in C(E)$ with minimal absolute value.

We need the following definition extending the similar notion in Li .
Definition 4.8. An object $E \in D^{b}(X)$ is called $\bar{\beta}_{B, H, A}$-stable if there is an open neighbourhood $U \subset \mathbb{R}^{2}$ containing $\left(A_{B}\left(\bar{\beta}_{B}(E)\right), \bar{\beta}_{B}(E)\right)$ such that for any $(\alpha, \beta) \in U$ with $\alpha>$ $A_{B}\left(\bar{\beta}_{B}(E)\right), E \in \mathcal{B}_{\sqrt{3} \alpha H, B+\beta H}$ is $v_{\sqrt{3} \alpha H, B+\beta H^{- \text {stable }}}$.

From the definition of $\bar{\beta}$-stability and Proposition 3.11, we have the following:
Proposition 4.9. Let E be an object in $\mathrm{D}^{\mathrm{b}}(\mathrm{X})$. Then E is $\bar{\beta}_{\mathrm{B}, \mathrm{H}, \mathrm{A}}$-stable with respect to some continuous function $A_{B}: \mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}$ if and only if $\mathrm{E}^{1}$ is $\bar{\beta}_{-\mathrm{B}, \mathrm{H}, \mathrm{A}}$-stable with respect to the continuous function $A_{-B}: \mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}$ satisfying $A_{-B}(-\beta)=A_{B}(\beta)$ for all $\beta \in \mathbb{R}$.

We conjecture the following for $X$.
Conjecture 4.10. There exist a constant $\xi \in \mathbb{R}_{\geqslant 0}$ and a smooth function $A_{B}: \mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}$ such that, any $\bar{\beta}_{\mathrm{B}, \mathrm{H}, \mathrm{A}}$-stable object $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ satisfies the inequality

$$
D_{A_{B}\left(\bar{\beta}_{B}(E)\right), \bar{\beta}_{B}(E)}^{B, \xi}(E) \leqslant 0
$$

Theorem 4.11. Conjectures 4.2 and 4.10 are equivalent.
We need few results to prove this theorem.
Note 4.12. Let $E$ be an object satisfying the conditions in above lemma. So $\mathrm{Hch}_{2}^{\mathrm{B}}(\mathrm{E})=$ $\alpha_{0}^{2} \mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E}) / 2$, and for $(\alpha, \beta) \in \mathrm{C}(\mathrm{E})$ we have

$$
\left(\beta^{2}-\alpha^{2}\right) H^{3} \operatorname{ch}_{0}(E)-2 \beta H^{2} \operatorname{ch}_{1}^{B}(E)+\alpha_{0}^{2} H^{3} \operatorname{ch}_{0}(E)=0
$$

Moreover, by Corollary 3.3 we have $\mathrm{H}^{2} \mathrm{ch}_{1}^{\mathrm{B} \pm \alpha_{0} \mathrm{H}}(\mathrm{E}) \geqslant 0$; so

$$
\left(\mathrm{H}^{2} \operatorname{ch}_{1}^{\mathrm{B}}(\mathrm{E})\right)^{2}-\alpha_{0}^{2}\left(\mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})\right)^{2} \geqslant 0
$$

When $\operatorname{ch}_{0}(E)=0, C(E)$ is a vertical line at $\beta=0$ from $\alpha=0$ to $\alpha_{0}$ in $(\beta, \alpha)$-plane.
Let us consider the case $\operatorname{ch}_{0}(E) \neq 0$. By (8) in Lemma 4.6, along $C(E)$ at $\beta=0$ we have

$$
\left(\frac{\mathrm{d} \alpha}{\mathrm{~d} \beta}\right)^{2}=\left(\frac{\mathrm{H}^{2} \operatorname{ch}_{1}^{\mathrm{B}}(\mathrm{E})}{\alpha \mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})}\right)^{2} \geqslant\left(\frac{\mathrm{H}^{2} \operatorname{ch}_{1}^{\mathrm{B}}(\mathrm{E})}{\alpha_{0} \mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})}\right)^{2} \geqslant 1
$$

Proposition 4.13. Let $\mathrm{E} \in \mathcal{B}_{\sqrt{3} \alpha_{0} \mathrm{H}, \mathrm{B}}$ be a tilt stable object with $\vee_{\sqrt{3} \alpha_{0} \mathrm{H}, \mathrm{B}}(\mathrm{E})=0$. Then $\mathrm{E} \in \mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}$ for $\beta \in\left[-\alpha_{0}, \alpha_{0}\right]$, any $\alpha \in \mathbb{R}_{>0}$; in particular $\mathrm{E} \in \mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}$ for all $(\alpha, \beta) \in C(E)$.

Proof. By Proposition 3.2, we have $\operatorname{ch}_{1}^{\mathrm{B}+\alpha_{0} \mathrm{H}}\left(\mathcal{H}^{0}(\mathrm{E})\right) \geqslant 0$ and $\mathrm{ch}_{1}^{\mathrm{B}-\alpha_{0} \mathrm{H}}\left(\mathcal{H}^{-1}(\mathrm{E})\right) \geqslant 0$. Therefore, $E \in \mathcal{B}_{\sqrt{3} \alpha H, B+\beta H}$ for all $\beta \in\left[-\alpha_{0}, \alpha_{0}\right], \alpha \in \mathbb{R}_{>0}$; in particular, from the discussion in Note 4.12, for any $(\alpha, \beta)$ on $C(E)$.
Proposition 4.14. Let $\mathrm{E} \in \mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}}$ be $v_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}}$-stable for all $0<\alpha \leqslant \alpha_{0}$ with $v_{\sqrt{3} \alpha_{0} \mathrm{H}, \mathrm{B}}(\mathrm{E})=$ 0 and $\bar{\Delta}_{\mathrm{H}, \mathrm{B}}(\mathrm{E})=0$. Then $\mathrm{E} \in \mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}}$ is $\nu_{\sqrt{3} \alpha \mathrm{H}, \mathrm{B}+\beta \mathrm{H}^{-s t a b l e}}$ for all $(\alpha, \beta) \in \mathrm{C}(\mathrm{E})$.
Proof. Assume the opposite for a contradiction: there exists $(\bar{\alpha}, \bar{\beta}) \in C(E)$ such that $E \in$ $\mathcal{B}_{\sqrt{3} \bar{\alpha} H, B+\bar{\beta} H}$ is strictly tilt semistable with zero tilt slope. By Proposition 3.4. E is strictly semistable for the complexified classes $(B+\beta)+i \sqrt{3} \alpha$ H satisfying $\alpha^{2}+(\beta-\bar{\beta})^{2}=\alpha_{0}^{2}$. By Note 4.12, $0<\bar{\beta}^{2}<\alpha_{0}^{2}$, and so there exists $0<\alpha<\alpha_{0}$ such that $\alpha^{2}+\bar{\beta}^{2}=\alpha_{0}^{2}$. That is $E \in \mathcal{B}_{\sqrt{3} \alpha H, B}$ is strictly semistable for some $\alpha \in\left(0, \alpha_{0}\right)$; this is the required contradiction.

### 4.3. Proof of the equivalences of the conjectures. Let us prove Theorem 4.11,

Proof. One implication in the theorem is obvious. Let us prove the other implication using contradiction method.

Assume Conjecture 4.10 holds for our 3 -fold $X$, and there is a counterexample for Conjecture 4.2. Let $E \in \mathcal{B}_{\sqrt{3} \alpha_{0} H, B}$ be a tilt stable object with $v_{\sqrt{3} \alpha_{0} H, B}(E)=0$. Suppose $D_{\alpha_{0}, 0}^{B, \xi}>0$ for a contradiction. By Proposition 4.13, $E$ stays in the same tilt category for all $(\alpha, \beta)$ in $C(E)$.

Let us consider the tilt stability of $E$ along $C(E)$ when $\alpha$ is decreasing from $P_{0}=\left(\alpha_{0}, 0\right)$. For a sequence of pairs $P_{j}=\left(\alpha_{j}, \beta_{j}\right), j \geqslant 0$ in $\mathbb{R}^{2}$ we simply write

$$
\begin{aligned}
\mathcal{B}_{P_{j}} & =\mathcal{B} \sqrt{3} \alpha_{j} H, B+\left(\beta_{1}+\cdots+\beta_{j}\right) H \\
\nu_{P_{j}} & =v_{\sqrt{3} \alpha_{j}} H, B+\left(\beta_{1}+\cdots+\beta_{j}\right) H .
\end{aligned}
$$

By Proposition 3.6, $\bar{\Delta}_{H, B}(E) \geqslant 0$. When $\bar{\Delta}_{H, B}(E)>0$, there might be a point $P_{1}=$ $\left(\alpha_{1}, \beta_{1}\right) \in C(E)$ such that $E \in \mathcal{B}_{P_{1}}$ becomes strictly $\gamma_{P_{1}}$-semistable. From Lemma 4.6, we have

$$
0<D_{\alpha_{0}, 0}^{B, \xi}(E)<D_{\alpha_{1}, \beta_{1}}^{B, \xi}(E)=D_{\alpha_{1}, 0}^{B+\beta_{1} H, \xi}(E)
$$

From the Jordan-Hölder filtration of $E$, there exists $\gamma_{P_{1}}$-stable factor $E_{1} \in \mathcal{B}_{P_{1}}$ of $E$ with $D_{\alpha_{1}, 0}^{\mathrm{B}+\beta_{1} \mathrm{H}, \mathrm{E}}\left(\mathrm{E}_{1}\right)>0$. Moreover, from Proposition 3.7

$$
\bar{\Delta}_{H, B}(E)>\bar{\Delta}_{H, B}\left(E_{1}\right)
$$

Now we take $E_{1} \in \mathcal{B}_{P_{1}}$ and consider the tilt stability along $C\left(E_{1}\right)$ in $\alpha$ decreasing direction from $\left(\alpha_{1}, 0\right)$. In this way there exists a sequence of points $P_{j}=\left(\alpha_{j}, \beta_{j}\right) \in C\left(E_{i-1}\right)$ with

$$
\begin{gathered}
\alpha_{0}>\alpha_{1}>\alpha_{2}>\cdots>\alpha_{j}>\cdots \\
D_{\alpha_{j}, 0}^{\mathrm{B}+\left(\beta_{1}+\cdots+\beta_{j}\right) H, \xi}\left(E_{j}\right)>0 \text { for all } \mathfrak{j}, \text { and } \\
\bar{\Delta}_{H, B}(E)>\bar{\Delta}_{H, B}\left(E_{1}\right)>\cdots>\bar{\Delta}_{H, B}\left(E_{j}\right)>\cdots \geqslant 0 .
\end{gathered}
$$

Since B is chosen to be rational, the image of $\bar{\Delta}_{\mathrm{H}, \mathrm{B}}$ forms a discrete set in $\mathbb{R}$; hence, this sequence terminates. That is there is $E_{j} \in \mathcal{B}_{P_{j}}$ which is $v_{P_{j}}$-stable, with $D_{\alpha_{j}, 0}^{B+\left(\beta_{1}+\cdots+\beta_{j}\right) H, \xi}\left(E_{j}\right)>$ 0 , and
(i) either $\Delta\left(\mathrm{E}_{\mathfrak{j}}\right)=0$,

 $C\left(E_{j}\right)$. From Proposition 4.6, we have

$$
0<D_{\alpha_{j}, 0}^{\mathrm{B}+\left(\beta_{1}+\cdots+\beta_{\mathfrak{j}}\right) H, \xi}\left(E_{\mathfrak{j}}\right) \leqslant D_{A, \bar{\beta}}^{\mathrm{B}+\left(\beta_{1}+\cdots+\beta_{\mathfrak{j}}\right) H, \xi}\left(\mathrm{E}_{\mathfrak{j}}\right),
$$

where $\bar{\beta}=\bar{\beta}_{B+\left(\beta_{1}+\cdots+\beta_{j}\right) H}\left(E_{j}\right)$, and $A=A_{B+\left(\beta_{1}+\cdots+\beta_{j}\right) H}(\bar{\beta})$. But this is not possible as we already assume Conjecture 4.10 holds for X . This completes the proof.

## 5. Some Properties of Tilt Stable Objects

5.1. Some Hom vanishing results for $\bar{\beta}$-stable objects. We follow the same notation in Section 4 for our smooth projective 3 -fold X . Let $\mathrm{H} \in \mathrm{NS}(\mathrm{X})$ be an ample divisor class. Let $B$ be a class proportional to $H$; so that $B=b H$ for some $b \in \mathbb{R}$.

We have the following vanishing results for $\bar{\beta}_{B, H, A}$-stable objects.
Proposition 5.1. Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be a $\bar{\beta}_{\mathrm{B}, \mathrm{H}, \mathrm{A}}$-stable object with $\mathrm{H}^{2} \mathrm{ch}_{1}^{(\mathrm{b}+\bar{\beta}) \mathrm{H}}(\mathrm{E}) \neq 0$; that is
 the following:
(i) if $\mathrm{k}<\mathrm{b}+\bar{\beta}-\mathrm{A}$ then

$$
\operatorname{Hom}_{\chi}\left(E, \mathcal{O}_{X}(\mathrm{kH})[1+j]\right)=0 \quad \text { for all } j \leqslant 0 .
$$

(ii) if $\mathrm{k}>\mathrm{b}+\bar{\beta}+\mathrm{A}$ then

$$
\operatorname{Hom}_{x}\left(0_{X}(k H), E[j]\right)=0 \text { for all } j \leqslant 0 .
$$

Proof. (i) Let $k$ be an integer such that $k<b+\bar{\beta}-A$. Since $E, \mathcal{O}_{X}(k H)[1] \in \mathcal{B}_{\sqrt{3} \alpha H,(b+\bar{\beta}) H}$ for all $\alpha \in \mathbb{R}_{>0}$, we have $\operatorname{Hom}_{x}\left(E, \mathcal{O}_{X}(k H)[1+j]\right)=0$ for all $j \leqslant-1$. Let us prove the Hom vanishing for $\mathfrak{j}=0$ case. Let

$$
0<\varepsilon<(\mathrm{b}+\bar{\beta}-\mathrm{A}-\mathrm{k}) / 2 .
$$

We have

$$
\mathcal{O}_{X}(\mathrm{kH})[1] \in \mathcal{B}_{\sqrt{3}(\mathrm{~A}+\varepsilon) \mathrm{H},(\mathrm{~b}+\bar{\beta}-\varepsilon) \mathrm{H}}
$$

is $V_{\sqrt{3}(\mathrm{~A}+\varepsilon) \mathrm{H},(\mathrm{b}+\bar{\beta}-\varepsilon) \mathrm{H}^{- \text {stable }} \text { (see Proposition 3.6) with }}$

$$
v_{\sqrt{3}(A+\varepsilon) H,(b+\bar{\beta}-\varepsilon) H}\left(\mathcal{O}_{X}(k H)[1]\right)=-\frac{(b+\bar{\beta}-A-k-2 \varepsilon)(b+\bar{\beta}+A-k)}{\sqrt{3}(b+\bar{\beta}-k-\varepsilon) \varepsilon}<0 .
$$

Since $\mathrm{H}_{\mathrm{ch}}^{2}{ }^{(\mathrm{b}+\bar{\beta}) \mathrm{H}}(\mathrm{E})-\mathcal{A}^{2} \mathrm{H}^{3} \mathrm{ch}_{0}(\mathrm{E}) / 2=0$,

$$
v_{\sqrt{3}(A+\varepsilon) H,(b+\bar{\beta}-\varepsilon) H}(E)=\frac{\varepsilon H^{2} \operatorname{ch}_{1}^{(\mathrm{b}+\bar{\beta}-\mathrm{A}) \mathrm{H}}(\mathrm{E})}{\sqrt{3}(\mathrm{~A}+\varepsilon) \mathrm{H}^{2} \mathrm{ch}_{1}^{(\mathrm{b}+\bar{\beta}-\varepsilon) \mathrm{H}}(\mathrm{E})} .
$$

Since $E$ is $\bar{\beta}_{B, H, A}$-stable with $H^{2} \operatorname{ch}_{1}^{(b+\bar{\beta}) H}(E)>0$, so for small enough $\varepsilon>0, H^{2} \operatorname{ch}_{1}^{(b+\bar{\beta}-\varepsilon) H}(E)>$ 0 . Also by Corollary 3.3, $\mathrm{H}^{2} \mathrm{ch}_{1}^{(\mathrm{b}+\bar{\beta}-\mathrm{A}) \mathrm{H}}(\mathrm{E}) \geqslant 0$. Therefore,

$$
\nu_{\sqrt{3}(\mathrm{~A}+\varepsilon) \mathrm{H},(\mathrm{~b}+\bar{\beta}-\varepsilon) \mathrm{H}}(\mathrm{E}) \geqslant 0,
$$

and hence, we have $\operatorname{Hom}_{\mathrm{X}}\left(\mathrm{E}, \mathcal{O}_{\mathrm{X}}(\mathrm{kH})[1]\right)=0$ as required.
(ii) Since $E, \mathcal{O}_{X}(k H) \in \mathcal{B}_{\sqrt{3} \alpha H,(b+\bar{\beta}) H}$ for all $\alpha \in \mathbb{R}_{>0}$, we have $\operatorname{Hom}_{X}\left(\mathcal{O}_{X}(k H), E[j]\right)=0$ for all $\mathfrak{j} \leqslant-1$. Let us prove the vanishing for $\mathfrak{j}=0$ case.
 $\bar{\beta}_{B}(E)$. So from part (i) for $-k<-b-\bar{\beta}-A$, we have $\operatorname{Hom}_{x}\left(E^{1}, \mathcal{O}_{X}(-k H)[1]\right)=0$. By Proposition 3.10, E fits into the short exact sequence:

$$
0 \rightarrow \mathrm{E} \rightarrow \mathrm{E}^{11} \rightarrow \mathrm{E}^{23} \rightarrow 0
$$

in $\mathcal{B} \sqrt{3} \alpha \mathrm{H},(\mathrm{b}+\bar{\beta}) \mathrm{H}$ with $\mathrm{E}^{23} \in \mathrm{Coh}_{0}(\mathrm{X})$. By applying the functor $\operatorname{Hom}_{\mathrm{X}}(\mathrm{O}(\mathrm{kH}),-)$ we get

$$
\operatorname{Hom}_{X}(\mathrm{O}(\mathrm{kH}), \mathrm{E}) \hookrightarrow \operatorname{Hom}_{\mathrm{X}}\left(\mathrm{O}(\mathrm{kH}), \mathrm{E}^{11}\right) \cong \operatorname{Hom}_{\mathrm{x}}\left(\mathrm{E}^{1}, \mathcal{O}_{X}(-\mathrm{kH})[1]\right)=0
$$

as required.
5.2. Strong bound for the discriminant of tilt stable objects. In this subsection we recall some of the results from Li with a slight generalization.

In this section we let $X$ be a Fano 3 -fold of index $r$. So $-K_{X}=r H$ for some ample divisor class $H$. Let $d=H^{3}$ be the degree of $X$.

The Chern character map ch : $\mathrm{K}(\mathrm{X}) \rightarrow \mathrm{H}_{\text {alg }}^{2 *}(\mathrm{X}, \mathbb{Q})$ defines the following linear map

$$
\mathrm{K}(\mathrm{X}) \rightarrow \mathbb{R}^{4}, \mathrm{E} \mapsto\left(\mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E}), \mathrm{H}^{2} \operatorname{ch}_{1}(\mathrm{E}), \mathrm{H}^{2} \operatorname{ch}_{2}(\mathrm{E}), \operatorname{ch}_{3}(\mathrm{E})\right)
$$

By truncating the last component, some objects $E$ in $K(X)$ map to $P\left(\mathbb{R}^{3}\right)$ by setting

$$
\widetilde{v}(E)=\left(H^{3} \operatorname{ch}_{0}(E): H^{2} \operatorname{ch}_{1}(E): H \operatorname{ch}_{2}(E)\right)
$$

more precisely, when $H^{3-\mathfrak{i}} \operatorname{ch}_{\mathfrak{i}}(E) \neq 0$ for some $\mathfrak{i}=0,1,2$. When $\operatorname{ch}_{0}(E) \neq 0$, we have

$$
\widetilde{v}(E)=\left(1, \frac{\mathrm{H}^{2} \operatorname{ch}_{1}(E)}{\mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})}, \frac{\mathrm{H}_{\mathrm{H}_{2}}(\mathrm{E})}{\mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})}\right)
$$

Definition 5.2. We call $P\left(\mathbb{R}^{3}\right) \backslash\left\{\left(a_{0}, a_{1}, a_{2}\right): a_{0} \neq 0\right\}$ the $\left\{1, \frac{\mathrm{H}^{2} \mathrm{ch}_{1}}{\mathrm{H}^{3} \mathrm{ch}_{0}}, \frac{\mathrm{Hch}}{\mathrm{H}^{3} \mathrm{ch}_{0}}\right\}$-plane.
We denote $\bar{\Delta}_{\mathrm{H}}=\bar{\Delta}_{\mathrm{H}, 0}=\left(\mathrm{H}^{2} \operatorname{ch}_{1}\right)^{2}-2 \mathrm{H}^{3} \operatorname{ch}_{0} \mathrm{H}_{\mathrm{ch}}^{2}$ (see Definition 10). So for any $\beta \in \mathbb{R}$ we have $\bar{\Delta}_{\mathrm{H}, \beta \mathrm{H}}=\bar{\Delta}_{\mathrm{H}}$. When $\mathrm{ch}_{0} \neq 0$ we define the reduced H -discriminant by

$$
\begin{equation*}
\widetilde{\Delta}_{\mathrm{H}}=\frac{\bar{\Delta}_{\mathrm{H}}}{\left(\mathrm{H}^{3} \mathrm{ch}_{0}\right)^{2}}=\left(\frac{\mathrm{H}^{2} \mathrm{ch}_{1}}{\mathrm{H}^{3} \mathrm{ch}_{0}}\right)^{2}-2\left(\frac{\mathrm{Hch}_{2}}{\mathrm{H}^{3} \mathrm{ch}_{0}}\right) \tag{10}
\end{equation*}
$$

Definition 5.3. We define $\widetilde{\Delta}_{\mathfrak{m}}$ as the parabola $\widetilde{\Delta}_{\mathrm{H}}=\mathrm{m}$ in $\left\{1, \frac{\mathrm{H}^{2} \mathrm{ch}_{1}}{\mathrm{H}^{3} \mathrm{ch}_{0}}, \frac{\mathrm{Hch}_{2}}{\mathrm{H}^{3} \mathrm{ch}_{0}}\right\}$-plane. Moreover, $\widetilde{\Delta}_{<\mathrm{m}}$ is the area defined by $\widetilde{\Delta}_{\mathrm{H}}<\mathrm{m}$.

The open region $\mathrm{R}_{\mathrm{m}}$ on $\left\{1, \frac{\mathrm{H}^{2} \mathrm{ch}_{1}}{\mathrm{H}^{3} \mathrm{ch}_{0}}, \frac{\mathrm{Hch}_{2}}{\mathrm{H}^{3} \mathrm{ch}_{0}}\right\}$-plane is defined as the set of points above the curve $\widetilde{\Delta}_{m}$ and above the tangent lines to the curve $\widetilde{\Delta}_{0}$ at $\widetilde{v}\left(\mathcal{O}_{X}(k H)\right)$ for all $k \in \mathbb{Z}$.

We have the following result for X .
Lemma 5.4. Let $\mathrm{E} \in \mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, \beta \mathrm{H}}$ be $v_{\sqrt{3} \alpha \mathrm{H}, \beta \mathrm{H}^{-}}$stable object with $\operatorname{ch}_{0}(\mathrm{E}) \neq 0$. Then $\widetilde{\mathcal{v}}(\mathrm{E})$ in $\left\{1, \frac{\mathrm{H}^{2} \mathrm{ch}_{1}}{\mathrm{H}^{3} \mathrm{ch}_{0}}, \frac{\mathrm{H} \mathrm{ch}_{2}}{\mathrm{H}^{3} \mathrm{ch}_{0}}\right\}$-plane is not in $\mathrm{R}_{3 /(2 \mathrm{rd})}$. If $\widetilde{\mathcal{v}}(\mathrm{E})$ is on the boundary of $\mathrm{R}_{3 /(2 \mathrm{rd})}$ then $\mathrm{ch}_{0}(\mathrm{E})$ is 1 or 2 .

Proof. The proof is identical to the proof of Picard rank one case [Li, Proposition 3.2].

## 6. BG Inequality for Fano 3-folds with Index 2

Let $X$ be a Fano 3 -fold of index 2 . So $-\mathrm{K}_{X}=2 \mathrm{H}$ for some ample divisor class $H$. Let the degree of $X$ be $d=H^{3}$. We carry the same notation in Section 4 for our Fano 3-fold X. Let $B$ be a class proportional to $H$; hence we can assume $B=0$ for Conjectures 4.2 and 4.10 .

By (3), the Todd class of $X$ is

$$
\operatorname{td}(\mathrm{X})=\left(1, \mathrm{H}, \mathrm{t}_{2}, 1\right), \quad \text { where } \mathrm{t}_{2}=\frac{\mathrm{H}^{2}}{3}+\frac{\mathrm{c}_{2}(\mathrm{X})}{12}
$$

Proposition 6.1. Let $\mathrm{E} \in \mathrm{D}^{\mathfrak{b}}(\mathrm{X})$. Then for any $\beta \in \mathbb{R}$, we have

$$
\chi(E(-H))=\operatorname{ch}_{3}^{\beta \mathrm{H}}(E)+\beta H \operatorname{ch}_{2}^{\beta \mathrm{H}}(E)+\gamma \operatorname{ch}_{1}^{\beta \mathrm{H}}(E)+\delta \operatorname{ch}_{0}(E)
$$

where

$$
\begin{aligned}
& \gamma=\frac{c_{2}(X)}{12}+\left(\frac{\beta^{2}}{2}-\frac{1}{6}\right) H^{2}, \\
& \delta=\frac{1}{6} \beta\left(d \beta^{2}+(6-d)\right) .
\end{aligned}
$$

Proof. From the Hirzebruch-Riemann-Roch theorem (2), we have

$$
\chi(\mathrm{E}(-\mathrm{H}))=\operatorname{ch}_{3}(\mathrm{E}(-\mathrm{H}))+\mathrm{Hch}_{2}(\mathrm{E}(-\mathrm{H}))+\mathrm{t}_{2} \operatorname{ch}_{1}(\mathrm{E}(-\mathrm{H}))+\operatorname{ch}_{0}(\mathrm{E}(-\mathrm{H}))
$$

From (ii) and (iii) of Proposition [2.3, we have $t_{2} \cdot H=\left(d r^{2} / 12\right)+(2 / r)$. Since $\operatorname{ch}(E(-H))=$ $\operatorname{ch}(E) \cdot e^{-H}$, we get $\chi(E(-H))=\operatorname{ch}_{3}(E)+\left(c_{2}(X) / 12-H^{2} / 6\right) \operatorname{ch}_{1}(E)$. We get the required expression by using $\operatorname{ch}(E)=\operatorname{ch}^{\beta \mathrm{H}}(E) \cdot e^{\beta \mathrm{H}}$ to write each components $\operatorname{ch}_{i}$ in terms of $\operatorname{ch}_{\mathfrak{i}}^{B}$ 's.
6.1. Case: degree $\leqslant 6$. Let us consider the case when the degree of $X$ is $d \leqslant 6$.

We prove that Conjecture 4.10 holds for $X$ with respect to

$$
\xi=0, \text { and }
$$

the function defined by

$$
A_{0}: \mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}, \beta \mapsto 0
$$

Theorem 6.2. Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be a $\bar{\beta}_{0, \mathrm{H}, \mathrm{A}}$-stable object. Then we have $\mathrm{D}_{0, \bar{\beta}_{0}(\mathrm{E})}^{0,0}(\mathrm{E}) \leqslant 0$.
Proof. Let us write $\bar{\beta}=\bar{\beta}_{0}(\mathrm{E})$. By using Proposition 4.9, and since tilt stability is preserved under tensoring by a line bundle, we can assume

$$
\operatorname{ch}_{0}(E) \geqslant 0 \text { and } \bar{\beta} \in[0,1)
$$

From Proposition 6.1, we have

$$
x(E(-H))=D_{0, \bar{\beta}}^{0,0}(E)+\bar{\beta} H \operatorname{ch}_{2}^{\bar{\beta}} H(E)+g(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H(E)+h(\bar{\beta}) \operatorname{ch}_{0}(E)
$$

where $g(\bar{\beta})=\bar{\beta}^{2} / 2+(6-d) /(6 d)$ and $h(\bar{\beta})=\bar{\beta}\left(d \bar{\beta}^{2}+(6-d)\right) / 6$. Since $d \leqslant 6$ and $\bar{\beta} \in[0,1)$, we have $g(\bar{\beta}), h(\bar{\beta}) \geqslant 0$. Also $H^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}(E) \geqslant 0$, and since $A_{0}=0$ we have $\mathrm{H}^{\bar{\beta}}{ }_{2}^{\bar{\beta}} \mathrm{H}(E)=0$. Therefore, $\chi(E(-H)) \geqslant D_{0, \bar{\beta}}^{0,0}(E)$.

From (ii) and (i) of Proposition 5.1, for any $j \leqslant 0$ we have

$$
\operatorname{Hom}_{X}\left(\mathcal{O}_{X}(H), E[j]\right)=0, \quad \text { and } \quad \operatorname{Hom}_{X}\left(E, \mathcal{O}_{X}(-H)[1+j]\right)=0
$$

By the Serre duality, $\operatorname{Hom}_{x}\left(E, \mathcal{O}_{X}(-H)[1+j]\right) \cong \operatorname{Hom}_{x}\left(\mathcal{O}_{X}(H), E[2-j]\right)^{*}$. Therefore,

$$
\chi\left(\mathcal{O}_{X}(H), E\right)=\sum_{i \in \mathbb{Z}}(-1)^{i} \operatorname{hom}_{X}\left(\mathcal{O}_{X}(H), E[i]\right)=-\operatorname{hom}_{X}\left(\mathcal{O}_{X}(H), E[1]\right) \leqslant 0
$$

Since $\chi(E(-H))=\chi\left(\mathcal{O}_{X}(H), E\right)$, we have the required inequality $D_{0, \bar{\beta}}^{0,0}(E) \leqslant 0$.
6.2. Blow-up of $\mathbb{P}^{3}$ at a point. Let us consider the case when the degree of $X$ is $d \geqslant 6$.

From Iskovskikh-Mori-Mukai classification of smooth Fano 3-folds (see [IP, Chapter 12]), X is isomorphic to the blow-up of $\mathbb{P}^{3}$ at a point.

We prove that Conjecture 4.10 holds for $X$ with respect to $\xi>0$ and $A_{0}: \mathbb{R} \rightarrow \mathbb{R} \geqslant 0$ defined as follows. Let $\Gamma$ be the area in $(\beta, \alpha)$-plane defined by

$$
\Gamma=\left\{(\beta, \alpha): \beta \in[-1 /(2 \sqrt{7}), 0], \beta+1 / \sqrt{7} \leqslant \alpha \leqslant \sqrt{1 / 7-3 \beta^{2}}\right\}
$$

Then clearly $\Gamma \subset[-1 /(2 \sqrt{7}), 0] \times[1 /(2 \sqrt{7}), 1 / \sqrt{7}]$. We define

$$
\begin{equation*}
\xi=\max _{(\beta, \alpha) \in \Gamma}\left\{\frac{-(\beta+\alpha)(\beta+\alpha-1 / \sqrt{7})(\beta+\alpha+1 / \sqrt{7})}{6 \alpha}\right\} \tag{11}
\end{equation*}
$$

Note 6.3. By definition we have

$$
\xi \leqslant \frac{\max _{(\beta, \alpha) \in \Gamma}-(\beta+\alpha)(\beta+\alpha-1 / \sqrt{7})(\beta+\alpha+1 / \sqrt{7})}{\min _{(\beta, \alpha) \in \Gamma} 6 \alpha}
$$

By differentiating one can show that the function defined by $\theta \mapsto-\theta(\theta-1 / \sqrt{7})(\theta+1 / \sqrt{7})$ has a maximum value $2 /(21 \sqrt{21})$ for $\theta \in[0,1 / \sqrt{7}]$. So we have

$$
\begin{equation*}
\xi \leqslant \frac{2}{63 \sqrt{3}} \tag{12}
\end{equation*}
$$

For $\beta \in[-1 / 2,0]$ we define

$$
A_{0}(\beta)= \begin{cases}-\beta-1 / \sqrt{7} & \text { if } \beta \in[-1 / 2,-1 / \sqrt{7})  \tag{13}\\ \beta+1 / \sqrt{7} & \text { if } \beta \in[-1 / \sqrt{7},-1 /(2 \sqrt{7})) \\ \sqrt{1 / 7-3 \beta^{2}} & \text { if } \beta \in[-1 /(2 \sqrt{7}), 0]\end{cases}
$$

and for other $\beta \in \mathbb{R}, A_{0}(\beta)$ is defined from the relations $A_{0}(-\beta)=A_{0}(\beta)$ and $A_{0}(\beta+1)=$ $A_{0}(\beta)$.

From the definition, for any $\beta \in \mathbb{R}$

$$
A_{0}(\beta) \leqslant 1 / \sqrt{7}<1 / 2
$$

Theorem 6.4. Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be a $\bar{\beta}_{0, \mathrm{H}, \mathrm{A}}$-stable object. Then $\mathrm{D}_{\mathrm{A}_{0}\left(\bar{\beta}_{0}(\mathrm{E})\right), \bar{\beta}_{0}(\mathrm{E})}(\mathrm{E}) \leqslant 0$.
Proof. Let us write $\bar{\beta}=\bar{\beta}_{0}(E)$ and $A=A_{0}\left(\bar{\beta}_{0}(E)\right)$; so $A=A_{0}(\bar{\beta})<1 / 2$.
Since tilt stability is preserved under tensoring by a line bundle, and also from Proposition 4.9 we can assume

$$
\operatorname{ch}_{0}(E) \geqslant 0, \quad \text { and } \bar{\beta} \in[-1 / 2,1 / 2)
$$

From (ii) and (i) of Proposition 5.1, for any $\mathfrak{j} \leqslant 0$ we have

$$
\operatorname{Hom}_{X}\left(\mathcal{O}_{X}(H), E[j]\right)=0, \quad \text { and } \quad \operatorname{Hom}_{X}\left(E, \mathcal{O}_{X}(-H)[1+j]\right)=0
$$

By the Serre duality, $\operatorname{Hom}_{\chi}\left(E, \mathcal{O}_{X}(-H)[1+j]\right) \cong \operatorname{Hom}_{\chi}\left(\mathcal{O}_{X}(H), E[2-j]\right)^{*}$. Therefore,

$$
\chi\left(\Theta_{X}(H), E\right)=\sum_{i \in \mathbb{Z}}(-1)^{i} \operatorname{hom}_{x}\left(\mathcal{O}_{X}(H), E[i]\right)=-\operatorname{hom}_{x}\left(\mathcal{O}_{X}(H), E[1]\right) \leqslant 0
$$

That is

$$
\begin{equation*}
\chi(\mathrm{E}(-\mathrm{H})) \leqslant 0 \tag{14}
\end{equation*}
$$

Since $H \operatorname{ch}_{2}^{\bar{\beta}} \mathrm{H}(E)=A^{2} \mathrm{H}^{3} \mathrm{ch}_{0}(E) / 2$ and from Proposition 6.1,

$$
\chi(\mathrm{E}(-\mathrm{H}))=\mathrm{D}_{\mathrm{A}, \bar{\beta}}^{0, \xi}(\mathrm{E})+\mathrm{pH}^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}(\mathrm{E})+\mathrm{qH}^{3} \operatorname{ch}_{0}(\mathrm{E}),
$$

where $p=\left(A^{2} / 6+\bar{\beta}^{2} / 2-1 / 42+\xi\right) \geqslant 0$ and $q=\bar{\beta}\left(A^{2} / 2+\bar{\beta}^{2} / 6-1 / 42\right)$.
First let us consider the case: $\bar{\beta} \in[-1 / 2,-1 / \sqrt{7}) \cup[-1 /(2 \sqrt{7}), 1 / 2)$. We can write

$$
\chi(E(-H))=D_{A, \bar{\beta}}^{0, \xi}(E)+\mathrm{pH}^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}+\mathrm{AH}(E)+\mathrm{q}_{1} \mathrm{H}^{3} \mathrm{ch}_{0}(E)
$$

where $q_{1}=\frac{1}{6}(\bar{\beta}+A)(\bar{\beta}+A-1 / \sqrt{7})(\bar{\beta}+A+1 / \sqrt{7})+\xi A$. From Definition (13) of $A_{0}$, for $\bar{\beta} \in$ $[-1 / 2,-1 / \sqrt{7}) \cup[-1 /(2 \sqrt{7}), 1 / 2)$ we have $\mathrm{q}_{1} \geqslant 0$. From Corollary 3.3, $\mathrm{H}^{2} \mathrm{ch}_{1}^{\bar{\beta}} \mathrm{H}+\mathrm{AH}(\mathrm{E}) \geqslant 0$ and so we have $\chi(E(-H)) \geqslant D_{A, \bar{\beta}}^{0, \xi}(E)$. Hence, by (14) we have the required inequality.

Let us consider the remaining case: $\bar{\beta} \in[-1 / \sqrt{7},-1 /(2 \sqrt{7}))$. We can write

$$
\chi(E(-H))=D_{A, \bar{\beta}}^{0, \xi}(E)+\xi H^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}^{(E)}+\mathrm{p}_{1} \mathrm{H}^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}-\mathrm{AH}(\mathrm{E})+\mathrm{q}_{2} \mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})
$$

where $p_{1}=\left(A^{2} / 6+\bar{\beta}^{2} / 2-1 / 42\right)$, and $q_{2}=\frac{1}{6}(\bar{\beta}-\mathcal{A})(\bar{\beta}-A-1 / \sqrt{7})(\bar{\beta}-A+1 / \sqrt{7})$. From Definition (13) of $A_{0}$, for $\bar{\beta} \in[-1 / \sqrt{7},-1 /(2 \sqrt{7}))$ we have $p_{1} \geqslant 0$ and $q_{2}=0$. From Corollary 3.3, $\mathrm{H}^{2} \mathrm{ch}_{1}^{\bar{\beta} H-A H}(E) \geqslant 0$ and so we have $\chi(E(-H)) \geqslant D_{A, \bar{\beta}}^{0, \xi}(E)$. Hence, by (14) we have the required inequality.

Remark 6.5. If we closely see the above proof, one can define a better function for $A_{0}$ as follows: for $\beta \in[-1 / 2,1 / 2]$

$$
\begin{aligned}
A_{0}(\beta)=\max \left\{A \in \mathbb{R}_{\geqslant 0}:\right. & A^{2} / 6+\beta^{2} / 2-1 / 42+\xi=0 \text { or } \\
& (\beta+A)(\beta+A-1 / \sqrt{7})(\beta+A+1 / \sqrt{7})+\xi A=0 \text { or } \\
& (\beta-A)(\beta-A-1 / \sqrt{7})(\beta-A+1 / \sqrt{7})-\xi A=0\},
\end{aligned}
$$

and for other $\beta \in \mathbb{R}$ from the relation $A_{0}(\beta+1)=A_{0}(\beta)$.

## 7. BG Inequality for Fano 3-folds with Index 1

Let X be a Fano 3 -fold of index 1 . So $-\mathrm{K}_{\mathrm{X}}=\mathrm{H}$ for some ample divisor class H . Let the degree of $X$ be $d=H^{3}$. We carry the same notation in Section 4 for our Fano 3 -fold X. Let $B$ be a class proportional to H ; hence we can assume $\mathrm{B}=0$ for Conjectures 4.2 and 4.10 ,

By (3), the Todd class of $X$ is

$$
\operatorname{td}(\mathrm{X})=\left(1, \mathrm{H}, \mathrm{t}_{2}, 1\right), \quad \text { where } \mathrm{t}_{2}=\frac{\mathrm{H}^{2}+\mathrm{c}_{2}(\mathrm{X})}{12}
$$

Proposition 7.1. Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$. Then for any $\beta \in \mathbb{R}$, we have

$$
\chi(E(-H))=\operatorname{ch}_{3}^{\beta \mathrm{H}}(E)+\left(\beta-\frac{1}{2}\right) \operatorname{Heh}_{2}^{\beta \mathrm{H}}(E)+\gamma \operatorname{ch}_{1}^{\beta \mathrm{H}}(E)+\delta \mathrm{H}^{3} \operatorname{ch}_{0}(E)
$$

where

$$
\begin{aligned}
& \gamma=\frac{c_{2}(X)}{12}+\left(\frac{\beta^{2}}{2}-\frac{\beta}{2}+\frac{1}{12}\right) H^{2} \\
& \delta=\frac{\beta^{3}}{6}-\frac{\beta^{2}}{4}+\left(\frac{2}{d}+\frac{1}{12}\right) \beta-\frac{1}{d} .
\end{aligned}
$$

Proof. Similar to the proof of Proposition 6.1.
We need the following result:
Proposition 7.2. Let d be a positive integer. Let $\mathrm{f}: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$
f(x)=\frac{1}{6}\left(x-\frac{1}{2}\right)^{3}+\frac{(48-d)}{24 d}\left(x-\frac{1}{2}\right) .
$$

Let g be a real valued function defined by

$$
g(x)=\sqrt{\frac{3}{2 d}} f^{\prime}(x)+f(x)
$$

Here $\mathrm{f}^{\prime}(\mathrm{x})$ is the derivative of $\mathrm{f}(\mathrm{x})$ with respect to x . Then we have the following:
(i) if $\mathrm{d} \leqslant 48$, then for $\mathrm{x} \in[0,1 / 2)$ we have $\mathrm{g}(\mathrm{x}) \geqslant 0$.
(ii) if $\mathrm{d}>48$, then for $\mathrm{x} \in[0,1 / 2-\sqrt{(\mathrm{d}-48) /(4 \mathrm{~d})})$ we have $\mathrm{g}(\mathrm{x}) \geqslant 0$.

Proof. By differentiating, we have $\mathrm{f}^{\prime}(\mathrm{x})=(\mathrm{x}-1 / 2)^{2} / 2+(48-\mathrm{d}) /(24 \mathrm{~d})$. Therefore,

$$
g(x)=\frac{1}{6}\left(x-\frac{1}{2}\right)^{3}+\frac{1}{2} \sqrt{\frac{3}{2 d}}\left(x-\frac{1}{2}\right)^{2}+\frac{(48-d)}{24 d}\left(x-\frac{1}{2}\right)+\sqrt{\frac{3}{2 d}} \frac{(48-d)}{24 d} .
$$

By differentiating we get

$$
\begin{aligned}
g^{\prime}(x) & =\frac{1}{2}\left(x-\frac{1}{2}\right)^{2}+\sqrt{\frac{3}{2 d}}\left(x-\frac{1}{2}\right)+\frac{(48-\mathrm{d})}{24 \mathrm{~d}} \\
& =\frac{1}{2}\left(x-\frac{1}{2}+\sqrt{\frac{3}{2 d}}\right)^{2}-\frac{(d-30)}{12 \mathrm{~d}} .
\end{aligned}
$$

By evaluating $g$ at 0 we get

$$
g(0)=\sqrt{\frac{3}{2 d}}\left(\frac{1}{12}+\frac{2}{d}\right)-\frac{1}{d}=\sqrt{\frac{6}{d}}\left(\frac{1}{\sqrt{d}}-\frac{1}{\sqrt{24}}\right)^{2} \geqslant 0
$$

When $d \leqslant 30$ we have $g^{\prime}(x) \geqslant 0$ for all $x \in \mathbb{R}$ with $g^{\prime}(0)>0$. Hence for $x \in[0,1 / 2), g(x) \geqslant 0$.
Let us consider the case $d>30$. The derivative $g^{\prime}(x)$ is vanishing at $x=\lambda_{1}, \lambda_{2}$ with $\lambda_{1}<\lambda_{2}$. Here

$$
\begin{aligned}
& \lambda_{1}=\frac{1}{2}-\sqrt{\frac{3}{2 d}}-\sqrt{\frac{(d-30)}{12 \mathrm{~d}}}=\frac{1}{2}-\frac{(\mathrm{d}-48)}{\sqrt{12 \mathrm{~d}}(\sqrt{\mathrm{~d}-30}-\sqrt{18})} \\
& \lambda_{2}=\frac{1}{2}-\sqrt{\frac{3}{2 \mathrm{~d}}}+\sqrt{\frac{(\mathrm{d}-30)}{12 \mathrm{~d}}}=\frac{1}{2}+\frac{(\mathrm{d}-48)}{\sqrt{12 \mathrm{~d}}(\sqrt{\mathrm{~d}-30}+\sqrt{18})}
\end{aligned}
$$

One can rearrange $g$ as

$$
g(x)=\frac{1}{6}\left(x-\frac{1}{2}+\sqrt{\frac{3}{2 d}}\right)^{3}+\frac{(30-d)}{24 d}\left(x-\frac{1}{2}\right)+\sqrt{\frac{3}{2 d}} \frac{(42-d)}{24 d} .
$$

Let us consider the case $30<d \leqslant 48$. The local minimum value of $g(x)$ at $x=\lambda_{2}$ is

$$
g\left(\lambda_{2}\right)=-\frac{1}{3}\left(\frac{d-30}{12 d}\right)^{3 / 2}+\frac{1}{2 d} \sqrt{\frac{3}{2 d}}=\frac{18^{3 / 2}-(d-30)^{3 / 2}}{72 \sqrt{3} d^{3 / 2}} \geqslant 0
$$

with equality when $d=48$. Moreover, $\lambda_{2}=1 / 2$ when $d=48$. Hence for $x \in[0,1 / 2)$ we have $g(x) \geqslant 0$.

Let us consider the remaining case $\mathrm{d}>48$. We have

$$
\begin{gathered}
0<1 / 2-\sqrt{(d-48) /(4 d)}<1 / 2<\lambda_{2} \\
g\left(\lambda_{2}\right)<0
\end{gathered}
$$

and $f(1 / 2-\sqrt{(d-48) /(4 d)})=0$, so

$$
g\left(\frac{1}{2}-\sqrt{\frac{d-48}{4 d}}\right)=\sqrt{\frac{3}{2 d}} f^{\prime}\left(\frac{1}{2}-\sqrt{\frac{d-48}{4 d}}\right)=\frac{(d-48)}{4 \sqrt{6} d^{3 / 2}}>0 .
$$

Hence, if $\mathrm{d}>48$ we have for $\mathrm{x} \in[0,1 / 2-\sqrt{(\mathrm{d}-48) /(4 \mathrm{~d})}), \mathrm{g}(\mathrm{x}) \geqslant 0$. This completes the proof.
7.1. Case: degree $d \leqslant 48$. Suppose the degree of $X$ is $d \leqslant 48$. We prove that Conjecture 4.10 holds for $X$ with respect to

$$
\xi=0, \text { and }
$$

the function defined by

$$
A_{0}: \mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}, \beta \mapsto 0
$$

Proposition 7.3. Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be a $\bar{\beta}_{0, \mathrm{H}, \mathrm{A}}$-stable object with $\operatorname{ch}_{0}(\mathrm{E}) \geqslant 0, \bar{\beta}_{0}(\mathrm{E}) \in[0,1)$ and $\chi(\mathrm{E}(-\mathrm{H})) \leqslant 0$. Then we have $\mathrm{D}_{0, \bar{\beta}_{0}(\mathrm{E})}^{0,0}(\mathrm{E}) \leqslant 0$.

Proof. The following proof is adapted from a part in the proof for Fano 3 -folds of index 1 with Picard rank one in [Li].

Let us write $\bar{\beta}=\bar{\beta}_{0}(E)$. From Proposition 7.1, we have

$$
\chi(E(-H))=D_{0, \bar{\beta}}^{0,0}(E)+\left(\beta-\frac{1}{2}\right) H \operatorname{ch}_{2}^{\beta H}(E)+f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H(E)+f(\bar{\beta}) H^{3} \operatorname{ch}_{0}(E)
$$

where

$$
\begin{aligned}
f(\bar{\beta}) & =\frac{\bar{\beta}^{3}}{6}-\frac{\bar{\beta}^{2}}{4}+\left(\frac{2}{d}+\frac{1}{12}\right) \bar{\beta}-\frac{1}{d}=\frac{1}{6}\left(\bar{\beta}-\frac{1}{2}\right)^{3}+\frac{(48-d)}{24 d}\left(\bar{\beta}-\frac{1}{2}\right) \\
f^{\prime}(\bar{\beta}) & =\frac{\bar{\beta}^{2}}{2}-\bar{\beta}^{2}+\left(\frac{2}{d}+\frac{1}{12}\right)=\frac{1}{2}\left(\bar{\beta}-\frac{1}{2}\right)^{2}+\frac{(48-d)}{24 d} .
\end{aligned}
$$

Since $A_{0}=0$ we have $\operatorname{Hch}_{2}^{\bar{\beta}} \mathrm{H}(E)=0$. Therefore,

$$
\begin{equation*}
\chi(E(-H))=D_{0, \bar{\beta}}^{0,0}(E)+f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H(E)+f(\bar{\beta}) H^{3} \operatorname{ch}_{0}(E) \tag{15}
\end{equation*}
$$

Since $f^{\prime}(\bar{\beta}) \geqslant 0$, when $\operatorname{ch}_{0}(E)=0$ from (15), we have $\chi(E(-H)) \leqslant D_{0, \bar{\beta}}^{0,0}(E)$; hence, we have the required inequality as $\chi(\mathrm{E}(-\mathrm{H})) \leqslant 0$. Therefore, we can assume

$$
\mathrm{ch}_{0}(\mathrm{E})>0 .
$$

Case 1: $\bar{\beta} \in[1 / 2,1)$. We have $f(\bar{\beta}), f^{\prime}(\bar{\beta}) \geqslant 0$. So from (15), we have $0 \geqslant \chi(E(-H)) \geqslant$ $D_{0, \bar{\beta}}^{0,0}(E)$ as required.
Case 2: $\bar{\beta} \in[0,1 / 2)$. We show that $f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H(E)+f(\bar{\beta}) H^{3} \operatorname{ch}_{0}(E) \geqslant 0$ by contradiction. Suppose the opposite for a contradiction. So we have

$$
\begin{aligned}
0 & >f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H^{(E)}+f(\bar{\beta}) H^{3} \operatorname{ch}_{0}(E) \\
& =f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{(1 / 2) H}(E)+\left(-\left(\beta-\frac{1}{2}\right) f^{\prime}(\bar{\beta})+f(\bar{\beta})\right) H^{3} \operatorname{ch}_{0}(E) \\
& =f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{(1 / 2) H}(E)-\frac{1}{3}\left(\beta-\frac{1}{2}\right)^{3} H^{3} \operatorname{ch}_{0}(E) \\
& \geqslant f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{(1 / 2) H}(E) .
\end{aligned}
$$

Since $f^{\prime}(\bar{\beta})>0, H^{2} \operatorname{ch}_{1}^{(1 / 2) H}(E)<0$; that is $\left(H^{2} \operatorname{ch}_{1}(E)\right) /\left(H^{3} \operatorname{ch}_{0}(E)\right)<1 / 2$. Also since $\bar{\beta} \in[0,1 / 2)$, we deduce that the corresponding point $\widetilde{\mathcal{v}}(\mathrm{E})$ of E in $\left\{1, \frac{\mathrm{H}^{2} \mathrm{ch}_{1}}{\mathrm{H}^{3} \mathrm{ch}_{0}}, \frac{\mathrm{Hch}_{2}}{\mathrm{H}^{3} \mathrm{ch}} \mathbf{0}\right.$ $\}$-plane is above the tangent line of $\widetilde{\Delta}_{0}$ at $\widetilde{v}\left(\mathcal{O}_{X}(\mathrm{H})\right)$ as shown in the Figure at page 15 of [Li].

Since $\mathrm{f}^{\prime}(\bar{\beta}) \mathrm{H}^{2} \mathrm{ch}_{1}^{\bar{\beta}} \mathrm{H}(\mathrm{E})+\mathrm{f}(\bar{\beta}) \mathrm{H}^{3} \operatorname{ch}_{0}(E)<0$ and $\mathrm{f}^{\prime}(\bar{\beta})>0$, we have

$$
0 \leqslant \frac{\mathrm{H}^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}(\mathrm{E})}{\mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})}<\frac{-\mathrm{f}(\bar{\beta})}{\mathrm{f}^{\prime}(\bar{\beta})} \leqslant \sqrt{\frac{3}{2 \mathrm{~d}}} .
$$

Here the last inequality follows from (i) of Proposition [7.2. Therefore,

$$
\widetilde{\Delta}_{\mathrm{H}}(\mathrm{E})=\frac{\bar{\Delta}_{\mathrm{H}, \overline{\mathrm{\beta}} \mathrm{H}}(\mathrm{E})}{\left(\mathrm{H}^{3} \mathrm{ch}_{0}(\mathrm{E})\right)^{2}}=\left(\frac{\mathrm{H}^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}(\mathrm{E})}{\mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})}\right)^{2}<\frac{3}{2 \mathrm{~d}} .
$$

Hence $\widetilde{v}(E)$ is in $R_{3 /(2 d)}$ region. But this in not possible from Lemma 5.4. So from (15), we have $0 \geqslant \chi(\mathrm{E}(-\mathrm{H})) \geqslant \mathrm{D}_{0, \bar{\beta}}^{0,0}(\mathrm{E})$ as required. This completes the proof.

Theorem 7.4. Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be a $\bar{\beta}_{0, \mathrm{H}, \mathrm{A}}$-stable object. Then we have $\mathrm{D}_{0, \bar{\beta}_{0}(\mathrm{E})}^{0,0}(\mathrm{E}) \leqslant 0$.
Proof. The following proof is very close to the proof for Fano 3 -folds of index 1 with Picard rank one in Li .

Let us write $\bar{\beta}=\bar{\beta}_{0}(\mathrm{E})$. By using Proposition 4.9, and since tilt stability is preserved under tensoring by a line bundle, we can assume

$$
\operatorname{ch}_{0}(E) \geqslant 0 \text { and } \bar{\beta} \in[0,1) .
$$

Case 1: $\bar{\beta} \in(0,1)$. From (ii) and (i) of Proposition 5.1, for any $j \leqslant 0$ we have

$$
\operatorname{Hom}_{X}\left(\mathcal{O}_{X}(H), E[j]\right)=0, \quad \text { and } \quad \operatorname{Hom}_{X}\left(E, \mathcal{O}_{X}[1+j]\right)=0 .
$$

By the Serre duality, $\operatorname{Hom}_{x}\left(E, \mathcal{O}_{\mathrm{X}}[1+j]\right) \cong \operatorname{Hom}_{\mathrm{x}}\left(\mathcal{O}_{\mathrm{X}}(\mathrm{H}), \mathrm{E}[2-j]\right)^{*}$. Therefore,

$$
\chi\left(\mathcal{O}_{X}(\mathrm{H}), \mathrm{E}\right)=\sum_{\mathfrak{i} \in \mathbb{Z}}(-1)^{\mathrm{i}} \operatorname{hom}_{\mathrm{X}}\left(\mathcal{O}_{\mathrm{X}}(\mathrm{H}), \mathrm{E}[i]\right)=-\operatorname{hom}_{\mathrm{X}}\left(\mathcal{O}_{\mathrm{X}}(\mathrm{H}), \mathrm{E}[1]\right) \leqslant 0 .
$$

That is $\chi(E(-H)) \leqslant 0$. So from Proposition 7.3, we have the required inequality.
Case 2: $\bar{\beta}=0$. If $\chi(E(-H)) \leqslant 0$ then the required inequality follows from Proposition 7.3. So let us assume $\chi(E(-H))>0$.

Suppose there is $E$ such that $D_{0, \bar{\beta}}^{0,0}(E)>0$ for a contradiction. Among all such objects, let E has the minimum $\bar{\Delta}_{\mathrm{H}}$; that is with with minimum $\left|\mathrm{H}^{2} \mathrm{ch}_{1}\right|$.

As in the previous case we have

$$
\chi\left(\mathcal{O}_{X}(H), E\right)=\sum_{i \in \mathbb{Z}}(-1)^{i} \operatorname{hom}_{X}\left(\mathcal{O}_{X}(H), E[i]\right)=-\operatorname{hom}_{X}\left(\mathcal{O}_{X}(H), E[1]\right)+\operatorname{hom}_{X}\left(\mathcal{O}_{X}(H), E[2]\right) .
$$

Since $\chi(E(-H))>0$, from the Serre duality we have $\operatorname{Hom}_{X}\left(\mathcal{O}_{X}(H), E[2]\right) \cong \operatorname{Hom}_{X}\left(E, \mathcal{O}_{X}[1]\right)^{*} \neq$ 0 . So there is a non-trivial map $E \rightarrow \mathcal{O}_{X}[1]$ in $D^{b}(X)$ and hence, we have the distinguished triangle

$$
\mathcal{O}_{X} \rightarrow \mathrm{~F} \rightarrow \mathrm{E} \rightarrow \mathcal{O}_{\mathrm{X}}[1]
$$

for some $F \in D^{b}(X)$. Here $E, \mathcal{O}_{X}[1] \in \mathcal{B}_{\sqrt{3} \alpha H, 0}$ for any $\alpha>0$, and also from Proposition 2.1. $\mathcal{O}_{\mathrm{X}}$ [1] is a minimal object. Therefore by considering the long exact sequence of $\mathcal{B}_{\sqrt{3} \alpha \mathrm{H}, 0^{-}}$ cohomologies we get $F \cong H_{\mathcal{B}_{\sqrt{3} \alpha H, 0}^{0}}^{0}(F)$ and the following short exact sequence in $\mathcal{B}_{\sqrt{3} \alpha H, 0}$ :

$$
0 \rightarrow \mathrm{~F} \rightarrow \mathrm{E} \rightarrow \mathcal{O}_{\mathrm{X}}[1] \rightarrow 0
$$

By similar arguments in the proof of [Li, Lemma 3.6] one can show that $F$ is $\bar{\beta}_{0, \mathrm{H}, \mathrm{A}^{-}}$-stable with $\bar{\beta}_{0}(F)=0$ with $\operatorname{ch}_{0}(F)=\operatorname{ch}_{0}(E)+1>0$. Also

$$
\chi(\mathrm{E}(-\mathrm{H}))=\operatorname{ch}(\mathrm{F}(-\mathrm{H}))+\chi\left(\mathcal{O}_{\chi}(-\mathrm{H})[1]\right)=\chi(\mathrm{F}(-\mathrm{H}))+1
$$

and $D_{0,0}^{0,0}(E)=D_{0,0}^{0,0}(F)$. Hence, if $\chi(F(-H)) \leqslant 0$ then from Proposition 7.3 we have the required contradiction. Otherwise, we can repeat the same process and we get a sequence of $\bar{\beta}_{0, \mathrm{H}, \mathrm{A}}$-stable objects $\mathrm{F}_{1}:=\mathrm{F}, \mathrm{F}_{2}, \mathrm{~F}_{3}, \cdots$ such that $\bar{\beta}_{0}\left(\mathrm{~F}_{\mathrm{i}}\right)=0$ with $\operatorname{ch}_{0}\left(\mathrm{~F}_{\mathrm{i}}\right)>0$ and

$$
+\infty>\chi(\mathrm{E}(-\mathrm{H}))>\chi\left(\mathrm{F}_{1}(-\mathrm{H})\right)>\chi\left(\mathrm{F}_{2}(-\mathrm{H})\right)>\ldots
$$

Hence, there is some $F_{i}$ with $\chi\left(F_{i}(-H)\right) \leqslant 0$; this gives the required contradiction. This completes the proof.
7.2. Case: degree $d>48$. Suppose the degree of $X$ is $d>48$. Let us denote

$$
\eta=\sqrt{\frac{d-48}{48 d}}
$$

We prove that Conjecture 4.10 holds for $X$ with respect to

$$
\begin{equation*}
\xi=\frac{2}{3} \eta^{2}, \text { and } \tag{16}
\end{equation*}
$$

the function $A_{0}: \mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}, \beta \mapsto A_{0}(\beta)$ defined by, when $\beta \in[0,1]$

$$
A_{0}(\beta)= \begin{cases}0 & \text { if } \beta \in[0,1 / 2-\sqrt{6} \eta] \cup[1 / 2+\sqrt{6} \eta, 1]  \tag{17}\\ \sqrt{2 \eta^{2}-\frac{1}{3}(\beta-1 / 2)^{2}} & \text { if } \beta \in[1 / 2-\sqrt{6} \eta], 1 / 2+\sqrt{6} \eta]\end{cases}
$$

and for other $\beta \in \mathbb{R}$ from the relation $A_{0}(\beta+1)=A_{0}(\beta)$.
Proposition 7.5. Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be a $\bar{\beta}_{0, \mathrm{H}, \mathrm{A}}$-stable object with $\operatorname{ch}_{0}(\mathrm{E}) \geqslant 0, \bar{\beta}_{0}(\mathrm{E}) \in[0,1)$ and $\chi(\mathrm{E}(-\mathrm{H})) \leqslant 0$. Then we have $\mathrm{D}_{\mathrm{A}_{0}\left(\bar{\beta}_{0}(\mathrm{E})\right), \bar{\beta}_{0}(\mathrm{E})}^{0, \xi}(\mathrm{E}) \leqslant 0$.

Proof. The following proof is very close to the proof of Proposition 7.3.
Let us write $\bar{\beta}=\bar{\beta}_{0}(E)$ and $A=A_{0}\left(\bar{\beta}_{0}(E)\right)$. Since $H h_{2}^{\bar{\beta}} H(E)=A^{2} H^{3} \operatorname{ch}_{0}(E) / 2$, from Proposition 7.1, we have

$$
\begin{align*}
& \chi(E(-H))=D_{A, \bar{\beta}}^{0, \xi}(E)+\left(f^{\prime}(\bar{\beta})+\xi+\frac{A^{2}}{6}\right) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H  \tag{18}\\
&(E)+ \\
&\left(f(\bar{\beta})+\frac{A^{2}}{2}\left(\bar{\beta}-\frac{1}{2}\right)\right) H^{3} \operatorname{ch}_{0}(E)
\end{align*}
$$

where

$$
\begin{aligned}
& f(\bar{\beta})=\frac{\bar{\beta}^{3}}{6}-\frac{\bar{\beta}^{2}}{4}+\left(\frac{2}{d}+\frac{1}{12}\right) \bar{\beta}-\frac{1}{d}=\frac{1}{6}\left(\bar{\beta}-\frac{1}{2}\right)^{3}-\eta^{2}\left(\bar{\beta}-\frac{1}{2}\right) \\
& f^{\prime}(\bar{\beta})=\frac{\bar{\beta}^{2}}{2}-\bar{\beta}^{2}+\left(\frac{2}{d}+\frac{1}{12}\right)=\frac{1}{2}\left(\bar{\beta}-\frac{1}{2}\right)^{2}-\eta^{2}
\end{aligned}
$$

From the definition of $A_{0}$, for $\bar{\beta} \in[0,1)$, we have

$$
f^{\prime}(\bar{\beta})+\xi+\frac{A^{2}}{6}=\frac{1}{2}\left(\bar{\beta}-\frac{1}{2}\right)^{2}+\frac{A^{2}}{6}-\frac{1}{3} \eta^{2} \geqslant 0
$$

Hence, when $\operatorname{ch}_{0}(E)=0$ from (18), we have $0 \geqslant \chi(E(-H)) \geqslant D_{A, \bar{\beta}}^{0, \xi}(E)$ as required. Therefore, we can assume

$$
\operatorname{ch}_{0}(E)>0
$$

Case 1: $\bar{\beta} \in[1 / 2-\sqrt{6} \eta, 1)$. We have $f^{\prime}(\bar{\beta})+\xi+A^{2} / 6 \geqslant 0$ and

$$
f(\bar{\beta})+\frac{A^{2}}{2}\left(\bar{\beta}-\frac{1}{2}\right)=\left(\bar{\beta}-\frac{1}{2}\right)\left(\frac{1}{6}\left(\bar{\beta}-\frac{1}{2}\right)^{2}+\frac{A^{2}}{2}-\eta^{2}\right) \geqslant 0
$$

So from (18), we have $0 \geqslant \chi(E(-H)) \geqslant D_{A, \bar{\beta}}^{0, \xi}(E)$ as required.
Case 2: $\bar{\beta} \in[0,1 / 2-\sqrt{6} \eta)$. In this case $A=0$, and so

$$
\begin{equation*}
\chi(E(-H))=D_{0, \bar{\beta}}^{0, \xi}(E)+\left(f^{\prime}(\bar{\beta})+\xi\right) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H(E)+f(\bar{\beta}) H^{3} \operatorname{ch}_{0}(E) \tag{19}
\end{equation*}
$$

We show that $f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H(E)+f(\bar{\beta}) H^{3} \operatorname{ch}_{0}(E) \geqslant 0$ by contradiction. Suppose the opposite for a contradiction. So we have

$$
\begin{aligned}
0 & >f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H^{(E)}+f(\bar{\beta}) H^{3} \operatorname{ch}_{0}(E) \\
& =f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{(1 / 2) H}(E)+\left(-\left(\beta-\frac{1}{2}\right) f^{\prime}(\bar{\beta})+f(\bar{\beta})\right) H^{3} \operatorname{ch}_{0}(E) \\
& =f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{(1 / 2) H}(E)-\frac{1}{3}\left(\beta-\frac{1}{2}\right)^{3} H^{3} \operatorname{ch}_{0}(E) \\
& \geqslant f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{(1 / 2) H}(E) .
\end{aligned}
$$

Since $f^{\prime}(\bar{\beta})>0, H^{2} \operatorname{ch}_{1}^{(1 / 2) H}(E)<0$; that is $\left(H^{2} \operatorname{ch}_{1}(E)\right) /\left(H^{3} \operatorname{ch}_{0}(E)\right)<1 / 2$. Also since $\bar{\beta} \in[0,1 / 2)$, we deduce that the corresponding point $\widetilde{\mathcal{V}}(E)$ of $E$ in $\left\{1, \frac{\mathrm{H}^{2} \mathrm{ch}_{1}}{\mathrm{H}^{3} \mathrm{ch}_{0}}, \frac{\mathrm{H} \mathrm{ch}_{2}}{\mathrm{H}^{3} \mathrm{ch}_{0}}\right\}$-plane is above the tangent line of $\widetilde{\Delta}_{0}$ at $\widetilde{v}\left(\mathcal{O}_{\mathrm{X}}(\mathrm{H})\right)$ as shown in the Figure at page 15 of [Li].

Since $f^{\prime}(\bar{\beta}) H^{2} \operatorname{ch}_{1}^{\bar{\beta}} H(E)+f(\bar{\beta}) H^{3} \operatorname{ch}_{0}(E)<0$ and $f^{\prime}(\bar{\beta})>0$, we have

$$
0 \leqslant \frac{H^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}(E)}{\mathrm{H}^{3} \operatorname{ch}_{0}(E)}<\frac{-f(\bar{\beta})}{\mathrm{f}^{\prime}(\bar{\beta})} \leqslant \sqrt{\frac{3}{2 \mathrm{~d}}}
$$

Here the last inequality follows from (ii) of Proposition 7.2. Therefore,

$$
\widetilde{\Delta}_{H}(E)=\frac{\bar{\Delta}_{H, \bar{\beta} H}(E)}{\left(\mathrm{H}^{3} \operatorname{ch}_{0}(E)\right)^{2}}=\left(\frac{\mathrm{H}^{2} \operatorname{ch}_{1}^{\bar{\beta}} \mathrm{H}(\mathrm{E})}{\mathrm{H}^{3} \operatorname{ch}_{0}(\mathrm{E})}\right)^{2}<\frac{3}{2 \mathrm{~d}}
$$

Hence $\widetilde{v}(E)$ is in $R_{3 /(2 d)}$ region. But this in not possible from Lemma 5.4. So from (19), we have $0 \geqslant \chi(E(-H)) \geqslant D_{0, \beta}^{0, \xi}(E)$ as required. This completes the proof.

Theorem 7.6. Let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ be a $\bar{\beta}_{0, \mathrm{H}, \mathrm{A}}$-stable object. Then $\mathrm{D}_{\mathcal{A}_{0}\left(\bar{\beta}_{0}(\mathrm{E})\right), \bar{\beta}_{0}(\mathrm{E})}^{0, \xi}(\mathrm{E}) \leqslant 0$.
Proof. By using Proposition 4.9, and since tilt stability is preserved under tensoring by a line bundle, we can assume

$$
\mathrm{ch}_{0}(\mathrm{E}) \geqslant 0 \text { and } \bar{\beta}_{0}(\mathrm{E}) \in[0,1) .
$$

Hence, from the definition of $A_{0}$

$$
\bar{\beta}_{0}(E)+A_{0}\left(\bar{\beta}_{0}(E)\right)<1, \text { and } \bar{\beta}_{0}(E)-A_{0}\left(\bar{\beta}_{0}(E)\right) \geqslant 0
$$

with equality when $\bar{\beta}_{0}(E)=0$. Also we have Proposition 7.5. Therefore, the proof is similar to that of Theorem 7.4.

## References

[BMS] A. Bayer, E. Macrì, and P. Stellari, The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds, preprint, arXiv:1410.1585
[BMT] A. Bayer, E. Macrì, and Y. Toda, Bridgeland stability conditions on 3-folds I: Bogomolov-Gieseker type inequalities, J. Algebraic Geom. 23 (2014), 117-163.
[BMSZ] M. Bernardara, E. Macrì, B. Schmidt, X. Zhao, Bridgeland Stability Conditions on Fano Threefolds, preprint arXiv:1607.08199
[Bri] T. Bridgeland, Stability conditions on triangulated categories, Ann. of Math 166 (2007), 317-345.
[Har] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.
[Huy1] D. Huybrechts, Derived and abelian equivalence of K3 surfaces, J. Algebraic Geom. 17 (2008), no. 2, 375-400.
[Huy2] , Introduction to stability conditions, Moduli spaces, 179-229, London Math. Soc. Lecture Note Ser., 411, Cambridge Univ. Press, Cambridge, 2014.
[HL] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, Second edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.
[IP] V.A. Iskovskikh and Yu G. Prokhorov, Algebraic geometry. V. Fano varieties. A translation of Algebraic geometry. 5 (Russian), Ross. Akad. Nauk, Vseross. Inst. Nauchn. i Tekhn. Inform., Moscow. Translation edited by A. N. Parshin and I. R. Shafarevich. Encyclopaedia of Mathematical Sciences, 47. Springer-Verlag, Berlin, 1999.
[Li] C. Li, Stability conditions on Fano threefolds of Picard number one, preprint arXiv:1510.04089v2.
$[\mathrm{LM}] \quad$ J. Lo and Y. More, Some examples of tilt-stable objects on threefolds, Comm. Algebra 44 (2016), no. 3, 1280-1301.
[Mac] E. Macrì, A generalized Bogomolov-Gieseker inequality for the three-dimensional projective space, Algebra Number Theory 8 (2014), 173-190.
[MP1] A. Maciocia and D. Piyaratne, Fourier-Mukai Transforms and Bridgeland Stability Conditions on Abelian Threefolds, Algebr. Geom. 2 (2015), no. 3, 270-297.
[MP2] , Fourier-Mukai Transforms and Bridgeland Stability Conditions on Abelian Threefolds II, Inter. J. Math. 27, 1 (2016), 1650007 ( 27 pages).
$[\mathrm{MM}] \quad$ S. Mori and S. Mukai, On Fano 3-folds with $\mathrm{B}_{2} \geqslant 0$, Algebraic varieties and analytic varieties (Tokyo, 1981), 101-129, Adv. Stud. Pure Math., 1, North-Holland, Amsterdam, 1983.
[OSS] C. Okonek, M. Schneider and H. Spindler, Vector bundles on complex projective spaces, Corrected reprint of the 1988 edition. With an appendix by S. I. Gelfand. Modern Birkhuser Classics. Birkhäuser/Springer Basel AG, Basel, 2011.
[Piy1] D. Piyaratne, Fourier-Mukai Transforms and Stability Conditions on Abelian Threefolds, PhD thesis, University of Edinburgh (2014), http://hdl.handle.net/1842/9635
[Piy2] , Fourier-Mukai Transforms and Stability Conditions on Abelian Varieties, To appear in the Proceedings of Kinosaki Symposium on Algebraic Geometry in 2015, preprint arXiv:1512.02034.
[PT] D. Piyaratne and Y. Toda, Moduli of Bridgeland semistable objects on 3-folds and Donaldson-Thomas invariants, preprint arXiv:1504.01177.
[Sch1] B. Schmidt, A generalized Bogomolov-Gieseker inequality for the smooth quadric threefold, Bull. Lond. Math. Soc. 46 (2014), 915-923.
, Counterexample to the Generalized Bogomolov-Gieseker Inequality for Threefolds, preprint arXiv:1602.05055.

Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo Institutes for Advanced Study, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan.

E-mail address: dulip.piyaratne@ipmu.jp


[^0]:    Date: July 29, 2016.
    2010 Mathematics Subject Classification. Primary 14F05; Secondary 14J30, 14J45, 14J60, 14K99, 18E10, 18E30, 18E40.
    Key words and phrases. Derived category, Bridgeland stability conditions, Bogomolov-Gieseker inequality, Fano 3-folds.

