REMARKS ON KAWAMATA'S EFFECTIVE NON-VANISHING CONJECTURE FOR MANIFOLDS WITH TRIVIAL FIRST CHERN CLASSES

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Abstract. Kawamata proposed a conjecture predicting that every nef and big line bundle on a smooth projective variety with trivial first Chern class has nontrivial global sections. We verify this conjecture for several cases, including (i) all hyperkähler varieties of dimension ≤ 6; (ii) all known hyperkähler varieties except for O'Grady's 10-dimensional example; (iii) general complete intersection Calabi–Yau varieties in certain Fano manifolds (e.g. toric ones). Moreover, we investigate the effectivity of Todd classes of hyperkähler varieties and Calabi–Yau varieties. We prove that the fourth Todd classes are "fakely effective" for all hyperkähler varieties and general complete intersection Calabi–Yau varieties in products of projective spaces.

1. Introduction

Throughout this paper, we work over the complex number field $\mathbb{C}$.

Kodaira vanishing theorem asserts that $H^i(X, K_X \otimes L) = 0$ for $i > 0$ and any ample line bundle L on a smooth projective variety X. It is natural to ask when $H^0(X, K_X \otimes L)$ does not vanish. For example, Fujita's freeness conjecture predicts that $|K_X \otimes L^{\otimes(\dim X+1)}|$ is base-point-free (hence has many global sections). Kawamata proposed the following conjecture in general settings.

Conjecture 1.1 (Effective Non-vanishing Conjecture, [19, Conjecture 2.1])**.** *Let X be a projective normal variety with at most log terminal singularities, L a Cartier divisor on X. Assume that L is nef and $L - K_X$ is nef and big. Then $H^0(X, L) \neq 0$.*

Recall that a line bundle L on X is said to be *nef* if $L \cdot C \geq 0$ for any curve $C \subset X$, moreover, it is said to be *big* if $L^{\dim X} > 0$.

We are interested in this conjecture for manifolds with trivial first Chern classes. In particular, Kawamata's Effective Non-vanishing Conjecture predicts the following conjecture:

Conjecture 1.2. *Let X be a smooth projective variety with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ and L a nef and big line bundle on X. Then $H^0(X, L) \neq 0$.*

Thanks to Yau's proof of Calabi conjecture [35], we have the so-called Beauville–Bogomolov decomposition ([1, 3]) for compact Kähler manifolds with trivial first Chern classes, which allows us to reduce Conjecture 1.2 to the cases of Calabi–Yau varieties and hyperkähler varieties.

Proposition 1.3 (=Proposition 2.2)**.** *Conjecture 1.2 is true if it holds for all Calabi–Yau varieties and hyperkähler varieties.*

In this paper, a *Calabi–Yau variety* is a compact Kähler manifold X of dimension $n \geq 3$ with trivial canonical bundle such that $h^0(\Omega_X^p) = 0$ for $0 < p < n$ (since $h^{0,2}(X) = 0$, X is automatically a smooth projective variety by Kodaira embedding and Chow lemma), and a *hyperkähler variety* is a simply connected smooth projective variety Y such that $H^0(Y, \Omega_Y^2)$ is spanned by a non-degenerate two form.

For hyperkähler varieties, the only known examples are (up to deformations): Hilbert schemes of points on K3 surfaces, generalized Kummer varieties (due to Beauville's construction [1]), and two examples in dimension 6 and 10 introduced by O'Grady [31, 32]. We can show

Theorem 1.4 (= Corollary 3.4 and Proposition 3.5)**.** *Conjecture 1.2 is true for the following cases:*

(1) hyperkähler varieties of dimension ≤ 6;
(2) hyperkähler varieties homeomorphic to Hilbert schemes of points on K3 surfaces;
(3) hyperkähler varieties homeomorphic to generalized Kummer varieties;

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(4) hyperkähler varieties homeomorphic to O'Grady's 6*-dimensional example.*

For Calabi–Yau varieties, we have lots of examples provided by smooth complete intersections in Fano manifolds. To state our result, we introduce a subclass of Fano manifolds, which we call *perfect* Fano manifolds, i.e. on which any nef line bundle has a nontrivial section (Definition 4.3). This subclass contains, for example, toric Fano manifolds, Fano manifolds of dimension $n \leq 4$, and their products (see Proposition 4.4). On the other hand, Kawamata's Effective Non-vanishing Conjecture predicts that every Fano manifold is perfect. Then we show that Conjecture 1.2 is true for general complete intersection Calabi–Yau varieties in perfect Fano manifolds.

Theorem 1.5 (=Theorem 4.5)**.** *Let Y be a perfect Fano manifold, $\{H_i\}_{i=1}^m$ a sequence of base-point-free ample line bundles on Y such that $\otimes_{i=1}^m H_i \cong K_Y^{-1}$. Let $X \subseteq Y$ be a general complete intersection of dimension $n \geq 3$ defined by common zero locus of sections of $\{H_i\}_{i=1}^m$. Then Conjecture 1.2 is true for X.*

Here "general" means that if we label the sections by $\{s_i\}_{i=1}^m$, we require $\{s_1 = s_2 = \cdots = s_i = 0\}$ are smooth for $i = 1, 2, ..., m$.

By Hirzebruch–Riemann–Roch formula, the existence of global sections is related to the effectivity of Todd classes. We define the following fake effectivity for cycles.

Definition 1.6. Let X be a smooth projective variety and γ an algebraic k-cycle. γ is said to be *fakely effective* if the intersection number $\gamma \cdot L^k \geq 0$ for any nef line bundle L.

We remark that to test fake effectivity, it suffices to check for only ample (or nef and big) line bundles L since nef line bundles can be viewed as "limits" of ample line bundles [24].

As explained above, we will consider the following question.

Question 1.7. *Are Todd classes of a Calabi–Yau variety or a hyperkähler variety fakely effective?*

It is well-known that the second Todd class of a smooth projective variety with $c_1 = 0$ is fakely effective (which is equivalent to pseudo-effectivity in this case) by the result of Miyaoka and Yau [26, 34]. We consider higher Todd classes and answer this question affirmatively in the following cases.

Theorem 1.8 (=Theorem 3.2, Proposition 3.5, and Theorem 4.6)**.**

(1) The fourth Todd classes of all hyperkähler varieties are fakely effective;
(2) All Todd classes of hyperkähler varieties homeomorphic to any known hyperkähler variety, except for O'Grady's 10-dimensional example remaining unknown, are fakely effective;
(3) The fourth Todd classes of general complete intersection Calabi–Yau in products of projective spaces are fakely effective.

We also prove a weaker version of Conjecture 1.2 which relates to a conjecture of Beltrametti and Sommese (see Höring's work [15]).

Theorem 1.9 (=Theorems 5.1, 5.2)**.** *Fix an integer $n \geq 2$. Let X be an n-dimensional smooth projective variety with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ and L a nef and big line bundle on X. Then there exists a positive integer $i \leq \lfloor \frac{n-1}{4} \rfloor + \lfloor \frac{n+2}{4} \rfloor$ such that $H^0(X, L^{\otimes i}) \neq 0$.*

In particular, Conjecture 1.2 is true in dimension ≤ 4.

Finally, we remark that one could also consider the Kähler version of Conjecture 1.2 and Question 1.7. However, they can be simply reduced to the projective case. As for Conjecture 1.2, the existence of a nef and big line bundle on a compact complex manifold forces the manifold to be Moishezon (e.g. [25, Theorem 2.2.15]), and hence projective since it is also Kähler. For Question 1.7, if we consider a non-projective hyperkähler manifold X, then $q_X(L) = 0$ for every nef line bundle L ([16, Theorem 3.11]) and hence the intersection numbers of nef line bundles with Todd classes are identically zero by Fujiki's result [7] (see Theorem 3.1).

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2. Reduction to Calabi–Yau and hyperkähler varieties

Recall the following Beauville–Bogomolov decomposition thanks to Yau's proof of Calabi conjecture [35].

Theorem 2.1 (Beauville–Bogomolov decomposition, [1, 3]). *Every smooth projective variety X with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ admits a finite cover isomorphic to a product of Abelian varieties, Calabi–Yau varieties, and hyperkähler varieities.*

Then it is easy to reduce Conjecture 1.2 to the cases of Calabi–Yau varieties and hyperkähler varieties.

Proposition 2.2. *Conjecture 1.2 is true if it holds for all Calabi–Yau varieties and hyperkähler varieties.*

Proof. Let X be a smooth projective variety with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ and L a nef and big line bundle on X. By Beauville–Bogomolov decomposition, there exists a finite cover $\pi : X' \to X$ such that

$$X' \cong A \times X_1 \times \cdots \times X_n \times Y_1 \times \cdots \times Y_m,$$

where A is an Abelian variety, X_i a Calabi–Yau variety, and Y_j a hyperkähler variety. After pull-back L to X',

$$\chi(X', \pi^*L) = \deg(\pi) \cdot \chi(X, L).$$

and π^*L is again nef and big on X'. Kawamata–Viehweg vanishing theorem ([18]) implies

$$h^0(X', \pi^*L) = \chi(X', \pi^*L),$$
$$h^0(X, L) = \chi(X, L).$$

Hence to prove $h^0(X, L) \neq 0$, it is equivalent to prove $h^0(X', \pi^*L) \neq 0$. On the other hand, since by definition $h^1(X_i, \mathcal{O}_{X_i}) = h^1(Y_j, \mathcal{O}_{Y_j}) = 0$ for all i, j, by [12, Ex III.12.6], there is a natural isomorphism

$$\operatorname{Pic}(X') \cong \operatorname{Pic}(A) \times \prod_{i=1}^{n} \operatorname{Pic}(X_i) \times \prod_{j=1}^{m} \operatorname{Pic}(Y_j),$$

i.e. a line bundle on X' is the box tensor of its restriction on each factors. Since the restriction of π^*L on each factor is obviously nef and big, to prove $h^0(X', \pi^*L) \neq 0$, it suffices to show for any nef and big line bundle on each factor, there exists a nontrivial global section, that is, to verify Conjecture 1.2 for Abelian varieties, Calabi–Yau varieties, and hyperkähler varieties. Note that Conjecture 1.2 is true for Abelian varieties by Kodaira vanishing and Hirzebruch–Riemann–Roch formula. □

3. Hyperkähler case

3.1. **Preliminaries.** Let X be a hyperkähler variety. Beauville [1] and Fujiki [7] proved that there exists a quadratic form $q_X : H^2(X, \mathbb{C}) \to \mathbb{C}$ and a constant $c_X \in \mathbb{Q}_+$ such that for all $\alpha \in H^2(X, \mathbb{C})$,

$$\int_X \alpha^{2n} = c_X \cdot q_X(\alpha)^n.$$

The above equation determines c_X and q_X uniquely if assuming:

(1) q_X is a primitive integral quadratic form on $H^2(X, \mathbb{Z})$;
(2) $q_X(\sigma + \overline{\sigma}) > 0$ for $0 \neq \sigma \in H^{2,0}(X)$.

Here q_X and c_X are called the *Beauville–Bogomolov–Fujiki form* and the *Fujiki constant* of X respectively. Note that condition (2) above is equivalent to the following condition (see [28, Propositions 8 and 11]):

(2') There exists an $\alpha \in H^2(X, \mathbb{R})$ with $q_X(\alpha) \neq 0$, and for all $\alpha \in H^2(X, \mathbb{R})$ with $q_X(\alpha) \neq 0$, we have that

$$\frac{\int_X \mathrm{p}_1(X)\alpha^{2n-2}}{q_X(\alpha)^{n-1}} < 0.$$

Here $\mathrm{p}_1(X)$ is the first Pontrjagin class. By the above definition, it is easy to see that q_X and c_X are actually topological invariants.

Recall a result by Fujiki [7] (see also [10, Corollary 23.17] for a generalization).

Theorem 3.1 ([7], [10, Corollary 23.17]). *Let X be a hyperkähler variety of dimension $2n$. Assume $\alpha \in H^{4j}(X, \mathbb{C})$ is of type $(2j, 2j)$ on all small deformation of X. Then there exists a constant $C(\alpha) \in \mathbb{C}$ depending on α such that*

$$\int_X \alpha \cdot \beta^{2n-2j} = C(\alpha) \cdot q_X(\beta)^{n-j},$$

for all $\beta \in H^2(X, \mathbb{C})$.

A direct application of this result (cf. [16, 1.11]) is that, for a line bundle L on X, Hirzebruch–Riemann–Roch formula gives

$$\chi(X, L) = \sum_{i=0}^{2n} \frac{1}{(2i)!} \int_X \mathrm{td}_{2n-2i}(X)(c_1(L))^{2i} = \sum_{i=0}^{2n} \frac{a_i}{(2i)!} q_X\big(c_1(L)\big)^i,$$

where

$$a_i = C(\mathrm{td}_{2n-2i}(X)).$$

Since rational Chern classes are determined by rational Pontrjagin classes (cf. [30, Proposition 1.13]), rational Chern classes (and hence Todd classes) are topological invariants of X by Novikov's theorem, hence a_i's in the above formula are constants depending only on the topology of X.

For a line bundle L on X, Nieper [29] defined the *characteristic value* of L,

$$\lambda(L) := \begin{cases} \frac{24n \int_X \mathrm{ch}(L)}{\int_X c_2(X)\mathrm{ch}(L)} & \text{if well-defined;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\lambda(L)$ is a positive (topological constant) multiple of $q_X(c_1(L))$ (cf. [29, Proposition 10]), more precisely,

$$\lambda(L) = \frac{12c_X}{(2n-1)C(c_2(X))} q_X(c_1(L)).$$

It is easy to see that if L is a nef line bundle, then $q_X(c_1(L)) \geq 0$ and $\lambda(L) \geq 0$; if L is a nef and big line bundle, then $q_X(c_1(L)) > 0$ and $\lambda(L) > 0$ (cf. [16, Corollary 6.4]).

3.2. **Fake effectivity of 4-th Todd classes of hyperkähler varieities.** In this subsection, we prove the following thereom.

Theorem 3.2. *Let X be a hyperkähler variety of dimension $2n$ with $n \geq 2$. Then $\mathrm{td}_4(X)$ is fakely effective, that is,*

$$\int_X \mathrm{td}_4(X) \cdot L^{2n-4} \geq 0$$

for any nef line bundle L on X. Moreover, the inequality is strict for nef and big line bundle L.

Firstly, we prove the following lemma.

Lemma 3.3 (cf. [23, Lemma 1], [11, Lemma 3]). *Let X be a hyperkähler variety of dimension $2n$ with $n \geq 2$. Then $c_2^2(X)$ is fakely effective, that is,*

$$\int_X c_2^2(X) \cdot L^{2n-4} \geq 0$$

for any nef line bundle L on X. Moreover, the inequality is strict for nef and big line bundle L.

Proof. By Theorem 3.1,

$$\int_X c_2^2(X) \cdot L^{2n-4} = C(c_2^2(X)) \cdot q_X(c_1(L))^{n-2}$$

for a line bundle L. Hence it is equivalent to prove that $C(c_2^2(X)) > 0$.

Fix $0 \neq \sigma \in H^{2,0}(X)$, by Theorem 3.1,

$$\binom{2n-4}{n-2} \int_X c_2^2(X) \cdot (\sigma\overline{\sigma})^{n-2} = C(c_2^2(X)) \cdot q_X(\sigma + \overline{\sigma})^{n-2}.$$

Hence it is equivalent to prove that $\int_X c_2^2(X) \cdot (\sigma\overline{\sigma})^{n-2} > 0$.

Take $Q \in \mathrm{Sym}^2 H^2(X)$ to be the dual of Beauville–Bogomolov–Fujiki form q_X. Taking the orthogonal decomposition of $c_2 \in H^4(X)$ induced by the projection to $\mathrm{Sym}^2 H^2(X)$, we may write $c_2 = \mu Q + p$, where $\mu > 0$ is a positive rational number (cf. [28, Proposition 12]) and $p \in H^4_{\mathrm{prim}}(X)$. Since p is a $(2, 2)$-form, by the Hodge–Riemann bilinear relations,

$$\int_X c_2^2(X) \cdot (\sigma\overline{\sigma})^{n-2} = \mu^2 \int_X Q^2 \cdot (\sigma\overline{\sigma})^{n-2} + 2\mu \int_X Qp \cdot (\sigma\overline{\sigma})^{n-2} + \int_X p^2 \cdot (\sigma\overline{\sigma})^{n-2}$$

$$
\begin{aligned}
&= \mu^2 \int_X Q^2 \cdot (\sigma\overline{\sigma})^{n-2} + \int_X p^2 \cdot (\sigma\overline{\sigma})^{n-2} \\
&\geq \mu^2 \int_X Q^2 \cdot (\sigma\overline{\sigma})^{n-2}.
\end{aligned}
$$

Hence it suffices to show that $\int_X Q^2 \cdot (\sigma\overline{\sigma})^{n-2} > 0$. Since q_X is a deformation invariant, so is Q^2. By Theorem 3.1, it suffices to show that $C(Q^2) > 0$.

Let $e_1, \dots, e_{b_2}$ be an orthonormal basis on $H^2(X, \mathbb{C})$ for which $Q = \sum_{i=1}^{b_2} e_i^2$. Then for distinct integers i, j, k, and formal parameters t, s,

$$\int_X (e_i + te_j + se_k)^{2n} = c_X \cdot q_X(e_i + te_j + se_k)^n = c_X \cdot (1 + t + s)^n.$$

Comparing the coefficients of t, s,

$$
\begin{aligned}
\int_X e_i^{2n} &= c_X, \\
\int_X e_i^{2n-2} e_j^2 &= \frac{c_X}{2n-1}, \\
\int_X e_i^{2n-4} e_j^4 &= \frac{3c_X}{(2n-1)(2n-3)}, \\
\int_X e_i^{2n-4} e_j^2 e_k^2 &= \frac{c_X}{(2n-1)(2n-3)}.
\end{aligned}
$$

Hence by Theorem 3.1 and $q_X(e_1) = 1$,

$$C(Q^2) = \int_X Q^2 e_1^{2n-4} = c_X + \frac{2(b_2-1)c_X}{2n-1} + \frac{3(b_2-1)c_X}{(2n-1)(2n-3)} + \frac{(b_2-1)(b_2-2)c_X}{(2n-1)(2n-3)} > 0.$$

We complete the proof. □

Proof of Theorem 3.2. By Nieper's formula (see [29, Remark after Definition 19] or [30, Corollary 3.7]), for any $\alpha \in H^2(X)$,

$$\int_X \sqrt{\mathrm{td}(X)} \exp(\alpha) = (1 + \lambda(\alpha))^n \int_X \sqrt{\mathrm{td}(X)}. \tag{3.1}$$

Here $\sqrt{\mathrm{td}(X)}$ is defined to be the formal power series in cohomology ring whose square is $\mathrm{td}(X)$. Note that

$$\sqrt{\mathrm{td}(X)} = 1 + \frac{1}{24} c_2(X) + \frac{1}{5760}(7c_2^2(X) - 4c_4(X)) + \cdots$$

and $\int_X \sqrt{\mathrm{td}(X)} > 0$ by a theorem of Hitchin and Sawon [14].

Fix a nef and big line bundle L on X (hence $\lambda(L) > 0$), take $\alpha = t \cdot c_1(L)$ where t is a formal parameter and compare the coefficients of t in (3.1), then

$$\int_X \sqrt{\mathrm{td}(X)} \cdot L^{2n-4} = (2n-4)! \binom{n}{n-2} \lambda(L)^{n-2} \int_X \sqrt{\mathrm{td}(X)} > 0.$$

Equivalently, this gives

$$\int_X (7c_2^2(X) - 4c_4(X)) \cdot L^{2n-4} > 0.$$

Combining with Lemma 3.3,

$$\int_X \mathrm{td}_4(X) \cdot L^{2n-4} = \frac{1}{2880} \int_X (7c_2^2(X) - 4c_4(X)) \cdot L^{2n-4} + \frac{1}{576} \int_X c_2^2(X) \cdot L^{2n-4} > 0.$$

We complete the proof. □

Corollary 3.4. *Let X be a 6-dimensional hyperkähler variety and L a nef and big line bundle on X. Then*

$$h^0(X, L) \geq 5.$$

Proof. By Kawamata–Viehweg vanishing theorem and Hirzubruch–Riemann–Roch formula,

$$h^0(X, L) = \chi(X, L) = \frac{1}{6!} \int_X c_1^6(L) + \frac{1}{4!} \int_X c_1^4(L) \cdot \mathrm{td}_2(X) + \frac{1}{2} \int_X c_1^2(L) \cdot \mathrm{td}_4(X) + \chi(X, \mathcal{O}_X).$$

As it is shown by Miyaoka and Yau [26, 34] that $\mathrm{td}_2(X) = \frac{1}{12} c_2(X)$ is fakely effective, along with Theorem 3.2 we get $h^0(X, L) > \chi(X, \mathcal{O}_X) = 4$. □

3.3. **Known hyperkähler varieties.** For hyperkähler varieties, the only known examples are (up to deformations) two families, Hilbert schemes of points on K3 surfaces and generalized Kummer varieties, due to Beauville's construction [1] and two examples introduced by O'Grady [31, 32]. We verify Conjecture 1.2 for these known examples except for O'Grady's 10-dimensional example.

Proposition 3.5. *Let L be a nef and big line bundle on a hyperkähler variety X of dimension $2n$.*

(1) If X is homeomorphic to a Hilbert scheme of points on K3 surfaces, then

$$h^0(X,L)\geq \frac{(n+2)(n+1)}{2}.$$

(2) If X is homeomorphic to a generalized Kummer variety with $n\geq 2$, or O'Grady's 6-dimensional example, then

$$h^0(X,L)\geq (n+1)^2.$$

Moreover, in the above cases, all Todd classes of X are fakely effective.

Proof. Let X be a hyperkähler manifold of dimension $2n$ and L a line bundle. By Theorem 3.1, we know that

$$\chi(X,L)=P_X(q_X(c_1(L)))=P'_X(\lambda(L)),$$

where P_X and P'_X are polynomials depending only on the topology of X. Note that all Todd classes of X are fakely effective if and only if all coefficients of P_X (or P'_X) are non-negative.

For a generalized Kummer variety K^nA of an Abelian surface A, and a line bundle H on K^nA, Britze–Nieper [4] showed that

$$\chi(K^nA,H)=(n+1)\binom{\frac{(n+1)\lambda(H)}{4}+n}{n}.$$

Thus

$$P'_{K^nA}(t)=(n+1)\binom{\frac{(n+1)t}{4}+n}{n}$$

is a polynomial with positive coefficients. Hence if X is homeomorphic to K^nA and L is a line bundle on X, then all Todd classes of X are fakely effective and

$$\chi(X,L)=P'_X(\lambda(L))=P'_{K^nA}(\lambda(L))=(n+1)\binom{\frac{(n+1)\lambda(L)}{4}+n}{n}.$$

When L is nef and big, $\lambda(L)\in\mathbb{Q}_{>0}$ and hence by Lemma 6.1 (assuming that $n\geq 2$), $\frac{(n+1)\lambda(L)}{4}$ is a positive integer and $h^0(X,L)=\chi(X,L)\geq (n+1)^2$.

For a Hilbert scheme of n-points $\mathrm{Hilb}^n(S)$ on a K3 surface S,

$$\mathrm{Pic}(\mathrm{Hilb}^n(S))\cong \mathrm{Pic}(S)\oplus\mathbb{Z}E$$

and any line bundle on $\mathrm{Hilb}^n(S)$ is of the form $H_n\otimes E^r$, where $r\in\mathbb{Z}$ and H_n is induced by a line bundle H on S (see [6, Section 5]). Ellingsrud–Göttsche–Lehn's formula ([6, Theorem 5.3]) gives

$$\chi(\mathrm{Hilb}^n(S),H_n\otimes E^r)=\binom{\chi(S,H)-(r^2-1)(n-1)}{n}.$$

With the Beauville–Bogomolov–Fujiki form q, we have

$$\big(H^2(\mathrm{Hilb}^n(S),\mathbb{Z}),q\big)\cong H^2(S,\mathbb{Z})_{(-,-)}\oplus_\perp\mathbb{Z}[E],$$

where $(-,-)$ is the cup product on S and $q([E])=-2(n-1)$ (see [16, Section 2.2]). Thus

$$q(c_1(H_n\otimes E^r))=(H)^2-2r^2(n-1),$$

and

$$\chi(\mathrm{Hilb}^n(S),H_n\otimes E^r)=\binom{\frac{1}{2}q(c_1(H_n\otimes E^r))+n+1}{n}.$$

Thus

$$P_{\mathrm{Hilb}^n(S)}(t)=\binom{\frac{1}{2}t+n+1}{n}$$

is a polynomial with positive coefficients. Hence if X is homeomorphic to $\mathrm{Hilb}^n(S)$ and L is a line bundle on X, then all Todd classes of X are fakely effective and

$$\chi(X,L)=P_X(q_X(c_1(L)))=P_{\mathrm{Hilb}^n(S)}(q_X(c_1(L)))=\binom{\frac{1}{2}q_X(c_1(L))+n+1}{n}.$$

When L is nef and big, $q_X(c_1(L)) \in \mathbb{Q}_{>0}$ and hence by Lemma 6.1, $\frac{1}{2}q_X(c_1(L))$ is a positive integer and $h^0(X,L) = \chi(X,L) \geq \binom{n+2}{n}$.

In general, for a hyperkähler variety X of dimension $2n$ and a line bundle L on X, Nieper [29] used Rozansky–Witten invariants to express those coefficients a_i in the expansion of $\chi(X,L)$ in terms of Chern numbers of X. More precisely, we have

$$\chi(X,L) = \int_X \exp\Big(-2\sum_{k=1}^{\infty}\frac{B_{2k}}{4k}\mathrm{ch}_{2k}(X)T_{2k}\Big(\sqrt{\frac{\lambda(L)}{4}+1}\Big)\Big), \tag{3.2}$$

where B_{2k} are the Bernoulli numbers with $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, and T_{2k} are even Chebyshev polynomials.

Now consider O'Grady's six dimensional hyperkähler manifold $\mathcal{M}_6$. By Mongardi–Rapagnetta–Saccà's computation ([27, Corollary 6.8]), we have

$$\int_{\mathcal{M}_6} c_2^3(\mathcal{M}_6) = 30720, \quad \int_{\mathcal{M}_6} c_2(\mathcal{M}_6)c_4(\mathcal{M}_6) = 7680, \quad \int_{\mathcal{M}_6} c_6(\mathcal{M}_6) = 1920.$$

After direct computations by (3.2), for a line bundle H on $\mathcal{M}_6$,

$$\chi(\mathcal{M}_6,H) = \frac{4}{27}\Big(T_6(\sqrt{\lambda(H)/4+1})+6T_2(\sqrt{\lambda(H)/4+1})\cdot T_4(\sqrt{\lambda(H)/4+1})+20T_2^3(\sqrt{\lambda(H)/4+1})\Big).$$

Plugging Chebyshev polynomials: $T_2(x) = 2x^2-1$, $T_4(x) = 8x^4-8x^2+1$, $T_6(x) = 32x^6-48x^4+18x^2-1$ into the formula, we get

$$\chi(\mathcal{M}_6,H) = 4\binom{\lambda(H)+3}{3}.$$

Thus

$$P'_{\mathcal{M}_6}(t) = 4\binom{t+3}{3}$$

is a polynomial with positive coefficients. Hence if X is homeomorphic to $\mathcal{M}_6$ and L is a line bundle on X, then all Todd classes of X are fakely effective and

$$\chi(X,L) = P'_X(\lambda(L)) = P'_{\mathcal{M}_6}(\lambda(L)) = 4\binom{\lambda(L)+3}{3}.$$

When L is nef and big, $\lambda(L) \in \mathbb{Q}_{>0}$ and hence by Lemma 6.1, $\lambda(L)$ is a positive integer and $h^0(X,L) = \chi(X,L) \geq 4\binom{4}{3} = 16$. $\square$

From the above computations, we may propose the following conjecture.

Conjecture 3.6. *Let X be a hyperkähler variety of dimension $2n$ and L a nef and big line bundle on X. Then*

(1) $h^0(X,L) \geq n+2$;
(2) more wildly, $\int_X \mathrm{td}_{2n-2i}(X)\cdot L^{2i} \geq 0$ for all $i \in \mathbb{Z}$.

4. Calabi–Yau case

4.1. **Complete intersections in Fano manifolds.** As for Calabi–Yau varieties, there is a huge amount of examples provided by complete intersections in Fano manifolds. In this subsection, we verify Conjecture 1.2 for these examples.

Let X be a projective variety, $N_1(X)$ be its set of numerical equivalent classes of 1-cycles in $\mathbb{R}$-coefficients. Set

$$NE(X) = \big\{\sum a_i[C_i] \in N_1(X) \mid C_i \subseteq X,\ 0 \leq a_i \in \mathbb{R}\big\}.$$

and $\overline{NE}(X)$ its closure in $N_1(X)$. Note that $\overline{NE}(X)$ is the dual of the cone of nef divisors on X.

As our testing examples are realized as hypersurfaces, or more generally, complete intersections in Fano manifolds, we recall the following comparison result for closed cone of curves.

Theorem 4.1 ([20], [2, Proposition 3.5])**.** *Let Y be a projective manifold of dimension $n \geq 4$, H be an ample line bundle on Y, and X be an effective smooth divisor in $|H|$. Assume $K_Y^{-1}\otimes H^{-1}$ is nef. Then $\overline{NE}(X) \cong \overline{NE}(Y)$ under the natural embedding $X \hookrightarrow Y$.*

In fact, by the Lefschetz hyperplane theorem (see e.g. [24, Example 3.1.25]), $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$ under the embedding $i : X \hookrightarrow Y$, and $i_* : \overline{NE}(X) \to \overline{NE}(Y)$ is an inclusion. The above theorem says that under certain condition, i_* is an isomorphism and nef line bundles on X and Y are identified under i^*. We then compare sections of those line bundles on X and Y.

Proposition 4.2. *Let Y be a projective manifold, H a line bundle such that $h^0(Y,H) \geq 2$. Suppose that the linear system $|H|$ contains an smooth element X. Assume that $K_Y^{-1} \otimes H^{-1}$ is nef. Then for any nef line bundle L on Y with $L \otimes (K_Y^{-1} \otimes H^{-1})$ big, we have*

$$h^0(Y,L) = h^0(X, L|_X) + h^0(Y, L \otimes H^{-1}).$$

Furthermore, if $h^0(Y,L) > 0$, then $h^0(X, L|_X) > 0$.

Proof. Twisting the exact sequence

$$0 \to \mathcal{O}_Y(-X) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

with L, and taking long exact sequence of the corresponding cohomology, we obtain

$$0 \to H^0(Y, L \otimes H^{-1}) \to H^0(Y,L) \to H^0(X, L|_X) \to H^1(Y, L \otimes H^{-1}).$$

Since $L \otimes (K_Y^{-1} \otimes H^{-1})$ is nef and big, $H^1(Y, L \otimes H^{-1}) = 0$ by Kawamata–Viehweg vanishing theorem [18]. Thus

$$h^0(X, L|_X) = h^0(Y,L) - h^0(Y, L \otimes H^{-1}).$$

Assume that $h^0(Y,L) > 0$. If $h^0(Y, L \otimes H^{-1}) = 0$, then

$$h^0(X, L|_X) = h^0(Y,L) > 0.$$

If $h^0(Y, L \otimes H^{-1}) > 0$, then by [21, Lemma 15.6.2],

$$h^0(Y,L) \geq h^0(Y,H) + h^0(Y, L \otimes H^{-1}) - 1 \geq h^0(Y, L \otimes H^{-1}) + 1,$$

and hence $h^0(X, L|_X) \geq 1$. □

Thus, to prove Conjecture 1.2 for complete intersections Calabi–Yau in Fano manifolds, we only need to prove that nef line bundles on those Fano manifolds have nontrivial global sections. Motivated by this, we define the following:

Definition 4.3. A Fano manifold Y is *perfect* if any nef line bundle on it has a nontrivial global section.

Kawamata's conjecture 1.1 predicts that any Fano manifold should be perfect. At the moment, we only show that there are lots of examples of perfect Fano manifolds.

Proposition 4.4. *The following Fano manifolds are perfect:*

(1) toric Fano manifolds;
(2) Fano manifolds of dimension $n \leq 4$;
(3) products of perfect Fano manifolds.

Proof. (1) This is obvious from the fact that any nef line bundle on a complete toric variety is base-point-free (see [5, Theorem 6.3.12]).

(2) We only deal with Fano 4-fold X here. By Kawamata–Viehweg vanishing theorem, $H^i(X,L) = 0$ for any nef line bundle L on X and $i > 0$. Then Hirzebruch–Riemann–Roch formula gives

$$\begin{aligned} h^0(X,L) = \chi(X,L) = {} & \frac{1}{24}\int_X c_1^4(L) + \frac{1}{12}\int_X c_1^3(L)c_1(X) + \frac{1}{24}\int_X c_1^2(L)c_1^2(X) \\ & + \frac{1}{24}\int_X c_1^2(L)c_2(X) + \frac{1}{24}\int_X c_1(L)c_1(X)c_2(X) + \chi(X, \mathcal{O}_X). \end{aligned}$$

Fano condition implies that $\chi(X, \mathcal{O}_X) = 1$. By the pseudo-effectiveness of second Chern classes for Fano manifolds (see [26, Theorem 6.1], [22, Theorem 2.4], and [33, Theorem 1.3]), those terms with $c_2(X)$ are non-negative. Hence $h^0(X,L) > 0$.

(3) For two Fano manifolds X and Y, as $H^1(X, \mathcal{O}_X) = 0$, we know $\mathrm{Pic}(X \times Y) \cong \mathrm{Pic}(X) \times \mathrm{Pic}(Y)$ (see e.g. [12, Ex. III.12.6]), i.e. a line bundle on $X \times Y$ is the box tensor of line bundles on X and Y. Furthermore, it is obvious that the box tensor is nef if and only if so is each factor, and the box tensor has a global section if and only if so does each factor. □

To sum up, we verify Conjecture 1.2 for general complete intersection Calabi–Yau in perfect Fano manifolds.

Theorem 4.5. *Let Y be a perfect Fano manifold, $\{H_i\}_{i=1}^m$ a sequence of base-point-free ample line bundles on Y such that $\otimes_{i=1}^m H_i \cong K_Y^{-1}$. Let $X \subseteq Y$ be a general complete intersection of dimension $n \geq 3$ defined by common zero locus of sections of $\{H_i\}_{i=1}^m$.*

Then Conjecture 1.2 is true for X.

Proof. There is a sequence of projective manifolds

$$X = X_m \subseteq X_{m-1} \subseteq \cdots \subseteq X_1 \subseteq Y$$

with $X_i = \cap_{j=1}^{i} s_j^{-1}(0)$, the common zero locus of sections of $\{H_j\}_{j=1}^{i}$. We inductively apply Theorem 4.1 and Proposition 4.2 to $(X_i, H_{i+1}|_{X_i})_{i=1}^{m}$, then any nef and big line bundle L on X comes from the restriction of a nef line bundle L_Y on Y and $h^0(X, L) > 0$ since $h^0(Y, L_Y) > 0$ by definition of perfect Fano manifolds. $\square$

4.2. Td_4 **of CICY.** In this subsection, we consider complete intersection Calabi–Yau varities (CICY, for short) in products of projective spaces. Fix positive integers $n_1, \dots, n_m$, a CICY X in $\mathcal{P}(\mathbf{n}) = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is provided by the *configuration matrix*

$$[\mathbf{n}|\mathbf{q}] = \left[\begin{array}{c|ccc} n_1 & q_1^1 & \cdots & q_K^1 \\ \vdots & \vdots & & \vdots \\ n_m & q_1^m & \cdots & q_K^m \end{array}\right]$$

which encodes the dimensions of the ambient projective spaces and the (multi)-degrees of the defining polynomials. Here $c_1(X) = 0$ implies that

$$\sum_{\alpha=1}^{K} q_\alpha^r = n_r + 1 \tag{4.1}$$

for all $1 \le r \le m$. We say that $\mathbf{q} > 0$ if $q_\alpha^r > 0$ for all α and r. Our goal is to prove the following:

Theorem 4.6. *Let $X = [\mathbf{n}|\mathbf{q}] \subset \mathcal{P}(\mathbf{n})$ be a general CICY with $\mathbf{q} > 0$. Then $\mathrm{td}_4(X)$ is fakely effective.*

From now, $X = [\mathbf{n}|\mathbf{q}] \subset \mathcal{P}(\mathbf{n})$ is a general CICY with $\mathbf{q} > 0$. For each r, denote H_r be the pull-back of hyperplane of $\mathbb{P}^{n_r}$ to $\mathcal{P}(\mathbf{n})$, and $J_r = H_r|_X$. It is easy to compute Chern classes of X in terms of J_r (cf. [13, B.2.1]), for example, we have

$$\begin{aligned} c_2(X) &= \sum_{r,s=1}^{m} c_2^{rs} J_r J_s \\ &= \sum_{r,s=1}^{m} \frac{1}{2}\left[-(n_r+1)\delta^{rs} + \sum_{\alpha=1}^{K} q_\alpha^r q_\alpha^s\right] J_r J_s, \end{aligned}$$

and

$$\begin{aligned} c_4(X) &= \sum_{r,s,t,u=1}^{m} c_4^{rstu} J_r J_s J_t J_u \\ &= \sum_{r,s,t,u=1}^{m} \frac{1}{4}\left[-(n_r+1)\delta^{rstu} + \sum_{\alpha=1}^{K} q_\alpha^r q_\alpha^s q_\alpha^t q_\alpha^u + 2c_2^{rs}c_2^{tu}\right] J_r J_s J_t J_u. \end{aligned}$$

Hence

$$\begin{aligned} 2880\mathrm{td}_4(X) &= 4(3c_2^2(X) - c_4(X)) \\ &= \sum_{r,s,t,u=1}^{m} (12c_2^{rs}c_2^{tu} - 4c_4^{rstu}) J_r J_s J_t J_u \\ &= \sum_{r,s,t,u=1}^{m} \left[10c_2^{rs}c_2^{tu} + (n_r+1)\delta^{rstu} - \sum_{\alpha=1}^{K} q_\alpha^r q_\alpha^s q_\alpha^t q_\alpha^u\right] J_r J_s J_t J_u \\ &= \sum_{r,s,t,u=1}^{m} \left[\frac{5}{2}\left(-(n_r+1)\delta^{rs} + \sum_{\alpha=1}^{K} q_\alpha^r q_\alpha^s\right)\left(-(n_t+1)\delta^{tu} + \sum_{\alpha=1}^{K} q_\alpha^t q_\alpha^u\right)\right. \\ &\qquad \left. + (n_r+1)\delta^{rstu} - \sum_{\alpha=1}^{K} q_\alpha^r q_\alpha^s q_\alpha^t q_\alpha^u\right] J_r J_s J_t J_u. \end{aligned}$$

Here we denote the coefficient of $J_r J_s J_t J_u$ in the above equation to be A^{rstu}. Rewrite the equation,

$$2880\mathrm{td}_4(X) = \sum_{r=1}^{m} B^{rrrr} J_r^4 + \sum_{\substack{1 \le r,s \le m \\ r \ne s}} B^{rrrs} J_r^3 J_s + \sum_{1 \le r < s \le m} B^{rrss} J_r^2 J_s^2 \tag{4.2}$$

$$+\sum_{\substack{1\le r,s,t\le m\\ r\neq s,r\neq t,s<t}} B^{rrst}J_r^2J_sJ_t+\sum_{1\le r<s<t<u\le m} B^{rstu}J_rJ_sJ_tJ_u.$$

Here by symmetry,

$$B^{rrrr}=A^{rrrr}=\frac{5}{2}\left(-(n_r+1)+\sum_{\alpha=1}^{K}(q_\alpha^r)^2\right)^2+(n_r+1)-\sum_{\alpha=1}^{K}(q_\alpha^r)^4;$$

$$B^{rrrs}=4A^{rrrs}=10\left(-(n_r+1)+\sum_{\alpha=1}^{K}(q_\alpha^r)^2\right)\left(\sum_{\alpha=1}^{K}q_\alpha^rq_\alpha^s\right)-4\sum_{\alpha=1}^{K}(q_\alpha^r)^3q_\alpha^s;$$

$$\begin{aligned}B^{rrss}&=2A^{rrss}+4A^{rsrs}\\&=10\left(\sum_{\alpha=1}^{K}q_\alpha^rq_\alpha^s\right)^2+5\left(-(n_r+1)+\sum_{\alpha=1}^{K}(q_\alpha^r)^2\right)\left(-(n_s+1)+\sum_{\alpha=1}^{K}(q_\alpha^s)^2\right)-6\sum_{\alpha=1}^{K}(q_\alpha^rq_\alpha^s)^2;\end{aligned}$$

$$\begin{aligned}B^{rrst}&=4A^{rrst}+8A^{rsrt}\\&=10\left(-(n_r+1)+\sum_{\alpha=1}^{K}(q_\alpha^r)^2\right)\left(\sum_{\alpha=1}^{K}q_\alpha^sq_\alpha^t\right)+20\left(\sum_{\alpha=1}^{K}q_\alpha^rq_\alpha^s\right)\left(\sum_{\alpha=1}^{K}q_\alpha^rq_\alpha^t\right)-12\sum_{\alpha=1}^{K}(q_\alpha^r)^2q_\alpha^sq_\alpha^t;\end{aligned}$$

$$B^{rstu}=8A^{rstu}+8A^{rtsu}+8A^{rust}.$$

We have the following inequalities for these coefficients.

Lemma 4.7. *In equation (4.2),*

(1) $B^{rrrr}\ge 0$ *unless* $q_{\alpha_0}^r=2$ *for some* α_0 *and* $q_\alpha^r=1$ *for all* $\alpha\neq\alpha_0$*;*
(2) $B^{rrrs}\ge 0$ *unless* $q_\alpha^r=1$ *for all* α*;*
(3) $B^{rrss}\ge 0$*;*
(4) $B^{rrst}\ge 0$*;*
(5) $B^{rstu}\ge 0$.

Proof. Keep in mind that q_α^r are all positive integers and we will often use (4.1).

(1) Divide the index set into three part:

$$\begin{aligned}S_1&=\{\alpha\mid q_\alpha^r=1\};\\S_2&=\{\alpha\mid q_\alpha^r=2\};\\S_3&=\{\alpha\mid q_\alpha^r\ge 3\}.\end{aligned}$$

The goal is to show that $B^{rrrr}\ge 0$ if and only if either $|S_3|>0$ or $|S_2|\neq 1$. The only if part is easy. We show the if part. We have

$$\begin{aligned}&B^{rrrr}\\&=\frac{5}{2}\left(-(n_r+1)+\sum_{\alpha=1}^{K}(q_\alpha^r)^2\right)^2+(n_r+1)-\sum_{\alpha=1}^{K}(q_\alpha^r)^4\\&=\frac{5}{2}\left(\sum_{\alpha\in S_2\cup S_3}\left((q_\alpha^r)^2-q_\alpha^r\right)\right)^2+\sum_{\alpha\in S_2\cup S_3}\left(q_\alpha^r-(q_\alpha^r)^4\right)\\&\ge\frac{5}{2}\sum_{\alpha\in S_3}\left((q_\alpha^r)^2-q_\alpha^r\right)^2+\frac{5}{2}\left(\sum_{\alpha\in S_2}\left((q_\alpha^r)^2-q_\alpha^r\right)\right)\left(\sum_{\alpha\in S_2\cup S_3}\left((q_\alpha^r)^2-q_\alpha^r\right)\right)+\sum_{\alpha\in S_2\cup S_3}\left(q_\alpha^r-(q_\alpha^r)^4\right)\\&\ge\frac{5}{2}\left(\sum_{\alpha\in S_2}\left((q_\alpha^r)^2-q_\alpha^r\right)\right)\left(\sum_{\alpha\in S_2\cup S_3}\left((q_\alpha^r)^2-q_\alpha^r\right)\right)+\sum_{\alpha\in S_2}\left(q_\alpha^r-(q_\alpha^r)^4\right)\\&=5|S_2|\left(2|S_2|+\sum_{\alpha\in S_3}\left((q_\alpha^r)^2-q_\alpha^r\right)\right)-14|S_2|.\end{aligned}$$

Here the first inequality is just easy computation and for the second inequality, we use the fact that if $q\ge 3$, then

$$\frac{5}{2}(q^2-q)^2+q-q^4=\frac{5}{2}q(q-1)(3q^2-7q-2)>0.$$

If $|S_3|>0$, then $\sum_{\alpha\in S_3}((q_\alpha^r)^2-q_\alpha^r)\ge 6$ and hence the above inequality gives

$$B^{rrrr}\ge 30|S_2|-14|S_2|\ge 0.$$

If $|S_2| \neq 1$, then the above inequality gives

$$B^{rrrr} \geq 10|S_2|^2 - 14|S_2| \geq 0.$$

(2) Divide the index set into two part:

$$\begin{aligned} S_1 &= \{\alpha \mid q^r_\alpha = 1\}; \\ S'_2 &= \{\alpha \mid q^r_\alpha \geq 2\}. \end{aligned}$$

The goal is to show that $B^{rrrs} \geq 0$ if and only if $|S'_2| > 0$. The only if part is easy. We show the if part. Assume that $|S'_2| > 0$, then

$$\begin{aligned} B^{rrrs} &= 10\left(-(n_r+1) + \sum_{\alpha=1}^K (q^r_\alpha)^2\right)\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right) - 4\sum_{\alpha=1}^K (q^r_\alpha)^3 q^s_\alpha \\ &= 10\left(\sum_{\alpha=1}^K \left((q^r_\alpha)^2 - q^r_\alpha\right)\right)\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right) - 4\sum_{\alpha=1}^K (q^r_\alpha)^3 q^s_\alpha \\ &= 10\left(\sum_{\alpha\in S'_2} \left((q^r_\alpha)^2 - q^r_\alpha\right)\right)\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right) - 4\sum_{\alpha=1}^K (q^r_\alpha)^3 q^s_\alpha \\ &\geq 5\left(\sum_{\alpha\in S'_2} (q^r_\alpha)^2\right)\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right) - 4\sum_{\alpha\in S_1\cup S'_2} (q^r_\alpha)^3 q^s_\alpha \\ &\geq \left(\sum_{\alpha\in S'_2} (q^r_\alpha)^2\right)\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right) - 4\sum_{\alpha\in S_1} (q^r_\alpha)^3 q^s_\alpha \\ &\geq 4\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right) - 4\sum_{\alpha\in S_1} q^s_\alpha \geq 0. \end{aligned}$$

Here for the first inequality we use the fact that $2(q^2 - q) \geq q^2$ for $q \geq 2$, the second is just easy computation, and for the last inequality we use the fact that $\sum_{\alpha\in S'_2} (q^r_\alpha)^2 \geq 4$ since $|S'_2| > 0$.

(3) Recall that

$$-(n_r+1) + \sum_{\alpha=1}^K (q^r_\alpha)^2 = \sum_{\alpha=1}^K \left((q^r_\alpha)^2 - q^r_\alpha\right) \geq 0.$$

It is easy to see that

$$\begin{aligned} B^{rrss} &\geq 10\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right)^2 - 6\sum_{\alpha=1}^K (q^r_\alpha q^s_\alpha)^2 \\ &\geq 4\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right)^2 \geq 0. \end{aligned}$$

(4) Similarly, it is easy to see that

$$\begin{aligned} B^{rrst} &\geq 20\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right)\left(\sum_{\alpha=1}^K q^r_\alpha q^t_\alpha\right) - 12\sum_{\alpha=1}^K (q^r_\alpha)^2 q^s_\alpha q^t_\alpha \\ &\geq 8\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right)\left(\sum_{\alpha=1}^K q^r_\alpha q^t_\alpha\right) \geq 0. \end{aligned}$$

(5) For $r < s < t < u$, it is easy to see that

$$\begin{aligned} A^{rstu} &= \frac{5}{2}\left(\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha\right)\left(\sum_{\alpha=1}^K q^t_\alpha q^u_\alpha\right) - \sum_{\alpha=1}^K q^r_\alpha q^s_\alpha q^t_\alpha q^u_\alpha \\ &\geq \frac{5}{2}\sum_{\alpha=1}^K q^r_\alpha q^s_\alpha q^t_\alpha q^u_\alpha - \sum_{\alpha=1}^K q^r_\alpha q^s_\alpha q^t_\alpha q^u_\alpha \\ &\geq 0. \end{aligned}$$

Similarly, $A^{rtsu} \geq 0$ and $A^{rust} \geq 0$. Hence $B^{rstu} \geq 0$. □

Note that, since J_r's are nef divisors by definition, it is easy to see that if all the coefficients in (4.2) are non-negative, then $\mathrm{td}_4(X)$ is fakely effective. We need to deal with the case when the coefficients are not non-negative in the following two lemmas. Also note that since we consider X to be a general CICY in $\mathcal{P}(\mathbf{n})$, the nef cones of X and $\mathcal{P}(\mathbf{n})$ concides by Theorem 4.1 (see proof of Proposition 4.5). Hence to check the fake effectivity of a cycle on X, we can view it as a cycle on $\mathcal{P}(\mathbf{n})$ since fake effectivity is tested by nef divisors. We will always use this observation.

For two cycles C and C', we will write $C \succeq C'$ if $C - C'$ is fakely effective. We denote the index set $\mathcal{R} := \{r \mid q^r_\alpha = 1 \text{ for all } \alpha\}$.

Lemma 4.8. *If $r \notin \mathcal{R}$,*

$$B^{rrrr}J_r^4 + \sum_{\substack{1\le s\le m\\ s\ne r}} B^{rrrs}J_r^3J_s \succeq 0.$$

Proof. By Lemma 4.7, $B^{rrrs} \ge 0$. Hence if $B^{rrrr} \ge 0$, there is nothing to prove. We may assume $B^{rrrr} < 0$. By Lemma 4.7, after reordering the index, we have $q^r_1 = 2$ and $q^r_\alpha = 1$ for all $\alpha \ge 2$. In this case $B^{rrrr} = -4$ and

$$B^{rrrs} = 8q_1^s + 16\sum_{\alpha=2}^K q^s_\alpha.$$

Also we have $n_r + 1 = \sum_{\alpha=1}^K q^r_\alpha = K + 1$. Viewing this cycle as a cycle on $\mathcal{P}(\mathbf{n})$, we have

$$\begin{aligned}
&B^{rrrr}J_r^4 + \sum_{\substack{1\le s\le m\\ s\ne r}} B^{rrrs}J_r^3J_s\\
=\;& -4J_r^4 + \sum_{\substack{1\le s\le m\\ s\ne r}}\left(8q_1^s + 16\sum_{\alpha=2}^K q^s_\alpha\right)J_r^3J_s\\
=\;& \left(-4H_r^4 + \sum_{\substack{1\le s\le m\\ s\ne r}}\left(8q_1^s + 16\sum_{\alpha=2}^K q^s_\alpha\right)H_r^3H_s\right)\cdot\prod_{\alpha=1}^K\left(\sum_{s=1}^m q^s_\alpha H_s\right)\\
=\;& 4\left(-H_r^4 + \sum_{\substack{1\le s\le m\\ s\ne r}}\left(2q_1^s + 4\sum_{\alpha=2}^K q^s_\alpha\right)H_r^3H_s\right)\cdot\left(2H_r + \sum_{s\ne r} q_1^sH_s\right)\cdot\prod_{\alpha=2}^K\left(H_r + \sum_{s\ne r} q^s_\alpha H_s\right)\\
=\;& 8H_r^3\left(-H_r + \sum_{\substack{1\le s\le m\\ s\ne r}}\left(2q_1^s + 4\sum_{\alpha=2}^K q^s_\alpha\right)H_s\right)\cdot\left(H_r + \sum_{s\ne r}\frac{q_1^s}{2}H_s\right)\cdot\prod_{\alpha=2}^K\left(H_r + \sum_{s\ne r} q^s_\alpha H_s\right)\\
\succeq\;& 8H_r^3\left(-H_r + \sum_{\substack{1\le s\le m\\ s\ne r}}\left(\frac{1}{2}q_1^s + \sum_{\alpha=2}^K q^s_\alpha\right)H_s\right)\cdot\left(H_r + \sum_{s\ne r}\frac{q_1^s}{2}H_s\right)\cdot\prod_{\alpha=2}^K\left(H_r + \sum_{s\ne r} q^s_\alpha H_s\right).
\end{aligned}$$

Note that by Lemma 6.2,

$$\left(-H_r + \sum_{\substack{1\le s\le m\\ s\ne r}}\left(\frac{1}{2}q_1^s + \sum_{\alpha=2}^K q^s_\alpha\right)H_s\right)\cdot\left(H_r + \sum_{s\ne r}\frac{q_1^s}{2}H_s\right)\cdot\prod_{\alpha=2}^K\left(H_r + \sum_{s\ne r} q^s_\alpha H_s\right) + H_r^{K+1}$$

is a polynomial in terms of $H_1, \dots, H_m$ with non-negative coefficients and note that

$$H_r^{K+1} = 0$$

since $K = n_r$ and H_r is the pullback of hyperplane on $\mathbb{P}^{n_r}$. Hence we have written

$$B^{rrrr}J_r^4 + \sum_{\substack{1\le s\le m\\ s\ne r}} B^{rrrs}J_r^3J_s$$

as a polynomial in terms of $H_1, \dots, H_m$ with non-negative coefficients, which is clearly fakely effective. $\square$

Lemma 4.9. *If $r \in \mathcal{R}$,*

$$\sum_{\substack{1\le s\le m\\ s\neq r}} B^{rrrs} J_r^3 J_s + \sum_{\substack{s\notin\mathcal{R}\\ s\neq r}} B^{rrss} J_r^2 J_s^2 + \sum_{\substack{s\in\mathcal{R}\\ s\neq r}} \frac{1}{2} B^{rrss} J_r^2 J_s^2 + \sum_{\substack{1\le s<t\le m\\ s\neq r, t\neq r}} B^{rrst} J_r^2 J_s J_t \succeq 0.$$

Here for convenience, we set $B^{rrss} = B^{ssrr}$ if $r > s$.

Proof. In this case, $n_r + 1 = \sum_{\alpha=1}^K q_\alpha^r = K$. Hence $J_r^K = (H_r|_X)^{n_r+1} = 0$, since H_r is the pullback of hyperplane on $\mathbb{P}^{n_r}$. We may assume $K \geq 3$ otherwise there is nothing to prove. We have

$$\begin{aligned}
B^{rrrs} &= -4\sum_{\alpha=1}^K q_\alpha^s;\\
B^{rrss} &= 10\left(\sum_{\alpha=1}^K q_\alpha^s\right)^2 - 6\sum_{\alpha=1}^K (q_\alpha^s)^2 \geq 4\left(\sum_{\alpha=1}^K q_\alpha^s\right)^2;\\
B^{rrst} &= 20\left(\sum_{\alpha=1}^K q_\alpha^s\right)\left(\sum_{\alpha=1}^K q_\alpha^t\right) - 12\sum_{\alpha=1}^K q_\alpha^s q_\alpha^t \geq 8\left(\sum_{\alpha=1}^K q_\alpha^s\right)\left(\sum_{\alpha=1}^K q_\alpha^t\right).
\end{aligned}$$

Moreover, since $K \geq 3$, if $s \in \mathcal{R}$ and $s \neq r$, then

$$B^{rrss} = 10\left(\sum_{\alpha=1}^K q_\alpha^s\right)^2 - 6\sum_{\alpha=1}^K (q_\alpha^s)^2 = 10K^2 - 6K \geq 8K^2 \geq 8\left(\sum_{\alpha=1}^K q_\alpha^s\right)^2.$$

Hence we have

$$\begin{aligned}
&\sum_{\substack{1\le s\le m\\ s\neq r}} B^{rrrs} J_r^3 J_s + \sum_{\substack{s\notin\mathcal{R}\\ s\neq r}} B^{rrss} J_r^2 J_s^2 + \sum_{\substack{s\in\mathcal{R}\\ s\neq r}} \frac{1}{2} B^{rrss} J_r^2 J_s^2 + \sum_{\substack{1\le s<t\le m\\ s\neq r, t\neq r}} B^{rrst} J_r^2 J_s J_t\\
\succeq & -\sum_{\substack{1\le s\le m\\ s\neq r}} 4\sum_{\alpha=1}^K q_\alpha^s J_r^3 J_s + \sum_{s\neq r} 4\left(\sum_{\alpha=1}^K q_\alpha^s\right)^2 J_r^2 J_s^2 + \sum_{\substack{1\le s<t\le m\\ s\neq r, t\neq r}} 8\left(\sum_{\alpha=1}^K q_\alpha^s\right)\left(\sum_{\alpha=1}^K q_\alpha^t\right) J_r^2 J_s J_t\\
= & -4\sum_{\substack{1\le s\le m\\ s\neq r}} \sum_{\alpha=1}^K q_\alpha^s J_r^3 J_s + 4\left(\sum_{\substack{1\le s\le m\\ s\neq r}} \sum_{\alpha=1}^K q_\alpha^s J_r J_s\right)^2\\
= & \, 4J_r\left(\sum_{\substack{1\le s\le m\\ s\neq r}} \sum_{\alpha=1}^K q_\alpha^s J_r J_s\right)\left(-J_r + \sum_{\substack{1\le s\le m\\ s\neq r}} \sum_{\alpha=1}^K q_\alpha^s J_s\right).
\end{aligned}$$

Viewing as a cycle on $\mathcal{P}(\mathbf{n})$ and use the fact that $H_r^{K+1} = 0$,

$$\begin{aligned}
& -J_r + \sum_{\substack{1\le s\le m\\ s\neq r}} \sum_{\alpha=1}^K q_\alpha^s J_s\\
= & \left(-H_r + \sum_{\substack{1\le s\le m\\ s\neq r}} \sum_{\alpha=1}^K q_\alpha^s H_s\right)\cdot \prod_{\alpha=1}^K \left(\sum_{s=1}^m q_\alpha^s H_s\right)\\
= & \left(-H_r + \sum_{\substack{1\le s\le m\\ s\neq r}} \sum_{\alpha=1}^K q_\alpha^s H_s\right)\cdot \prod_{\alpha=1}^K \left(H_r + \sum_{s\neq r} q_\alpha^s H_s\right)\\
= & \left(-H_r + \sum_{\substack{1\le s\le m\\ s\neq r}} \sum_{\alpha=1}^K q_\alpha^s H_s\right)\cdot \prod_{\alpha=1}^K \left(H_r + \sum_{s\neq r} q_\alpha^s H_s\right) + H_r^{K+1}
\end{aligned}$$

can be expressed as a polynomial in terms of $H_1,\dots,H_m$ with non-negative coefficients by Lemma 6.2. Hence we may express

$$\sum_{\substack{1\le s\le m\\ s\ne r}} B^{rrrs}J_r^3J_s+\sum_{\substack{s\notin\mathcal{R}\\ s\ne r}} B^{rrss}J_r^2J_s^2+\sum_{\substack{s\in\mathcal{R}\\ s\ne r}}\frac{1}{2}B^{rrss}J_r^2J_s^2+\sum_{\substack{1\le s<t\le m\\ s\ne r,t\ne r}} B^{rrst}J_r^2J_sJ_t$$

as a polynomial in terms of $H_1,\dots,H_m$ with non-negative coefficients, which is clearly fakely effective. □

Proof of Theorem 4.6. By (4.2), Lemmas 4.7, 4.8, and 4.9,

$$\begin{aligned}
&2880\mathrm{td}_4(X)\\
&=\sum_{r=1}^m B^{rrrr}J_r^4+\sum_{\substack{1\le r,s\le m\\ r\ne s}} B^{rrrs}J_r^3J_s+\sum_{1\le r<s\le m} B^{rrss}J_r^2J_s^2\\
&\quad+\sum_{\substack{1\le r,s,t\le m\\ r\ne s,r\ne t,s<t}} B^{rrst}J_r^2J_sJ_t+\sum_{1\le r<s<t<u\le m} B^{rstu}J_rJ_sJ_tJ_u\\
&\succeq\sum_{r\notin\mathcal{R}} B^{rrrr}J_r^4+\sum_{\substack{1\le r,s\le m\\ r\ne s}} B^{rrrs}J_r^3J_s+\sum_{1\le r<s\le m} B^{rrss}J_r^2J_s^2+\sum_{\substack{r\in\mathcal{R}\\ 1\le s,t\le m\\ r\ne s,r\ne t,s<t}} B^{rrst}J_r^2J_sJ_t\\
&=\sum_{r\notin\mathcal{R}} B^{rrrr}J_r^4+\sum_{r\notin\mathcal{R}}\sum_{\substack{1\le s\le m\\ s\ne r}} B^{rrrs}J_r^3J_s\\
&\quad+\sum_{r\in\mathcal{R}}\sum_{\substack{1\le s\le m\\ s\ne r}} B^{rrrs}J_r^3J_s+\sum_{1\le r<s\le m} B^{rrss}J_r^2J_s^2+\sum_{\substack{r\in\mathcal{R}\\ 1\le s,t\le m\\ r\ne s,r\ne t,s<t}} B^{rrst}J_r^2J_sJ_t\\
&\succeq\sum_{r\notin\mathcal{R}}\left[B^{rrrr}J_r^4+\sum_{s\ne r} B^{rrrs}J_r^3J_s\right]\\
&\quad+\sum_{r\in\mathcal{R}}\left[\sum_{s\ne r} B^{rrrs}J_r^3J_s+\sum_{\substack{s\notin\mathcal{R}\\ s\ne r}} B^{rrss}J_r^2J_s^2+\sum_{\substack{s\in\mathcal{R}\\ s\ne r}}\frac{1}{2}B^{rrss}J_r^2J_s^2+\sum_{\substack{1\le s<t\le m\\ s\ne r,t\ne r}} B^{rrst}J_r^2J_sJ_t\right]\\
&\succeq 0.
\end{aligned}$$

Here for the first and second inequalities we use Lemma 4.7, and for the last inequality we use Lemmas 4.8 and 4.9. □

5. Some effective results

In this section, using Hirzebruch–Riemann–Roch formula and Miyaoka–Yau inequality [26, 34], we prove a weaker version of Conjecture 1.2 in all dimensions (which is related to a conjecture of Beltrametti and Sommese, see for instance Höring's work [15]).

We first consider odd dimensions.

Theorem 5.1. *Let X be a smooth projective variety of dimension $2k+1$ ($k\ge 1$) with $c_1(X)=0$ in $H^2(X,\mathbb{R})$ and L a nef and big line bundle on X. Then there exists $i\in\{1,2,\dots,k\}$ such that $H^0(X,L^{\otimes i})\ne 0$.*

Proof. By contradiction, we assume $h^0(X,L^{\otimes i})=0$ for $i=1,2,\dots,k$, then $\chi(X,L^{\otimes i})=0$ by Kawamata–Viehweg vanishing theorem. Hirzebruch–Riemann–Roch formula gives

$$f(t)\triangleq\chi(X,L^{\otimes t})=\int_X\frac{L^{2k+1}}{(2k+1)!}t^{2k+1}+\int_X\frac{L^{2k-1}\cdot\mathrm{td}_2(X)}{(2k-1)!}t^{2k-1}+\cdots+\int_X(L\cdot\mathrm{td}_{2k}(X))t.$$

as $\mathrm{td}_{\mathrm{odd}}(X)=0$. Then $f(-t)=-f(t)$ and degree $(2k+1)$-polynomial $f(t)$ has roots $\{0,\pm1,\pm2,\dots,\pm k\}$. Then we can write

$$f(t)=\alpha t(t^2-1)(t^2-2^2)\cdots(t^2-k^2),$$

where $\alpha=\int_X\frac{L^{2k+1}}{(2k+1)!}>0$. The coefficient of t^{2k-1} is $-\alpha\cdot(\sum_{i=1}^k i^2)=\int_X\frac{L^{2k-1}\cdot c_2(X)}{12(2k-1)!}$, where we get a contradiction as the RHS is non-negative by the Miyaoka–Yau inequality [26, 34]. □

Then we consider even dimensions.

Theorem 5.2. *Let X be a smooth projective variety of dimension $4k+2$ or $4k+4$ $(k \geq 0)$ with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ and L a nef and big line bundle on X. Then there exists $i \in \{1, 2, \ldots, 2k+1\}$ such that $H^0(X, L^i) \neq 0$.*

Proof. If $\dim X = 4k+2$, assume $h^0(X, L^{\otimes i}) = 0$ for $i = 1, 2, \ldots, 2k+1$, then $\chi(X, L^{\otimes i}) = 0$. Hirzebruch–Riemann–Roch formula formula gives

$$f(t) \triangleq \chi(X, L^{\otimes t}) = \int_X \frac{L^{4k+2}}{(4k+2)!} t^{4k+2} + \int_X \frac{L^{4k} \cdot \mathrm{td}_2(X)}{(4k)!} t^{4k} + \cdots + \chi(X, \mathcal{O}_X).$$

as $\mathrm{td}_{\mathrm{odd}}(X) = 0$. Then $f(-t) = f(t)$ and degree $(4k+2)$-polynomial $f(t)$ has roots $\{\pm 1, \pm 2, \ldots, \pm(2k+1)\}$. Then we can write

$$f(t) = \alpha(t^2 - 1)(t^2 - 2^2) \cdots (t^2 - (2k+1)^2),$$

where $\alpha = \int_X \frac{L^{4k+2}}{(4k+2)!} > 0$. The coefficient of t^{4k} is $-\alpha \cdot (\sum_{i=1}^{2k+1} i^2) = \int_X \frac{L^{4k} \cdot c_2(X)}{12(4k)!}$, where we get a contradiction as the RHS is non-negative by the Miyaoka–Yau inequality [26, 34].

If $\dim X = 4k+4$, we similarly assume $f(t) = \chi(X, L^{\otimes t})$ has roots $\{\pm 1, \pm 2, \ldots, \pm(2k+1)\}$, and then

$$f(t) = \alpha(t^2 - 1)(t^2 - 2^2) \cdots (t^2 - (2k+1)^2)(t^2 - \beta)$$

for some $\beta \in \mathbb{C}$ and $\alpha = \int_X \frac{L^{4k+4}}{(4k+4)!} > 0$. The coefficient of t^{4k+2} is $-\alpha \cdot (\beta + \sum_{i=1}^{2k+1} i^2) = \int_X \frac{L^{4k+2} \cdot c_2(X)}{12(4k+2)!}$, and the constant term is $\alpha \cdot \beta \cdot ((2k+1)!)^2 = \chi(X, \mathcal{O}_X)$. Miyaoka–Yau inequality gives $\beta < 0$, which contradicts to $\chi(X, \mathcal{O}_X) \geq 0$ by Theorem 2.1. □

Remark 5.3.
1. As a corollary, Conjecture 1.2 holds true in dimension $n \leq 4$.
2. For a hyperkähler variety X of dimension $2n$ $(n \geq 2)$ and L a nef and big line bundle on X, we can enhance the above result by using the effectiveness of fourth Todd class (Theorem 3.2), and show that there exists a positive integer $i \leq \lfloor \frac{n-2}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ such that $H^0(X, L^{\otimes i}) \neq 0$. We leave the detail to the readers.

6. Appendix

In the appendix, we prove some basic lemmas.

Lemma 6.1. *Let $n \in \mathbb{Z}_{>0}$ be a positive integer and $\lambda \in \mathbb{Q}$ be a rational number.*
(1) If $n \geq 2$ and $(n+1) \cdot \binom{\lambda+n}{n}$ is an integer, then $\lambda \in \mathbb{Z}$.
(2) If $\binom{\lambda+n+1}{n}$ is an integer, then $\lambda \in \mathbb{Z}$.

Proof. (1) Write $\lambda = p/q$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ with $\mathrm{lcm}(p, q) = 1$. By contrary, we may assume $q > 1$. Then $(n+1) \cdot \binom{\lambda+n}{n} \in \mathbb{Z}$ implies that

$$n! q^n \mid (n+1)(p+nq) \cdot (p+q). \tag{6.1}$$

We claim that either $(n+1)$ is prime or $(n+1) \mid 2 \cdot n!$. If $n \leq 4$, it is obvious. Assume that $n \geq 5$. If $(n+1)$ is not prime, we have a factorization $(n+1) = a \cdot b$, for intergers $2 \leq a, b \leq n$. If $a \neq b$, they both appear in $n!$ as factors, and hence $(n+1) \mid n!$. If $a = b$, then $a = \sqrt{n+1} \leq \frac{n}{2}$, so $2a \leq n$. Hence a and $2a$ appear in $n!$ as factors, and hence $a^2 \mid n!$.

For the case when $(n+1)$ is prime, (6.1) implies that

$$q^n \mid (n+1)(p+nq) \cdots (p+q),$$

which implies that $q \mid (n+1)p^n$ and $q^2 \mid (n+1)(p^n + \frac{n(n+1)}{2} p^{n-1} q)$. As $(p, q) = 1$ and $(n+1)$ is prime, we conclude from the first dividing relation that $q = n+1$. Combining with the second relation, we get $q^2 \mid qp^n$, which contradicts with $(p, q) = 1$.

For the case when $(n+1) \mid 2 \cdot n!$, (6.1) implies that

$$q^n \mid 2(p+nq) \cdots (p+q).$$

which implies that $q \mid 2p^n$. As $(p, q) = 1$, we conclude that $q = 2$ and p is odd. Hence $(p+nq) \cdots (p+q)$ is odd and (6.1) implies that $2^n \mid n+1$, which is absurd for $n \geq 2$.

(2) Write $\lambda = p/q$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ with $\mathrm{lcm}(p, q) = 1$ Then $\binom{\lambda+n+1}{n} \in \mathbb{Z}$ implies that

$$q^n \mid (p+(n+1)q)(p+nq) \cdots (p+2q),$$

and hence $q \mid p^n$. Since $(p, q) = 1$, this implies that $q = 1$ and λ is an integer. □

Lemma 6.2. *Let m, K be two positive integers and $\{q_\alpha^s \mid 1 \leq \alpha \leq K, 1 \leq s \leq m\}$ a set of non-negative numbers. Consider the homogenous polynomial*

$$f(x_1, \ldots, x_m) = \left(-x_1 + \sum_{\alpha=1}^{K}\sum_{s=2}^{m} q_\alpha^s x_s\right) \cdot \prod_{\alpha=1}^{K}\left(x_1 + \sum_{s=2}^{m} q_\alpha^s x_s\right) + x_1^{K+1}.$$

Then all coefficients of f are non-negative.

Proof. Consider f as a polynomial of x_1 with coefficients in terms of $x_2, \ldots, x_m$. We need to show that for $1 \leq k \leq K+1$, the coefficient of x_1^k is a polynomial in terms of $x_2, \ldots, x_m$ with non-negative coefficients. It is easy to see that the coefficient of x_1^{K+1} and x_1^K are 0. Fix $1 \leq k \leq K$, then the coefficient of x_1^{K-k} is

$$\begin{aligned}
&\left(\sum_{\alpha=1}^{K}\sum_{s=2}^{m} q_\alpha^s x_s\right) \cdot \left(\sum_{\alpha_1<\cdots<\alpha_k}\prod_{j=1}^{k}\sum_{s=2}^{m} q_{\alpha_j}^s x_s\right) - \sum_{\alpha_1<\cdots<\alpha_{k+1}}\prod_{j=1}^{k+1}\sum_{s=2}^{m} q_{\alpha_j}^s x_s\\
=&\sum_{\alpha_1<\cdots<\alpha_k}\left(\left(\sum_{\alpha=1}^{K}\sum_{s=2}^{m} q_\alpha^s x_s\right)\cdot\left(\prod_{j=1}^{k}\sum_{s=2}^{m} q_{\alpha_j}^s x_s\right) - \sum_{\alpha_{k+1}>\alpha_k}\prod_{j=1}^{k+1}\sum_{s=2}^{m} q_{\alpha_j}^s x_s\right)\\
=&\sum_{\alpha_1<\cdots<\alpha_k}\left(\left(\sum_{\alpha=1}^{K}\sum_{s=2}^{m} q_\alpha^s x_s\right)\cdot\left(\prod_{j=1}^{k}\sum_{s=2}^{m} q_{\alpha_j}^s x_s\right) - \left(\sum_{\alpha_{k+1}>\alpha_k}\sum_{s=2}^{m} q_{\alpha_{k+1}}^s x_s\right)\cdot\left(\prod_{j=1}^{k}\sum_{s=2}^{m} q_{\alpha_j}^s x_s\right)\right)\\
=&\sum_{\alpha_1<\cdots<\alpha_k}\left(\left(\sum_{\alpha\leq\alpha_k}\sum_{s=2}^{m} q_\alpha^s x_s\right)\cdot\left(\prod_{j=1}^{k}\sum_{s=2}^{m} q_{\alpha_j}^s x_s\right)\right)
\end{aligned}$$

which is a polynomial in terms of $x_2, \ldots, x_m$ with non-negative coefficients. □

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