

# Tempered homogeneous spaces

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## Abstract

Let  $G$  be a semisimple real Lie group with finite center and  $H$  a connected closed subgroup. We establish a geometric criterion which detects whether the representation of  $G$  in  $L^2(G/H)$  is tempered.

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# 1 Introduction

This article is the sequel of our paper [3] dealing with harmonic analysis on homogeneous spaces  $G/H$  of semisimple Lie groups  $G$ . In the first paper [3] we studied the regular representation of  $G$  in  $L^2(G/H)$  when both  $G$  and  $H$  are semisimple groups. The main result of [3] is a geometric criterion which detects whether the representation of  $G$  in  $L^2(G/H)$  is tempered. The aim of the present paper is to extend this geometric criterion to the whole generality that  $H$  is an arbitrary closed connected subgroup.

In this introduction we will discuss the following questions:

- Why do we care about tempered representations of semisimple Lie groups?
- What is our temperedness criterion for the homogeneous space  $G/H$ ?
- What are the main ideas and ingredients in the proof of this criterion?

## 1.1 Tempered representations

Let  $G$  be a semisimple Lie group with finite center and  $K$  a maximal compact subgroup of  $G$ . Understanding unitary representations of  $G$  in Hilbert spaces is a major topic of research since the beginning of the 20<sup>th</sup> century. Its history includes the works of Cartan, Weyl, Gelfand, Harish-Chandra, Helgason, Langlands, Vogan and many others. The main motivations came from quantum physics, analysis and number theory.

Among unitary representations of  $G$  a smaller class called *tempered representations* plays a crucial role. Let us recall why they are so useful.

- By definition tempered representations are those which are weakly contained in the regular representation of  $G$  in  $L^2(G)$ . Therefore a unitary representation of  $G$  is tempered if and only if its disintegration into irreducible unitary representations involves only tempered representations.
- Tempered representations are those for which  $K$ -finite matrix coefficients belong to  $L^{2+\varepsilon}(G)$  for all  $\varepsilon > 0$ . This definition does not look so enlightening but equivalently these matrix coefficients are bounded by an explicit multiple of an explicit spherical function  $\Xi$ , see [8].
- Classification of irreducible tempered representations of  $G$  was accomplished by Knapp and Zuckerman in [14], while non-tempered irreducible unitary representations have not yet been completely understood.

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- Tempered representations are a cornerstone of the Langlands classification of admissible irreducible representations of  $G$  in [17], see also [13].
- Irreducible tempered representations  $\pi$  can be characterized in term of the *leading exponents*: these exponents must be a positive linear combination of negative roots, see [13, Chap. 8].
- One can also characterize them in terms of the *distribution character* of  $\pi$ : this character must be a tempered distribution on  $G$ , see [13, Chap. 12].
- Tempered representations are closed under induction, restriction, tensor product, and direct integral of unitary representations.
- The Kirillov–Kostant orbit methods works fairly well for tempered representations, see [11] for example.

## 1.2 The regular representation in $L^2(G/H)$

One of the most studied representations of  $G$  are the natural unitary representations of  $G$  in  $L^2(G/H)$  where  $H$  is a closed subgroup of  $G$ . When  $H$  is unimodular, the space  $G/H$  is implicitly endowed with a  $G$ -invariant measure and  $G$  acts naturally by translation on  $L^2$ -functions. When  $H$  is not unimodular, the representation of  $G$  in  $L^2(G/H)$  involves an extra factor (2.2) and is nothing but the (unitarily) induced representation  $\text{Ind}_H^G(\mathbf{1})$ .

The disintegration of  $L^2(G/H)$  is sometimes called *Plancherel formula* or  $L^2$ -harmonic analysis on  $G/H$ . For instance, Harish-Chandra’s celebrated Plancherel formula [10] deals with the case where  $H = \{e\}$ . Another case which attracted a lot of activities in the late 20<sup>th</sup> century is the case where  $G/H$  is a symmetric space, for which the disintegration of  $L^2(G/H)$  is proved up to a classification of (singular) discrete series representations for symmetric spaces [9, 20]. The Whittaker Plancherel theorem is the disintegration of  $L^2(G/H, \chi)$  for a unitary character  $\chi$  of a maximal unipotent subgroup  $H$ , [23]. Even when  $H$  is not unimodular, the regular representation of  $G$  in  $L^2(G/H)$  is still interesting. For instance, by Bruhat’s theory, when  $H$  is a parabolic subgroup of  $G$ , this representation is known to be a finite sum of irreducible representations of  $G$ . The decomposition of the tensor product (*fusion rule*) is sometimes equivalent to the Plancherel formula for  $G/H$  where  $H$  is not necessarily unimodular (see Section 5.3).

We refer to [3, Intro.] for more remarks on historical developments of the disintegration of the regular representation of  $G$  in  $L^2(G/H)$ . Getting *a priori* information on this disintegration was one of the main motivation in our research of such a general criterion. Up to our knowledge there does

not exist yet any general theorem involving simultaneously all these regular representations in  $L^2(G/H)$  with  $H$  connected. Our temperedness criterion below seems to be the first one in that direction.

In this series of papers, we address the following question: *What kind of unitary representations occur in the disintegration of  $L^2(G/H)$ ? More precisely, when are all of them tempered?*

As noted in [3], this question has not been completely solved even for reductive symmetric spaces  $G/H$ , because the Plancherel formula involves a delicate algebraic problem on discrete series representations for (sub)symmetric spaces with singular infinitesimal characters (see Example 5.5).

We give a geometric necessary and sufficient condition on  $G/H$  under which all these irreducible unitary representations in the Plancherel formula are tempered, or equivalently under which the regular representation of  $G$  in  $L^2(G/H)$  is tempered. This criterion was first discovered in our paper [3] in the special case where  $H$  is a reductive subgroup of  $G$ .

In the present paper we extend this criterion to any closed subgroup  $H$  with finitely many connected components.

Formally the extended criterion is exactly the same as for  $H$  reductive. Here is a short way to state our criterion (See Theorem 2.9).

$$L^2(G/H) \text{ is tempered} \iff \rho_{\mathfrak{h}}(Y) \leq \rho_{\mathfrak{g}/\mathfrak{h}}(Y) \text{ for all } Y \in \mathfrak{h}. \quad (1.1)$$

Here  $\mathfrak{g}$  and  $\mathfrak{h}$  are the Lie algebras of  $G$  and  $H$ , and, for a  $\mathfrak{h}$ -module  $V$  and  $Y \in \mathfrak{h}$ , the quantity  $\rho_V(Y)$  is half the sum of the absolute values of the real part of the eigenvalues of  $Y$  in  $V$  (see Section 2.3). We note that our criterion holds beyond the (real) spherical case ([16], cf. [21] for non archimedean field) so that the disintegration of  $L^2(G/H)$  may involve infinite multiplicities. We will give an explicit example of calculations of functions  $\rho_V$  in Corollary 5.8.

Our criterion (1.1) detects also whether or not  $L^2(X)$  is tempered for any real algebraic  $G$ -variety  $X$ :  $L^2(X)$  is unitarily equivalent to the direct integral of the regular representations for generic orbits, and one just has to check (1.1) at almost all orbits.

### 1.3 Strategy of proof

The proof of (1.1) relies on the uniform decay of matrix coefficients as in [3], but the main techniques are different from those in [3].

To avoid any confusion we will sometimes say *G-tempered* for *tempered as a representation of G*.

Dealing with non-unimodular subgroups  $H$  and dealing with the finitely many components of  $H$  will not be a problem because one proves in Corollary 3.3 the equivalence

$$\begin{aligned} L^2(G/H) \text{ is tempered} &\iff L^2(G/[H, H]) \text{ is tempered} \\ &\iff L^2(G/H_e) \text{ is tempered ,} \end{aligned}$$

where  $[H, H]$  is the derived subgroup and  $H_e$  is the identity component of  $H$ . Therefore the temperedness of  $L^2(G/H)$  depends only on  $\mathfrak{h}$  and we can assume that  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ . Then the homogeneous space  $G/H$  admits a  $G$ -invariant Radon measure  $\text{vol}$  and, according to Corollary 2.7, the temperedness of  $L^2(G/H)$  means that, for any compact subset  $C$  of  $G/H$ , the volume  $\text{vol}(gC \cap C)$  is bounded by a multiple of the Harish-Chandra function  $\Xi$ :

$$\text{vol}(gC \cap C) \leq M_C \Xi(g) \quad \text{for all } g \text{ in } G. \quad (1.2)$$

To prove the direct implication “ $L^2(G/H)$  tempered  $\implies \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$ ”, we will just estimate in Proposition 3.7 this volume (1.2) when  $C$  is a small neighborhood of the base point of the space  $G/H$ .

As in [3], the converse implication “ $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}} \implies L^2(G/H)$  tempered”, is much harder to prove. Using again Corollary 3.3, we can also assume that both  $G$  and  $H$  are Zariski connected algebraic groups. We proceed by induction on the dimension of  $G$ . We introduce in Definition-Lemma 4.1 two intermediate subgroups  $F$  and  $P$ :

$$H \subset F \subset P \subset G. \quad (1.3)$$

The group  $P$  is a parabolic subgroup of minimal dimension that contains  $H$ . We write  $P = LU$  and  $H = SV$  where  $U$  is the unipotent radical of  $P$ ,  $L$  is a maximal reductive subgroup of  $P$ ,  $V \subset U$  is the unipotent radical of  $H$ , and  $S \subset L$  a maximal semisimple subgroup of  $H$ . The group  $F$  is given by  $F := SU$ .

When  $P$  is equal to  $G$ , the group  $H$  is semisimple and we apply the main result of [3]. We now assume that  $P$  is a proper parabolic subgroup of  $G$ . Let  $Z$  be the homogeneous space  $Z = F/H \simeq U/V$  endowed with the natural  $F$ -action. We denote

- by  $\tau$  the regular representation of  $F$  in  $L^2(F/H)$ ,

- by  $\pi$  the regular representation of  $P$  in  $L^2(P/H)$ ,
- by  $\Pi$  the regular representation of  $G$  in  $L^2(G/H)$ .

Let  $Z_0$  be the same space  $Z_0 = U/V$  but endowed with another  $F$ -action, where  $U$  acts trivially and where  $S$  acts by conjugation.

We denote

- by  $\tau_0$  the regular representation of  $F$  in  $L^2(Z_0)$ ,
- by  $\pi_0$  the regular representation of  $P$  in  $L^2(P \times_F Z_0)$ ,
- by  $\Pi_0$  the regular representation of  $G$  in  $L^2(G \times_F Z_0)$ .

Note that both the  $G$ -manifolds  $X := G/H \simeq G \times_F Z$  and  $X_0 := G \times_F Z_0$  have a  $G$ -equivariant fiber bundle structure over the same space  $G/F$  with fiber  $Z \simeq Z_0$ , but that the unitary representations of  $G$  in  $L^2(X)$  and in  $L^2(X_0)$  are different. We want to study the representation of  $G$  in  $L^2(X)$ , whereas the induction hypothesis will give us information on  $L^2(X_0)$ . This is why we will need to compare in (1.5) below the volumes in the fibers  $Z$  and  $Z_0$ .

More precisely, the induction hypothesis combined with a reformulation of our criterion in Proposition 3.8 and a simple computation in Lemma 4.2 tell us that the representation  $\pi$  is  $L$ -tempered. If we knew that  $\pi$  were a  $P$ -tempered representation it would be easy to conclude, using Lemma 2.3, that the representation  $\Pi = \text{Ind}_P^G \pi$  is  $G$ -tempered. However, what we know is the temperedness of  $\pi_0$  and  $\text{Ind}_P^G \pi_0$ , and Corollary 2.7 gives us, for any compact subset  $C_0$  of  $G \times_F Z_0$ , a bound:

$$\text{vol}(g C_0 \cap C_0) \leq M_{C_0} \Xi(g) \quad \text{for all } g \text{ in } G. \quad (1.4)$$

In order to deduce (1.2) from (1.4), we first focus on the representations  $\tau$  in  $L^2(Z)$  and  $\tau_0$  in  $L^2(Z_0)$ . We prove in Proposition 4.4, for every compact subset  $D$  of the fiber  $Z = F/H$ , a uniform estimate of  $\text{vol}(fD \cap D)$  with respect to translates of the element  $f \in F$  by elements of the unipotent subgroup  $U$ . Namely, there exists a compact subset  $D_0 \subset Z_0$  such that

$$\text{vol}(fD \cap D) \leq \text{vol}(fD_0 \cap D_0) \quad \text{for all } f \text{ in } F. \quad (1.5)$$

Since  $\Pi = \text{Ind}_F^G \tau$  and  $\Pi_0 = \text{Ind}_F^G \tau_0$ , we deduce from (1.5) in Proposition 4.9 that for every compact subset  $C \subset G/H \simeq G \times_F Z$ , there exists a compact subset  $C_0 \subset G \times_F Z_0$  such that

$$\text{vol}(g C \cap C) \leq \text{vol}(g C_0 \cap C_0) \quad \text{for all } g \text{ in } G. \quad (1.6)$$

And (1.2) follows from (1.4) and (1.6). This ends the sketch of the proof of the criterion (1.1). The details are explained below.

Here is the organization of the paper.

In Section 2 we recall the basic definition and state precisely our criterion.

In Sections 3 we collect the parts of the proof which do not involve the intermediate parabolic subgroup  $P$ . It includes the proof for the necessity of the inequality  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$ , and a formulation of the temperedness criterion for  $\text{Ind}_H^G(L^2(V))$  (Theorem 3.6).

In Section 4 we introduce the intermediate subgroups  $F$  and  $P$  in (1.3), and detail how they work to conclude the proof for the hard part, *i.e.*, the sufficiency of the inequality  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$ .

In Section 5 we give a few examples which enlight the efficiency of our criterion.

## 2 Definition and main result

We collect in this chapter a few well-known facts on tempered representations, on almost  $L^2$  representations, and on uniform decay of matrix coefficients.

### 2.1 Regular and induced representations

We first recall the construction of the regular representations and the induced representations.

#### 2.1.1 Regular representations

Let  $G$  be a locally compact group,  $X$  be a locally compact space endowed with a continuous action of  $G$  and let  $\nu_X$  be a Radon measure on  $X$ .

When the  $G$ -action preserves the measure  $\nu_X$ , one has a natural unitary representation  $\lambda_X$  of  $G$  in the Hilbert space  $L^2(X) := L^2(X, \nu_X)$  called the *regular representation* and given by

$$(\lambda_X(g)\varphi)(x) = \varphi(g^{-1}x) \quad \text{for } g \text{ in } G, \varphi \text{ in } L^2(X) \text{ and } x \text{ in } X.$$

When the class of the measure  $\nu_X$  is  $G$ -invariant, one still has a natural unitary representation  $\pi$  of  $G$  in the Hilbert space  $L^2(X) := L^2(X, \nu_X)$

also called the *regular representation*. The formula will involve the Radon–Nikodym cocycle  $c(g, x)$  which is defined, for all  $g$  in  $G$  and  $\nu_X$ -almost all  $x$  in  $X$  by the equality

$$g_*\nu_X = c(g^{-1}, x) \nu_X. \quad (2.1)$$

The regular representation of  $G$  in  $L^2(X)$  is then given by

$$(\lambda_X(g)\varphi)(x) = c(g^{-1}, x)^{1/2} \varphi(g^{-1}x) \quad \text{for } g \text{ in } G, \varphi \text{ in } L^2(X), x \text{ in } X. \quad (2.2)$$

### 2.1.2 Induced representations

Assume now that  $X$  is a homogeneous space  $G/H$  where  $H$  is a closed subgroup of  $G$ . One can choose a  $G$ -invariant Radon measure on  $G/H$  if and only if the modular function of  $G$  coincides on  $H$  with that of  $H$ ,

$$\Delta_G(h) = \Delta_H(h) \quad \text{for all } h \text{ in } H.$$

In general there always exists a measure  $\nu$  on  $G/H$  whose class is  $G$ -invariant, and the regular representation of  $G$  in  $L^2(G/H)$  is the induced representation of the trivial representation of  $H$

$$\lambda_{G/H} = \text{Ind}_H^G(\mathbf{1}).$$

More generally, for any unitary representation  $\pi$  of  $H$ , one defines the (unitarily) induced representation  $\Pi := \text{Ind}_H^G(\pi)$  in the following way. The projection

$$G \longrightarrow G/H$$

is a principal bundle with structure group  $H$ . We fix a  $G$ -equivariant Borel measurable trivialization of this principal bundle

$$G \simeq G/H \times H \quad (2.3)$$

which sends relatively compact subsets to relatively compact subsets. The action of  $G$  by left multiplication through this trivialization can be read as

$$g(x, h) = (gx, \sigma(g, x)h) \quad \text{for all } g \in G, x \in G/H \text{ and } h \in H,$$

where  $\sigma: G \times G/H \rightarrow H$  is a Borel measurable cocycle.

The space of the representation  $\Pi$  is the space  $\mathcal{H}_\Pi := L^2(G/H; \mathcal{H}_\pi)$  of  $\mathcal{H}_\pi$ -valued  $L^2$ -functions on  $G/H$  and the action of  $G$  is given, for  $g$  in  $G$ ,  $\psi$  in  $\mathcal{H}_\Pi$ ,  $x$  in  $G/H$ , by

$$(\Pi(g)\psi)(x) = c(g^{-1}, x)^{1/2} \pi(\sigma(g, g^{-1}x))\psi(g^{-1}x),$$

where  $c$  is again the Radon–Nikodym cocycle (2.1).



### 2.1.3 Induced actions

When the closed subgroup  $H$  of  $G$  is acting continuously on a locally compact space  $Z$  one can define the induced action of  $G$  on the fibered space

$$G \times_H Z := (G \times Z)/H$$

where the quotient is taken for the right  $H$ -action  $(g, z)h = (gh, h^{-1}z)$  and where the  $G$ -action is given by  $g_0(g, z) = (g_0g, z)$ , for all  $g_0, g$  in  $G$ ,  $z$  in  $Z$  and  $h$  in  $H$ . Using (2.3), one gets a  $G$ -equivariant Borel measurable trivialization of this fibered space

$$G \times_H Z \simeq G/H \times Z.$$

Through this identification, the  $G$ -action is given by

$$g(x, z) = (gx, \sigma(g, x)z) \quad \text{for all } g \in G, x \in G/H \text{ and } z \in Z.$$

When the  $H$ -action preserves the class of a measure  $\nu_Z$  on  $Z$ , the  $G$ -action preserves the class of the measure  $\nu_X := \nu \otimes \nu_Z$ . In this case the regular representation of  $G$  in  $L^2(G \times_H Z)$  is unitarily equivalent to the induced representation of the regular representation of  $H$  in  $Z$ :

$$L^2(G \times_H Z) \simeq \text{Ind}_H^G(L^2(Z)) \quad \text{as unitary representations of } G. \quad (2.4)$$

## 2.2 Decay of matrix coefficients

We now recall the control of the matrix coefficients of tempered representations of a semisimple Lie group.

### 2.2.1 Tempered representations

Let  $G$  be a locally compact group and  $\pi$  be a unitary representation of  $G$  in a Hilbert space  $\mathcal{H}_\pi$ . All representations  $\pi$  of  $G$  will be assumed to be continuous *i.e.* the map  $G \rightarrow \mathcal{H}_\pi, g \mapsto \pi(g)v$  is continuous for all  $v$  in  $\mathcal{H}_\pi$ . The notion of tempered representation is due to Harish-Chandra.

**Definition 2.1.** The unitary representation  $\pi$  is said to be tempered or  $G$ -tempered if  $\pi$  is weakly contained in the regular representation  $\lambda_G$  of  $G$  in  $L^2(G)$  *i.e.* if every matrix coefficient of  $\pi$  is a uniform limit on every compact subset of  $G$  of a sequence of sums of matrix coefficients of  $\lambda_G$ .

We refer to [1, Appendix F] for more details on weak containments.

*Remark 2.2.* The notion of temperedness is stable by passage to a finite index subgroup  $G'$  of  $G$ , *i.e.* a unitary representation  $\pi$  of  $G$  is tempered if and only if  $\pi$  is tempered as a representation of  $G'$ .

This notion is also preserved by induction.

**Lemma 2.3.** *Let  $G$  be a locally compact group,  $H$  be a closed subgroup of  $G$  and  $\pi$  be a unitary representation of  $H$ . If  $\pi$  is  $H$ -tempered then the induced representation  $\text{Ind}_H^G(\pi)$  is  $G$ -tempered.*

*Proof.* Since the  $H$ -representation  $\pi$  is weakly contained in the regular representation  $\lambda_H$  of  $H$ , the  $G$ -representation  $\text{Ind}_H^G(\pi)$  is weakly contained in the regular representation  $\lambda_G = \text{Ind}_H^G(\lambda_H)$ , and hence is  $G$ -tempered.  $\square$

*Remark 2.4.* 1) When  $G$  is amenable, according to the Hulanicki–Reiter theorem in [1, Th. G.3.2], every unitary representation of  $G$  is tempered.

2) When  $G$  is a product of two closed subgroups  $G = SZ$  with  $Z$  central, a unitary representation  $\pi$  of  $G$  is  $G$ -tempered if and only if it is  $S$ -tempered. Indeed the regular representation of  $G$  in  $L^2(G)$  is clearly  $S$ -tempered. Conversely, we want to prove that any unitary representation  $\pi$  of  $G$  which is  $S$ -tempered is also  $G$ -tempered. We can assume that  $\pi$  is  $G$ -irreducible. The action of  $Z$  in this representation is given by a unitary character  $\chi$  and  $\pi$  is weakly contained in the representation  $\text{Ind}_Z^G \chi$ . Since  $\chi$  is  $Z$ -tempered, this representation is  $G$ -tempered.

### 2.2.2 Matrix coefficients

Let now  $G$  be a semisimple Lie group (always implicitly assumed to be real Lie groups with finitely many connected components and whose identity component has finite center).

**Definition 2.5.** A unitary representation  $\pi$  of  $G$  is said to be almost  $L^2$  if there exists a dense subset  $\mathcal{D} \subset \mathcal{H}_\pi$  for which the matrix coefficients  $g \mapsto \langle \pi(g)v_1, v_2 \rangle$  are in  $L^{2+\varepsilon}(G)$  for all  $\varepsilon > 0$  and all  $v_1, v_2$  in  $\mathcal{D}$ .

We fix a maximal compact subgroup  $K$  of  $G$ . Let  $\Xi$  be the Harish-Chandra spherical function on  $G$  (see [8]). By definition,  $\Xi$  is the matrix

coefficient of a normalized  $K$ -invariant vector  $v_0$  of the spherical unitary principal representation  $\pi_0 = \text{Ind}_{P_{\min}}^G(\mathbf{1}_{P_{\min}})$  where  $P_{\min}$  is a minimal parabolic subgroup of  $G$ . That is

$$\Xi(g) = \langle \pi_0(g)v_0, v_0 \rangle \text{ for all } g \text{ in } G. \quad (2.5)$$

Since  $P_{\min}$  is amenable, the representation  $\pi_0$  is  $G$ -tempered. Moreover, the function  $\Xi$  belongs to  $L^{2+\varepsilon}(G)$  for all  $\varepsilon > 0$  (see [13, Prop. 7.15]). We will need the following much more precise version of this fact.

**Proposition 2.6** (Cowling, Haagerup and Howe [8]). *Let  $G$  be a connected semisimple Lie group with finite center and  $\pi$  be a unitary representation of  $G$ . The following are equivalent:*

- (i) *the representation  $\pi$  is tempered,*
- (ii) *the representation  $\pi$  is almost  $L^2$ ,*
- (iii) *for every  $K$ -finite vectors  $v, w$  in  $\mathcal{H}_\pi$ , for every  $g$  in  $G$ , one has*

$$|\langle \pi(g)v, w \rangle| \leq \Xi(g) \|v\| \|w\| (\dim \langle Kv \rangle)^{\frac{1}{2}} (\dim \langle Kw \rangle)^{\frac{1}{2}}.$$

See [8, Thms. 1, 2 and Cor.]. See also [12], [18] and [19] for other applications of Proposition 2.6.

For regular representations this proposition becomes:

**Corollary 2.7.** *Let  $G$  be a connected semisimple Lie group with finite center and  $X$  a locally compact space endowed with a continuous action of  $G$  preserving a Radon measure  $\text{vol}$ . The regular representation of  $G$  in  $L^2(X)$  is tempered if and only if, for any compact subset  $C$  of  $X$ , the function  $g \mapsto \text{vol}(gC \cap C)$  belongs to  $L^{2+\varepsilon}(G)$  for all  $\varepsilon > 0$ .*

*In this case, when  $C$  is  $K$ -invariant, one has*

$$\text{vol}(gC \cap C) \leq \text{vol}(C) \Xi(g) \text{ for all } g \text{ in } G. \quad (2.6)$$

The notation  $gC$  denotes the set  $gC := \{gx \in X : x \in C\}$ .

*Proof.* Note that a compact subset  $C$  of  $X$  is always included in a  $K$ -invariant compact subset  $C_0$ , that the function  $\mathbf{1}_{C_0}$  is a  $K$ -invariant vector in  $L^2(X)$  and that

$$\langle \lambda_X(g)\mathbf{1}_{C_0}, \mathbf{1}_{C_0} \rangle = \text{vol}(gC_0 \cap C_0) \text{ for all } g \text{ in } G.$$

Note also that the functions  $\mathbf{1}_C$  span a dense subspace in  $L^2(X)$ . □

### 2.3 The function $\rho_V$

We now define the functions  $\rho_{\mathfrak{h}}$  and  $\rho_{\mathfrak{g}/\mathfrak{h}}$  occurring in the temperedness criterion, explain how to compute them and emphasize their geometric meaning.

When  $H$  is a Lie group we denote by the corresponding gothic letter  $\mathfrak{h}$  the Lie algebra of  $H$ . Let  $V$  be a real finite-dimensional representation of  $H$ . For an element  $Y$  in  $\mathfrak{h}$ , we consider the eigenvalues of  $Y$  in  $V$  (more precisely in the complexification  $V_{\mathbb{C}}$ ) and we denote by  $V_+$ ,  $V_0$  and  $V_-$  the largest vector subspaces of  $V$  on which the real part of all the eigenvalues of  $Y$  are respectively positive, zero and negative. One has the decomposition  $V = V_+ \oplus V_0 \oplus V_-$ . We define the non-negative functions  $\rho_V^+$  and  $\rho_V$  on  $\mathfrak{h}$  by

$$\begin{aligned}\rho_V^+(Y) &:= \text{Tr}(Y|_{V_+}), \\ \rho_V(Y) &:= \frac{1}{2}\rho_V^+(Y) + \frac{1}{2}\rho_V^+(-Y),\end{aligned}$$

where  $\text{Tr}$  denotes the trace of a matrix. Note that one has the equality  $\text{Tr}(Y|_{V_-}) = -\rho_V^+(-Y)$ .

By definition, one always has the equality  $\rho_V(-Y) = \rho_V(Y)$ . Moreover, when the action of  $H$  on  $V$  is volume preserving one has the equality

$$\rho_V(Y) = \rho_V^+(Y).$$

The function called  $\rho_V$  in [3, Sec. 3.1] is what we call now  $\rho_V^+$ . It coincides with our  $\rho_V$  since in [3] we only need to consider volume preserving actions.

Since this function  $\rho_V: \mathfrak{h} \rightarrow \mathbb{R}_{\geq 0}$  plays a crucial role in our criterion, we begin by a few trivial but useful comments, which make it easy to compute when dealing with examples. To simplify these comments, we assume that  $H$  is an algebraic subgroup of  $\text{GL}(V)$ . Let  $\mathfrak{a} = \mathfrak{a}_{\mathfrak{h}}$  be a maximal split abelian Lie subalgebra of  $\mathfrak{h}$  *i.e.* the Lie subalgebra of a maximal split torus  $A$  of  $H$ . Any element  $Y$  in  $\mathfrak{h}$  admits a unique Jordan decomposition  $Y = Y_e + Y_h + Y_n$  as a sum of three commuting elements of  $\mathfrak{h}$  where  $Y_e$  is a semisimple matrix with imaginary eigenvalues,  $Y_h$  is a semisimple matrix with real eigenvalues and  $Y_n$  is a nilpotent matrix. Moreover there exists an element  $\lambda_Y$  in  $\mathfrak{a}$  which is  $H$  conjugate to  $Y_h$ . Then one has the equality

$$\rho_V(Y) = \rho_V(\lambda_Y) \text{ for all } Y \text{ in } \mathfrak{h}.$$

This equality tells us that the function  $\rho_V$  is completely determined by its restriction to  $\mathfrak{a}$ .

This function  $\rho_V: \mathfrak{a} \rightarrow \mathbb{R}_{\geq 0}$  is continuous and is piecewise linear *i.e.* there exist finitely many convex polyhedral cones which cover  $\mathfrak{a}$  and on which  $\rho_V$  is linear. Indeed, let  $P_V$  be the set of weights of  $\mathfrak{a}$  in  $V$  and, for all  $\alpha$  in  $P_V$ , let  $m_\alpha := \dim V_\alpha$  be the dimension of the corresponding weight space. Then one has the equality

$$\rho_V(Y) = \frac{1}{2} \sum_{\alpha \in P_V} m_\alpha |\alpha(Y)| \quad \text{for all } Y \text{ in } \mathfrak{a}. \quad (2.7)$$

For example, when  $\mathfrak{h}$  is semisimple and  $V = \mathfrak{h}$  via the adjoint action, our function  $\rho_{\mathfrak{h}}$  is equal on each positive Weyl chamber  $\mathfrak{a}_+$  of  $\mathfrak{a}$  to the sum of the corresponding positive roots *i.e.* to twice the usual “ $\rho$ ” linear form. For other representations  $V$ , the maximal convex polyhedral cones on which  $\rho_V$  is linear are most often much smaller than the Weyl chambers. Explicit computations of the functions  $\rho_V$  will be given in Section 5.

The geometric meaning of this function  $\rho_V$  is given by the following elementary Lemma as in [3, Prop. 3.6].

**Lemma 2.8.** *Let  $V = \mathbb{R}^d$ . Let  $\mathfrak{a}$  be an abelian split Lie subalgebra of  $\text{End}(V)$  and  $C$  be a compact neighborhood of 0 in  $V$ . Then there exist constants  $m_C > 0$ ,  $M_C > 0$  such that*

$$m_C e^{-\rho_V(Y)} \leq e^{-\text{Trace}(Y)/2} \text{vol}(e^Y C \cap C) \leq M_C e^{-\rho_V(Y)} \quad \text{for all } Y \in \mathfrak{a}.$$

Such a factor  $e^{-\text{Trace}(Y)/2}$  occurs in computing the matrix coefficient of the vector  $\mathbf{1}_C$  in the regular representation  $L^2(V)$  when the action on  $V$  does not preserve the volume. Here  $\text{vol}$  denotes the volume with respect to the Lebesgue measure on  $V$ . The proof of Lemma 2.8 goes similarly to that of [3, Prop. 3.6] which deals with the case where the action is volume preserving.

## 2.4 Temperedness criterion for $L^2(G/H)$

We can now state precisely our temperedness criterion.

Let  $G$  be a semisimple Lie group and  $H$  a closed subgroup of  $G$ . Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be the Lie algebras of  $G$  and  $H$ . The temperedness criterion for the regular representation of  $G$  in  $L^2(G/H)$  will involve the functions  $\rho_{\mathfrak{h}}$  and  $\rho_{\mathfrak{g}/\mathfrak{h}}$  for the  $H$ -modules  $V = \mathfrak{h}$  and  $V = \mathfrak{g}/\mathfrak{h}$ .

**Theorem 2.9.** *Let  $G$  be a connected semisimple Lie group with finite center,  $H$  a closed connected subgroup of  $G$ . Then, one has the equivalence :*

$$L^2(G/H) \text{ is } G\text{-tempered} \iff \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}.$$

*Remark 2.10.* The assumption that  $G$  and  $H$  are connected are not very important. As we shall explain in Corollary 3.3, Theorem 2.9 is still true when  $G$  and  $H$  have finitely many connected components as soon as the identity component  $G_e$  has finite center.

*Remark 2.11.* When  $H$  is algebraic and  $\mathfrak{a}$  is a maximal abelian split Lie subalgebra of  $\mathfrak{h}$ , Inequality  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$  holds on  $\mathfrak{h}$  if and only if it holds on  $\mathfrak{a}$ .

*Remark 2.12.* When  $H$  is a minimal parabolic subgroup of  $G$  the representation of  $G$  in  $L^2(G/H)$  is tempered because the group  $H$  is amenable. Our criterion is easy to check in this case since the functions  $\rho_{\mathfrak{h}}$  and  $\rho_{\mathfrak{g}/\mathfrak{h}}$  are equal. This example explains why, when  $H$  is non-unimodular, our temperedness criterion involves the functions  $\rho_V$  instead of the functions  $\rho_V^+$ .

### 3 Preliminary proofs

In this section we state a useful reformulation of Theorem 2.9 and prove the direct implication in Theorem 2.9.

#### 3.1 The Herz majoration principle

We first explain how to reduce the proof of Theorem 2.9 to the case where both  $G$  and  $H$  are algebraic and how to deal with groups having finitely many connected components.

**Proposition 3.1.** *Let  $G$  be a semisimple Lie group with finitely many components such that the identity component  $G_e$  has finite center and  $H' \subset H$  two closed subgroups of  $G$ .*

- 1) *If  $L^2(G/H)$  is  $G$ -tempered then  $L^2(G/H')$  is  $G$ -tempered.*
- 2) *The converse is true when  $H'$  is normal in  $H$  and  $H/H'$  is amenable (for instance finite, compact, or abelian).*

**Lemma 3.2.** *Let  $G$  be a semisimple Lie group with finitely many connected components such that  $G_e$  has finite center, and  $H$  be a closed subgroup of  $G$ . If the regular representation in  $L^2(G/H)$  is  $G$ -tempered then the induced representation  $\Pi = \text{Ind}_H^G(\pi)$  is also  $G$ -tempered for any unitary representation  $\pi$  of  $H$ .*

*Proof of Lemma 3.2.* This classical lemma is called “Herz majoration principle” (see [2, Chap. 6]). We recall the short argument since it will be very useful in Proposition 4.9. We use freely the notation of Section 2. For a function  $\varphi$  in the space  $L^2(G/H, \mathcal{H}_\pi)$  of the induced representation  $\Pi = \text{Ind}_H^G(\pi)$ , we denote by  $|\varphi|$  the function in the space  $L^2(G/H)$  of the regular representation  $\Pi_0 = \text{Ind}_H^G(\mathbf{1})$  given by  $|\varphi|(x) := \|\varphi(x)\|$  for  $x$  in  $G/H$ . The space  $\mathcal{D}$  of bounded functions with compact support is dense in  $L^2(G/H, \mathcal{H}_\pi)$ . For  $\varphi$  and  $\psi$  in  $\mathcal{D}$ , one can compute the matrix coefficients

$$\begin{aligned} \langle \Pi(g)\varphi, \psi \rangle &= \int_{G/H} c(g^{-1}, x)^{1/2} \langle \pi(\sigma(g, g^{-1}x))\varphi(g^{-1}x), \psi(x) \rangle d\nu(x), \\ |\langle \Pi(g)\varphi, \psi \rangle| &\leq \int_{G/H} c(g^{-1}, x)^{1/2} \|\varphi(g^{-1}x)\| \|\psi(x)\| d\nu(x) \\ &\leq \langle \Pi_0(g)|\varphi|, |\psi| \rangle. \end{aligned}$$

Since  $\Pi_0$  is tempered, these matrix coefficients belong to  $L^{2+\varepsilon}(G)$  for all  $\varepsilon > 0$ . Therefore the representation  $\Pi$  is almost  $L^2$  and hence is  $G$ -tempered by Proposition 2.6.  $\square$

*Proof of Proposition 3.1.* 1) This follows from Lemma 3.2 applied to the regular representation  $\pi$  of  $H$  in  $L^2(H/H')$ .

2) Since  $H/H'$  is amenable, the trivial representation of  $H$  is weakly contained in the regular representation of  $H$  in  $L^2(H/H')$ . Therefore, inducing to  $G$ , the regular representation of  $G$  in  $L^2(G/H)$  is weakly contained in the regular representation of  $G$  in  $L^2(G/H')$  and hence is  $G$ -tempered.  $\square$

The following corollary tells us that the temperedness of  $L^2(G/H)$  depends only on the Lie algebras  $\mathfrak{g}$ ,  $\mathfrak{h}$  and does not change if we replace  $\mathfrak{h}$  by its derived Lie algebra  $[\mathfrak{h}, \mathfrak{h}]$ .

**Corollary 3.3.** *Let  $G$  be a semisimple Lie group with finitely many connected components such that  $G_e$  has a finite center  $Z_G$  and  $H$  be a closed subgroup with finitely many connected components. Then the following are equivalent.*

- (i)  $L^2(G/H)$  is  $G$ -tempered  $\iff$  (ii)  $L^2(G_e/H_e)$  is  $G_e$ -tempered  $\iff$
- (iii)  $L^2(G/HZ_G)$  is  $G/Z_G$ -tempered  $\iff$  (iv)  $L^2(G/[H, H])$  is  $G$ -tempered.

*Proof.* This follows from Proposition 3.1 and Remark 2.2 since the quotients  $H/H_e$ ,  $HZ_G/H$  and  $H/[H, H]$  are amenable groups.  $\square$

*Remark 3.4.* Corollary 3.3 is useful to reduce the proof of Theorem 2.9 to the case where both  $G$  and  $H$  are algebraic groups.

Indeed, every semisimple Lie algebra  $\mathfrak{g}$  is the Lie algebra of an algebraic group: the group  $\text{Aut}(\mathfrak{g})$ . Therefore, using  $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ , we can assume that  $G$  is algebraic.

Moreover, by Chevalley's "théorie des repiques" in [6], for any closed subgroup  $H$  of an algebraic group  $G$ , there exists two algebraic subgroups  $H_1$  and  $H_2$  of  $G$  whose Lie algebras satisfy

$$\mathfrak{h}_1 \subset \mathfrak{h} \subset \mathfrak{h}_2 \quad \text{and} \quad \mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}] = [\mathfrak{h}_2, \mathfrak{h}_2].$$

Therefore, using  $(i) \Leftrightarrow (iv)$ , we can assume that  $H$  is an algebraic subgroup.

*Remark 3.5.* Since the group  $[H, H]$  is unimodular, Corollary 3.3 is also useful to reduce the proof of Theorem 2.9 to the case where  $H$  is unimodular.

### 3.2 A strengthening of the main theorem

Theorem 2.9 will be proven by induction on the dimension of  $G$ . This induction process forces us to prove simultaneously an apparently stronger theorem which involves  $L^2(V)$ -valued sections over  $G/H$  associated to a finite-dimensional  $H$ -module  $V$ .

**Theorem 3.6.** *Let  $G$  be an algebraic semisimple Lie group,  $H$  an algebraic subgroup of  $G$  and  $V$  a real finite-dimensional algebraic representation of  $H$ . Then, one has the equivalence :*

$$\text{Ind}_H^G(L^2(V)) \text{ is } G\text{-tempered} \iff \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}} + 2\rho_V.$$

Again, we only need to check this inequality on a maximal split abelian Lie subalgebra  $\mathfrak{a}$  of  $\mathfrak{h}$ . Note also that, by Remark 3.4, Theorem 2.9 is the special case of Theorem 3.6 where  $V = \{0\}$ .

### 3.3 The direct implication

We first prove the direct implication in Theorems 2.9 and 3.6.

From now on, we will set  $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$ .

**Proposition 3.7.** *Let  $G$  be an algebraic semisimple Lie group,  $H$  an algebraic subgroup of  $G$  and  $V$  an algebraic representation of  $H$ . If the representation  $\Pi = \text{Ind}_H^G(L^2(V))$  is  $G$ -tempered then one has  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}} + 2\rho_V$ .*



*Proof.* By (2.4) this representation  $\Pi$  is also the regular representation of the  $G$ -space  $X := G \times_H V$ . Let  $A$  be a maximal split torus of  $H$  and  $\mathfrak{a}$  be the Lie algebra of  $A$ . We choose an  $A$ -invariant decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}_0$  and small closed balls  $B_0 \subset \mathfrak{q}_0$  and  $B_V \subset V$  centered at 0. We can see  $B_V$  as a subset of  $X$  and the map

$$B_0 \times B_V \longrightarrow G \times_H V, \quad (u, v) \mapsto \exp(u)v$$

is a homeomorphism onto its image  $C$ . Since  $\Pi$  is tempered one has a bound as in (2.6)

$$\langle \Pi(g)1_C, 1_C \rangle \leq M_C \Xi(g) \quad \text{for all } g \text{ in } G. \quad (3.1)$$

We will exploit this bound for elements  $g = e^Y$  with  $Y$  in  $\mathfrak{a}$ . In our coordinate system (3.1) we can choose the measure  $\nu_X$  to coincide with the Lebesgue measure on  $\mathfrak{q}_0 \oplus V$ . Taking into account the Radon–Nykodim derivative and the  $A$ -invariance of  $\mathfrak{q}_0$ , one computes

$$\langle \Pi(e^Y)1_C, 1_C \rangle \geq e^{-\text{Trace}_{\mathfrak{q}_0}(Y)/2} e^{-\text{Trace}_V(Y)/2} \text{vol}_{\mathfrak{q}_0}(e^Y B_0 \cap B_0) \text{vol}_V(e^Y B_V \cap B_V),$$

and therefore, using Lemma 2.8, one deduces

$$\langle \Pi(e^Y)1_C, 1_C \rangle \geq m_C e^{-\rho_{\mathfrak{q}}(Y)} e^{-\rho_V(Y)} \quad \text{for all } Y \text{ in } \mathfrak{a}. \quad (3.2)$$

Combining (3.1) and (3.2) with known bounds for the spherical function  $\Xi$  as in [13, Prop 7.15], one gets, for suitable positive constants  $d, C$ ,

$$\frac{m_C}{M_C} e^{-\rho_{\mathfrak{q}}(Y) - \rho_V(Y)} \leq \Xi(e^Y) \leq M_0 (1 + \|Y\|)^d e^{-\rho_{\mathfrak{g}}(Y)/2} \quad \text{for all } Y \text{ in } \mathfrak{a}.$$

Therefore one has  $\rho_{\mathfrak{g}} \leq 2\rho_{\mathfrak{q}} + 2\rho_V$ , and hence  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}} + 2\rho_V$  as required.  $\square$

### 3.4 Equivalence of the main theorems

We have already noticed that Theorem 2.9 is a special case of Theorem 3.6. We explain now why Theorem 3.6 is a consequence of Theorem 2.9.

**Proposition 3.8.** *Let  $G$  be an algebraic semisimple Lie group. If the conclusion of Theorem 2.9 is true for all algebraic subgroups  $H$  of  $G$ , then the conclusion of Theorem 3.6 is also true for all algebraic subgroups  $H$  of  $G$ .*

The proof relies on the following lemma.

**Lemma 3.9.** *Let  $H$  be a Lie group and  $V$  a finite-dimensional representation of  $H$ . Let  $v \in V$  be a point whose orbit  $Hv$  has maximal dimension and  $H_v$  be the stabilizer of  $v$  in  $H$ . Then the action of  $H_v$  on  $V/\mathfrak{h}v$  is trivial.*

*Proof of Lemma 3.9.* Assume by contradiction that there exist  $Y$  in  $\mathfrak{h}$  and  $w$  in  $V$  such that the vector  $Yw$  does not belong to  $\mathfrak{h}v$ .

Choose a complementary subspace  $\mathfrak{m}$  of  $\mathfrak{h}_v$  in  $\mathfrak{h}$  so that  $\mathfrak{h} = \mathfrak{h}_v \oplus \mathfrak{m}$ . Choose also a point  $v_\varepsilon = v + \varepsilon w$  near  $v$ . For  $\varepsilon$  small, the tangent space  $\mathfrak{h}v_\varepsilon$  to the orbit  $Hv_\varepsilon$  contains both the subspace  $\mathfrak{m}v_\varepsilon$  which is near  $\mathfrak{m}v = \mathfrak{h}v$  and the vector  $\varepsilon^{-1}Yv_\varepsilon = Yw$ . Therefore, for  $\varepsilon$  small, one has the inequality  $\dim \mathfrak{h}v_\varepsilon > \dim \mathfrak{h}v$  which gives us a contradiction.  $\square$

*Proof of Proposition 3.8.* We assume that  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}} + 2\rho_V$  and we want to prove, using Theorem 2.9, that the regular representation of  $G$  in  $L^2(G \times_H V)$  is tempered. Since the action is algebraic, there exists a Borel measurable subset  $T \subset V$  which meets each of these  $H$ -orbits in exactly one point. Let  $\nu_V$  be a probability measure on  $V$  with positive density and  $\nu_T$  be the probability measure on  $T \simeq H \backslash V$  given as the image of  $\nu_V$ . One has a direct integral decomposition of the regular representation

$$L^2(G \times_H V) = \int_T^{\oplus} L^2(G/H_v) d\nu_T(v)$$

where  $H_v$  is the stabilizer of  $v$  in  $H$ . Since the direct integral of tempered representations is tempered, we only need to prove that, for  $\nu_T$ -almost all  $v$  in  $T$ ,

$$L^2(G/H_v) \text{ is } G\text{-tempered.} \quad (3.3)$$

Our assumption implies that

$$\rho_{\mathfrak{h}}(Y) \leq \rho_{\mathfrak{q}}(Y) + 2\rho_V(Y) \text{ for all } Y \text{ in } \mathfrak{h}_v \quad (3.4)$$

For  $\nu_T$ -almost all  $v$  in  $T$ , the orbit  $Hv$  has maximal dimension, hence, by Lemma 3.9, the action of  $\mathfrak{h}$  on the quotient  $V/\mathfrak{h}v$  is trivial, and therefore one has the equality

$$\rho_V(Y) = \rho_{\mathfrak{h}}(Y) - \rho_{\mathfrak{h}_v}(Y) \text{ for all } Y \text{ in } \mathfrak{h}_v \quad (3.5)$$

Combining (3.4) and (3.5), one gets, for  $\nu_T$ -almost all  $v$  in  $T$ ,

$$2\rho_{\mathfrak{h}_v}(Y) \leq \rho_{\mathfrak{q}}(Y) + \rho_{\mathfrak{h}}(Y) = \rho_{\mathfrak{g}}(Y) \text{ for all } Y \text{ in } \mathfrak{h}_v$$

which can be rewritten as the temperedness criterion  $\rho_{\mathfrak{h}_v}(Y) \leq \rho_{\mathfrak{q}_v}(Y)$  for  $L^2(G/H_v)$  in Theorem 2.9 and hence proves (3.3).  $\square$

## 4 Using parabolic subgroups

The aim of this section is to prove the converse implication in Theorem 2.9. As we have seen in Remarks 3.4 and 3.5, we can assume that  $G$  is a Zariski connected algebraic group and that  $H$  is a Zariski connected algebraic subgroup such that  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ .

The proof relies on the presence of two nice intermediate subgroups

$$H \subset F \subset P \subset G.$$

### 4.1 The intermediate subgroups

We first explain the construction of these intermediate subgroups  $F$  and  $P$ .

Let  $G$  be an algebraic semisimple Lie group and  $H$  a Zariski connected algebraic subgroup of  $G$  such that  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ .

**Lemma-Definition 4.1.** *We fix a parabolic subgroup  $P$  of  $G$  of minimal dimension that contains  $H$  and denote by  $U$  the unipotent radical of  $P$ . There exists a reductive subgroup  $L \subset P$  such that  $P = LU$  and  $H = (L \cap H)(U \cap H)$ . Moreover the group  $S := L \cap H$  is semisimple and the group  $V := U \cap H$  is the unipotent radical of  $H$ . We denote by  $F$  the group  $F = SU$ .*

*Proof.* The group  $V := U \cap H$  is a unipotent normal subgroup of  $H$ . The quotient  $S' := H/V$  is a Zariski connected subgroup of the reductive group  $P/U$  which is not contained in any proper parabolic subgroup of  $P/U$ . Therefore, by [5, Sec. VIII.10] this group  $S'$  is reductive. Since  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ , this group  $S'$  is semisimple and there exists a semisimple subgroup  $S \subset H$  such that  $H = SV$ . Since  $S$  is semisimple, the group  $V$  is the unipotent radical of  $H$ . Since maximal reductive subgroups  $L$  of  $P$  are  $U$ -conjugate, one can choose  $L$  containing  $S$  and therefore one has  $S = L \cap H$ .  $\square$

The following two lemmas will be useful in our induction process.

**Lemma 4.2.** *With the notation of Definition 4.1, the following two functions on  $\mathfrak{s}$  are equal :*

$$\rho_{\mathfrak{g}/\mathfrak{h}} - \rho_{\mathfrak{h}} = \rho_{\mathfrak{l}/\mathfrak{s}} + 2\rho_{\mathfrak{u}/\mathfrak{v}} - \rho_{\mathfrak{s}}. \quad (4.1)$$

*Proof.* Since  $\rho_{\mathfrak{g}/\mathfrak{p}} = \rho_{\mathfrak{u}}$ , one has the equalities of functions on  $\mathfrak{s}$ ,

$$\rho_{\mathfrak{g}/\mathfrak{h}} = \rho_{\mathfrak{u}} + \rho_{\mathfrak{l}/\mathfrak{s}} + \rho_{\mathfrak{u}/\mathfrak{v}} \quad \text{and} \quad \rho_{\mathfrak{h}} = \rho_{\mathfrak{s}} + \rho_{\mathfrak{v}}. \quad \square$$

**Lemma 4.3.** *Let  $P = LU$  be a real algebraic group which is a semidirect product of a reductive subgroup  $L$  and its unipotent radical  $U$ . Let  $\pi_0$  be a unitary representation of  $P$  which is  $L$ -tempered and trivial on  $U$ . Then the representation  $\pi_0$  is also  $P$ -tempered.*

*Proof.* The weak containment  $\pi_0 \prec L^2(L)$  as unitary representations of  $L$  implies the weak containment  $\pi_0 \prec L^2(P/U)$  as unitary representations of  $P$  because  $U$  acts trivially on the both sides. Since  $U$  is amenable, the trivial representation of  $U$  is  $U$ -tempered, therefore by Lemma 2.3 the regular representation of  $P$  in  $L^2(P/U)$  is  $P$ -tempered, and  $\pi_0$  is also  $P$ -tempered.  $\square$

## 4.2 Bounding volume of compact sets

The proof of Theorem 2.9 relies on a control of the volume of the intersection of translates of compact sets in  $X = G/H$ . We first explain how to bound such volumes in  $Z = F/H$ . This bound is quite general.

**Proposition 4.4.** *Let  $F = SU$  be a real algebraic group which is a semidirect product of a reductive subgroup  $S$  and its unipotent radical  $U$ . Let  $H = SV$  be an algebraic subgroup of  $F$  containing  $S$  where  $V = U \cap H$ . Let  $Z$  be the  $F$ -space  $Z = F/H = U/V$  endowed with a  $U$ -invariant Radon measure. Then for every compact subset  $D \subset Z$ , there exists a compact subset  $D_0 \subset Z$  such that for all  $s \in S$  and  $u \in U$ , one has*

$$\text{vol}(suD \cap D) \leq \text{vol}(sD_0 \cap D_0). \quad (4.2)$$

Here is the reformulation of Proposition 4.4 that will be used later on.

**Definition 4.5.** Let  $Z_0$  be the same space  $Z_0 = U/V$  as  $Z$  but endowed with another  $F$ -action where  $U$  acts trivially and where  $S$  acts by conjugation.

**Corollary 4.6.** *Same notation as in Proposition 4.4. Then for every compact subset  $D \subset Z$ , there exists a compact subset  $D_0 \subset Z_0$  such that for every  $f \in F$ , one has*

$$\text{vol}(fD \cap D) \leq \text{vol}(fD_0 \cap D_0). \quad (4.3)$$

The proof of Proposition 4.4 is by induction on the dimension of  $Z$ . It relies only on geometric arguments and uses no representation theory.

Before studying the proof of Proposition 4.4 the reader could as an exercise focus on the following very simple example where  $Z = \mathbb{R}^2$  is the affine 2-plane and  $F$  is the group of affine bijections  $\begin{pmatrix} a & r \\ 0 & b \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix}$  that preserve the horizontal foliation. In this case  $S$  is the 2-dimensional group of diagonal matrices and  $U$  is the 3-dimensional Heisenberg group. The proof for this example relies on the same ideas while being very concrete.

*Proof of Proposition 4.4. First case:*  $S$  is a split torus.

We denote by  $C$  the center of  $U$  and  $C_V = V \cap C$ . Let  $W$  be the closed subgroup  $W := VC \subset U$ . The projection

$$Z = U/V \longrightarrow Z' := U/W$$

is a principal bundle of group  $C_W := C/C_V = W/V$ . According to Lemma 4.7 below, there exists a continuous trivialization of this principal bundle

$$Z \simeq Z' \times C_W \quad (4.4)$$

such that the action of  $U$  and  $S$  through this trivialization can be read as

$$su(z', c) = (su z', s c + s c_0(u, z')) \quad (4.5)$$

for all  $s \in S$ ,  $u \in U$ ,  $z' \in Z'$ ,  $c \in C_W$ , where  $c_0$  is a continuous cocycle  $c_0 : U \times Z' \rightarrow C_W$ . We fix three compatible invariant measures  $\text{vol}_Z$ ,  $\text{vol}_{Z'}$  and  $\text{vol}$  on  $Z$ ,  $Z'$ , and  $C_W$ .

We start with a compact set  $D \subset Z$ . Through the trivialization (4.4), this set  $D$  is included in a product of two compact sets  $D' \subset Z'$  and  $B \subset C_W$

$$D \subset D' \times B,$$

where  $B$  is a symmetric convex set in the group  $C_W$  seen as a real vector space. By the induction hypothesis, there exists a compact set  $D'_0 \subset Z'$  which satisfies the bound (4.2) for  $D'$ , *i.e.* such that

$$\text{vol}_{Z'}(suD' \cap D') \leq \text{vol}_{Z'}(sD'_0 \cap D'_0) \quad \text{for all } s \in S, u \in U. \quad (4.6)$$

We compute using (4.5) and Lemma 4.8 below, for all  $s \in S$  and  $u \in U$ ,

$$\begin{aligned} \text{vol}_Z(suD \cap D) &\leq \int_{suD' \cap D'} \text{vol}((sB + s c_0(u, (su)^{-1}z')) \cap B) dz' \\ &\leq \int_{suD' \cap D'} \text{vol}(sB \cap B) dz' \end{aligned}$$

where  $dz'$  also denotes the  $U$ -invariant measure on  $Z'$ . Hence, using (4.6), we go on

$$\begin{aligned} \text{vol}_Z(suD \cap D) &\leq \text{vol}_{Z'}(suD' \cap D') \text{vol}(sB \cap B) \\ &\leq \text{vol}_{Z'}(sD'_0 \cap D'_0) \text{vol}(sB \cap B) \\ &= \text{vol}_Z(sD_0 \cap D_0), \end{aligned}$$

where  $D_0$  is the compact subset of  $Z$  given by  $D_0 := D'_0 \times B$ .

**Second case:**  $S$  is a reductive group.

This general case will be deduced from the first case. Indeed any reductive group admits a Cartan decomposition  $S = K_S A_S K_S$  where  $K_S$  is a maximal compact subgroup of  $S$  and where  $A_S$  is a maximal split torus of  $S$ . We start with a compact set  $D$  of  $Z$ . According to the first case, there exists a  $K_S$ -invariant compact set  $D_0 \subset Z$  such that, for all  $a \in A_S$  and  $u \in U$ , one has

$$\text{vol}(auK_S D \cap K_S D) \leq \text{vol}(aD_0 \cap D_0).$$

Therefore, for all  $s$  in  $S$  and  $u$  in  $U$ , writing  $s = k_1 a k_2$  with  $k_1, k_2$  in  $K_S$  and  $a$  in  $A_S$ , one has

$$\begin{aligned} \text{vol}(suD \cap D) &\leq \text{vol}(a(k_2 u k_2^{-1})k_2 D \cap k_1^{-1} D) \\ &\leq \text{vol}(aD_0 \cap D_0) \\ &= \text{vol}(sD_0 \cap D_0), \end{aligned}$$

as required. □

In the proof of Proposition 4.4, we have used the following two lemmas.

**Lemma 4.7.** *Let  $U$  be a unipotent group,  $V \subset U$  a unipotent subgroup,  $C$  be the center of  $U$ ,  $W := VC$  and  $C_V := C \cap V$ . Let  $S \subset \text{Aut}(U)$  be a split torus which preserves  $V$ . Then there exists a continuous trivialization of the  $U$ -equivariant principal bundle  $U/V \rightarrow U/W$  with structure group  $C/C_V$*

$$U/V \simeq U/W \times C/C_V$$

such that the action of  $U$  and  $S$  through this trivialization can be read as

$$su(y, c) = (su y, s c + s c_0(u, y))$$

for all  $u \in U$ ,  $s \in S$ ,  $y \in U/W$ ,  $c \in C/C_V$ , where  $c_0$  is a continuous cocycle  $c_0: U \times U/W \rightarrow C/C_V$ .

*Proof of Lemma 4.7.* These claims are a variation of a classical result of Chevalley–Rosenlicht (See for instance [7, Thm. 3.1.4]). The proof relies on the existence of “an adapted basis in a nilpotent Lie algebra”. Here is a sketch of proof of these claims.

As usual, let  $\mathfrak{u}$ ,  $\mathfrak{v}$ ,  $\mathfrak{c}$  and  $\mathfrak{w}$  be the Lie algebras of the groups  $U$ ,  $V$ ,  $C$  and  $W$ . Let  $I$  be the ordered set  $I = \{1, \dots, n\}$ , where  $n = \dim \mathfrak{u}$ . We fix a basis  $(e_i)_{i \in I}$  of  $\mathfrak{u}$ , such that

- for every  $i \geq 1$ , the vector space spanned by the  $e_j$  for  $j \geq i$  is an ideal;
- for every  $i \geq 1$ , the line  $\mathbb{R}e_i$  is invariant by  $S$ ;
- there exists a subset  $I_V \subset I$  such that  $\mathfrak{v}$  is spanned by  $e_i$  for  $i \in I_V$ ;
- there exists a subset  $I_C \subset I$  such that  $\mathfrak{c}$  is spanned by  $e_i$  for  $i \in I_C$ ;
- the Lie algebra  $\mathfrak{w}$  is spanned by  $e_i$  for  $i \in I_W := I_C \cup I_V$ .

Then, the map

$$\Psi : \mathbb{R}^I \longrightarrow U \quad (t_i)_{i \in I} \mapsto \prod_{i \in I} \exp(t_i e_i),$$

where the product is performed using the order on  $I$ , is a diffeomorphism and one has

$$\Psi(\mathbb{R}^{I_V}) = V, \quad \Psi(\mathbb{R}^{I_C}) = C \quad \text{and} \quad \Psi(\mathbb{R}^{I_W}) = W.$$

Setting  $J_V := I \setminus I_V$  and  $J_W := I \setminus I_W$ , the map

$$\Psi_V : \mathbb{R}^{J_V} \longrightarrow U/V, \quad (t_i)_{i \in J_V} \mapsto \prod_{i \in J_V} \exp(t_i e_i) V$$

is also a diffeomorphism and the restriction of this map to the subset  $\mathbb{R}^{J_W}$  gives an  $S$ -equivariant section of the bundle  $U/V \rightarrow U/W$ .  $\square$

Here is the second basic lemma used in the proof of Proposition 4.4.

**Lemma 4.8.** *Let  $B, B'$  be two symmetric convex sets of  $\mathbb{R}^d$ , then one has*

$$\text{vol}((B + v) \cap B') \leq \text{vol}(B \cap B') \quad \text{for all } v \in \mathbb{R}^d.$$

*Proof.* By the Brunn–Minkowski inequality (see [4, Sec. 11]), the map  $v \mapsto \text{vol}((B + v) \cap B')^{1/d}$  is concave on the convex set  $B' - B$  and hence achieves its maximum value at  $v = 0$ .  $\square$

### 4.3 Matrix coefficients of induced representations

We now explain how to control the volume of the intersection of translates of compact sets in the  $G$ -space  $X = G/H$  with those in  $X_0 := G \times_F Z_0$ .

**Proposition 4.9.** *Let  $G$  be an algebraic semisimple Lie group and  $H$  a Zariski connected algebraic subgroup such that  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ . Let  $P = LU$ ,  $F = SU$  and  $H = SV$  be the groups introduced in Definition 4.1. Let  $Z_0$  be the  $F$ -space introduced in Definition 4.5 and  $X_0$  the  $G$ -space  $X_0 := G \times_F Z_0$ . Then, for every compact subset  $C \subset G/H$ , there exists a compact subset  $C_0 \subset X_0$  such that*

$$\text{vol}(gC \cap C) \leq \text{vol}(gC_0 \cap C_0) \quad \text{for all } g \text{ in } G. \quad (4.7)$$

In Proposition 4.9 the assumption  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$  can be removed but the conclusion (4.7) becomes slightly more technical when there is no  $G$ -invariant measure on  $G/H$ . Indeed, when  $\mathfrak{h} \neq [\mathfrak{h}, \mathfrak{h}]$ , one has to replace the bound (4.7) by a bound of  $K$ -finite matrix coefficients of the induced representation  $\Pi = \text{Ind}_F^G(L^2(F/H))$  thanks to  $K$ -finite matrix coefficients of the induced representation  $\Pi_0 = \text{Ind}_F^G(L^2(Z_0))$ .

*Proof of Proposition 4.9.* The projection

$$G \rightarrow X' := G/F$$

is a  $G$ -equivariant principal bundle with structure group  $F$ . As in Section 2.1, we fix a Borel measurable trivialization of this principal bundle

$$G \simeq X' \times F \quad (4.8)$$

which sends relatively compact subsets to relatively compact subsets. The action of  $G$  by left multiplication through this trivialization can be read as

$$g(x', f) = (gx', \sigma_F(g, x')f) \quad \text{for all } g \in G, x' \in X' \text{ and } f \in F,$$

where  $\sigma_F: G \times X' \rightarrow F$  is a Borel measurable cocycle. This trivialization (4.8) induces a trivialization of the associated bundles

$$\begin{aligned} X &= G \times_F Z \simeq X' \times Z, \\ X_0 &= G \times_F Z_0 \simeq X' \times Z_0. \end{aligned}$$



We start with a compact set  $C$  of  $X$ . Through the first trivialization, this compact set is included in a product of two compact sets  $C' \subset X'$  and  $D \subset Z$

$$C \subset C' \times D. \quad (4.9)$$

We denote by  $D_0 \subset Z_0$  the compact set given by Corollary 4.6 and we compute using (4.3), for  $g$  in  $G$ ,

$$\begin{aligned} \text{vol}_X(g C \cap C) &\leq \int_{g C' \cap C'} \text{vol}_Z(\sigma_F(g, g^{-1} x') D \cap D) dx' \\ &\leq \int_{g C' \cap C'} \text{vol}_{Z_0}(\sigma_F(g, g^{-1} x') D_0 \cap D_0) dx' \\ &\leq \text{vol}_{X_0}(g C_0 \cap C_0), \end{aligned}$$

where  $dx'$  is a  $G$ -invariant measure on  $X'$  and  $C_0$  is a compact subset of  $X_0 \simeq X' \times Z_0$  which contains  $C' \times D_0$ .  $\square$

#### 4.4 Proof of the temperedness criterion

We conclude the proof of Theorem 2.9.

*Proof of the converse implication in Theorem 2.9.* We prove it by induction on the dimension of  $G$ . By Remarks 3.4 and 3.5, we can assume that  $G$  is a Zariski connected semisimple algebraic group and that  $H$  is a Zariski connected algebraic subgroup such that  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ . Let

$$H = SV \subset F = SU \subset P = LU \subset G$$

be the groups introduced in Definition 4.1. Let  $Z_0 = U/V$  be the  $F$ -space introduced in Definition 4.5 and  $X_0$  be the  $G$ -space  $X_0 = G \times_F Z_0$ .

When  $P$  is equal to  $G$ , the group  $H$  is semisimple and we apply [3, Thm 3.1]. We now assume that  $P$  is a proper parabolic subgroup of  $G$ .

By assumption one has  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$  on  $\mathfrak{h}$ . Therefore, by Lemma 4.2, one has the inequality on  $\mathfrak{s}$

$$\rho_{\mathfrak{s}} \leq \rho_{\mathfrak{l}/\mathfrak{s}} + 2\rho_{\mathfrak{u}/\mathfrak{v}}. \quad (4.10)$$

We introduce the regular representation  $\pi_0$  of  $P$  in  $L^2(P \times_F Z_0)$  which is unitarily equivalent to  $\text{Ind}_F^P(L^2(\mathfrak{u}/\mathfrak{v}))$  by the isomorphism (2.4). As a representation of  $L$ , one has

$$\pi_0|_L = \text{Ind}_S^L(L^2(\mathfrak{u}/\mathfrak{v})).$$

Using our induction hypothesis on the dimension of  $G$  to the derived subgroup of  $L$ , Proposition 3.8 and Remark 2.4 tell us that the representation  $\pi_0$  is  $L$ -tempered by (2.4). Therefore, by Lemma 4.3, the representation  $\pi_0$  is  $P$ -tempered. The regular representation  $\Pi_0$  of  $G$  in  $L^2(G \times_F Z_0)$  is unitarily equivalent to  $\Pi_0 = \text{Ind}_P^G(\pi_0)$  because  $\Pi_0 \simeq \text{Ind}_P^G(\text{Ind}_F^P(L^2(\mathfrak{u}/\mathfrak{v})))$ . Now, Lemma 2.3 implies that this representation  $\Pi_0$  is  $G$ -tempered. Therefore, by Corollary 2.7, for any  $K$ -invariant compact subset  $C_0$  of  $G \times_F Z_0$ , one has a bound:

$$\text{vol}(g C_0 \cap C_0) \leq \text{vol}(C_0) \Xi(g) \quad \text{for all } g \text{ in } G.$$

Hence, by Proposition 4.9, for any compact subset  $C$  of  $G/H$ , one also has a bound

$$\text{vol}(g C \cap C) \leq M_C \Xi(g) \quad \text{for all } g \text{ in } G.$$

Again by Corollary 2.7, this tells us that the representation of  $G$  in  $L^2(G/H)$  is  $G$ -tempered.  $\square$

## 5 Examples

The criterion given in Theorem 2.9 allows to easily detect for a given homogeneous space  $G/H$  whether the unitary representation of a semisimple Lie group  $G$  in  $L^2(G/H)$  is tempered or not. We collect in this chapter a few examples, omitting the details of the computational verifications.

### 5.1 Examples of tempered homogeneous spaces

We first recall a few examples extracted from [3] where  $H$  is reductive.

**Example 5.1.**  $L^2(SL(p+q, \mathbb{R})/SO(p, q))$  is always tempered.

$L^2(SL(2m, \mathbb{R})/Sp(m, \mathbb{R}))$  is never tempered.

$L^2(SL(m+n, \mathbb{C})/SL(m, \mathbb{C}) \times SL(n, \mathbb{C}))$  is tempered iff  $|m-n| \leq 1$ .

$L^2(SO(m+n, \mathbb{C})/SO(m, \mathbb{C}) \times SO(n, \mathbb{C}))$  is tempered iff  $|m-n| \leq 2$ .

$L^2(Sp(m+n, \mathbb{C})/Sp(m, \mathbb{C}) \times Sp(n, \mathbb{C}))$  is tempered iff  $m = n$ .

**Example 5.2.** Let  $n = n_1 + \dots + n_r$  with  $n_1 \geq \dots \geq n_r \geq 1$ ,  $r \geq 2$ .

$L^2(SL(n, \mathbb{R})/\prod SL(n_i, \mathbb{R}))$  is tempered iff  $2n_1 \leq n+1$ .

$L^2(Sp(n, \mathbb{R})/\prod Sp(n_i, \mathbb{R}))$  is tempered iff  $2n_1 \leq n$ .

Let  $p = p_1 + p_2$ ,  $q = q_1 + q_2$  with  $p_1, p_2, q_1, q_2 \geq 1$ .

$L^2(SO(p, q)/SO(p_1, q_1) \times SO(p_2, q_2))$  is tempered iff  $|p_1 + q_1 - p_2 - q_2| \leq 2$ .

**Example 5.3.** Let  $G$  be an algebraic semisimple Lie group and  $K$  a maximal compact subgroup.  $L^2(G_{\mathbb{C}}/K_{\mathbb{C}})$  is  $G_{\mathbb{C}}$ -tempered iff  $G$  is quasisplit.

*Remark 5.4.* A way to justify this last example is to notice that our criterion  $2\rho_{\mathfrak{k}_{\mathbb{C}}} \leq \rho_{\mathfrak{g}_{\mathbb{C}}}$  means that *the trivial  $K$ -type is a small  $K$ -type of  $G$*  in the terminology of Vogan's paper [22, Def. 6.1], see also in Knapp's book [13, Chap. XV], and to use the following equivalences due to Vogan in the same paper [22, Thm. 6.4]:

$G$  has a small  $K$ -type  $\iff$  the trivial  $K$ -type is small  $\iff G$  is quasisplit.

Here is a delicate example for semisimple symmetric spaces.

**Example 5.5.** Let  $G/H := Sp(2, 1)/Sp(1) \times Sp(1, 1)$ . The Plancherel formula [9, 20] tells that both the continuous part and a “generic portion” of the discrete part of  $L^2(G/H)$  are tempered, however, our criterion (1.1) tells that  $L^2(G/H)$  is nontempered because  $\rho_{\mathfrak{h}}(Y) = \frac{3}{2}\rho_{\mathfrak{q}}(Y) > \rho_{\mathfrak{q}}(Y)$  if  $Y$  is a nonzero hyperbolic element of  $\mathfrak{h}$ . In fact, the discrete part of  $L^2(G/H)$  consist of Harish-Chandra's discrete series representations, say  $\pi_n$  ( $n = 1, 2, \dots$ ), and two more non-vanishing representations  $\pi_0$  and  $\pi_{-1}$  in the coherent family, where  $\pi_0$  is still tempered but  $\pi_{-1}$  is nontempered ([15, Thm. 1]).

Here is another direct application of our criterion (1.1) where  $H$  is not anymore assumed to be reductive.

**Corollary 5.6.** Let  $G$  be an algebraic semisimple Lie group, and  $H$  an algebraic subgroup.

- 1) If the representation of  $G_{\mathbb{C}}$  in  $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$  is tempered, then the representation of  $G$  in  $L^2(G/H)$  is tempered.
- 2) The converse is true under if  $H$  contains a maximal torus which is split.

## 5.2 Subgroups of $SL(n, \mathbb{R})$

We now explain how to check our criterion (1.1) on a very concrete example.

In Table 1, we specify our criterion (1.1) when  $G = SL(\mathbb{R}^p \oplus \mathbb{R}^q)$  and  $H$  is a subgroup of  $G$  normalized by the group  $SL(\mathbb{R}^p) \times SL(\mathbb{R}^q)$ .

In Table 2, we specify our criterion (1.1) when  $G = SL(\mathbb{R}^p \oplus \mathbb{R}^q \oplus \mathbb{R}^r)$  and  $H$  is a subgroup of  $G$  normalized by the group  $SL(\mathbb{R}^p) \times SL(\mathbb{R}^q) \times SL(\mathbb{R}^r)$ . Note that in these two tables, the center of the diagonal blocks is not important by Corollary 3.3.

$H_1: \begin{pmatrix} * & 0 \\ 0 & I \end{pmatrix}$	$H_2: \begin{pmatrix} * & * \\ 0 & I \end{pmatrix}$	$H_3: \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$	$H_4: \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$
$p \leq q + 1$	$p = 1$	$p = q = 1$	$p \leq q + 1$ $q \leq p + 1$

Table 1: The criterion  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$  when  $G = \mathrm{SL}(p + q, \mathbb{R})$

*Remark 5.7.* It is rather easy to guess the inequalities in Table 2. Here is the heuristic recipe: there is one inequality for each non-identity diagonal block. The left-hand side of this inequality is given by the size of this diagonal block, while the right-hand side can be guessed by looking at the size of the zero blocks on the right and on the top of it.

We will just explain the proof for the group  $H = H_{11}$  in Table 2. The other cases are similar.

**Corollary 5.8.** *Let  $G = \mathrm{SL}(p + q + r, \mathbb{R})$  and  $H$  the subgroup of matrices*

*$\begin{pmatrix} \alpha & 0 & z \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$  with  $\alpha \in \mathrm{GL}(p, \mathbb{R})$ ,  $\beta \in \mathrm{GL}(q, \mathbb{R})$ ,  $\gamma \in \mathrm{GL}(r, \mathbb{R})$ ,  $z \in \mathrm{M}(p, r; \mathbb{R})$ . Then  $L^2(G/H)$  is  $G$ -tempered if and only if  $p \leq q + 1$ ,  $q \leq p + r + 1$ ,  $r \leq q + 1$ .*

*Proof of Corollary 5.8.* We denote by  $\mathfrak{a}$  the Lie algebra of diagonal matrices

$$\mathfrak{a} = \{Y = (x, y, z) \in \mathbb{R}^p \oplus \mathbb{R}^q \oplus \mathbb{R}^r \mid \mathrm{Trace}(Y) = 0\}.$$

We only need to check the criterion (1.1) on the chamber

$$\mathfrak{a}_+ = \{Y = (x, y, z) \in \mathfrak{a} \mid x, y \text{ and } z \text{ have non-decreasing coordinates}\}.$$

We recall that  $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$  and we compute for  $Y \in \mathfrak{a}_+$ ,

$$\rho_{\mathfrak{h}}(Y) = \sum_{i=1}^p a_i x_i + \sum_{j=1}^q b_j y_j + \sum_{k=1}^r c_k z_k,$$

where  $a_i := 2i - p - 1$ ,  $b_j := 2j - q - 1$ ,  $c_k := 2k - r - 1$ , and

$$\rho_{\mathfrak{q}}(Y) = \sum_{i,j} |x_i - y_j| + \sum_{j,k} |y_j - z_k|.$$

Assume first that the criterion  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$  is satisfied on  $\mathfrak{a}$ . It is then also satisfied on  $\mathbb{R}^{p+q+r}$ . Applying it successively to the three vectors  $Y = e_p$ ,

$H_1 : \begin{pmatrix} * & 0 & * \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ $p \leq q + 1$	$H_2 : \begin{pmatrix} I & 0 & * \\ 0 & * & 0 \\ 0 & 0 & I \end{pmatrix}$ $q \leq p + r + 1$	$H_3 : \begin{pmatrix} I & * & * \\ 0 & * & 0 \\ 0 & 0 & I \end{pmatrix}$ $q \leq r + 1$	$H_4 : \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$ $p = q = r = 1$
$H_5 : \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & I \end{pmatrix}$ $p \leq q + r + 1$ $q \leq p + r + 1$	$H_6 : \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & I \end{pmatrix}$ $p \leq q + 1$ $q \leq p + r + 1$	$H_7 : \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & I \end{pmatrix}$ $p = 1$ $q \leq r + 1$	$H_8 : \begin{pmatrix} * & 0 & * \\ 0 & I & 0 \\ 0 & 0 & * \end{pmatrix}$ $p \leq q + 1$ $r \leq q + 1$
$H_9 : \begin{pmatrix} I & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$ $q \leq r + 1$ $r \leq q + 1$	$H_{10} : \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$ $p \leq q + r + 1$ $q \leq p + r + 1$ $r \leq p + q + 1$	$H_{11} : \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$ $p \leq q + 1$ $q \leq p + r + 1$ $r \leq q + 1$	$H_{12} : \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$ $p = 1$ $q \leq r + 1$ $r \leq q + 1$

Table 2: The criterion  $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$  when  $G = \mathrm{SL}(p + q + r, \mathbb{R})$

$Y = e_{p+q}$  and  $Y = e_{p+q+r}$  of the standard basis  $e_1, \dots, e_{p+q+r}$  of  $\mathbb{R}^{p+q+r}$ , one gets successively the three inequalities  $p \leq q + 1$ ,  $q \leq p + r + 1$ ,  $r \leq q + 1$ .

Assume now that these three inequalities are satisfied. Note that

$$\begin{aligned} \rho_{\mathfrak{q}}(Y) &\geq \sum_{a_i > b_j} (x_i - y_j) + \sum_{b_j > a_i} (y_j - x_i) + \sum_{b_j > c_k} (y_j - z_k) + \sum_{c_k > b_j} (z_k - y_j) \\ &= \sum_{i=1}^p \ell_i x_i + \sum_{j=1}^q m_j y_j + \sum_{k=1}^r n_k z_k, \quad \text{where} \end{aligned}$$

$$\begin{aligned} \ell_i &= |\{j \mid b_j < a_i\}| - |\{j \mid b_j > a_i\}| \\ m_j &= |\{i \mid a_i < b_j\}| - |\{i \mid a_i > b_j\}| + |\{k \mid c_k < b_j\}| - |\{k \mid c_k > b_j\}| \\ n_k &= |\{j \mid b_j < c_k\}| - |\{j \mid b_j > c_k\}|. \end{aligned}$$

Since  $p \leq q + 1$ , one has  $\ell_i = a_i$  for all  $1 \leq i \leq p$ .

Since  $r \leq q + 1$ , one has  $n_k = c_k$  for all  $1 \leq k \leq r$ .

For  $1 \leq j \leq q$ , one has  $m_{q+1-j} = -m_j$  and, when  $j > q/2$ ,

$$m_j = \min(b_j, p) + \min(b_j, r).$$

Since  $q \leq p + r + 1$ , one has  $m_j \geq b_j$  for all  $j > q/2$ . Then using the fact that the  $y_j$ s are non-decreasing functions of  $j$ , one gets, for  $Y$  in  $\mathfrak{a}_+$ ,

$$\rho_q(Y) - \rho_{\mathfrak{h}}(Y) = \sum_{j=1}^q (m_j - b_j) y_j \geq 0.$$

This proves that the criterion  $\rho_{\mathfrak{h}} \leq \rho_q$  is satisfied.  $\square$

Some of the subgroups in Table 2 appear naturally in analyzing the tensor product representations of  $SL(n, \mathbb{R})$  as below.

### 5.3 Tensor product of nontempered representations

Suppose  $\Pi$  and  $\Pi'$  are unitary representations of  $G$ . The tensor product representation  $\Pi \otimes \Pi'$  is tempered if  $\Pi$  or  $\Pi'$  is tempered. In contrast,  $\Pi \otimes \Pi'$  may be and may not be tempered when both  $\Pi$  and  $\Pi'$  are nontempered.

For instance, let  $n = n_1 + \cdots + n_k$  be a partition, and we consider the (degenerate) principal series representation  $\Pi_{n_1, \dots, n_k} := \text{Ind}_{P_{n_1, \dots, n_k}}^G(\mathbf{1})$  of  $G = SL(n, \mathbb{R})$ , where  $P_{n_1, \dots, n_k}$  is the standard parabolic subgroup with Levi subgroup  $S(GL(n_1, \mathbb{R}) \times \cdots \times GL(n_k, \mathbb{R}))$ . Then  $\Pi_{n_1, \dots, n_k}$  is tempered iff  $k = n$  and  $n_1 = \cdots = n_k = 1$ . Here are some examples of the temperedness criterion (1.1) applied to the tensor product of two such representations.

**Proposition 5.9.** *Let  $0 \leq k, l \leq n$  and  $a + b + c = n$ .*

- (1)  $\Pi_{k, n-k} \otimes \Pi_{n-l, l}$  is tempered iff  $|k - l| \leq 1$  and  $|k + l - n| \leq 1$ .
- (2)  $\Pi_{a, b, c} \otimes \Pi_{b+c, a}$  is tempered iff  $\max(b, c) - 1 \leq a \leq b + c + 1$ .
- (3)  $\Pi_{a, b, c} \otimes \Pi_{c, b, a}$  is tempered iff  $2 \max(a, b, c) \leq n + 1$ .

*Proof.* For any parabolic subgroups  $P$  and  $P'$  of  $G$ , there exists an element  $w \in G$  such that  $PwP'$  is open dense in  $G$ , and thus the tensor product  $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_{P'}^G(\mathbf{1})$  is unitarily equivalent to the regular representation in  $L^2(G/H)$  by the Mackey theory, where  $H = w^{-1}Pw \cap P'$ . In the above cases, we have the following unitary equivalences:

$$\begin{aligned} \Pi_{k, n-k} \otimes \Pi_{n-l, l} &\simeq L^2(G/H_{12}) && \text{with } (p, q, r) = (|k - l|, \min(k, l), n - \max(k, l)), \\ \Pi_{a, b, c} \otimes \Pi_{b+c, a} &\simeq L^2(G/H_{11}) && \text{with } (p, q, r) = (b, a, c), \\ \Pi_{a, b, c} \otimes \Pi_{c, b, a} &\simeq L^2(G/H_{10}) && \text{with } (p, q, r) = (a, b, c), \end{aligned}$$

whence Proposition follows from Table 2 in Section 5.2.  $\square$

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