

BIRATIONAL BOUNDEDNESS OF RATIONALLY CONNECTED CALABI–YAU 3-FOLDS

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ABSTRACT. We prove that rationally connected Calabi–Yau 3-folds with kawamata log terminal (klt) singularities form a birationally bounded family, or more generally, rationally connected 3-folds of ϵ -CY type form a birationally bounded family for $\epsilon > 0$. Moreover, we show that the set of ϵ -lc log Calabi–Yau pairs (X, B) with coefficients of B bounded away from zero is log bounded modulo flops. As a consequence, we deduce that rationally connected klt Calabi–Yau 3-folds with mld bounded away from 1 are bounded modulo flops.

1. INTRODUCTION

Throughout this paper, we work over an uncountable algebraically closed field of characteristic 0, for instance, the complex number field \mathbb{C} .

A normal projective variety X is a *Fano* (resp. *Calabi–Yau*) variety if $-K_X$ is ample (resp. $K_X \equiv 0$). According to the Minimal Model Program, Fano varieties and Calabi–Yau varieties form fundamental classes in birational geometry as building blocks of algebraic varieties. Hence, it is interesting to ask whether such kinds of varieties satisfy any finiteness properties, namely, whether they can be parametrized by finitely many families. In this regard, Birkar [Bir16b, Bir16c] recently showed that the set of ϵ -lc Fano varieties of dimension d forms a bounded family for fixed $\epsilon > 0$ and d . This is known as the Borisov–Alexeev–Borisov (BAB) Conjecture.

However, Calabi–Yau varieties in general are not bounded in the category of algebraic varieties: for example, it is well-known that there are infinitely many algebraic families of projective K3 surfaces. Nonetheless, K3 surfaces all fit into a unique topological family, once we consider also the non-algebraic ones. A similar picture holds for abelian varieties. In higher dimension, the situation is even more varied and there are classes of Calabi–Yau varieties for which boundedness is still a hard unresolved question.

Therefore, in this article, rather than considering Calabi–Yau varieties in full generality, we focus on a special class: that of rationally connected Calabi–Yau varieties. Recall that a variety is *rationally connected* if any two general points can be connected by a rational curve. Rationally connected varieties are very close to Fano varieties and all klt Fano varieties are rationally connected by Zhang [Zha06]. Hence, rationally connected

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Calabi–Yau varieties can be viewed as those Calabi–Yau varieties which behave most like Fano varieties. This class of varieties has received scarce attention so far due to the technical difficulties involved in its treatment. The recent developments in the study of boundedness of algebraic varieties, e.g., [HMX14, Jia15, Bir16b, Bir16c, DS16], provide new tools to approach the issue for this class of Calabi–Yau varieties.

In dimension two, klt Calabi–Yau surfaces (also known as log Enriques surfaces) with worse than du Val singularities are rationally connected Calabi–Yau varieties (see [AM04, Proof of Lemma 1.4] or [Bla95, Theorem D(1)]) and form a bounded family by a result of Alexeev [Ale94, Corollary 6.10]. The works of Blache and Zhang [Bla95, Zha91, Zha93] provide a systematical study of such surfaces together with many examples. Interesting examples of rationally connected klt Calabi–Yau 3-fold can also be found in [OT15] where the authors show that E^3/G is a rational klt Calabi–Yau 3-fold. Here E is the elliptic curve corresponding to the regular hexagonal lattice in \mathbb{C} and G is the group generated by an automorphism corresponding to multiplication by a primitive third root of unity on the lattice, see [OT15, Remark 2.8] for details. For more examples, we refer to [Cam11, COT14, CT15, COV15]. Unfortunately, there are not many known examples of rationally connected klt Calabi–Yau varieties of dimension $d \geq 3$. We expect that one can construct more examples by considering finite quotients of smooth Calabi–Yau varieties (cf. [KL09]). In the Appendix A, we provide a sufficient condition for a dlt log Calabi–Yau pair being rationally connected.

Another reason for interest is that rationally connected Calabi–Yau varieties appear as bases of elliptic Calabi–Yau manifolds. For example, in [Ogu93], Oguiso proved that the base of an elliptic Calabi–Yau 3-fold X is a rationally connected Calabi–Yau surface when a semiample divisor D defining the fibration satisfies the extra condition $c_2(X) \cdot D = 0$. Oguiso provided explicit examples of such fibrations. In general, the structure theorem proved in [DS16, Theorem 3.2] shows that they appear as bases of elliptic Calabi–Yau varieties is some rather special situation. Roughly speaking, if the base is a rationally connected Calabi–Yau variety then the fibration is isotrivial, up to a birational modification, and the total space behaves like a product, or more precisely it is of product type. See [DS16] for more details.

Motivated by Alexeev’s work in dimension two, we may expect that boundedness holds for rationally connected klt Calabi–Yau varieties and consider the following conjecture.

Conjecture 1.1. *Fix a positive integer d . The set of all rationally connected klt Calabi–Yau varieties of dimension d forms a bounded family.*

As it is not hard to show that the singularities of rationally klt Calabi–Yau varieties have bounded discrepancies (cf. Lemma 3.12), Conjecture 1.1 is a special case of the following conjecture generalizing the BAB Conjecture.

Conjecture 1.2 (cf. [Ale94], [MP04, Conjecture 3.9]). *Fix a positive real number ϵ and a positive integer d . The set of all X satisfying*

- (1) $\dim X = d$,
- (2) *there exists a boundary B such that (X, B) is ϵ -klt,*
- (3) $-(K_X + B)$ *is nef, and*

(4) X is rationally connected,
forms a bounded family.

In dimension two, Conjecture 1.2 was proven by Alexeev [Ale94, Theorem 6.9]. However, it still remains open even in dimension three. Conjecture 1.2 is already very interesting in the case when $K_X + B \equiv 0$ which can be formulated separately as the following statement.

Conjecture 1.3. *Fix a positive real number ϵ and a positive integer d . The set of rationally connected d -dimensional varieties of ϵ -CY type forms a bounded family.*

Here a normal projective variety X is of ϵ -CY type if there exists an effective \mathbb{R} -divisor B such that (X, B) is an ϵ -klt log Calabi–Yau pair.

Di Cerbo and Svaldi [DS16, Theorem 1.3] recently gave a partial answer to the birational boundedness for Conjecture 1.2 in the case when $K_X + B \equiv 0$, coefficients of B belong to a fixed DCC set, $B \neq 0$, and $\dim X \leq 4$. However, to conclude Conjecture 1.1 from Conjecture 1.2, it is necessary to consider the case when coefficients of B do not belong to a fixed DCC set or $B = 0$.

The goal of this article is to study Conjecture 1.3 in dimension three and to establish several birational boundedness results.

The first result provides an affirmative answer to the birational boundedness in Conjecture 1.1 for dimension three.

Theorem 1.4. *The set of all rationally connected klt Calabi–Yau 3-folds forms a birationally bounded family.*

Theorem 1.4 is a special case of the following theorem, which gives an affirmative answer to the birational boundedness for Conjecture 1.3 in dimension three.

Theorem 1.5. *Fix a positive real number ϵ . The set of rationally connected 3-folds of ϵ -CY type forms a birationally bounded family.*

Theorem 1.5 can be viewed as a generalization of [DS16, Theorem 1.3] in dimension 3, and also a generalization of the birational BAB conjecture in dimension 3 [Jia15].

We moreover focus on ϵ -lc log Calabi–Yau pairs (X, B) such that coefficients of B are bounded from below. For such pairs, we show that log boundedness modulo flops holds: this is a stronger version of (log) birational boundedness, see Section 2.4 for the definition.

Theorem 1.6 (=Corollary 4.2). *Fix positive real numbers ϵ, δ . Then, the set of pairs (X, B) satisfying*

- (1) (X, B) is an ϵ -lc log Calabi–Yau pair of dimension 3,
- (2) X is rationally connected,
- (3) $B > 0$, and the coefficients of B are at least δ

forms a log bounded family modulo flops.

In fact, this result can be generalized to any dimension modulo Conjecture 1.3 in lower dimensions, see Theorem 4.1. It is a consequence of a relative version of the Special BAB [Bir16b, Theorem 1.4] (see Theorem 4.6).

We apply this result to show that rationally connected klt Calabi–Yau 3-folds with mld bounded away from 1 are bounded modulo flops.

Theorem 1.7 (=Theorem 5.1). *Fix $0 < c < 1$. Let \mathcal{D} be the set of varieties X such that*

- (1) *X is a rationally connected Calabi–Yau 3-fold, and*
- (2) *$0 < \text{mld}(X) < c$.*

Then \mathcal{D} is bounded modulo flops.

This theorem has several interesting immediate applications to the boundedness problem. We show that for rationally connected klt Calabi–Yau 3-folds, the boundedness modulo flops is equivalent to the boundedness of global indices (Corollary 5.2) and that the boundedness modulo flops holds modulo 1-Gap conjecture for minimal log discrepancies on 3-folds, which is a special case of Shokurov’s ACC conjecture (Corollary 5.5). Finally, we establish that the boundedness modulo flops holds for those rationally connected klt Calabi–Yau varieties which are quasi-étale quotients of irregular varieties (Corollary 5.9). We refer the reader to Section 5 for details.

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2. PRELIMINARIES

We adopt the standard notation and definitions in [KMM85] and [KM98], and will freely use them.

2.1. Pairs, singularities, and mld. A *log pair* (X, B) consists of a normal projective variety X and an effective \mathbb{R} -divisor B on X such that $K_X + B$ is \mathbb{R} -Cartier.

Let $f: Y \rightarrow X$ be a log resolution of the log pair (X, B) , write

$$K_Y = f^*(K_X + B) + \sum a_i F_i,$$

where $\{F_i\}$ are distinct prime divisors. For a non-negative real number ϵ , the log pair (X, B) is called

- (a) *ϵ -kawamata log terminal* (ϵ -klt, for short) if $a_i > -1 + \epsilon$ for all i ;
- (b) *ϵ -log canonical* (ϵ -lc, for short) if $a_i \geq -1 + \epsilon$ for all i ;
- (c) *terminal* if $a_i > 0$ for all f -exceptional divisors F_i and all f ;
- (d) *canonical* if $a_i \geq 0$ for all f -exceptional divisors F_i and all f .

Usually we write X instead of $(X, 0)$ in the case $B = 0$. Note that 0-klt (resp., 0-lc) is just klt (resp., lc) in the usual sense. Also note that ϵ -lc singularities only make sense if $\epsilon \in [0, 1]$, and ϵ -klt singularities only make sense if $\epsilon \in [0, 1)$,

The *log discrepancy* of the divisor F_i is defined to be $a(F_i, X, B) = 1 + a_i$. It does not depend on the choice of the log resolution f . F_i is called a *non-lc place* of (X, B) if $a_i < -1$. A subvariety $V \subset X$ is called a *non-lc center* of (X, B) if it is the image of a non-lc place. The *non-lc locus* $\text{Nlc}(X, B)$ is the union of all non-lc centers of (X, B) .

Let (X, B) be an lc pair and $Z \subset X$ an irreducible closed subset with η_Z the generic point of Z . The *minimal log discrepancy* of (X, B) over Z is defined as

$$\text{mld}_Z(X, B) = \inf\{a(E, X, B) \mid \text{center}_X(E) \subset Z\},$$

and the minimal log discrepancy of (X, B) at η_Z is defined as

$$\text{mld}_{\eta_Z}(X, B) = \inf\{a(E, X, B) \mid \text{center}_X(E) = Z\}.$$

For simplicity, we just write $\text{mld}(X, B)$ instead of $\text{mld}_X(X, B)$.

2.2. Log Calabi–Yau pairs. The log pair (X, B) is called a *log Calabi–Yau pair* if $K_X + B \equiv 0$. Recall that if (X, B) is lc, this is equivalent to $K_X + B \sim_{\mathbb{R}} 0$ by [Gon13].

A normal projective variety X is of ϵ -CY type if there exists an effective \mathbb{R} -divisor B such that (X, B) is an ϵ -klt log Calabi–Yau pair.

2.3. Terminal Mori fibrations. A projective morphism $f: X \rightarrow Z$ between normal projective varieties is called a *terminal Mori fibration* (or *terminal Mori fiber space*) if

- (1) X is \mathbb{Q} -factorial with terminal singularities;
- (2) f is a *contraction*, i.e., $f_*\mathcal{O}_X = \mathcal{O}_Z$;
- (3) $-K_X$ is ample over Z ;
- (4) $\rho(X/Z) = 1$;
- (5) $\dim X > \dim Z$.

We say that X is endowed with a *terminal Mori fibration structure* if there exists a terminal Mori fibration $X \rightarrow Z$. In particular, in this situation, X has at most \mathbb{Q} -factorial terminal singularities by definition.

2.4. Bounded pairs. A collection of varieties \mathcal{D} is said to be *bounded* (resp., *rationally bounded*, or *bounded in codimension one*) if there exists $h: \mathcal{Z} \rightarrow S$ a projective morphism of schemes of finite type such that each $X \in \mathcal{D}$ is isomorphic (resp., birational, or isomorphic in codimension one) to \mathcal{Z}_s for some closed point $s \in S$.

We say that a collection of log pairs \mathcal{D} is *log rationally bounded* (resp., *log bounded*, or *log bounded in codimension one*) if there is a quasi-projective scheme \mathcal{Z} , a reduced divisor \mathcal{E} on \mathcal{Z} , and a projective morphism $h: \mathcal{Z} \rightarrow S$, where S is of finite type and \mathcal{E} does not contain any fiber, such that for every $(X, B) \in \mathcal{D}$, there is a closed point $s \in S$ and a birational map $f: \mathcal{Z}_s \dashrightarrow X$ (resp., isomorphic, or isomorphic in codimension one) such that \mathcal{E}_s contains the support of $f_*^{-1}B$ and any f -exceptional divisor (resp., \mathcal{E}_s coincides with the support of $f_*^{-1}B$, \mathcal{E}_s coincides with the support of $f_*^{-1}B$).

Moreover, if \mathcal{D} is a set of klt Calabi–Yau varieties (resp., klt log Calabi–Yau pairs), then it is said to be *bounded modulo flops* (resp., *log bounded modulo flops*) if it is (log) bounded in codimension one, and each fiber \mathcal{Z}_s corresponding to X in the definition is normal projective, and $K_{\mathcal{Z}_s}$ is \mathbb{Q} -Cartier (resp., $K_{\mathcal{Z}_s} + f_*^{-1}B$ is \mathbb{R} -Cartier).

Note that if \mathcal{D} is a set of klt log Calabi–Yau pairs which is log bounded modulo flops, and $(X, B) \in \mathcal{D}$ with a birational map $f: \mathcal{Z}_s \dashrightarrow X$ isomorphic in codimension one as in the definition, then $(\mathcal{Z}_s, f_*^{-1}B)$ is again a klt log Calabi–Yau pair by the Negativity Lemma. Moreover, (X, B) is ϵ -lc if and only if $(\mathcal{Z}_s, f_*^{-1}B)$ is so. A similar statement holds for \mathcal{D} a set of klt Calabi–Yau varieties.

Here the name “modulo flops” comes from the fact that, if we assume that X and \mathcal{Z}_s are both \mathbb{Q} -factorial, then they are connected by flops by running a $(K_X + B + \delta f_*H)$ -MMP where H is an ample divisor on \mathcal{Z}_s and δ is a sufficiently small positive number (cf. [BCHM10, Kaw08]).

2.5. Volume. Let X be a d -dimensional projective variety and D a Cartier divisor on X . The *volume* of D is the real number

$$\mathrm{Vol}(X, D) = \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.$$

For more backgrounds on the volume, see [Laz04, 2.2.C]. By the homogenous property and continuity of the volume, we can extend the definition to \mathbb{R} -Cartier \mathbb{R} -divisors. Moreover, if D is a nef \mathbb{R} -divisor, then $\mathrm{Vol}(X, D) = D^d$. If D is a non- \mathbb{R} -Cartier \mathbb{R} -divisor, we may take a \mathbb{Q} -factorialization of X , i.e., a birational morphism $\phi: Y \rightarrow X$ which is isomorphic in codimension one and Y is \mathbb{Q} -factorial, then $\mathrm{Vol}(X, D) := \mathrm{Vol}(Y, \phi_*^{-1}D)$. Note that \mathbb{Q} -factorializations always exist for varieties that admit klt pairs (cf. [BCHM10, Corollary 1.4.3]).

It is easy to see the following inequality for volumes by comparing global sections by exact sequences.

Lemma 2.1 ([Jia15, Lemma 2.5], [DS16, Lemma 4.2]). *Let X be a projective normal variety, D a \mathbb{R} -Cartier \mathbb{R} -divisor, and S a base-point free normal Cartier prime divisor. Then for any real number $q > 0$,*

$$\mathrm{Vol}(X, D + qS) \leq \mathrm{Vol}(X, D) + q(\dim X) \mathrm{Vol}(S, D|_S + qS|_S).$$

2.6. Length of extremal rays. Recall the following result on the length of extremal rays due to Kawamata.

Theorem 2.2 ([Kaw91]). *Let (X, B) be a klt pair. Then every $(K_X + B)$ -negative extremal ray R is generated by the class of a rational curve C such that*

$$0 < -(K_X + B) \cdot C \leq 2 \dim X.$$

However, as we need to deal with non-klt pairs in the applications, we will use the following generalization of this theorem for log pairs which was proven by Fujino.

Theorem 2.3 ([Fuj11, Theorem 1.1(5)]). *Let (X, B) be a log pair. Let $i: \text{Nlc}(X, B) \rightarrow X$ be the inclusion of the non-lc locus in X . Fix a $(K_X + B)$ -negative extremal ray R . Assume that*

$$R \cap \overline{NE}(X)_{\text{Nlc}(X, B)} = \{0\},$$

where

$$\overline{NE}(X)_{\text{Nlc}(X, B)} = \text{Im}(i_*: \overline{NE}(\text{Nlc}(X, B)) \rightarrow \overline{NE}(X)).$$

Then R is generated by a rational curve C such that

$$0 < -(K_X + B) \cdot C \leq 2 \dim X.$$

3. BIRATIONAL BOUNDEDNESS OF RATIONALLY CONNECTED CALABI–YAU 3-FOLDS

The goal of this section is to prove Theorems 1.4 and 1.5.

3.1. Sketch of the proof. The strategy of proof for Theorems 1.4 and 1.5 originates from [Jia15]. Using the minimal model program, it suffices to work with varieties of CY-type endowed with a terminal Mori fibration structure. The precise result, whose proof will be given in Section 3.2, is the following.

Proposition 3.1 (cf. [Jia14, Proof of Theorem 2.3]). *Fix a positive real number ϵ and a positive integer d . Every d -dimensional rationally connected variety X of ϵ -CY type is birational to a d -dimensional rationally connected variety X' of ϵ -CY type with a terminal Mori fibration structure.*

Now let X be a 3-fold of ϵ -CY type with a terminal Mori fibration $f: X \rightarrow Z$. If $-K_X$ is big, then X is of Fano type, and the (birational) boundedness follows from the BAB conjecture in dimension 3, see [Jia15, Corollary 1.8] or [Bir16c, Corollary 1.2]. Thus, we only need to consider the case when $-K_X$ is not big. Since $-K_X$ is ample over Z , this implies that $\dim Z > 0$. In this case, we prove the following theorem.

Theorem 3.2. *Fix a positive real number ϵ . Then there exist positive integers $n = n(\epsilon)$, $c = c(\epsilon)$, and $v = v(\epsilon)$ depending only on ϵ , with the following property:*

Assume X is a rationally connected 3-fold of ϵ -CY type endowed with a terminal Mori fibration $f: X \rightarrow Z$ such that $-K_X$ is not big.

- (1) *If $\dim Z = 1$ (i.e. $Z = \mathbb{P}^1$), take a general fiber F of f , then*
 - (1.1) $-K_X + nF$ *is ample,*
 - (1.2) $|-3K_X + 8nF|$ *defines a birational map, and*
 - (1.3) $(-K_X + nF)^3 \leq v$.
- (2) *If $\dim Z = 2$, then there exists a very ample divisor H on Z such that*
 - (2.1) $H^2 \leq c$,
 - (2.2) $-K_X + nf^*H$ *is ample,*
 - (2.3) $|-3K_X + 8nf^*H|$ *defines a birational map, and*
 - (2.4) $(-K_X + nf^*H)^3 \leq v$.

The proof of Theorem 3.2 will be given in Section 3.3, while the proof of Theorems 1.4 and 1.5 will be given in Section 3.4.

3.2. Proof of Proposition 3.1. In this subsection, for the reader's convenience, we recall the proof of Proposition 3.1.

Proof of Proposition 3.1. Fix a positive real number ϵ and a positive integer d . Let X be a rationally connected variety of ϵ -CY type of dimension d . By [BCHM10, Corollary 1.4.3], taking a terminalization of (X, B) , we have a birational morphism $\pi: X_1 \rightarrow X$ where $K_{X_1} + B_1 = \pi^*(K_X + B)$, $B_1 > 0$ is an effective \mathbb{R} -divisor, and X_1 is \mathbb{Q} -factorial terminal. Here $K_{X_1} + B_1 \equiv 0$ and (X_1, B_1) is ϵ -klt; moreover, X_1 is again rationally connected. In particular, K_{X_1} is not pseudo-effective since X_1 is terminal.

We can run a K_{X_1} -MMP with scaling of an ample divisor on X_1 , which terminates with a Mori fiber space $X' \rightarrow T$, cf. [BCHM10, Corollary 1.3.3]. As we run a K_{X_1} -MMP, X' is again \mathbb{Q} -factorial terminal and rationally connected. By the Negativity Lemma, $K_{X'} + B' \equiv 0$ and (X', B') is ϵ -klt where B' is the strict transform of B_1 on X' . Now X' is an d -dimensional rationally connected variety of ϵ -CY type with a terminal Mori fibration structure by construction, which is birational to X . This concludes the proof. \square

3.3. Proof of Theorem 3.2. In this subsection, we prove Theorem 3.2. That will follow directly from Lemmas 3.7, 3.9, and 3.10 below.

3.3.1. Setting. Fix a positive real number ϵ . Let X be a rationally connected 3-fold of ϵ -CY type with a terminal Mori fibration $f: X \rightarrow Z$ such that $-K_X$ is not big and $\dim Z > 0$. Suppose (X, B) is an ϵ -klt log Calabi–Yau pair. We define a base-point free divisor G on X , coming from the boundedness of the base Z , in the following way.

When $\dim Z = 1$, then $Z = \mathbb{P}^1$. In this case, G is defined to be a general fiber of f , which is a smooth del Pezzo surface since X is terminal. $(G, B|_G)$ is an ϵ -klt log Calabi–Yau pair by adjunction.

If $\dim Z = 2$, then the collection of such Z forms a bounded family. In fact, since X is of ϵ -CY type, there exists an effective \mathbb{R} -divisor Δ such that (Z, Δ) is δ -klt, $K_Z + \Delta \sim_{\mathbb{R}} 0$, see [Bir16a, Corollary 1.7]. Here $\delta = \delta(\epsilon)$ is a positive number depending only on ϵ . Hence Z is rationally connected and of δ -CY type. The boundedness then follows from the solution to the BAB Conjecture in dimension 2 (see [Ale94, Theorem 6.9] or [AM04, Lemma 1.4]). This implies that there is a positive integer $c = c(\epsilon)$ depending only on ϵ and we can find a general very ample divisor H on Z satisfying $H^2 \leq c$. We take $G = f^*H$, then G is a conic bundle over the curve H (i.e. $-K_G$ is ample). Note that H and G are smooth since H is general and X is terminal. Also $(G, B|_G)$ is ϵ -klt and $-(K_G + B|_G) + G|_G \sim_{\mathbb{R}} 0$ by adjunction. Moreover, $G|_G = f|_G^*(H|_H) \equiv (H^2)F$ where F is a general fiber of $f|_G$. Finally, since $\rho(X/Z) = 1$, $B^v \sim_{\mathbb{R}, f} 0$, where B^v is the f -vertical part of B , and hence $B^v|_G \sim_{\mathbb{R}, f|_G} 0$.

3.3.2. A boundedness theorem on surfaces. We recall the following boundedness theorem for surfaces from [Jia15]. The ideas behind its proof are inspired by the solution to the BAB Conjecture in dimension two given by Alexeev–Mori [AM04].

Theorem 3.3 ([Jia15, Theorem 5.1]). *Fix a positive integer m and a positive real number ϵ . Then there exists a number $\lambda' = \lambda'(m, \epsilon) > 0$ depending only on m and ϵ satisfying the following property:*

Assume that T is a projective smooth surface and $B = \sum_i b_i B^i$ an effective \mathbb{R} -divisor on T where each B^i is a prime divisor such that

- (1) (T, B) is ϵ -klt, but $(T, (1+t)B)$ is not klt for some $t > 0$,
- (2) $K_T + B \equiv N - A$ where A is an ample \mathbb{R} -divisor and N is a nef \mathbb{R} -divisor on T ,
- (3) $\sum_i b_i \leq m$,
- (4) $B^2 \leq m$, $(B \cdot N) \leq m$.

Then $t > \lambda'$.

For the proof, we refer to [Jia15, Theorem 5.1]. By applying Theorem 3.3 to our situation, we can show the following theorem, which is a simple modification of [Jia15, Theorem 1.7].

Theorem 3.4 (cf. [Jia15, Theorem 1.7]). *Fix a positive real number ϵ . Then there exists a number $\lambda = \lambda(\epsilon) > 0$ depending only on ϵ , satisfying the following property:*

- (1) *If (G, B) is an ϵ -klt log Calabi–Yau pair and G is a smooth del Pezzo surface, then $(G, (1+t)B)$ is klt for $0 < t \leq \lambda$.*
- (2) *If $f: G \rightarrow H$ is a conic bundle from a smooth surface G to a smooth curve, (G, B) is an ϵ -klt pair, $-(K_G + B) + kF \equiv 0$ for some integer $k \leq c$, and $B^v \sim_{\mathbb{R}, f} 0$, then $(X, (1+t)B)$ is klt for $0 < t \leq \lambda$. Here F is a general fiber of f , B^v is the f -vertical part of B , and c is the number depending only on ϵ defined in Section 3.3.1.*

Proof. (1) As G is a del Pezzo surface, it follows that $-K_G$ is ample, $-3K_G$ is very ample, and $(-K_G)^2 \leq 9$. Write $B = \sum_i b_i B^i$, then

$$\sum_i b_i \leq B \cdot (-K_G) = (-K_G)^2 \leq 9;$$

$$B^2 = (-K_G)^2 \leq 9.$$

As (G, B) is ϵ -klt, we can apply Theorem 3.3 for $A = N = -K_G$ and obtain that $(G, (1+t)B)$ is klt for all $0 < t \leq \lambda'(9, \epsilon)$.

(2) Suppose that $f: G \rightarrow H$ is a conic bundle from a smooth surface G to a smooth curve, (G, B) is an ϵ -klt pair, $-(K_G + B) + kF \equiv 0$ for some integer $k \leq c$, and $B^v \sim_{\mathbb{R}, f} 0$. The assumption $B^v \sim_{\mathbb{R}, f} 0$ implies that we may write $B = \sum_i b_i B^i + \sum_j c_j F^j$, where B^i is a curve not contained in a fiber of f for all i , and F^j is a fiber of f for any j . This condition is crucial in the following claim. Note that each F^j is reduced and contains at most 2 irreducible components since f is a conic bundle. Moreover, recall that $B \cdot F = (-K_G) \cdot F = 2$ and $(-K_G)^2 \leq 8$ for the conic bundle G .

Claim 3.5. *The sum of coefficients of B is bounded from above by*

$$\sum_i b_i + \sum_j 2c_j \leq 8 + 2k \leq 8 + 2c.$$

Proof of Claim 3.5. First of all, we have

$$\sum_i b_i \leq \sum_i b_i (B^i \cdot F) = (B \cdot F) = 2.$$

Hence, it suffices to show that $\sum_j c_j \leq 3 + k$. Assume, to the contrary, that $w = \sum_j c_j > 3 + k$. Then, for any choice of three sufficiently general fibers F_1, F_2, F_3 of f , consider the pair

$$K_G + \sum_i b_i B^i + \left(1 - \frac{3+k}{w}\right) \sum_j c_j F^j + F_1 + F_2 + F_3 \sim_{\mathbb{R}} 0.$$

Applying [Kol13, Theorem 4.37] to $X = G$, Z a point, and $D = F_1 + F_2 + F_3$, we conclude that D has 2 connected components, which is obviously absurd. \square

Moreover, we have

$$\begin{aligned} B^2 &= (kF - K_G)^2 = 4k + (-K_G)^2 \leq 4c + 8; \\ (B \cdot kF) &= (-K_G) \cdot kF = 2k \leq 2c. \end{aligned}$$

Applying Theorem 3.3 for $N = kF + A$, where A is a sufficiently small ample \mathbb{Q} -divisor such that $(A \cdot B) \leq c$, and fixing $m = 4c + 8$, we obtain that $(G, (1+t)B)$ is klt for all $0 < t \leq \lambda'(4c + 8, \epsilon)$. \square

We propose the following conjecture generalizing Theorem 3.4 to higher dimension.

Conjecture 3.6 (cf. [Jia15, Conjecture 1.13]). *Fix a positive real number ϵ and a positive integer d . There exists a positive number $t = t(d, \epsilon)$ depending only on d and ϵ , such that for any d -dimensional ϵ -klt log Calabi–Yau pair (X, B) , $(X, (1+t)B)$ is klt.*

3.3.3. Effective construction of an ample divisor.

Lemma 3.7. *Under the setting introduced in Section 3.3.1, there exists a positive integer $n = n(\epsilon)$ depending only on ϵ such that $-K_X + kG$ is ample for all $k \geq n$.*

Proof. By construction, $(G, B|_G)$ satisfies one of the two conditions in Theorem 3.4. Hence $(G, (1+\lambda)B|_G)$ is klt for $\lambda > 0$ and λ depends only on ϵ by Theorem 3.4.

Therefore, in either case, every curve in $\text{Nlc}(X, (1+\lambda)B)$ is contracted by f , by inversion of adjunction. That means that $f(\text{Nlc}(X, (1+\lambda)B))$ is a set of finitely many points. In particular, every curve C_0 supported in $\text{Nlc}(X, (1+\lambda)B)$ satisfies the equality $G \cdot C_0 = 0$, since G is the pull-back of an ample divisor on Z . This implies that every class $C \in \overline{NE}(X)_{\text{Nlc}(X, (1+\lambda)B)}$ satisfies $G \cdot C = 0$.

Let us consider an extremal ray R of $\overline{NE}(X)$. Note that $G \cdot R \geq 0$.

If $G \cdot R = 0$, then R is contracted by f since G is the pull-back of an ample divisor on Z and $-K_X \cdot R > 0$, as $-K_X$ is ample over Z .

If $G \cdot R > 0$ and R is $(K_X + (1+\lambda)B)$ -non-negative, then

$$\left(-K_X + \frac{7}{\lambda}G\right) \cdot R = \frac{1}{\lambda}(K_X + (1+\lambda)B) \cdot R + \frac{7}{\lambda}G \cdot R > 0,$$

as $K_X + B \equiv 0$.

If $G \cdot R > 0$ and R is $(K_X + (1 + \lambda)B)$ -negative, then

$$R \cap \overline{NE}(X)_{\text{Nlc}(X, (1+\lambda)B)} = \{0\},$$

since we showed that $G \cdot C = 0$ for any class $C \in \overline{NE}(X)_{\text{Nlc}(X, (1+\lambda)B)}$. By Theorem 2.3, R is generated by a rational curve C' such that

$$(K_X + (1 + \lambda)B) \cdot C' \geq -6.$$

On the other hand, $G \cdot C' \geq 1$ since $G \cdot C' > 0$ and G is Cartier. Hence,

$$\begin{aligned} & \left(-K_X + \frac{7}{\lambda}G \right) \cdot C' \\ &= \frac{1}{\lambda}(K_X + (1 + \lambda)B) \cdot C' + \frac{7}{\lambda}G \cdot C' > 0. \end{aligned}$$

In summary, the inequality

$$(-K_X + kG) \cdot R > 0$$

holds for any extremal ray R and for any $k \geq \frac{7}{\lambda}$, as G is nef. By Kleiman's Ampleness Criterion, $-K_X + kG$ is ample for all $k \geq \frac{7}{\lambda}$. We may take $n = \lceil 7/\lambda \rceil$ to complete the proof. \square

Remark 3.8. If Conjecture 3.6 holds in dimension 3, then Lemma 3.7 is an easy consequence of Kawamata's estimates on the length of extremal rays. However, the conjecture is still widely open, so we need to use the result for surfaces contained in Theorem 3.3.

3.3.4. Boundedness of birationality.

Lemma 3.9. *Under the setting of Section 3.3.1, the linear system $|-3K_X + kG|$ defines a birational map for $k \geq 4n + 4$, where n is the natural number given in Lemma 3.7.*

Proof. By Lemma 3.7, $-K_X + nG$ is ample.

If $\dim Z = 1$, then G is a smooth del Pezzo surface. It is well-known that $|-3K_G|$ gives a birational map (in fact, an embedding). For two general fibers G_1 and G_2 of f , and for an integer $k \geq 4n + 2$, consider the short exact sequence

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_X(-3K_X + kG - G_1 - G_2) \rightarrow \mathcal{O}_X(-3K_X + kG) \\ &\rightarrow \mathcal{O}_{G_1}(-3K_{G_1}) \oplus \mathcal{O}_{G_2}(-3K_{G_2}) \rightarrow 0. \end{aligned}$$

Since $k \geq 4n + 2$, $-4K_X + kG - G_1 - G_2$ is ample, by Kawamata–Viehweg vanishing theorem

$$\begin{aligned} & H^1(X, \mathcal{O}_X(-3K_X + kG - G_1 - G_2)) \\ &= H^1(X, \mathcal{O}_X(K_X - 4K_X + kG - G_1 - G_2)) = 0. \end{aligned}$$

Hence, the map

$$H^0(X, \mathcal{O}_X(-3K_X + kG)) \rightarrow H^0(G_1, \mathcal{O}_{G_1}(-3K_{G_1})) \oplus H^0(G_2, \mathcal{O}_{G_2}(-3K_{G_2}))$$

is surjective. Since $|-3K_{G_i}|$ gives a birational map on G_i for $i = 1, 2$, $|-3K_X + kG|$ gives a birational map on X for all $k \geq 4n + 2$.

Suppose now that $\dim Z = 2$. Note that $-K_X + kG$ is ample for $k \geq n$, and so is $-K_X|_G + kG|_G$. For two general fibers F_1 and F_2 of $f|_G$, and for an integer $k \geq 4n + 3$, let us consider the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_G(-3K_X|_G + kG|_G - F_1 - F_2) &\rightarrow \mathcal{O}_G(-3K_X|_G + kG|_G) \\ &\rightarrow \mathcal{O}_{F_1}(-3K_{F_1}) \oplus \mathcal{O}_{F_2}(-3K_{F_2}) \rightarrow 0. \end{aligned}$$

Since $k \geq 4n + 3$,

$$-3K_X|_G + kG|_G - F_1 - F_2 - K_G \equiv -4K_X|_G + (k-1)G|_G - F_1 - F_2$$

is ample. Again, by Kawamata–Viehweg vanishing theorem,

$$H^1(G, \mathcal{O}_G(-3K_X|_G + kG|_G - F_1 - F_2)) = 0.$$

Hence,

$$\begin{aligned} &H^0(G, \mathcal{O}_G(-3K_X|_G + kG|_G)) \\ &\rightarrow H^0(F_1, \mathcal{O}_{F_1}(-3K_{F_1})) \oplus H^0(F_2, \mathcal{O}_{F_2}(-3K_{F_2})) \end{aligned}$$

is surjective. Since $|-3K_{F_i}|$ gives a birational map on $F_i \simeq \mathbb{P}^1$ for $i = 1, 2$, $|-3K_X|_G + kG|_G|$ gives a birational map on G for all $k \geq 4n + 3$.

For an integer $k \geq 4n + 3$, consider the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(-3K_X + (k-1)G) &\rightarrow \mathcal{O}_X(-3K_X + kG) \\ &\rightarrow \mathcal{O}_G(-3K_X|_G + kG|_G) \rightarrow 0. \end{aligned}$$

Since $-4K_X + (k-1)G$ is ample, by Kawamata–Viehweg vanishing theorem,

$$\begin{aligned} &H^1(X, \mathcal{O}_X(-3K_X + (k-1)G)) \\ &= H^1(X, \mathcal{O}_X(K_X - 4K_X + (k-1)G)) = 0. \end{aligned}$$

Hence

$$H^0(X, \mathcal{O}_X(-3K_X + kG)) \rightarrow H^0(G, \mathcal{O}_G(-3K_X|_G + kG|_G))$$

is surjective. We showed that $|-3K_X|_G + kG|_G|$ gives a birational map on G since $k \geq 4n + 3$. In particular, $|-3K_X + kG| \neq \emptyset$. Hence $|-3K_X + (k+1)G|$ can separate general elements in $|G|$, and

$$H^0(X, \mathcal{O}_X(-3K_X + (k+1)G)) \rightarrow H^0(G, \mathcal{O}_G(-3K_X|_G + (k+1)G|_G))$$

is surjective, which gives a birational map on G . This implies that $|-3K_X + (k+1)G|$ gives a birational map for all $k \geq 4n + 3$. The proof is then complete. \square

3.3.5. Boundedness of the volume.

Lemma 3.10. *In the same setting as Section 3.3.1, there exists a positive integer $v = v(\epsilon)$ depending only on ϵ such that $(-K_X + nG)^3 \leq v$, where n is the natural number given in Lemma 3.7.*

Proof. If $\dim Z = 1$, G is a smooth del Pezzo surface. Note that $\text{Vol}(G, -K_G) = K_G^2 \leq 9$. By Lemma 2.1 and the fact that $-K_X$ is not big,

$$0 = \text{Vol}(X, -K_X) \geq \text{Vol}(X, -K_X + nG) - 3n\text{Vol}(G, -K_G).$$

This implies that $(-K_X + nG)^3 \leq 27n$.

Now suppose that $\dim Z = 2$. As constructed in Section 3.3.1, the fibration $f|_G: G \rightarrow H$ is a conic bundle from a smooth surface to a smooth curve.

Claim 3.11. $\text{Vol}(G, -K_X|_G + nG|_G) \leq 8 + 4(n+1)c$.

Proof of Claim 3.11. As $-K_X + nG$ is ample, so is $-K_X|_G + nG|_G$. Also note that $H^2 \leq c$ and $G|_G \equiv (H^2)F$ where $F \simeq \mathbb{P}^1$ is a general fiber of $f|_G$. Hence

$$\begin{aligned} \text{Vol}(G, -K_X|_G + nG|_G) &= (-K_X|_G + nG|_G)^2 \\ &= (-K_G + (n+1)G|_G)^2 \\ &= (K_G)^2 - 2(n+1)K_G \cdot (H^2)F \\ &\leq 8 + 4(n+1)c, \end{aligned}$$

where we used the fact that for the conic bundle G , $K_G^2 \leq 8$. \square

By Lemma 2.1 and Claim 3.11,

$$0 = \text{Vol}(X, -K_X) \geq \text{Vol}(X, -K_X + nG) - 3n\text{Vol}(G, -K_X|_G + nG|_G).$$

In particular, $(-K_X + nG)^3 \leq 3n(8 + 4(n+1)c)$. \square

3.4. Proof of Theorems 1.4 and 1.5.

Proof of Theorem 1.5. According to Proposition 3.1, after a birational modification, we may assume that X has a terminal Mori fibration structure $f: X \rightarrow Z$. If $-K_X$ is big, then X is of Fano type, and the (birational) boundedness follows from the solution of (birational) BAB conjecture in dimension 3, see [Jia15, Corollary 1.8] or [Bir16c, Corollary 1.2]. So we only need to consider the case when $-K_X$ is not big (and $\dim Z > 0$). By Theorem 3.2, there exist positive integers $n = n(\epsilon)$ and $v = v(\epsilon)$ depending only on ϵ , and an effective Cartier divisor G on X , such that $|-3K_X + 8nG|$ defines a birational map, and $(-K_X + nG)^3 \leq v$. Moreover, $\text{Vol}(X, -3K_X + 8nG)$ is bounded from above by

$$\text{Vol}(X, -3K_X + 8nG) \leq \text{Vol}(X, -8K_X + 8nG) \leq 512v.$$

Therefore, X belongs to a birationally bounded family by the boundedness of Chow varieties (see, for example, [HMX13, Lemma 2.4.2(2)]). \square

Proof of Theorem 1.4. By Theorem 1.5, it suffices to prove that there exists $\epsilon > 0$ independent of X , such that X is ϵ -klt. This is shown in Lemma 3.12, as an easy consequence of the Global ACC and may be well-known to the experts. \square

Lemma 3.12. *Fix a positive integer d . Then there exists a positive real number $\epsilon = \epsilon(d)$ depending only on d , such that every d -dimensional klt Calabi–Yau variety is ϵ -klt.*

Proof. Assume by contradiction that $\{X_i\}$ is a sequence of d -dimensional klt Calabi–Yau variety with $\lim_{i \rightarrow +\infty} \epsilon_i = 0$, where $\epsilon_i > 0$ is the minimal log discrepancy of X_i . Passing to a subsequence, we may assume that ϵ_i is decreasing and $\epsilon_i < 1$. Let $(X'_i, (1 - \epsilon_i)D'_i) \rightarrow X_i$ be the klt pair obtained by extracting a prime divisor of log discrepancy ϵ_i . Then $K_{X'_i} + (1 - \epsilon_i)D'_i \equiv 0$

and the coefficient of $(1 - \epsilon_i)D'_i$ belong to the set $\{1 - \epsilon_i \mid i \in \mathbb{N}\}$, which satisfies the descending chain condition and is infinite. This contradicts the Global ACC [HMX14, Theorem 1.5]. \square

4. BOUNDEDNESS OF MORI FIBRATIONS WITH BOUNDED COEFFICIENTS

The aim of this section is to prove the following result.

Theorem 4.1. *Fix positive real numbers ϵ , δ , and a positive integer d . Assume that Conjecture 1.3 holds in dimension $\leq d - 1$. Then, the set of log pairs (X, B) satisfying*

- (1) (X, B) is an ϵ -lc log Calabi–Yau pair of dimension d ,
- (2) X is rationally connected,
- (3) $B > 0$, and the coefficients of B are at least δ ,

forms a log bounded family modulo flops.

As Conjecture 1.3 holds in dimension ≤ 2 , we have the following corollary.

Corollary 4.2. *Fix positive real numbers ϵ , δ . Then, the set of log pairs (X, B) satisfying*

- (1) (X, B) is an ϵ -lc log Calabi–Yau pair of dimension 3,
- (2) X is rationally connected,
- (3) $B > 0$, and the coefficients of B are at least δ ,

forms a log bounded family modulo flops.

Before proceeding to the proof of Theorem 4.1, we first need some technical results extending those contained in [HMX14, DS16]. Such extensions are made possible by recent work of Birkar solving the BAB Conjecture [Bir16b, Bir16c].

The following lemma can be viewed as a generalization of [HMX14, Lemma 6.1]

Lemma 4.3. *Fix $0 < \epsilon' < \epsilon \leq 1$, and a positive integer d . Then there is a positive number $t = t(d, \epsilon, \epsilon')$ depending only on d , ϵ , and ϵ' , such that if (X, B) is an ϵ -lc pair of dimension d and (X, Φ) is a log pair such that $\Phi \geq (1 - t)B$ and $K_X + B \equiv K_X + \Phi \equiv 0$, then the pair (X, Φ) is ϵ' -klt.*

Proof. Assume, to the contrary, that there is a sequence of d -dimensional ϵ -lc pairs (X_i, B_i) and log pairs (X_i, Φ_i) such that $\Phi_i \geq (1 - \frac{1}{i})B_i$ and $K_{X_i} + B_i \equiv K_{X_i} + \Phi_i \equiv 0$, but (X_i, Φ_i) is not ϵ' -klt, for all integers $i > 0$. Replacing Φ_i by $(1 - \lambda_i)B_i + \lambda_i\Phi_i$ for some suitable $\lambda_i \in [0, 1)$, we can assume that (X, Φ_i) is ϵ' -lc but not ϵ' -klt.

By [BCHM10, Corollary 1.4.3], we can take a \mathbb{Q} -factorial birational modification $\phi_i : Y_i \rightarrow X_i$ extracting precisely one irreducible divisor S_i with $a(S_i, X_i, \Phi_i) = \epsilon'$. We can write

$$K_{Y_i} + \phi_{i,*}^{-1}\Phi_i + (1 - \epsilon')S_i = \phi_i^*(K_{X_i} + \Phi_i) \equiv 0, \quad (4.1)$$

and $K_{Y_i} + \phi_{i,*}^{-1}B_i + a_i S_i = \phi_i^*(K_{X_i} + B_i) \equiv 0$, where $a_i \leq 1 - \epsilon$. Now $K_{Y_i} + \phi_{i,*}^{-1}\Phi_i$ is not pseudo-effective, so we can run a $(K_{Y_i} + \phi_{i,*}^{-1}\Phi_i)$ -MMP with scaling of an ample divisor, which ends with a Mori fiber space $\pi_i : W_i \rightarrow Z_i$. As this MMP is also a $(-S_i)$ -MMP by (4.1), S_i dominates Z_i .

Denote by Φ'_i , B'_i , and S'_i the strict transform of $\phi_{i,*}^{-1}\Phi_i$, $\phi_{i,*}^{-1}B_i$, and S_i on W_i respectively. Then by construction, $(W_i, \Phi'_i + (1 - \epsilon')S'_i)$ is an ϵ' -lc log Calabi–Yau pair, $K_{W_i} + B'_i + a_i S'_i \equiv 0$ with $a_i \leq 1 - \epsilon$, and $\Phi'_i \geq (1 - \frac{1}{i})B'_i$.

Denote by F_i a general fiber of π_i , then F_i is an ϵ' -lc Fano variety of dimension $\leq d$. The family of such F_i is bounded by the BAB theorem [Bir16c], hence there exist positive integers r and M such that rK_{F_i} is Cartier and $(-K_{F_i})^{d_i} \leq M$ where $d_i = \dim F_i$. Now consider

$$\begin{aligned} 0 &\leq \Phi'_i|_{F_i} - \left(1 - \frac{1}{i}\right) B'_i|_{F_i} \\ &\equiv -K_{F_i} - (1 - \epsilon')S'_i|_{F_i} + \left(1 - \frac{1}{i}\right) K_{F_i} + \left(1 - \frac{1}{i}\right) a_i S'_i|_{F_i} \\ &\equiv -\frac{1}{i}K_{F_i} - \left(1 - \epsilon' - \left(1 - \frac{1}{i}\right)a_i\right) S'_i|_{F_i}, \end{aligned}$$

we have

$$\begin{aligned} \frac{M}{i} &\geq \frac{1}{i}(-K_{F_i})^{d_i} \\ &\geq \left(1 - \epsilon' - \left(1 - \frac{1}{i}\right)a_i\right) S'_i|_{F_i} \cdot (-K_{F_i})^{d_i-1} \\ &\geq (\epsilon - \epsilon') S'_i|_{F_i} \cdot (-K_{F_i})^{d_i-1} \\ &\geq \frac{\epsilon - \epsilon'}{r^{d_i-1}} \geq \frac{\epsilon - \epsilon'}{r^{d-1}}. \end{aligned}$$

Note that $\frac{\epsilon - \epsilon'}{r^{d-1}}$ is a constant positive number, this is absurd. \square

Lemma 4.4. *Fix a positive real number ϵ and a positive integer d . Then there exists a number $M = M(\epsilon, d)$ depending only on ϵ and d , such that if (X, B) is an ϵ -lc log Calabi–Yau pair of dimension d , then $\text{Vol}(X, B) \leq M$.*

Proof. We may assume that $\text{Vol}(X, B) > 0$; otherwise it is clear. Then B is big and this follows from [Bir16c, Corollary 1.2]. \square

Theorem 4.5. *Fix a positive real number ϵ and a positive integer d . Then there exists a positive number $k = k(\epsilon, d)$ depending only on ϵ and d satisfying the following: if (X, B) is a log pair such that*

- (1) (X, B) is ϵ -lc of dimension d ,
- (2) there exists a contraction of normal varieties $f: X \rightarrow Y$ such that $0 < \dim Y < d$,
- (3) $K_X + B \sim_{\mathbb{R}} f^*H$ for some very ample divisor H on Y , and
- (4) $K_Y + H$ is big,

then $\text{Vol}(X, B) \leq k\text{Vol}(Y, H)$.

Proof. Let $t = t(d, \epsilon, \frac{\epsilon}{2})$ as in Lemma 4.3. Let F be a general fiber of f . We may assume that H is general in its linear system.

First, we claim that $\text{Vol}(X, tB - 2f^*H) = 0$. Assume not, then $tB - 2f^*H$ is big. So there exists an effective \mathbb{R} -divisor $E \sim_{\mathbb{R}} tB - 2f^*H$. Let $\Phi = (1 - t)B + E$. Then $\Phi|_F \geq (1 - t)B|_F$ and $K_X + \Phi \sim_{\mathbb{R}} -f^*H$. By Lemma 4.3, (X, Φ) is klt over the generic point of Y . So by Ambro’s canonical bundle formula (see [DS16, Lemma 2.14]), there are pseudo-effective divisors

B_Y and M_Y such that $-f^*H \sim_{\mathbb{R}} K_X + \Phi \sim_{\mathbb{R}} f^*(K_Y + B_Y + M_Y)$. This immediately gives a contradiction since $K_Y + H$ is big.

Now by Lemma 2.1, we have

$$0 = \text{Vol}(X, tB - 2f^*H) \geq \text{Vol}(X, tB) - 2d\text{Vol}(f^*H, tB|_{f^*H}).$$

Hence, $\text{Vol}(X, B) \leq 2dt^{-1}\text{Vol}(f^*H, B|_{f^*H})$.

If $\dim Y = 1$, then $f^*H = \sum_{i=1}^h F_i$ where $h = \deg(H) = \text{Vol}(Y, H)$ and for each i , F_i is a general fiber of f and hence $(F_i, B|_{F_i})$ is an ϵ -lc log Calabi–Yau pair of dimension $d - 1$. Hence by Lemma 4.4, we have

$$\text{Vol}(f^*H, B|_{f^*H}) = \sum_{i=1}^h \text{Vol}(F_i, B|_{F_i}) \leq M(\epsilon, d - 1)\text{Vol}(Y, H).$$

Hence we may take $k(\epsilon, d) = 2dt^{-1}M(\epsilon, d - 1)$ in this case. In particular, this settles the case that $\dim X = 2$.

Finally we will use induction on the dimension of X to show the case $\dim Y > 1$. If $\dim Y > 1$, consider the map $f_1 = f|_{f^*H}: f^*H \rightarrow H$. Then $(f^*H, B|_{f^*H})$ is ϵ -lc of dimension $d - 1$, $K_{f^*H} + B|_{f^*H} \sim_{\mathbb{R}} 2f^*H|_{f^*H} = 2(f_1)^*(H|_H)$, and $K_H + 2H|_H$ is big. Hence by induction hypothesis, we have

$$\text{Vol}(f^*H, B|_{f^*H}) \leq k(\epsilon, d - 1)\text{Vol}(H, 2H|_H) = 2^{\dim Y - 1}k(\epsilon, d - 1)\text{Vol}(Y, H).$$

Taking $k(\epsilon, d) := 2^d dt^{-1}k(\epsilon, d - 1)$, the proof is complete. \square

The following theorem can be viewed as a relative version of the Special BAB [Bir16b, Theorem 1.4].

Theorem 4.6. *Fix a positive integer d and positive numbers ϵ , δ , and M . Then the set of log pairs (X, B) satisfying*

- (1) *(X, B) is an ϵ -lc log Calabi–Yau pair of dimension d such that K_X is \mathbb{Q} -Cartier,*
- (2) *there is a contraction of normal varieties $f: X \rightarrow Y$ such that $-K_X$ is ample over Y and $0 < \dim Y < d$,*
- (3) *there is a very ample Cartier divisor H with $\text{Vol}(Y, H) \leq M$, and*
- (4) *the coefficients of B are at least δ*

forms a log bounded family.

Proof. We may assume that $\delta < 1$. Fix a positive integer $n > \frac{1}{\epsilon}$. After replacing H by a fixed multiple depending only on $\dim Y$, we may assume that $K_Y + \delta H$ is big (cf. [HMX13, Lemma 2.3.4(2)]). We will always consider H as a general member of its linear system. As f^*H is base-point free, we can find an effective \mathbb{Q} -divisor $H' \sim_{\mathbb{Q}} f^*H$ with all coefficients equal to $\frac{1}{n}$ such that $(X, B + H')$ is still ϵ -lc. Let $\pi: \tilde{X} \rightarrow X$ be a log resolution of $(X, B + H')$ and write

$$K_{\tilde{X}} + \tilde{B} + \tilde{H} + E = \pi^*(K_X + B + H') + \sum a_i E_i,$$

where \tilde{B} and \tilde{H} are the strict transform of B and H respectively, E_i are prime π -exceptional divisors and $E = \sum E_i$. We can choose a sufficiently small positive number s (depending on X) such that $(X, (1 + s)B + H')$ is

$\frac{\epsilon}{2}$ -lc and $K_X + (1 + s)B + H' \equiv -sK_X + H'$ is ample, note that here we need the assumption that K_X is \mathbb{Q} -Cartier. So

$$\begin{aligned} 0 &< \text{Vol}(K_X + (1 + s)B + H') \leq \text{Vol}(K_{\tilde{X}} + [\tilde{B}] + \tilde{H} + E) \\ &\leq \text{Vol}(K_X + [B] + H') \leq \text{Vol}\left(\frac{1}{\delta}B + H'\right) \\ &= \frac{1}{\delta^d} \text{Vol}(B + \delta H') < \frac{1}{\delta^d} \text{Vol}(B + H') \end{aligned}$$

Note that $K_X + B + H' \sim_{\mathbb{R}} f^*(H)$, hence $\frac{1}{\delta^d} \text{Vol}(B + H')$ is bounded from above by Lemma 4.4.

In summary, $(\tilde{X}, [\tilde{B}] + \tilde{H} + E)$ is an lc pair such that $K_{\tilde{X}} + [\tilde{B}] + \tilde{H} + E$ is big with bounded volume and coefficients of $[\tilde{B}] + \tilde{H} + E$ are in the fixed finite set $\{\frac{1}{n}, 1\}$ independent of X . The set of such pairs forms a log birationally bounded set by [HMX13, Lemma 2.3.4(2), Theorem 3.1] and [HMX14, Theorem 1.3]. Therefore, the set $\{(X, (1 + s)B + H')\}$ is also log birationally bounded, and hence log bounded by [HMX14, Theorem 1.6]. So the set $\{(X, B)\}$ is log bounded. \square

It would be interesting to ask whether Theorem 4.6 still holds true if we relax the condition “ $-K_X$ is ample over Y ” to “ $-K_X$ is big over Y ”. Here, as a corollary, we can prove a weak version, namely, (X, B) forms a log bounded family modulo flops.

Corollary 4.7. *Fix a positive integer d and positive numbers ϵ , δ , and M . Then the set of log pairs (X, B) satisfying*

- (1) (X, B) is an ϵ -lc log Calabi–Yau pair of dimension d ,
- (2) there is a contraction of normal varieties $f: X \rightarrow Y$ such that $-K_X$ is big over Y and $0 < \dim Y < d$,
- (3) there is a very ample Cartier divisor H with $\text{Vol}(Y, H) \leq M$, and
- (4) the coefficients of B are at least δ

forms a log bounded family modulo flops.

Proof. We may replace X by its \mathbb{Q} -factorialization and assume that X is \mathbb{Q} -factorial. Since (X, B) is an ϵ -lc pair, there exists a sufficiently small $t > 0$ such that $(X, (1 + t)B)$ is also klt. Since B is big over Y , we may run a $(K_X + (1 + t)B)$ -MMP over Y (which is also a B -MMP over Y) with scaling of an ample divisor, and finally reach a relative log canonical model $f': X' \rightarrow Y$. Denote by B' the strict transform of B . It follows that $-K_{X'} \equiv B'$ is ample over Y . Also note that (X', B') is again an ϵ -lc log Calabi–Yau pair and coefficients of B' are at least δ . Then, by Theorem 4.6, (X', B') belongs to a log bounded family. The conclusion then follows from Proposition 4.8, as for any prime divisor E on X which is exceptional over X' , we have

$$a(E, X', B') = a(E, X, B) \leq a(E, X, 0) = 1,$$

where the first equality follows from the fact that (X, B) and (X', B') are crepant birational log Calabi–Yau pairs. \square

Proof of Theorem 4.1. We follow the strategy of [DS16]. We may replace X by its \mathbb{Q} -factorialization and assume that X is \mathbb{Q} -factorial. Since $K_X + B \equiv 0$

and $B > 0$, we can run a K_X -MMP with scaling of an ample divisor which ends with a Mori fiber space $f : Y \rightarrow Z$ with general fiber F . Denote by B_Y the strict transform of B . Since $B > 0$, $K_X + B \equiv 0$, and we are running a K_X -MMP, it follows that $B_Y > 0$. Also note that (Y, B_Y) is again an ϵ -lc log Calabi–Yau pair, Y is rationally connected, and coefficients of B_Y are at least δ .

If $\dim Z = 0$, then Y is an ϵ -lc log Fano variety, hence it is bounded by [Bir16c, Theorem 1.1]. It is then easy to show that the support of B_Y is also bounded, since there exist positive integers r and M such that $-rK_Y$ is very ample and $(-K_Y)^d \leq M$, and therefore

$$\text{Supp}(B_Y) \cdot (-rK_Y)^{d-1} \leq \frac{1}{\delta} B_Y \cdot (-rK_Y)^{d-1} \leq \frac{r^{d-1}M}{\delta}.$$

Hence (Y, B_Y) is log bounded.

If $\dim Z > 0$, by Ambro’s canonical bundle formula [FG12, Theorem 3.1], Z is naturally endowed with a log Calabi–Yau structure, that is, there exists an effective \mathbb{R} -divisor Γ on Z such that (Z, Γ) is klt and $K_Z + \Gamma \equiv 0$. Denote by F a general fiber of f . Then F is an ϵ -lc log Fano variety of dimension at most d , $K_F + B_Y|_F \equiv 0$ and the coefficients of $B_Y|_F$ are at least δ . Again by [Bir16c, Theorem 1.1], $(F, B_Y|_F)$ is log bounded. Hence by [Bir16a, Theorem 1.4], Γ can be chosen so that (Z, Γ) is ϵ' -lc for some ϵ' which only depends on ϵ and d . In particular, by Conjecture 1.3, it follows that Z belongs to a bounded family since Z is rationally connected. By Theorem 4.6, (Y, B_Y) is log bounded.

In summary, (Y, B_Y) belongs to a log bounded family. For any prime divisor E on X which is exceptional over Y , we have

$$a(E, Y, B_Y) = a(E, X, B) \leq a(E, X, 0) = 1.$$

The conclusion then follows from Proposition 4.8. \square

The following proposition is a generalization of [HX15, Proposition 2.5].

Proposition 4.8. *Fix a positive real number ϵ and a positive integer d . Let \mathcal{D} be a log bounded family of log pairs such that any $(X, B) \in \mathcal{D}$ is an d -dimensional ϵ -lc pair.*

Then there exist finitely many quasi-projective normal \mathbb{Q} -factorial varieties \mathcal{Y}_i , a reduced divisor \mathcal{F}_i on \mathcal{Y}_i , and a projective morphism $\mathcal{Y}_i \rightarrow T_i$, where T_i is a normal variety of finite type and \mathcal{F}_i does not contain any fiber, such that for any $(X, B) \in \mathcal{D}$, and any set of divisors $\{E_j\}$ exceptional over X such that the log discrepancy $a(E_j, X, B) \leq 1$, there exists an index i , a closed point $t \in T_i$, and a birational morphism $\mu_t : \mathcal{Y}_{i,t} \rightarrow X_i$ which extracts precisely the divisors $\{E_j\}$, and $\mathcal{F}_{i,t}$ coincides with the support of strict transform of B and all E_i .

Proof. By definition, there is a quasi-projective scheme \mathcal{Z} , a reduced divisor \mathcal{E} on \mathcal{Z} , and a projective morphism $h : \mathcal{Z} \rightarrow T$, where T is of finite type and \mathcal{E} does not contain any fiber, such that for every $(X, B) \in \mathcal{D}$, there is a closed point $t \in T$ and an isomorphism $f : \mathcal{Z}_t \rightarrow X$ such that \mathcal{E}_t coincides with the support of $f_*^{-1}B$.

We may assume that T is reduced. Blowing up $(\mathcal{Z}, \mathcal{E})$ and T and decomposing T into a finite union of locally closed subsets, we may assume that

there exists a pair $(\mathcal{Z}', \mathcal{E}')$ that has simple normal crossings support with the following diagram:

$$\begin{array}{ccc} (\mathcal{Z}', \mathcal{E}') & \xrightarrow{\quad} & (\mathcal{Z}, \mathcal{E}) \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

Passing to an open dense subset of T , we may assume that the fibers of $(\mathcal{Z}', \mathcal{E}') \rightarrow T$ are log smooth pairs, passing to a finite cover of T , we may assume that every stratum of $(\mathcal{Z}', \mathcal{E}')$ has irreducible fibers over T ; decomposing T into a finite union of locally closed subsets, we may assume that T is smooth; finally passing to a connected component of T , we may assume that T is integral. Note that all these operations will still yield a bounded family, thanks to Noetherian induction.

As $(\mathcal{Z}', (1 - \epsilon)\mathcal{E}')$ is klt, it follows that there are only finitely many valuations of log discrepancy at most one with respect to $(\mathcal{Z}', (1 - \epsilon)\mathcal{E}')$. As $(\mathcal{Z}', (1 - \epsilon)\mathcal{E}')$ has simple normal crossings over T , there is a sequence of blow ups $\phi: \mathcal{Z}'' \rightarrow \mathcal{Z}'$ of strata, which extracts every divisor of log discrepancy at most one with respect to $(\mathcal{Z}', (1 - \epsilon)\mathcal{E}')$. Note also that as $(\mathcal{Z}', (1 - \epsilon)\mathcal{E}')$ has simple normal crossings over T , it follows that if $t \in T$ is a closed point then every valuation of log discrepancy at most one with respect to $(\mathcal{Z}'_t, (1 - \epsilon)\mathcal{E}'_t)$ has center on a divisor on \mathcal{Z}''_t .

Denote by \mathcal{E}'' the support of the strict transform of \mathcal{E}' and all ϕ -exceptional divisors on \mathcal{Z}'' . Note that we can write $K_{\mathcal{Z}''} \sim_{\mathbb{Q}} \mathcal{A} + \mathcal{C}$, where \mathcal{A} is a general \mathbb{Q} -Cartier divisor ample over \mathcal{Z} , and \mathcal{C} is an effective \mathbb{Q} -divisor. We can find a positive real number λ such that $(\mathcal{Z}'', (1 + \lambda)(1 - \frac{\epsilon}{2})\mathcal{E}'' + \lambda(\mathcal{A} + \mathcal{C}))$ is still klt.

Let \mathcal{E}''_k for $1 \leq k \leq m$ be the components of \mathcal{E}'' . We may assume that \mathcal{E}''_k for $1 \leq k \leq r$ are the components which are the strict transform of components of \mathcal{E} on \mathcal{Z}'' . Consider the polytope

$$P = \left\{ \sum_{k=1}^m h_k \mathcal{E}''_k \mid 0 \leq h_k \leq 1 - \frac{\epsilon}{2} \text{ for } 1 \leq k \leq m \right\}.$$

Note that for any element Φ of P , a $(K_{\mathcal{Z}''} + \Phi)$ -minimal model over \mathcal{Z} is the same as a $(K_{\mathcal{Z}''} + (1 + \lambda)\Phi + \lambda(\mathcal{A} + \mathcal{C}))$ -minimal model over \mathcal{Z} . Hence by the finiteness of models [BCHM10, Corollary 1.1.5], there are just finitely many possible minimal models over \mathcal{Z} for this polytope P , that is, there exist finitely many quasi-projective normal \mathbb{Q} -factorial varieties $\mathcal{Y}_i \rightarrow T$, such that for any element Φ of P , there exists an index i and a closed point $t \in T$ such that $\mathcal{Y}_{i,t}$ is a $(K_{\mathcal{Z}''} + \Phi)$ -minimal model over \mathcal{Z} . Denote \mathcal{F}_i to be the strict transform of \mathcal{E}'' on \mathcal{Y}_i .

We now claim that such $(\mathcal{Y}_i, \mathcal{F}_i)$ are what we need. For $(X, B) \in \mathfrak{D}$, there is a closed point $t \in T$ and a birational map $f'': \mathcal{Z}''_t \rightarrow X$ such that the support of \mathcal{E}''_t contains the support of the strict transform of B and all f''^{-1} -exceptional divisors. As (X, B) is ϵ -lc, we have

$$K_{\mathcal{Z}'_t} + \Delta'_t = f'^*(K_X + B),$$

$$K_{\mathcal{Z}''_t} + \Delta''_t = f''^*(K_X + B)$$

where $\Delta'_t \leq (1 - \epsilon)\mathcal{E}'_t$ and $\Delta''_t \leq (1 - \epsilon)\mathcal{E}''_t$. Now consider a set of divisors $\{E_j\}$ exceptional over X such that $a(E_j, X, B) \leq 1$, then

$$1 \geq a(E_j, X, B) = a(E_j, \mathcal{Z}'_t, \Delta'_t) \geq a(E_j, \mathcal{Z}'_t, (1 - \epsilon)\mathcal{E}'_t)$$

which means that each E_j is a divisor on \mathcal{Z}''_t and appears as a component of \mathcal{E}''_t by the construction of \mathcal{Z}'' . We may assume that $\mathcal{E}''_{k,t}$ for $r < k \leq n$ correspond to the divisors $\{E_j\}$. Now we may write

$$\Delta''_t = \sum_{k=1}^m b_k \mathcal{E}''_k$$

where $b_k \in [0, 1 - \epsilon]$ for $1 \leq k \leq n$ and $b_k \leq 1 - \epsilon$ for $k > n$. We can consider the boundary

$$\Phi = \sum_{k=1}^n b_k \mathcal{E}''_k + \sum_{n < k \leq m} \left(1 - \frac{\epsilon}{2}\right) \mathcal{E}''_k \in P$$

on \mathcal{Z}'' . Then by construction, there exists an index i such that \mathcal{Y}_i is a $(K_{\mathcal{Z}''} + \Phi)$ -minimal model over \mathcal{Z} , and hence $\mathcal{Y}_{i,t}$ is a $(K_{\mathcal{Z}''_t} + \Phi_t)$ -minimal model over $\mathcal{Z}_t \cong X$, which extracts precisely the divisors $\{\mathcal{E}''_{k,t} \mid r < k \leq n\} = \{E_j\}$. \square

5. TOWARDS BOUNDEDNESS OF RATIONALLY CONNECTED KLT CALABI–YAU’S

We have seen so far that rationally connected klt Calabi–Yau 3-folds are birationally bounded in Section 3. By applying results in Section 4, in a good number of cases, it is actually possible to prove that boundedness holds under slightly stronger assumptions.

Firstly we show that rationally connected klt Calabi–Yau 3-folds with mld bounded away from 1 are bounded modulo flops.

Theorem 5.1. *Fix $0 < c < 1$. Let \mathcal{D} be the set of varieties X such that*

- (1) *X is a rationally connected klt Calabi–Yau 3-fold, and*
- (2) *$0 < \text{mld}(X) < c$.*

Then \mathcal{D} is bounded modulo flops.

Proof. Take $X \in \mathcal{D}$. By [BCHM10, Corollary 1.4.3], we may take a birational morphism $\pi: Y \rightarrow X$ extracting only one exceptional divisor E of log discrepancy $a = a(E, X) \in (0, c)$. Then

$$K_Y + (1 - a)E = \pi^* K_X.$$

Also by Global ACC (see Lemma 3.12), there exists a constant $\epsilon \in (0, \frac{1}{2})$ such that X is (2ϵ) -lc, and therefore $(Y, (1 - a)E)$ is a (2ϵ) -lc log Calabi–Yau pair with $1 - a > 1 - c > 0$ and Y rationally connected.

Now by Corollary 4.2, the pairs (Y, E) are log bounded modulo flops. That is, there are finitely many quasi-projective normal varieties \mathcal{W}_i , a reduced divisor \mathcal{E}_i on \mathcal{W}_i , and a projective morphism $\mathcal{W}_i \rightarrow S_i$, where S_i is a normal variety of finite type and \mathcal{E}_i does not contain any fiber, such that for every (Y, E) , there is an index i , a closed point $s \in S_i$, and a small birational map $f: \mathcal{W}_{i,s} \dashrightarrow Y$ such that $\mathcal{E}_{i,s} = f_*^{-1}E$. We may assume that the set of points

s corresponding to such Y is dense in each S_i . We may just consider a fixed index i and ignore the index in the following argument.

For the point s corresponding to (Y, E) ,

$$K_{\mathcal{W}_s} + (1 - a)f_*^{-1}E \equiv f_*^{-1}(K_Y + (1 - a)E) \equiv 0$$

and therefore $(\mathcal{W}_s, (1 - a)f_*^{-1}E)$ is a (2ϵ) -lc log Calabi–Yau pair. Now consider a log resolution $g : \mathcal{W}' \rightarrow \mathcal{W}$ of $(\mathcal{W}, \mathcal{E})$ and denote by \mathcal{E}' the strict transform of \mathcal{E} and all the sum of g -exceptional reduced divisors on \mathcal{W}' . Consider the log canonical pair $(\mathcal{W}', (1 - \epsilon)\mathcal{E}')$. There exists an open dense set $U \subset S$ such that for the point $s \in U$ corresponding to (Y, E) , $g_s : \mathcal{W}'_s \rightarrow \mathcal{W}_s$ is a log resolution and we can write

$$K_{\mathcal{W}'_s} + B_s = g_s^*(K_{\mathcal{W}_s} + (1 - a)f_*^{-1}E) \equiv 0$$

where the coefficients of B_s are $\leq 1 - 2\epsilon$ and its support is contained in $\mathcal{E}'_s = \mathcal{E}'|_{\mathcal{W}'_s}$. We have

$$\begin{aligned} & (K_{\mathcal{W}'} + (1 - \epsilon)\mathcal{E}')|_{\mathcal{W}'_s} \\ & \equiv K_{\mathcal{W}'_s} + (1 - \epsilon)\mathcal{E}'_s \\ & \equiv (1 - \epsilon)\mathcal{E}'_s - B_s. \end{aligned}$$

Note that the support of $(1 - \epsilon)\mathcal{E}'_s - B_s$ coincides with the support of \mathcal{E}'_s which are precisely the divisors on \mathcal{W}'_s exceptional over X . Hence $(K_{\mathcal{W}'} + (1 - \epsilon)\mathcal{E}')$ is of Kodaira dimension zero on the fiber \mathcal{W}'_s and we can run a $(K_{\mathcal{W}'} + (1 - \epsilon)\mathcal{E}')$ -MMP with scaling of an ample divisor over S to get a relative minimal model $\tilde{\mathcal{W}}$ over S . Such MMP terminates by [HX13, Corollary 2.9, Theorem 2.12]. Note that for the point $s \in U$ corresponding to (Y, E) , \mathcal{E}'_s is contracted and hence $\tilde{\mathcal{W}}_s$ is isomorphic to X in codimension one. This gives a bounded family modulo flops, over U . Applying Noetherian induction on S , the family of all such X is bounded modulo flops. \square

As an interesting application, we can show that for rationally connected klt Calabi–Yau 3-folds, the boundedness modulo flops is equivalent to the boundedness of global index.

Corollary 5.2. *Let \mathcal{D} be a set of rationally connected klt Calabi–Yau 3-folds. Then \mathcal{D} is bounded modulo flops if and only if there exists a positive integer r such that $rK_X \sim 0$ for any $X \in \mathcal{D}$.*

Proof. Assume that \mathcal{D} is bounded modulo flops, then there exists a bounded family \mathcal{D}' of normal projective varieties such that for every $X \in \mathcal{D}$, there exists $Y \in \mathcal{D}'$ and a small birational morphism $f : Y \dashrightarrow X$. Moreover, K_Y is \mathbb{Q} -Cartier by the definition. Hence we have $K_Y = f_*^{-1}K_X \sim_{\mathbb{Q}} 0$. Take a common resolution $p : W \rightarrow Y$ and $q : W \rightarrow X$, by the Negativity Lemma, we have $p^*K_Y = q^*K_X$ and Y is klt. Since \mathcal{D}' is bounded, there exists a constant $r \in \mathbb{N}_{>0}$ such that rK_Y is Cartier, which means that $p^*(rK_Y) \sim_{\mathbb{Q}} 0$ is a Cartier divisor. Since W is rationally connected, it is simply connected and hence $p^*(rK_Y) \sim 0$. Therefore $rK_X = q_*q^*(rK_X) \sim 0$. This proves the ‘only if’ part.

If there exists a positive integer r such that $rK_X \sim 0$ for any $X \in \mathcal{D}$, then it is clear that $r \cdot \text{mld}(X)$ is a positive integer. Since X is rationally connected, X has worse than canonical singularities, that is, $\text{mld}(X) < 1$.

Therefore $\text{mld}(X) \leq 1 - \frac{1}{r} < 1 - \frac{1}{2r}$. The ‘if’ part follows directly from Theorem 5.1. \square

As another application, Theorem 5.1 relates Conjecture 1.1 to conjectures for minimal log discrepancies. Recall the following deep conjecture regarding the behavior of minimal log discrepancy proposed by Shokurov.

Conjecture 5.3 (ACC for mld, cf. [Sho88, Problem 5], [Sho96, Conjecture 4.2]). *Fix a positive integer d and a DCC set $I \subset [0, 1]$. Then the set*

$$\{\text{mld}_{\eta_Z}(X, \Delta) \mid (X, \Delta) \text{ is lc, } \dim X \leq d, Z \subset X, \text{coeff}(\Delta) \in I\}$$

satisfies the ACC.

ACC stands for ascending chain condition whilst DCC stands for descending chain condition.

Here we only need a very weak version of Conjecture 5.3.

Conjecture 5.4 (1-Gap conjecture for mld). *Fix a positive integer d . Then 1 is not an accumulation point from below for the set*

$$\{\text{mld}(X) \mid \dim X \leq d\}.$$

Corollary 5.5. *Assume that Conjecture 5.4 holds for rationally connected klt Calabi–Yau 3-folds. Then the set of rationally connected klt Calabi–Yau 3-folds is bounded modulo flops.*

Proof. Let X be a rationally connected klt Calabi–Yau 3-fold. Note that X has worse than canonical singularities, that is, $\text{mld}(X) < 1$. By Conjecture 5.4, there exists a constant $c \in (0, 1)$ such that $\text{mld}(X) < c$. Hence the corollary follows directly from Theorem 5.1. \square

Finally we consider rationally connected klt Calabi–Yau 3-folds with positive augmented irregularity. For klt Calabi–Yau varieties, Greb, Guenancia, and Kebekus proved that there is a decomposition, after quasi-étale covering, that is analogous to the classical Beauville–Bogomolov decomposition in the smooth case, [Bea83].

Theorem 5.6 ([GGK17, Theorem B]). *Let X be a klt variety with $K_X \equiv 0$. Then there are normal projective varieties A, Z and a quasi-étale cover $\gamma: A \times Z \rightarrow X$ such that*

- (1) *A is an abelian variety of dimension $\tilde{q}(X)$,*
- (2) *Z has canonical singularities, $K_Z \sim 0$ and $\tilde{q}(Z) = 0$, and*
- (3) *there exists a flat Kähler form ω_A on A and a singular Ricci-flat Kähler metric ω_Z on Z such that $\gamma^*\omega_H = \text{pr}_1^*\omega_A + \text{pr}_2^*\omega_Z$ and such that the holonomy group of the corresponding Riemann metric on $A \times Z_{\text{reg}}$ is connected.*

Here we recall the notion of augmented irregularity.

Definition 5.7 (cf. [GGK17, Definition 2.20]). Let X be a normal projective variety. The *irregularity* of X is defined as $q(X) := h^1(X, \mathcal{O}_X)$. The *augmented irregularity* of X is defined as $\tilde{q}(X) = \sup\{q(Y) \mid Y \rightarrow X \text{ quasi-étale cover}\} \in \mathbb{N} \cup \{\infty\}$.

Remark 5.8. When X has rational singularities (as it will be the case in this section), then $q(X) = q(Z)$ for any $Z \rightarrow X$ resolution of singularities.

As a consequence of the decomposition for singular Calabi–Yau varieties and Theorem 5.1, we can prove boundedness for those rationally connected klt Calabi–Yau 3-folds that contain an abelian factor.

Corollary 5.9. *Let \mathcal{D}_{ab} be the set of varieties X such that*

- (1) *X is rationally connected klt Calabi–Yau 3-fold, and*
- (2) *$\tilde{q}(X) > 0$.*

Then \mathcal{D}_{ab} is bounded modulo flops.

Proof. Step 1. We will construct a quasi-étale Galois cover $\gamma : Y \rightarrow X = Y/G$ such that Y is either an abelian 3-fold or a product of an elliptic curve and a weak K3 surface (i.e., a normal projective surface S with canonical singularities, $K_S \sim 0$ and $\tilde{q}(S) = 0$, cf. [NZ10]).

Let $\gamma : W_1 \rightarrow X$ be the quasi-étale cover given in Theorem 5.6. Since $\tilde{q}(X) > 0$, it follows that W_1 is either an abelian 3-fold or a product of an elliptic curve and a weak K3 surface. Note that W_1 has canonical singularities.

In view of [GKP16, Theorem 3.7], there exists a finite surjective morphism $\phi_1 : Y_1 \rightarrow W_1$ such that $\gamma_1 = \gamma \circ \phi_1$ is Galois and the branch loci of γ and γ_1 are the same. The latter property implies that γ_1 is quasi-étale as well as ϕ_1 is.

If W_1 is an abelian 3-fold, then by purity of the branch locus it follows that ϕ_1 is étale and hence Y_1 is also an abelian 3-fold. In this case we complete Step 1 by setting $Y = Y_1$.

Now assume that W_1 is a product of an elliptic curve and a weak K3 surface. Note that $\tilde{q}(W) = 1$. Now we construct, by induction, a sequence of finite quasi-étale surjective morphism

$$\cdots \longrightarrow Y_i \xrightarrow{\gamma_i} Y_{i-1} \xrightarrow{\gamma_{i-1}} \cdots \xrightarrow{\gamma_2} Y_1 \xrightarrow{\gamma_1} Y_0 = X,$$

such that for each i , the compositions $Y_i \rightarrow X$ are Galois, and $\gamma_i : Y_i \rightarrow Y_{i-1}$ factors as

$$Y_i \xrightarrow{\phi_i} W_i \xrightarrow{\psi_i} Y_{i-1},$$

where W_i is a product of an elliptic curve and a weak K3 surface. Moreover, ψ_i is étale if $i > 1$.

The construction is as the following: assume that we constructed Y_{i-1} , then $K_{Y_{i-1}} \sim 0$ and Y_{i-1} has canonical singularities. By [Kaw85, Theorem 8.3, Corollary 8.4], there exists an étale covering $\psi_i : W_i \rightarrow Y_{i-1}$ where $W_i = C \times Z$, C is an abelian variety and Z is a canonical variety with $K_Z \sim 0$. We only need to show that in our case, C is an elliptic curve and Z is a surface with $\tilde{q}(Z) = 0$. Assume not, then it is easy to see that $\tilde{q}(C \times Z) > 1 = \tilde{q}(W_1)$, which is absurd since \tilde{q} is invariant under quasi-étale covers. Then $\psi_i : W_i \rightarrow Y_{i-1}$ is constructed by applying Theorem 5.6, and $\phi_i : Y_i \rightarrow W_i$ is constructed by applying [GKP16, Theorem 3.7] to the composition $W_i \rightarrow X$.

By [GKP16, Theorem 1.1], it follows that there exists $i > 1$ such that γ_i is étale. It follows that the morphism $Y_i \rightarrow W_i$ is étale (see [Mil80, Corollary 3.6]). We may write $W_i = E \times S$. As S is simply connected, see [Kol93, Theorem 7.8], it follows that $Y_i \simeq E' \times S$ for some elliptic curve E' . Hence we complete Step 2 by taking $Y = Y_i$.

Step 2. We will show that the group action of G on Y is bounded.

As $X = Y/G$ and Y is Gorenstein, if we bound the order of the representation of G on the vector space $H^0(Y, K_Y)$ by a universal constant C , then the index of K_X is at most $C!$. The boundedness modulo flops of \mathcal{D}_{ab} then follows from Theorem 5.1, as the discrepancies of X will all be contained in $\frac{1}{C!}\mathbb{Z}$.

Step 2.1. When Y is an abelian 3-fold, the above claim follows immediately from the fact that $\text{rk } H^3(Y, \mathbb{Z}) = 20$ and for any $g \in G$, g^* is an automorphism of $H^3(Y, \mathbb{Z})$ defined over the integers, hence its minimal polynomial has degree ≤ 20 . That implies that any eigenvalue of g^* is a root of unity of bounded order.

Step 2.2. When Y is a product of an elliptic curve E and a weak K3 surface S the claim follows in the same vein. In fact, take $S' \rightarrow S$ the minimal resolution, then S' is a K3 surface, and g lifts to an automorphism of $E \times S'$. This implies that any eigenvalue of g^* is a root of unity of bounded order since $\text{rk } H^3(E \times S', \mathbb{Z}) = 44$. \square

APPENDIX A. ON RATIONALLY CONNECTEDNESS OF VARIETIES OF CY-TYPE

The goal of this appendix is to give a sufficient condition for a dlt log Calabi–Yau pair being rationally connected. It was suggested by Chenyang Xu and carried out during a discussion with Zhiyu Tian.

Given a log pair (X, B) , recall that a subvariety $C \subset X$ is said to be a *log center* or *non-canonical center* if there is a prime divisor E over X with center C such that the log discrepancy $a(E, X, B) < 1$.

The following is the main result of this appendix.

Theorem A.1. *Let (X, B) be an lc log Calabi–Yau pair with a log center C , then X is rationally chain connected modulo C . In particular, if C is rationally chain connected, then X is rationally chain connected.*

As a simple corollary, we have the following result.

Corollary A.2. *Let (X, B) be a dlt log Calabi–Yau pair with a 0-dimensional log center. Then X is rationally connected.*

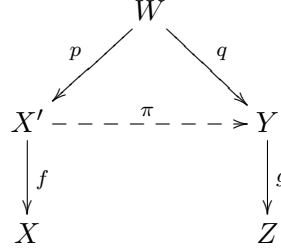
Proof of Theorem A.1. By assumption, there exists a prime divisor E over X , such that the center of E on X is C , and $a := a(E, X, B) < 1$. By [BCHM10, Corollary 1.4.3], after taking a \mathbb{Q} -factorial dlt modification of (X, B) , there exists a birational morphism $f : X' \rightarrow X$, such that

$$K_{X'} + B' + (1 - a)E = f^*(K_X + B) \equiv 0,$$

where B' is the sum of the birational transform of B on X' and the exceptional divisors of f except for E . It suffices to show that X' is rationally chain connected modulo E .

Run a $(K_{X'} + B')$ -MMP with scaling of an ample divisor on X' , according to [BCHM10, Corollary 1.3.3], this MMP ends up with a Mori fiber space

$Y \rightarrow Z$.



Let $W \subset X' \times Y$ be the closure of the graph of π , and p and q are the projections from W to X' and Y , respectively. Since $-K_Y$ is ample over Z and Y is klt, according to [HM07, Theorem 1.2], every fiber of $W \rightarrow Z$ is rationally chain connected.

Let E_Y, E_W be the strict transforms of E on Y and W , respectively. Since this MMP is also a $(-E)$ -MMP, E_Y dominates Z and so does E_W . Thus, W is rationally chain connected modulo E_W , and hence X' is rationally chain connected modulo E . We complete the proof. \square

Proof of Corollary A.2. Follows easily from Theorem A.1 and [HM07, Corollary 1.5(2)]. \square

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