Curve counting theories via stable objects II: DT/ncDT/flop formula

Yukinobu Toda

Abstract

The goal of the present paper is to show the transformation formula of Donaldson-Thomas invariants on smooth projective Calabi-Yau 3-folds under birational transformations via categorical method. We also generalize the non-commutative Donaldson-Thomas invariants, introduced by B. Szendrői in a local $(-1,-1)$-curve example, to an arbitrary flopping contraction from a smooth projective Calabi-Yau 3-fold. The transformation formula between such invariants and the usual Donaldson-Thomas invariants are also established. These formulas will be deduced from the wall-crossing formula in the space of weak stability conditions on the derived category.

1 Introduction

This paper is a sequel of the author's previous paper [30], and study the generating series of Donaldson-Thomas (DT for short) type invariants via categorical method. The main result is to show the transformation formula of our generating series under birational transformations of Calabi-Yau 3-folds, and the generalized McKay correspondence introduced by Van den Bergh [7]. We use the space of weak stability conditions on triangulated categories, which generalizes Bridgeland's stability conditions [5], and the wall-crossing formula of the generating series due to Joyce and Song [11], Kontsevich and Soibelman [15].

1.1 Motivation

Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e. the canonical line bundle $\wedge^3 T_X^*$ is trivial. Let

$$\phi\colon X^+ \dashrightarrow X,$$

be a birational map between smooth projective Calabi-Yau 3-folds. The purpose of this paper is to compare curve counting theories on X and X^+ via categorical method, i.e. effectively use an equivalence of bounded derived categories of coherent sheaves by Bridgeland [4],

$$\Phi\colon D^b(\mathrm{Coh}(X^+)) \xrightarrow{\sim} D^b(\mathrm{Coh}(X)). \tag{1}$$

The problem of comparing curve counting invariants under birational transformations has been studied in [22], [17], [20], [14] for Gromov-Witten invariants and in [9] for DT invariants, via explicit calculations or using J. Li's degeneration formula. The categorical approach for the above problem is studied by the author in [31] for (a kind of approximation of) Gopakumar-Vafa invariants. In this paper, we give a categorical understanding of transformation formula of DT invariants under birational maps using the equivalence (1).

Recall that a flop is a birational map $\phi\colon X^{+} \dashrightarrow X$ which fits into a diagram,

$$\begin{array}{ccc} X^{+} & \overset{\phi}{\dashrightarrow} & X \\ & {\scriptstyle f^{+}}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y, & \end{array} \tag{2}$$

where Y is a projective 3-fold with only Gorenstein singularities, f, f^{+} are birational morphisms isomorphic in codimension one, and the relative Picard numbers of f, f^{+} are one respectively. (cf. Definition 2.13.) It is well-known that any birational map $\phi\colon X^{+} \dashrightarrow X$ between smooth projective Calabi-Yau 3-folds is decomposed into a composition of flops, thus our problem is reduced to the case of a flop. In this case, M. Van den Bergh [7] shows that there is a sheaf of non-commutative algebras A_Y on Y and a derived equivalence,

$$\Psi\colon D^{b}(\mathrm{Coh}(A_Y)) \xrightarrow{\sim} D^{b}(\mathrm{Coh}(X)). \tag{3}$$

(In fact there are two such sheaves of non-commutative algebras ${}^{p}A_Y$ for $p = 0, -1$. Here we put $A_Y = {}^{0}A_Y$. See Theorem 2.22.) We also introduce an analogue of DT-invariant for the non-commutative scheme (Y, A_Y), which generalizes Szendrői's non-commutative DT (ncDT for short) invariant for a local $(-1,-1)$-curve example. It is introduced in [27] and some other local examples are studied in [35], [23], [24]. Our invariant is interpreted as a globalization of the local ncDT-invariant. We consider the generating series of our invariants, and establish the formula which relates global ncDT-invariants of (Y, A_Y) to usual DT-invariants of X and X^{+}. This result answers the problem addressed by Szendrői [27, Section 3.5].

1.2 Donaldson-Thomas theory

Let us briefly recall the Donaldson-Thomas theory. For a smooth projective Calabi-Yau 3-fold X, take $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$. Let $I_n(X, \beta)$ be the Hilbert scheme of curves on X,

$$I_n(X, \beta) = \left\{ \begin{array}{l} \text{subschemes } C \subset X, \dim C \le 1 \\ \text{with } [C] = \beta,\ \chi(\mathcal{O}_C) = n. \end{array} \right\}.$$

The moduli space $I_n(X, \beta)$ is projective and has a symmetric obstruction theory [28]. The associated virtual fundamental cycle has virtual dimension zero, and the integration along it defines the DT-invariant,

$$I_{n,\beta} = \int_{[I_n(X,\beta)^{\mathrm{vir}}]} 1 \in \mathbb{Z}.$$

Another way of defining DT-invariant is to use Behrend's microlocal function [1]. For an arbitrary scheme M, Behrend associates a constructible function,

$$\nu_M \colon M \to \mathbb{Z}.$$

The function ν_M has the property that if M has a symmetric obstruction theory, then the integration of the virtual fundamental cycle coincides with the weighted Euler characteristic,

$$\int_{[M^{\mathrm{vir}}]} 1 = \chi(M, \nu_M) := \sum_{n \in \mathbb{Z}} n \chi(\nu_M^{-1}(n)). \tag{4}$$

We consider the generating series,

$$\begin{aligned} \mathrm{DT}(X) &:= \sum_{n,\beta} I_{n,\beta} x^n y^\beta, \\ \mathrm{DT}_0(X) &:= \sum_{n} I_{n,0} x^n = M(-x)^{\chi(X)}, \\ \mathrm{DT}(X/Y) &:= \sum_{n, f_*\beta = 0} I_{n,\beta} x^n y^\beta, \end{aligned} \tag{5}$$

where $f \colon X \to Y$ is a flopping contraction as in the diagram (21), and $M(x)$ is the MacMahon function,

$$M(x) = \prod_{k \geq 1} (1 - x^k)^{-k}.$$

The formula (5) for $\mathrm{DT}_0(X)$ is established in [2], [18], [16]. By the MNOP conjecture [21], the reduced series

$$\mathrm{DT}'(X) = \frac{\mathrm{DT}(X)}{\mathrm{DT}_0(X)}, \quad \mathrm{DT}'(X/Y) = \frac{\mathrm{DT}(X/Y)}{\mathrm{DT}_0(X)},$$

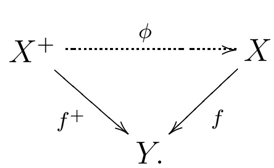
are expected to coincide with the generating series of Gromov-Witten invariants after a suitable variable change.

1.3 Non-commutative Donaldson-Thomas theory

The following example is worked out by Szendrői [27]. Let Y be the conifold singularity,

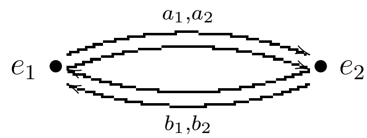
$$Y = (xy + zw = 0) \subset \mathbb{C}^4,$$

and $f \colon X \to Y$, $f^+ \colon X^+ \to Y$ blow-ups at ideals $(x, z) \subset \mathcal{O}_Y$, $(x, w) \subset \mathcal{O}_Y$ respectively. This gives an example of a (local) flop.



(6)

The exceptional loci of f, f^+ are smooth rational curves $C \subset X$, $C^+ \subset X^+$ whose normal bundles are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. There is a local version of the equivalence (3), and the $\mathcal{O}_Y$-algebra A_Y is the path algebra of the following quiver,



with relation defined by the derivations of the potential W,

$$W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1.$$

For a dimension vector $v = (v_1, v_2) \in \mathbb{Z}^{\oplus 2}$, the moduli space of framed A_Y-representations $M = (M_\bullet, a_\bullet, b_\bullet, u)$ is denoted by $\mathcal{M}_v$. Here $v = (\dim M_1, \dim M_2)$ and $u \in M_1$ generates $M_1 \oplus M_2$ as an A_Y-module. The integration of the virtual fundamental cycle yields the ncDT invariant,

$$A_{n,m[C]} = \int_{[\mathcal{M}^{\mathrm{vir}}_{(n,m+n)}]} 1 \in \mathbb{Z}. \tag{7}$$

The transformation rule between numerical classes on X and dimension vectors on A_Y is determined by the equivalence (3). We have the associated generating series,

$$\mathrm{DT}_0(A_Y) = \sum_{n,m} A_{n,m[C]} x^n y^m.$$

The following formula is conjectured by Szendrői [27] and proved by Young [36], Nagao and Nakajima [25].

Theorem 1.1. [36], [25] *We have the formula*

$$\begin{aligned} \mathrm{DT}_0(A_Y) &= M(-x)^2 \prod_{k \geq 1} (1 - (-x)^k y)^k \prod_{k \geq 1} (1 - (-x)^k y^{-1})^k, \\ &= \mathrm{DT}(X/Y) \cdot \phi_* \mathrm{DT}'(X^+/Y). \end{aligned} \tag{8}$$

Let us return to the situation of an arbitrary flopping contraction $f \colon X \to Y$ from a smooth projective Calabi-Yau 3-fold X. For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we will introduce a global version of the ncDT invariant,

$$A_{n,\beta} \in \mathbb{Z},$$

in Definition 2.37 as a generalization of the invariant (7). The invariant $A_{n,\beta}$ counts cyclic A_Y-modules F, satisfying $\dim \operatorname{Supp} \Psi(F) \leq 1$ and

$$[\Psi(F)] = \beta, \quad \chi(\Psi(F)) = n,$$

via the equivalence (3). If $f_*\beta \neq 0$, then such a cyclic A_Y-module is not of finite dimensional as a $\mathbb{C}$-vector space. The associated generating series are defined by

$$\mathrm{DT}(A_Y) := \sum_{n,\beta} A_{n,\beta} x^n y^\beta,$$

$$\mathrm{DT}_0(A_Y) := \sum_{n, f_*\beta = 0} A_{n,\beta} x^n y^\beta.$$

1.4 Main result

Let $f\colon X \to Y$ be a flopping contraction from a smooth projective Calabi-Yau 3-fold X to a singular 3-fold Y, and $\phi\colon X^+ \dashrightarrow X$ its flop as in the diagram (21). Using the results of [30, Section 8], which relies on the results of [11] and [3], we will prove the following. [1]

Theorem 1.2. **[Theorem 5.6, Theorem 5.7, Theorem 5.8]** *We have the following formula,*

$$\mathrm{DT}(X/Y) = i \circ \phi_* \mathrm{DT}(X^+/Y), \tag{9}$$

$$\mathrm{DT}_0(A_Y) = \mathrm{DT}(X/Y) \cdot \phi_* \mathrm{DT}'(X^+/Y), \tag{10}$$

$$\frac{\mathrm{DT}(X)}{\mathrm{DT}(X/Y)} = \frac{\mathrm{DT}(A_Y)}{\mathrm{DT}_0(A_Y)} = \phi_* \frac{\mathrm{DT}(X^+)}{\mathrm{DT}(X^+/Y)}. \tag{11}$$

Here the variable change is $\phi_*(\beta, n) = (\phi_*\beta, n)$ and $i(\beta, n) = (-\beta, n)$.

Note that (9) and the equality

$$\frac{\mathrm{DT}(X)}{\mathrm{DT}(X/Y)} = \phi_* \frac{\mathrm{DT}(X^+)}{\mathrm{DT}(X^+/Y)},$$

given in (11) are proved by J. Hu and W. P. Li [9] in the case of a flop at a $(-1,-1)$-curve, using J. Li's degeneration formula. Also in this case, the equality (10) is just the formula (8). The equality (10) together with the first equality of (11) yields

$$\mathrm{DT}(A_Y) = \mathrm{DT}(X) \cdot \phi_* \mathrm{DT}'(X^+/Y),$$

which is interpreted as a global version of the formula (8).

Our proof is based on the analysis of weak stability conditions on the triangulated category,

$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\le 1}(X) \rangle_{\mathrm{tr}} \subset D^b(\mathrm{Coh}(X)),$$

i.e. $\mathcal{D}_X$ is the smallest triangulated subcategory of $D^b(\mathrm{Coh}(X))$, which contains $\mathcal{O}_X$ and $E \in \mathrm{Coh}(X)$ with $\dim \mathrm{Supp}(E) \le 1$. We will construct the generating series of counting invariants of semistable objects, and see how such series vary under change of weak stability conditions. We will construct weak stability conditions on $\mathcal{D}_X$ explicitly, and apply the wall-crossing formula given in [30, Section 5, Section 8].

1.5 Content of the paper

In Section 2, we introduce some notions which is used in this paper. In Section 3, we construct weak stability conditions on the triangulated category $\mathcal{D}_X$. In Section 4, we investigate relevant semistable objects. In Section 5, we give a proof of Theorem 1.2. In Section 6, we prove some technical lemmas.

[1] At the moment the author writes the first draft of this paper, the result of [3] is not yet written.

1.6 Acknowledgement

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1.7 Notation and convention

In this paper, all the varieties are defined over $\mathbb{C}$. For a triangulated category $\mathcal{D}$, the shift functor is denoted by [1]. For a set of objects $S \subset \mathcal{D}$, we denote by $\langle S \rangle_{\mathrm{tr}} \subset \mathcal{D}$ the smallest triangulated subcategory of $\mathcal{D}$ which contains S. Also we denote by $\langle S \rangle_{\mathrm{ex}} \subset \mathcal{D}$ the smallest extension closed subcategory of $\mathcal{D}$ which contains S. For an abelian category $\mathcal{A}$ and a set of objects $S \subset \mathcal{A}$, the subcategory $\langle S \rangle_{\mathrm{ex}} \subset \mathcal{A}$ is also defined to be the smallest extension closed subcategory of $\mathcal{A}$ which contains S. The abelian category of coherent sheaves is denoted by $\mathrm{Coh}(X)$. We say $F \in \mathrm{Coh}(X)$ is d-dimensional if its support is d-dimensional. The bounded derived category of coherent sheaves is denoted by $D^b(\mathrm{Coh}(X))$. For an object $E \in D^b(\mathrm{Coh}(X))$ and $i \in \mathbb{Z}$, we denote by $\mathcal{H}^i(E) \in \mathrm{Coh}(X)$ the i-th cohomology of E.

2 Preliminaries

In this section, we introduce some notions which will be used in later sections.

2.1 Generalities on weak stability conditions

Here we collect definitions and properties of weak stability conditions on triangulated categories introduced in [30, Section 2]. This is a generalized notion of Bridgeland's stability conditions on triangulated categories [5]. Let $\mathcal{D}$ be a triangulated category, and $K(\mathcal{D})$ the Grothendieck group of $\mathcal{D}$. We fix a finitely generated free abelian group Γ, and its filtration,

$$0 \subsetneq \Gamma_0 \subsetneq \Gamma_1 \cdots \subsetneq \Gamma_N = \Gamma, \tag{12}$$

with each subquotient

$$\mathbb{H}_i = \Gamma_i / \Gamma_{i-1}, \quad (0 \le i \le N)$$

a free abelian group. We also fix a group homomorphism,

$$\mathrm{cl} \colon K(\mathcal{D}) \longrightarrow \Gamma.$$

We set $\mathbb{H}_i^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{H}_i, \mathbb{C})$ and fix a norm $\|*\|_i$ on $\mathbb{H}_i \otimes_{\mathbb{Z}} \mathbb{R}$. For an element

$$Z = \{Z_i\}_{i=0}^{N} \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee},$$

and $v \in \Gamma$, we set

$$Z(v) := Z_m([v]) \in \mathbb{C},$$

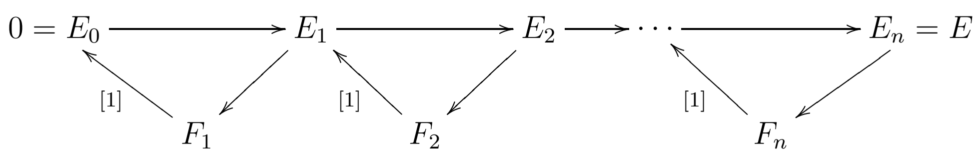
where $0 \le m \le N$ satisfies $v \in \Gamma_m \setminus \Gamma_{m-1}$, and $[v]$ is the class of v in $\mathbb{H}_m$. Here we set $\Gamma_{-1} = \emptyset$. Also we define $\|v\| := \|[v]\|_m$. Below we write $\mathrm{cl}(E) \in \Gamma$ just as $E \in \Gamma$ when there is no confusion.

Definition 2.1. A *weak stability condition* on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$,

$$Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}, \quad \mathcal{P}(\phi) \subset \mathcal{D}, \quad (\phi \in \mathbb{R}), \tag{13}$$

where $\mathcal{P}(\phi)$ is a full additive subcategory of $\mathcal{D}$, which satisfies the following axiom.

- For any $\phi \in \mathbb{R}$, we have $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$.
- For $E_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$, we have $\mathrm{Hom}(E_1, E_2) = 0$.
- **(Harder-Narasimhan property):** For any object $E \in \mathcal{D}$, we have the following collection of triangles:



  such that $F_j \in \mathcal{P}(\phi_j)$ with $\phi_1 > \phi_2 > \cdots > \phi_n$.

- For any non-zero $E \in \mathcal{P}(\phi)$, we have

$$Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi). \tag{14}$$

- **(Support property):** There is a constant $C > 0$ such that for any non-zero $E \in \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$, we have

$$\|E\| \le C|Z(E)|. \tag{15}$$

- **(Local finiteness condition):** There exists $\eta > 0$ such that for any $\phi \in \mathbb{R}$, the quasi-abelian category $\mathcal{P}((\phi - \eta, \phi + \eta))$ is of finite length.

Here for an interval $I \subset \mathbb{R}$, the subcategory $\mathcal{P}(I) \subset \mathcal{D}$ and the subset $C_\sigma(I) \subset \Gamma$ are defined to be

$$\begin{aligned} \mathcal{P}(I) &= \langle \mathcal{P}(\phi) : \phi \in I \rangle_{\mathrm{ex}} \subset \mathcal{D}, \\ C_\sigma(I) &= \mathrm{im}\{\mathrm{cl} \colon \mathcal{P}(I) \to \Gamma\}. \end{aligned} \tag{16}$$

If $I = (a, b)$ with $b - a < 1$, then $\mathcal{P}(I)$ is a quasi-abelian category (cf. [5, Definition 4.1],) and $\mathcal{P}(I)$ is said to be of finite length if $\mathcal{P}(I)$ is noetherian and artinian with respect to strict epimorphisms and strict morphisms. See [5, Section 4] for more detail.

Another way of defining weak stability conditions is using t-structures. The readers can refer [5] for bounded t-structures, and their hearts.

Definition 2.2. Let $\mathcal{A} \subset \mathcal{D}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{D}$. We say $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ is a *weak stability function* on $\mathcal{A}$ if for any non-zero $E \in \mathcal{A}$, we have

$$Z(E) \in \mathfrak{H} := \{r \exp(i\pi\phi) : r > 0,\ 0 < \phi \leq 1\}. \tag{17}$$

By (17), we can uniquely determine the argument,

$$\arg Z(E) \in (0, \pi],$$

for any $0 \neq E \in \mathcal{A}$. For an exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}$, one of the following equalities holds.

$$\arg Z(F) \leq \arg Z(E) \leq \arg Z(G),$$
$$\arg Z(F) \geq \arg Z(E) \geq \arg Z(G).$$

Definition 2.3. Let $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ be a weak stability function on $\mathcal{A}$. We say $0 \neq E \in \mathcal{A}$ is *Z-semistable* (resp. *stable*) if for any exact sequence $0 \to F \to E \to G \to 0$ we have

$$\arg Z(F) \leq \arg Z(G), \quad (\text{resp. } \arg Z(F) < \arg Z(G).) \tag{18}$$

The notion of Harder-Narasimhan filtration is defined in a similar way to usual stability conditions.

Definition 2.4. Let $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ be a weak stability function on $\mathcal{A}$. A *Harder-Narasimhan filtration* of an object $E \in \mathcal{A}$ is a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{k-1} \subset E_k = E,$$

such that each subquotient $F_j = E_j/E_{j-1}$ is Z-semistable with

$$\arg Z(F_1) > \arg Z(F_2) > \cdots > \arg Z(F_k).$$

A weak stability function Z is said to have the *Harder-Narasimhan property* if any object $E \in \mathcal{A}$ has a Harder-Narasimhan filtration.

We will use the following proposition. (cf. [30, Proposition 2.12].)

Proposition 2.5. *Let $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ be a weak stability function on $\mathcal{A}$. Suppose that the following chain conditions are satisfied.*

(a) There are no infinite sequences of subobjects in $\mathcal{A}$,

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

with $\arg Z(E_{j+1}) > \arg Z(E_j/E_{j+1})$ *for all j.*

(b) There are no infinite sequences of quotients in $\mathcal{A}$,

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \stackrel{\pi_j}{\twoheadrightarrow} E_{j+1} \twoheadrightarrow \cdots$$

with $\arg Z(\ker \pi_j) > \arg Z(E_{j+1})$ *for all j.*

Then Z has the Harder-Narasimhan property.

We have the following proposition. (cf. [5, Proposition 5.3], [30, Proposition 2.13].)

Proposition 2.6. *Giving a pair $(Z, \mathcal{P})$ as in (13) satisfying (14) is equivalent to giving a bounded t-structure $\mathcal{A}$ on $\mathcal{D}$ and a weak stability function on its heart with the Harder-Narasimhan property.*

The correspondence in the above proposition is given as follows. Given a pair $(Z, \mathcal{P})$ satisfying (14), we set $\mathcal{A}$ to be

$$\mathcal{A} = \mathcal{P}((0,1]) \subset \mathcal{D}.$$

Conversely given the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$ and a weak stability function Z on $\mathcal{A}$, we set $\mathcal{P}(\phi)$ to be the following full additive subcategory of $\mathcal{A}$,

$$\mathcal{P}(\phi) = \left\{ E \in \mathcal{A} : \begin{array}{l} E \text{ is } Z\text{-semistable with} \\ Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi) \end{array} \right\}.$$

Below we write an element of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ either as $(Z, \mathcal{P})$ or $(Z, \mathcal{A})$, where $(Z, \mathcal{P})$ is a pair (13) and $(Z, \mathcal{A})$ is given as in Definition 2.2. Let $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ be the set of weak stability conditions on $\mathcal{D}$. The following theorem is given in [30, Theorem 2.15], as an analogue of [5, Theorem 7.1].

Theorem 2.7. [30, Theorem 2.15] *The map*

$$\Pi \colon \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}) \ni (Z, \mathcal{P}) \longmapsto Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee},$$

is a local homeomorphism. In particular each connected component of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ *is a complex manifold.*

We will need the following two lemmas. The first one is proved in [30, Lemma 7.1].

Lemma 2.8. [30, Lemma 7.1] *Let $\mathcal{A}$ be the heart of a bounded t-structure on $\mathcal{D}$, and $(\mathcal{T}, \mathcal{F})$ a torsion pair (cf. Definition 2.14,) on $\mathcal{A}$. Let $\mathcal{B} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}$ the associated tilting. (cf. (25).) Let*

$$[0,1) \ni t \longmapsto Z_t \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee},$$

be a continuous map such that $\sigma_t = (Z_t, \mathcal{A})$ for $0 < t < 1$ and $\sigma_0 = (Z_0, \mathcal{B})$ determine points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$. *Then we have* $\lim_{t \to 0} \sigma_t = \sigma_0$.

The second one is a compatibility of the weak stability conditions via equivalences of triangulated categories. The proof is straightforward, and we omit the proof.

Lemma 2.9. *Let $\mathcal{D}'$ be another triangulated category together with similar additional data* $\mathrm{cl}' \colon K(\mathcal{D}') \to \Gamma'$ *and a filtration $\Gamma'_\bullet$ as in (12). Suppose that $\Phi \colon \mathcal{D} \to \mathcal{D}'$ gives an* *equivalence of triangulated categories such that there is a filtration preserving isomorphism* $\Phi_\Gamma \colon \Gamma_\bullet \to \Gamma'_\bullet$ *which fits into the following commutative diagram,*

$$\begin{array}{ccc} K(\mathcal{D}) & \xrightarrow{\Phi} & K(\mathcal{D}') \\ \downarrow \mathrm{cl} & & \downarrow \mathrm{cl}' \\ \Gamma & \xrightarrow{\Phi_\Gamma} & \Gamma'. \end{array} \tag{19}$$

Then there is an isomorphism $\Phi_* \colon \operatorname{Stab}_{\Gamma_\bullet}(\mathcal{D}) \to \operatorname{Stab}_{\Gamma'_\bullet}(\mathcal{D}')$ *such that the following diagram commutes,*

$$\begin{array}{ccc} \operatorname{Stab}_{\Gamma_\bullet}(\mathcal{D}) & \xrightarrow{\Phi_*} & \operatorname{Stab}_{\Gamma'_\bullet}(\mathcal{D}') \\ \downarrow \Pi & & \downarrow \Pi' \\ \prod_{i=0}^N \mathbb{H}_i^\vee & \xrightarrow{(\mathrm{gr}\,\Phi_\Gamma^{-1})^\vee} & \prod_{i=0}^N \mathbb{H}_i'^\vee. \end{array}$$

2.2 Terminology from birational geometry

In what follows, we assume that X is a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e. the canonical line bundle $\wedge^3 T_X^*$ is trivial. (We do not assume the simply connectedness of X.) Here we introduce standard terminology in birational geometry, for example used in [12, Definition 1.1].

Let S be a projective variety with a morphism $f \colon X \to S$. Two divisors D_1, D_2 on X are called *numerically equivalent* over S if and only if $D_1 \cdot C = D_2 \cdot C$ for any curve $C \subset X$ with $f_*[C] = 0$. Similarly, one-cycles C_1, C_2 on X contracted by f are *numerically equivalent* if and only if $D \cdot C_1 = D \cdot C_2$ for every divisor D on X.

Definition 2.10. We define abelian groups $N^1(X/S)$, $N_1(X/S)$ to be

$$\begin{aligned} N^1(X/S) &:= \{\text{Divisors on } X\}/(\text{numerical equivalence over } S), \\ N_1(X/S) &:= \{\text{One-cycles on } X \text{ contracted by } f\}/(\text{numerical equivalence}). \end{aligned}$$

By the definition, there is the perfect pairing,

$$N^1(X/S)_\mathbb{R} \times N_1(X/S)_\mathbb{R} \ni (D, C) \longmapsto D \cdot C \in \mathbb{R}.$$

Definition 2.11. We define the *ample cone* $A(X/S)$, the *complexified ample cone* $A(X/S)_\mathbb{C}$, and the *semigroup of effective one-cycles* $\mathrm{NE}(X/S)$ to be

$$\begin{aligned} A(X/S) &:= \{\text{Numerical classes of } f\text{-ample } \mathbb{R}\text{-divisors }\} \subset N^1(X/S)_\mathbb{R}, \\ A(X/S)_\mathbb{C} &:= \{B + i\omega \in N^1(X/S)_\mathbb{C} : \omega \in A(X/S)\}, \\ \mathrm{NE}(X/S) &:= \{\text{Effective one-cycles contracted by } f\} \subset N_1(X/S). \end{aligned}$$

For $\beta, \beta' \in N_1(X/S)$, we write $\beta \ge \beta'$ if $\beta - \beta' \in \mathrm{NE}(X/S)$. When $S = \operatorname{Spec}\mathbb{C}$, we write

$$N^1(X) := N^1(X/\operatorname{Spec}\mathbb{C}), \quad N_1(X) := N_1(X/\operatorname{Spec}\mathbb{C}),$$

etc, for simplicity. We set $N_{\le 1}(X)$ to be

$$N_{\le 1}(X) := \mathbb{Z} \oplus N_1(X).$$

Definition 2.12. A birational morphism $f\colon X \to Y$ is called a *flopping contraction* if the following conditions are satisfied.

- f is isomorphic in codimension one, and Y has only Gorenstein singularities.
- We have $\dim_{\mathbb{R}} N^1(X/Y)_{\mathbb{R}} = 1$.

Let $f\colon X \to Y$ be a flopping contraction. The exceptional locus $C \subset X$ is a tree of rational curves,

$$C = C_1 \cup C_2 \cup \cdots \cup C_N, \quad C_i \cong \mathbb{P}^1.$$

(See for example [7, Lemma 3.4.1].) By the second condition of Definition 2.12, there is a relative ample divisor H on X such that

$$N^1(X/Y) = \mathbb{R}[H], \quad A(X/Y) = \mathbb{R}_{>0}[H]. \tag{20}$$

Definition 2.13. Let $f\colon X \to Y$ be a flopping contraction. A *flop* of f is a birational map $\phi\colon X^+ \dashrightarrow X$, which fits into a diagram

$$\begin{array}{ccc} X^+ & \stackrel{\phi}{\dashrightarrow} & X \\ & {\scriptstyle f^+}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y, & \end{array} \tag{21}$$

such that f^+ is also a flopping contraction with $f \circ \phi = f^+$, and ϕ is not an isomorphism.

It is well-known that a flop is unique if it exists, and any birational map between smooth projective Calabi-Yau 3-folds is decomposed into a finite number of flops. (cf. [13, Theorem 1].) For a flop $\phi\colon X^+ \dashrightarrow X$, we have the linear isomorphisms,

$$\phi_*\colon N^1(X^+/Y)_{\mathbb{R}} \xrightarrow{\cong} N^1(X/Y)_{\mathbb{R}}, \tag{22}$$

$$\phi_*\colon N_1(X^+/Y)_{\mathbb{R}} \xrightarrow{\cong} N_1(X/Y)_{\mathbb{R}}, \tag{23}$$

where (22) is the strict transform of divisors, and (23) is the inverse of the dual of (22). Note that ϕ_* takes $A(X^+/Y)$ to $-A(X/Y)$ and takes $\mathrm{NE}(X^+/Y)$ to $-\mathrm{NE}(X/Y)$.

2.3 t-structures and tilting

Let $\mathcal{D}$ be a triangulated category, and $\mathcal{A} \subset \mathcal{D}$ the heart of a bounded t-structure on $\mathcal{D}$. Here we recall the notion of torsion pairs and tilting.

Definition 2.14. [8] Let $(\mathcal{T}, \mathcal{F})$ be a pair of full subcategories of $\mathcal{A}$. We say $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* if the following conditions hold.

- $\mathrm{Hom}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
- Any object $E \in \mathcal{A}$ fits into an exact sequence,

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0, \tag{24}$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$, its *tilting* is defined by

$$\mathcal{A}^{\dagger} := \left\{ E \in \mathcal{D} : \begin{array}{l} \mathcal{H}^{-1}_{\mathcal{A}}(E) \in \mathcal{F},\ \mathcal{H}^{0}_{\mathcal{A}}(E) \in \mathcal{T}, \\ \mathcal{H}^{i}_{\mathcal{A}}(E) = 0 \text{ for } i \notin \{-1, 0\}. \end{array} \right\}, \tag{25}$$

i.e. $\mathcal{A}^{\dagger} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}$ in $\mathcal{D}$. Here $\mathcal{H}^{i}_{\mathcal{A}}(*)$ is the i-th cohomology functor with respect to the t-structure with heart $\mathcal{A}$. It is known that $\mathcal{A}^{\dagger}$ is the heart of a bounded t-structure on $\mathcal{D}$. (cf. [8, Proposition 2.1].) Later we will need the following lemma.

Lemma 2.15. *Let $\mathcal{A} \subset \mathcal{D}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{D}$. Suppose that $\mathcal{A}$ is a noetherian abelian category.*

(i) Let $\mathcal{T} \subset \mathcal{A}$ be a full subcategory which is closed under extensions and quotients in $\mathcal{A}$. Then for $\mathcal{F} = \{E \in \mathcal{A} : \mathrm{Hom}(\mathcal{T}, E) = 0\}$, the pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\mathcal{A}$.

(ii) Let $\mathcal{F} \subset \mathcal{A}$ be a full subcategory which is closed under extensions and subobjects in $\mathcal{A}$. Then for $\mathcal{T} = \{E \in \mathcal{A} : \mathrm{Hom}(E, \mathcal{F}) = 0\}$, the pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\mathcal{A}$.

Proof. We only show (i), as the proof of (ii) is similar. Take $E \in \mathcal{A}$ with $E \notin \mathcal{F}$. Then there is $T \in \mathcal{T}$ and a non-zero morphism $T \to E$. Since $\mathcal{T}$ is closed under quotients, we may assume that $T \to E$ is a monomorphism in $\mathcal{A}$. Take an exact sequence in $\mathcal{A}$,

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0. \tag{26}$$

By the noetherian property of $\mathcal{A}$ and the assumption that $\mathcal{T}$ is closed under extensions, we may assume that there is no $T \subsetneq T' \subset E$ with $T' \in \mathcal{T}$. Then we have $F \in \mathcal{F}$ and (26) gives the desired sequence (24). □

2.4 Notation of abelian categories

Here we give some notation of abelian categories which will be used in this paper.

Definition 2.16. Let A be a sheaf of $\mathcal{O}_X$-algebras on a variety X, which is coherent as an $\mathcal{O}_X$-module. We denote by $\mathrm{Coh}(A)$ the abelian category of right coherent A-modules. For an object $E \in \mathrm{Coh}(A)$, the support of E is defined to be the support of E as an $\mathcal{O}_X$-module. We set

$$\begin{aligned} \mathrm{Coh}_0(A) &= \{E \in \mathrm{Coh}(A) : \dim \mathrm{Supp}(E) = 0\}, \\ \mathrm{Coh}_{\le 1}(A) &= \{E \in \mathrm{Coh}(A) : \dim \mathrm{Supp}(E) \le 1\}, \\ \mathrm{Coh}_{\ge 2}(A) &= \{E \in \mathrm{Coh}(A) : \mathrm{Hom}(\mathrm{Coh}_{\le 1}(A), E) = 0\}. \end{aligned}$$

If $A = \mathcal{O}_X$, we write $\mathrm{Coh}_{\bullet}(\mathcal{O}_X)$ as $\mathrm{Coh}_{\bullet}(X)$.

By Lemma 2.15, the pair $(\mathrm{Coh}_{\le 1}(A), \mathrm{Coh}_{\ge 2}(A))$ is a torsion pair on $\mathrm{Coh}(A)$.

Definition 2.17. We define $\mathrm{Coh}^{\dagger}(A)$ to be the tilting with respect to $(\mathrm{Coh}_{\le 1}(A), \mathrm{Coh}_{\ge 2}(A))$, i.e.

$$\mathrm{Coh}^{\dagger}(A) = \langle \mathrm{Coh}_{\ge 2}(A)[1], \mathrm{Coh}_{\le 1}(A) \rangle_{\mathrm{ex}}.$$

If $A = \mathcal{O}_X$, we write $\mathrm{Coh}^{\dagger}(\mathcal{O}_X)$ as $\mathrm{Coh}^{\dagger}(X)$.

2.5 Derived equivalence under flops

Let $f\colon X \to Y$ be a flopping contraction from a smooth projective Calabi-Yau 3-fold X. (cf. Definition 2.12.) In this situation, Bridgeland [4] associates the subcategories ${}^p\mathrm{Per}(X/Y)$ on $D^b(\mathrm{Coh}(X))$ for $p = 0, -1$, as follows.

Definition 2.18. We define ${}^p\mathrm{Per}(X/Y) \subset D^b(\mathrm{Coh}(X))$ for $p = 0, -1$ to be

$$
{}^p\mathrm{Per}(X/Y) = \left\{ E \in D^b(\mathrm{Coh}(X)) : \begin{array}{l} \mathbf{R}f_*E \in \mathrm{Coh}(Y), \\ \mathrm{Hom}^{<-p}(E, \mathcal{C}) = \mathrm{Hom}^{<-p}(\mathcal{C}, E) = 0. \end{array} \right\},
$$

where $\mathcal{C} = \{F \in \mathrm{Coh}(X) \mid \mathbf{R}f_*F = 0\}$. We also define ${}^p\mathrm{Per}_0(X/Y)$ and ${}^p\mathrm{Per}_{\le 1}(X/Y)$ to be

$$
\begin{aligned}
{}^p\mathrm{Per}_0(X/Y) &= \{E \in {}^p\mathrm{Per}(X/Y) : \dim \mathrm{Supp}\, \mathbf{R}f_*E = 0\}, \\
{}^p\mathrm{Per}_{\le 1}(X/Y) &= \{E \in {}^p\mathrm{Per}(X/Y) : \dim \mathrm{Supp}(E) \le 1\}.
\end{aligned}
$$

Remark 2.19. *By the definition, it is easy to see that*

$$
\mathcal{O}_X \in {}^p\mathrm{Per}(X/Y), \quad p = 0, -1.
$$

It is proved in [4] that ${}^p\mathrm{Per}(X/Y)$ are the hearts of bounded t-structures on $D^b(\mathrm{Coh}(X))$, hence they are abelian categories. The categories ${}^p\mathrm{Per}_{\le 1}(X/Y)$ are also the hearts of bounded t-structures on $D^b(\mathrm{Coh}_{\le 1}(X))$. (cf. [31, Proposition 5.2].) The generators of ${}^p\mathrm{Per}_{\le 1}(X/Y)$ are described as follows. Let $C_1, \cdots, C_N \subset X$ be the irreducible components of the exceptional locus of f. We have the following.

Lemma 2.20. *The abelian categories ${}^p\mathrm{Per}_{\le 1}(X/Y)$ are described as*

$$
{}^0\mathrm{Per}_{\le 1}(X/Y) = \langle \omega_{f^{-1}(y)}[1], \mathcal{O}_{C_i}(-1), \widetilde{\mathrm{Coh}}_{\le 1}(X) \rangle_{\mathrm{ex}}, \tag{27}
$$

$$
{}^{-1}\mathrm{Per}_{\le 1}(X/Y) = \langle \mathcal{O}_{f^{-1}(y)}, \mathcal{O}_{C_i}(-1)[1], \widetilde{\mathrm{Coh}}_{\le 1}(X) \rangle_{\mathrm{ex}}. \tag{28}
$$

Here $y \in \mathrm{Sing}(Y)$, $1 \le i \le N$, and $\widetilde{\mathrm{Coh}}_{\le 1}(X)$ is defined to be

$$
\widetilde{\mathrm{Coh}}_{\le 1}(X) := \{F \in \mathrm{Coh}_{\le 1}(X) \mid C_i \not\subseteq \mathrm{Supp}(F) \text{ for all } i\}.
$$

Proof. This is a straightforward generalization of [7] and the proof is written in [31, Proposition 5.2]. □

Let $\phi\colon X^+ \dashrightarrow X$ be the flop of f. (cf. Definition 2.13.) The following theorem is proved in [4].

Theorem 2.21. [4] *There is an equivalence of bounded derived categories of coherent sheaves,*

$$
\Phi\colon D^b(\mathrm{Coh}(X^+)) \xrightarrow{\sim} D^b(\mathrm{Coh}(X)), \tag{29}
$$

which takes ${}^{-1}\mathrm{Per}(X^+/Y)$ to ${}^0\mathrm{Per}(X/Y)$.

2.6 Flops and non-commutative algebras

Let $f\colon X \to Y$ be a flopping contraction as in Definition 2.12. By Van den Bergh [7], the abelian categories ${}^p\mathrm{Per}(X/Y)$ are related to sheaves of non-commutative algebras on Y.

Theorem 2.22. [7] *There are vector bundles ${}^p\mathcal{E}$ on X for $p = 0, -1$, which admit derived equivalences,*

$$ {}^p\Phi = \mathbf{R}f_*\mathbf{R}\mathcal{H}om({}^p\mathcal{E}, *) \colon D^b(\mathrm{Coh}(X)) \xrightarrow{\cong} D^b(\mathrm{Coh}({}^pA_Y)). \tag{30} $$

Here ${}^pA_Y = f_*\mathcal{E}nd({}^p\mathcal{E})$ are sheaves of non-commutative algebras on Y. The equivalences (30) restrict to equivalences

$$ {}^p\Phi \colon {}^p\mathrm{Per}(X/Y) \xrightarrow{\sim} \mathrm{Coh}({}^pA_Y). \tag{31} $$

Proof. Here we briefly recall how to construct ${}^p\mathcal{E}$, which will be needed in the later section. We treat the case of $p = 0$ for simplicity. Let $\mathcal{L}_X$ be a globally generated ample line bundle on X. We have a surjection of sheaves,

$$ (\mathcal{L}_Y^{-1})^{\oplus a} \twoheadrightarrow R^1 f_* \mathcal{L}_X^{-1}, $$

for a sufficiently ample line bundle $\mathcal{L}_Y$ on Y and $a > 0$. Taking the adjunction, we obtain the short exact sequence,

$$ 0 \longrightarrow \mathcal{L}_X^{-1} \longrightarrow {}^0\mathcal{E}' \longrightarrow f^*(\mathcal{L}_Y^{-1})^{\oplus a} \longrightarrow 0. \tag{32} $$

Then ${}^0\mathcal{E}$ is defined to be

$$ {}^0\mathcal{E} = \mathcal{O}_X \oplus {}^0\mathcal{E}'. $$

The constructions of ${}^{-1}\mathcal{E}'$ and ${}^{-1}\mathcal{E}$ are similar. (See [7] for the detail.) □

Remark 2.23. *By the construction, the sheaves of algebras ${}^pA_Y$ are direct sums of locally projective ${}^pA_Y$-modules,*

$$ {}^pA_Y = {}^pA'_Y \oplus {}^pA''_Y, \tag{33} $$

where ${}^pA'_Y = {}^p\Phi(\mathcal{O}_X)$ and ${}^pA''_Y = {}^p\Phi({}^p\mathcal{E}')$.

Note that the torsion pair $(\mathrm{Coh}_{\le 1}({}^pA_Y), \mathrm{Coh}_{\ge 2}({}^pA_Y))$ induces the torsion pair

$$ ({}^p\mathrm{Per}_{\le 1}(X/Y), {}^p\mathrm{Per}_{\ge 2}(X/Y)), $$

on ${}^p\mathrm{Per}(X/Y)$ via the equivalence ${}^p\Phi \colon {}^p\mathrm{Per}(X/Y) \xrightarrow{\sim} \mathrm{Coh}({}^pA_Y)$.

Definition 2.24. We define the abelian category ${}^p\mathrm{Per}^{\dagger}(X/Y)$ to be the tilting with respect to the torsion pair $({}^p\mathrm{Per}_{\le 1}(X/Y), {}^p\mathrm{Per}_{\ge 2}(X/Y))$, i.e.

$$ {}^p\mathrm{Per}^{\dagger}(X/Y) = \langle {}^p\mathrm{Per}_{\ge 2}(X/Y)[1], {}^p\mathrm{Per}_{\le 1}(X/Y) \rangle_{\mathrm{ex}}. $$

Remark 2.25. *By the construction, the equivalence ${}^p\Phi \colon D^b(\mathrm{Coh}(X)) \xrightarrow{\sim} D^b(\mathrm{Coh}({}^pA_Y))$ restricts to the equivalence between ${}^p\mathrm{Per}^{\dagger}(X/Y)$ and $\mathrm{Coh}^{\dagger}({}^pA_Y)$.*

2.7 Donaldson-Thomas theory

Here we introduce Donaldson-Thomas invariants. For $(n, \beta) \in \mathbb{Z} \oplus N_1(X)$, let $I_n(X, \beta)$ be the moduli space of subschemes $C \subset X$ with

$$\dim C \leq 1, \quad [C] = \beta, \quad \text{and} \quad \chi(\mathcal{O}_C) = n.$$

There is a symmetric perfect obstruction theory on $I_n(X, \beta)$ [29], and the associated virtual cycle,

$$[I_n(X, \beta)^{\mathrm{vir}}] \in A_0(I_n(X, \beta)).$$

Definition 2.26. The *Donaldson-Thomas invariant* is defined by

$$I_{n,\beta} := \int_{[I_n(X,\beta)^{\mathrm{vir}}]} 1 \in \mathbb{Z}.$$

Recall that for any scheme M, Behrend [1] associates a canonical constructible function,

$$\nu_M \colon M \to \mathbb{Z}, \tag{34}$$

such that if M has a symmetric perfect obstruction theory, we have

$$\int_{[M^{\mathrm{vir}}]} 1 = \sum_{i \in \mathbb{Z}} i\chi(\nu_M^{-1}(i)).$$

Here $\chi(*)$ is the topological Euler characteristic. In this way, the invariant $I_{n,\beta}$ is also defined as a weighted Euler characteristic with respect to the Behrend function on $I_n(X, \beta)$. The relevant generating series are defined as follows.

Definition 2.27. Let $f \colon X \to Y$ be a flopping contraction. We define the generating series $\mathrm{DT}(X)$ and $\mathrm{DT}(X/Y)$ to be

$$\mathrm{DT}(X) := \sum_{n,\beta} I_{n,\beta} x^n y^\beta,$$
$$\mathrm{DT}(X/Y) := \sum_{n, f_*\beta = 0} I_{n,\beta} x^n y^\beta.$$

The reduced series are defined by

$$\mathrm{DT}'(X) := \frac{\mathrm{DT}(X)}{\mathrm{DT}_0(X)}, \quad \mathrm{DT}'(X/Y) := \frac{\mathrm{DT}(X/Y)}{\mathrm{DT}_0(X)}.$$

Here $\mathrm{DT}_0(X)$ is given by [2], [18], [16],

$$\mathrm{DT}_0(X) := \sum_n I_{n,0} x^n = M(-x)^{\chi(X)},$$

for the MacMahon function,

$$M(x) = \prod_{k \geq 1} (1 - x^k)^{-k}.$$

2.8 Pandharipande-Thomas theory

The notion of stable pairs and the associated counting invariants are introduced by Pandharipande and Thomas [26] in order to give a geometric interpretation of the reduced DT theory.

Definition 2.28. [26] A pair (F, s) is a *stable pair* if it satisfies the following conditions.

- $F \in \mathrm{Coh}_{\leq 1}(X)$ is a pure 1-dimensional sheaf.
- $s \colon \mathcal{O}_X \to F$ is a morphism with 0-dimensional cokernel.

As a convention, the pair $(0, 0)$ is also a stable pair. For $(n, \beta) \in \mathbb{Z} \oplus N_1(X)$, we denote by $P_n(X, \beta)$ the moduli space of stable pairs (F, s) with

$$[F] = \beta, \quad \chi(F) = n.$$

It is proved in [26] that $P_n(X, \beta)$ is a projective scheme with a symmetric perfect obstruction theory, by viewing a stable pair (F, s) as a two term complex,

$$I^{\bullet} = \cdots \to 0 \to \mathcal{O}_X \xrightarrow{s} F \to 0 \to \cdots \in D^b(X). \tag{35}$$

We also call the two term complex (35) as a stable pair. There is an associated virtual fundamental cycle,

$$[P_n(X, \beta)^{\mathrm{vir}}] \in A_0(P_n(X, \beta)).$$

Definition 2.29. The *Pandharipande-Thomas invariant* $P_{n,\beta}$ is defined as

$$P_{n,\beta} = \int_{[P_n(X,\beta)^{\mathrm{vir}}]} 1 \in \mathbb{Z}.$$

The relevant generating series are defined as follows.

Definition 2.30. Let $f \colon X \to Y$ be a flopping contraction. We define the generating series $\mathrm{PT}(X)$ and $\mathrm{PT}(X/Y)$ to be

$$\mathrm{PT}(X) = \sum_{n,\beta} P_{n,\beta} x^n y^{\beta},$$

$$\mathrm{PT}(X/Y) = \sum_{n, f_*\beta = 0} P_{n,\beta} x^n y^{\beta}.$$

The following result, which is conjectured in [26, Conjecture 3.3], is proved in [30, Section 8] using the results of [11] and [3].

Theorem 2.31. [30, Theorem 8.11] *We have the formula,*

$$\mathrm{DT}'(X) = \mathrm{PT}(X),$$

$$\mathrm{DT}'(X/Y) = \mathrm{PT}(X/Y).$$

In particular, we have the equality of the generating series,

$$\frac{\mathrm{DT}(X)}{\mathrm{DT}(X/Y)} = \frac{\mathrm{PT}(X)}{\mathrm{PT}(X/Y)}.$$

2.9 Non-commutative Donaldson-Thomas theory

Here we introduce (global) non-commutative DT invariants associated to an arbitrary flopping contraction $f\colon X \to Y$. Recall the definition of ${}^p\mathrm{Per}(X/Y)$ in Definition 2.18.

Definition 2.32. An object $I \in {}^p\mathrm{Per}(X/Y)$ is called a *perverse ideal sheaf* if there is an injection $I \hookrightarrow \mathcal{O}_X$ in ${}^p\mathrm{Per}(X/Y)$.

The moduli theory of perverse ideal sheaves is studied by Bridgeland [4].

Theorem 2.33. **[4, Theorem 5.5]** *For $(n, \beta) \in \mathbb{Z} \oplus N_1(X)$, the functor of families of perverse ideal sheaves $I \in {}^p\mathrm{Per}(X/Y)$ which fit into the exact sequence in ${}^p\mathrm{Per}(X/Y)$,*

$$0 \longrightarrow I \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow 0, \tag{36}$$

satisfying

$$F \in {}^p\mathrm{Per}_{\le 1}(X/Y), \quad [F] = \beta \quad \text{and} \quad \chi(F) = n, \tag{37}$$

is representable by a projective scheme $I_n({}^pA_Y, \beta)$.

Remark 2.34. *In [4, Theorem 5.5], Bridgeland constructs the moduli space $I_n({}^pA_Y, \beta)$ only in the case of $p = -1$. However the case of $p = 0$ is reduced to the case of $p = -1$ by passing to the flop via the equivalence (29).*

Remark 2.35. *The object $F \in {}^p\mathrm{Per}_{\le 1}(X/Y)$ in the sequence (36) corresponds to an ${}^pA_Y$-module F' which admits surjections,*

$${}^pA_Y \twoheadrightarrow {}^pA'_Y \twoheadrightarrow F', \tag{38}$$

in $\mathrm{Coh}({}^pA_Y)$ via the equivalence (31). In this way, $I_n({}^pA_Y, \beta)$ is also interpreted as a moduli space of cyclic ${}^pA_Y$-modules of a given numerical type.

By [10], there is a symmetric perfect obstruction theory on $I_n({}^pA_Y, \beta)$, and the associated virtual fundamental cycle,

$$[I_n({}^pA_Y, \beta)^{\mathrm{vir}}] \in A_0(I_n({}^pA_Y, \beta)).$$

Definition 2.36. The *(global) non-commutative Donaldson-Thomas invariant* ${}^pA_{n,\beta}$ is defined by

$${}^pA_{n,\beta} = \int_{[I_n({}^pA_Y, \beta)^{\mathrm{vir}}]} 1 \in \mathbb{Z}.$$

The generating series are defined as follows.

Definition 2.37. We define the generating series $\mathrm{DT}({}^pA_Y)$ and $\mathrm{DT}_0({}^pA_Y)$ to be

$$\mathrm{DT}({}^pA_Y) = \sum_{n,\beta} {}^pA_{n,\beta}\, x^n y^\beta,$$

$$\mathrm{DT}_0({}^pA_Y) = \sum_{n, f_*\beta = 0} {}^pA_{n,\beta}\, x^n y^\beta.$$

Remark 2.38. *If $f\colon X \to Y$ contracts only single rational curve $C \subset X$ with normal bundle $N_{C/X} = \mathcal{O}_C(-1)^{\oplus 2}$, then the series $\mathrm{DT}_0({}^pA_Y)$ coincides with the one introduced by Szendrői [27] by Remark 2.35.*

3 Weak stability conditions on $\mathcal{D}_X$

In what follows, we use the notation introduced in the previous section. Let X be a smooth projective Calabi-Yau 3-fold with a flopping contraction $f\colon X \to Y$. (cf. Definition 2.12.) In this section, we study the space of weak stability conditions on the triangulated subcategory,

$$\mathcal{D}_X := \langle \mathcal{O}_X, \mathrm{Coh}_{\le 1}(X) \rangle_{\mathrm{tr}} \subset D^b(\mathrm{Coh}(X)).$$

We set Γ to be

$$\Gamma = \mathbb{Z} \oplus N_1(X) \oplus \mathbb{Z},$$

and a group homomorphism $\mathrm{cl}\colon K(\mathcal{D}_X) \to \mathbb{C}$ to be

$$\mathrm{cl}(E) = (\mathrm{ch}_3(E), \mathrm{ch}_2(E), \mathrm{ch}_0(E)).$$

By the definition of $\mathcal{D}_X$, it is obvious that $\mathrm{ch}_\bullet(E)$ has integer coefficients for $E \in \mathcal{D}_X$, thus cl is well-defined.

Remark 3.1. *Let $I_C \subset \mathcal{O}_X$ be an ideal sheaf of a 1-dimensional subscheme $C \subset X$. We have $I_C \in \mathcal{D}_X$, and*

$$\mathrm{cl}(I_C) = (-n, -\beta, 1) \text{ if and only if } [C] = \beta, \chi(\mathcal{O}_C) = n,$$

by Riemann-Roch theorem. The similar statement also holds for stable pairs (35) and perverse ideal sheaves.

We denote by rk the projection onto the third factor,

$$\mathrm{rk}\colon \Gamma \ni (s, l, r) \mapsto r \in \mathbb{Z}.$$

We set

$$\Gamma_0 = \mathbb{Z} \oplus N_1(X/Y),$$
$$\Gamma_1 = \mathbb{Z} \oplus N_1(X).$$

We have the filtration,

$$\Gamma_0 \overset{i}{\hookrightarrow} \Gamma_1 \overset{j}{\hookrightarrow} \Gamma_2 = \Gamma, \tag{39}$$

via $i(s, l) = (s, l)$ and $j(s, l) = (s, l, 0)$. Each subquotient $\mathbb{H}_i = \Gamma_i/\Gamma_{i-1}$ is

$$\mathbb{H}_0 = \mathbb{Z} \oplus N_1(X/Y), \quad \mathbb{H}_1 = N_1(Y), \quad \mathbb{H}_2 = \mathbb{Z},$$

and there is a natural isomorphism,

$$\left(\mathbb{C} \times N^1(X/Y)_{\mathbb{C}}\right) \times N^1(Y)_{\mathbb{C}} \times \mathbb{C} \xrightarrow{\sim} \prod_{i=0}^{2} \mathbb{H}_i^\vee. \tag{40}$$

Hence we have the local homeomorphism by Theorem 2.7,

$$\Pi\colon \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X) \to \left(\mathbb{C} \times N^1(X/Y)_{\mathbb{C}}\right) \times N^1(Y)_{\mathbb{C}} \times \mathbb{C}. \tag{41}$$

3.1 t-structures on $\mathcal{D}_X$

In this subsection, we construct t-structures on $\mathcal{D}_X$. The notation used here is introduced in subsection 2.5.

Lemma 3.2. *(i) There is the heart of a bounded t-structure $\mathcal{A}_X \subset \mathcal{D}_X$, written as*

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\le 1}(X)[-1] \rangle_{\mathrm{ex}}. \tag{42}$$

(ii) There are hearts of bounded t-structures ${}^p\mathcal{B}_{X/Y} \subset \mathcal{D}_X$ for $p = 0, -1$, written as

$${}^p\mathcal{B}_{X/Y} = \langle \mathcal{O}_X, {}^p\mathrm{Per}_{\le 1}(X/Y)[-1] \rangle_{\mathrm{ex}}. \tag{43}$$

Proof. (i) is proved in [30, Lemma 3.5], so we prove (ii). Let us consider the heart of a bounded t-structure ${}^p\mathrm{Per}^{\dagger}(X/Y) \subset D^b(\mathrm{Coh}(X))$, given in Definition 2.24. By the construction, it is obvious that

$${}^p\mathrm{Per}^{\dagger}(X/Y)[-1] \cap D^b(\mathrm{Coh}_{\le 1}(X)) = {}^p\mathrm{Per}_{\le 1}(X/Y)[-1].$$

Take $F \in {}^p\mathrm{Per}_{\le 1}(X/Y)$. Since $\mathcal{O}_X \in {}^p\mathrm{Per}(X/Y)$, we have $\mathrm{Hom}(\mathcal{O}_X, F[-1]) = 0$. Also we have

$$\begin{aligned}
\mathrm{Hom}(F[i], \mathcal{O}_X) &\cong \mathrm{Hom}(\mathcal{O}_X, F[3+i])^{\vee} \\
&\cong \mathrm{Hom}(\mathcal{O}_Y, \mathbf{R}f_*F[3+i])^{\vee} \\
&= 0,
\end{aligned}$$

for $i \ge -1$. Here the first isomorphism follows from the Serre duality, the second one is an adjunction, and the last one is a consequence of $\mathbf{R}f_*F \in \mathrm{Coh}(Y)$ by the definition of ${}^p\mathrm{Per}(X/Y)$. In particular, we have

$$\mathcal{O}_X \in {}^p\mathrm{Per}_{\ge 2}(X/Y) \subset {}^p\mathrm{Per}^{\dagger}(X/Y)[-1].$$

Applying Proposition 3.3 below by setting $\mathcal{D} = D^b(\mathrm{Coh}(X))$, $\mathcal{D}' = D^b(\mathrm{Coh}_{\le 1}(X))$, $\mathcal{A} = {}^p\mathrm{Per}^{\dagger}(X/Y)[-1]$ and $E = \mathcal{O}_X$, we obtain the result. □

We have used the following proposition, which is proved in [30, Proposition 3.6].

Proposition 3.3. **[30, Proposition 3.6]** *Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category and $\mathcal{A} \subset \mathcal{D}$ the heart of a bounded t-structure on $\mathcal{D}$. Take $E \in \mathcal{A}$ with* $\mathrm{End}(E) = \mathbb{C}$ *and a full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$, which satisfy the following conditions.*

- *The category $\mathcal{A}' := \mathcal{A} \cap \mathcal{D}'$ is the heart of a bounded t-structure on $\mathcal{D}'$, which is closed under subobjects and quotients in the abelian category $\mathcal{A}$.*
- *For any object $F \in \mathcal{A}'$, we have*

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(F, E) = 0. \tag{44}$$

Let $\mathcal{D}_E$ be the triangulated category,

$$\mathcal{D}_E := \langle E, \mathcal{D}' \rangle_{\mathrm{tr}} \subset \mathcal{D}.$$

Then $\mathcal{A}_E := \mathcal{D}_E \cap \mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}_E$, which satisfies

$$\mathcal{A}_E = \langle E, \mathcal{A}' \rangle_{\mathrm{ex}}.$$

Remark 3.4. *By Lemma 2.20 and (43), the abelian categories ${}^p\mathcal{B}_{X/Y}$ are written as*

$$ {}^0\mathcal{B}_{X/Y} = \langle \mathcal{O}_X, \omega_{f^{-1}(y)}, \mathcal{O}_{C_i}(-1)[-1], \widetilde{\mathrm{Coh}}_{\le 1}(X)[-1] \rangle_{\mathrm{ex}}, \tag{45}$$

$$ {}^{-1}\mathcal{B}_{X/Y} = \langle \mathcal{O}_X, \mathcal{O}_{f^{-1}(y)}[-1], \mathcal{O}_{C_i}(-1), \widetilde{\mathrm{Coh}}_{\le 1}(X)[-1] \rangle_{\mathrm{ex}}. \tag{46}$$

We have the following lemma, whose proof will be given in Section 6.

Lemma 3.5. *(i) The abelian categories $\mathcal{A}_X$, ${}^p\mathcal{B}_{X/Y}$ are noetherian.*
(ii) Any infinite chain of monomorphisms in $\mathcal{A}_X$, (resp. ${}^p\mathcal{B}_{X/Y}$,)

$$E_0 \hookleftarrow E_1 \hookleftarrow \cdots E_j \hookleftarrow E_{j+1} \hookleftarrow \cdots, \tag{47}$$

with $E_j/E_{j+1} \notin \mathrm{Coh}_0(X)[-1]$, (resp. $E_j/E_{j+1} \notin {}^p\mathrm{Per}_0(X/Y)[-1]$,) terminates.

Let us see that ${}^p\mathcal{B}_{X/Y}$ is obtained from $\mathcal{A}_X$ via tilting. Let ${}^p\mathcal{F}$ for $p = 0, -1$ be

$$\begin{aligned} {}^0\mathcal{F} &:= \{ F \in \mathrm{Coh}_{\le 1}(X) \mid f_*F = 0, \ \mathrm{Hom}(\mathcal{C}, F) = 0 \}, \\ {}^{-1}\mathcal{F} &:= \{ F \in \mathrm{Coh}_{\le 1}(X) \mid f_*F = 0 \}. \end{aligned}$$

(See Definition 2.18 for $\mathcal{C} \subset \mathrm{Coh}(X)$.) Then ${}^p\mathcal{F}$ fit into torsion pairs

$$({}^p\mathcal{T}, {}^p\mathcal{F}), \tag{48}$$

on $\mathrm{Coh}_{\le 1}(X)$ such that ${}^p\mathrm{Per}_{\le 1}(X/Y)$ is the associated tilting. (cf. [7, Section 3].) By Lemma 2.15 and Lemma 3.5, the subcategories ${}^p\mathcal{F}[-1] \subset \mathcal{A}_X$ also fit into torsion pairs on $\mathcal{A}_X$, denoted by

$$({}^p\mathcal{T}', {}^p\mathcal{F}[-1]). \tag{49}$$

Lemma 3.6. *The abelian category ${}^p\mathcal{B}_{X/Y}$ is the tilting with respect to $({}^p\mathcal{T}', {}^p\mathcal{F}[-1])$, i.e.*

$${}^p\mathcal{B}_{X/Y} = \langle {}^p\mathcal{F}, {}^p\mathcal{T}' \rangle_{\mathrm{ex}}. \tag{50}$$

Proof. We show the case of $p = 0$. It is enough to show that the LHS of (50) is contained in the RHS of (50), since both are hearts of bounded t-structures on $\mathcal{D}_X$. By Remark 3.4, any object in the LHS of (50) is given by a successive extensions of objects $\mathcal{O}_X$, $\omega_{f^{-1}(y)}$ for $y \in \mathrm{Sing}(Y)$, $\mathcal{O}_{C_i}(-1)[-1]$ and objects in $\widetilde{\mathrm{Coh}}_{\le 1}(X)[-1]$. Thus it suffices to show that these objects are contained in the RHS of (50). We have

$$\mathrm{Hom}(\mathcal{O}_X, {}^0\mathcal{F}[-1]) = 0, \quad \Rightarrow \quad \mathcal{O}_X \in {}^0\mathcal{T}', \tag{51}$$

$$\omega_{f^{-1}(y)}[1] \in {}^0\mathrm{Per}_{\le 1}(X/Y), \quad \Rightarrow \quad \omega_{f^{-1}(y)} \in {}^0\mathcal{F} \tag{52}$$

$$\mathrm{Hom}(\mathcal{O}_{C_i}(-1)[-1], {}^0\mathcal{F}[-1]) = 0, \quad \Rightarrow \quad \mathcal{O}_{C_i}(-1)[-1] \in {}^0\mathcal{T}', \tag{53}$$

$$\mathrm{Hom}(\widetilde{\mathrm{Coh}}_{\le 1}(X)[-1], {}^0\mathcal{F}[-1]) = 0, \quad \Rightarrow \quad \widetilde{\mathrm{Coh}}_{\le 1}(X)[-1] \subset {}^0\mathcal{T}'. \tag{54}$$

Here (53) follows from the definition of ${}^0\mathcal{F}$, and (54) follows from $f_*F = 0$ for any $F \in {}^0\mathcal{F}$. Hence (50) holds. ∎

3.2 Constructions of weak stability conditions (neighborhood of the large volume limit)

Here we construct weak stability conditions on $\mathcal{D}_X$, whose corresponding heart of bounded t-structure is $\mathcal{A}_X$. (cf. Lemma 3.2.) The set of weak stability conditions constructed here is interpreted as a neighborhood of the large volume limit at X in terms of string theory. Let us take the elements,

$$B + i\omega \in A(X/Y)_{\mathbb{C}}, \quad \omega' \in A(Y), \quad z \in \mathfrak{H} \text{ with } \arg z \in (\pi/2, \pi).$$

The data

$$\xi = (1, -(B+i\omega), -i\omega', z), \tag{55}$$

in the LHS of (40) determines the element $Z_\xi \in \prod_{i=0}^{2} \mathbb{H}_i^\vee$ via the isomorphism (40). It is written as

$$\begin{aligned}
&Z_{0,\xi} \colon \mathbb{Z} \oplus N_1(X/Y) \ni (s,l) \mapsto s - (B+i\omega)l, \\
&Z_{1,\xi} \colon N_1(Y) \ni l' \mapsto -i\omega' \cdot l', \\
&Z_{2,\xi} \colon \mathbb{Z} \ni r \mapsto zr.
\end{aligned}$$

Lemma 3.7. *The pairs*

$$\sigma_\xi = (Z_\xi, \mathcal{A}_X), \quad \xi \text{ is given by } (55), \tag{56}$$

determine points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$.

Proof. We check that (17) holds for any non-zero $E \in \mathcal{A}_X$. We write $\mathrm{cl}(E) = (-n, -\beta, r)$ for $n \in \mathbb{Z}$, $\beta \in N_1(X)$ and $r \in \mathbb{Z}$. By (42), we have one of the following.

- We have $r > 0$. In this case, we have
$$Z_\xi(E) = zr \in \mathfrak{H}.$$
- We have $r = 0$, $\beta \in \mathrm{NE}(X)$ and $f_*\beta \neq 0$. In this case, we have
$$Z_\xi(E) = i\omega' \cdot f_*\beta \in \mathfrak{H}.$$
- We have $r = 0$ and $\beta \in \mathrm{NE}(X/Y)$. In this case, we have
$$Z_\xi(E) = -n + (B+i\omega)\beta \in \mathfrak{H}.$$

The proofs to check other properties, i.e. Harder-Narasimhan property, support property and local finiteness will be given in Section 6. □

We define the subspace $\mathcal{U}_X \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ as follows.

Definition 3.8. We define $\mathcal{U}_X \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ to be

$$\mathcal{U}_X := \{\sigma_\xi \colon \sigma_\xi \text{ is given by (56)}\}.$$

For a fixed $B_0 \in N^1(X/Y)$, we set

$$\mathcal{U}_{X,B_0} := \{\sigma_\xi \in \mathcal{U}_X : \xi \text{ is given by (56) with } B = B_0\}.$$

By Lemma 2.15, the map $\xi \mapsto \sigma_\xi$ is continuous. In particular $\mathcal{U}_X$ and $\mathcal{U}_{X,B_0}$ are connected subspaces. The map (41) restricts to the homeomorphisms,

$$\begin{aligned} &\Pi \colon \mathcal{U}_X \stackrel{\sim}{\to} \{1\} \times \{-A(X/Y)_{\mathbb{C}}\} \times \{-iA(Y)\} \times \mathfrak{H}', \\ &\Pi \colon \mathcal{U}_{X,B_0} \stackrel{\sim}{\to} \{1\} \times \{-(B_0 + iA(X/Y))\} \times \{-iA(Y)\} \times \mathfrak{H}'. \end{aligned} \tag{57}$$

where $\mathfrak{H}'$ is

$$\mathfrak{H}' = \{z \in \mathfrak{H} : \arg z \in (\pi/2, \pi)\}.$$

Remark 3.9. *The subspace $\mathcal{U}_X \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ is interpreted as a kind of limiting degeneration of the neighborhood of the large volume limit in string theory. In fact for $B + i\omega \in A(X)_{\mathbb{C}}$, let $Z_{(B,\omega)} \colon K(X) \to \mathbb{C}$ be*

$$Z_{(B,\omega)}(E) = \int e^{-(B+i\omega)} \operatorname{ch}(E)\sqrt{\operatorname{td}_X} \in \mathbb{C}.$$

If $E \in \mathrm{Coh}_{\le 1}(X)[-1]$ with $\mathrm{cl}(E) = (-n, -\beta, 0)$, we have

$$Z_{(B,\omega)}(E) = -n + (B + i\omega)\beta,$$

which coincides with $Z_{0,\xi}(\mathrm{cl}(E))$.

3.3 Construction of weak stability conditions (non-commutative points)

Here we construct another weak stability conditions, whose corresponding hearts of bounded t-structures are ${}^p\mathcal{B}_{X/Y}$. (cf. Lemma 3.2.) Let $C_1, \cdots, C_N$ be the irreducible components of the exceptional locus of a flopping contraction $f \colon X \to Y$. We denote by Z_y the fundamental cycle of the scheme theoretic fiber of f at $y \in \mathrm{Sing}(Y)$. For $p = 0, -1$, we set ${}^pV(X/Y)$ as follows,

$${}^pV(X/Y) := \left\{ B \in N^1(X/Y)_{\mathbb{R}} : \begin{array}{c} (-1)^p B \cdot C_i < 0, \ (-1)^p B \cdot Z_y > -1, \\ \text{for all } 1 \le i \le N \text{ and } y \in \mathrm{Sing}(Y) \end{array} \right\}. \tag{58}$$

For the elements,

$$\begin{aligned} &B \in {}^pV(X/Y), \quad \omega' \in A(Y), \\ &z_0, z_1 \in \mathfrak{H} \text{ with } \arg z_i \in (\pi/2, \pi], \ z_1 \neq -1, \end{aligned} \tag{59}$$

the data

$$\xi = (-z_0, z_0 B, -i\omega', z_1), \tag{60}$$

in the LHS of (40) determines the element $Z_\xi \in \prod_{i=0}^{2} \mathbb{H}_i^\vee$ via the isomorphism (40). It is written as

$$\begin{aligned} &Z_{0,\xi} \colon \mathbb{Z} \oplus N_1(X/Y) \ni (s, l) \mapsto z_0(-s + Bl), \\ &Z_{1,\xi} \colon N_1(Y) \ni l' \mapsto -i\omega' \cdot l', \\ &Z_{2,\xi} \colon \mathbb{Z} \ni r \mapsto z_1 r. \end{aligned}$$

Lemma 3.10. *The pairs*

$$\sigma_\xi = (Z_\xi, {}^p\mathcal{B}_{X/Y}), \quad \xi \text{ is given by } (60), \tag{61}$$

determine points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$.

Proof. For simplicity we show the case of $p = 0$. In order to check (17), it is enough to show this for generators of ${}^0\mathcal{B}_{X/Y}$, given in (45). We have

$$\begin{aligned} Z_{2,\xi}(\mathcal{O}_X) &= z_1 \in \mathfrak{H}, \\ Z_{1,\xi}(F[-1]) &= i\omega' \cdot f_*\beta \in \mathfrak{H}, \\ Z_{0,\xi}(\mathcal{O}_{C_i}(-1)[-1]) &= -z_0 B \cdot C_i \in \mathfrak{H}, \\ Z_{0,\xi}(\omega_{f^{-1}(y)}) &= z_0(1 + B \cdot Z_y) \in \mathfrak{H}, \end{aligned}$$

by our choice of z_i and B. Here $0 \neq F \in \widetilde{\mathrm{Coh}}_{\le 1}(X)$ satisfies $\mathrm{cl}(F) = (n, \beta, 0)$. Note that $f_*\beta \in N_1(Y)$ is a non-zero effective class by the definition of $\widetilde{\mathrm{Coh}}_{\le 1}(X)$. Therefore the pair $(Z_\xi, {}^p\mathcal{B}_{X/Y})$ satisfies (17). The Harder-Narasimhan property, the local finiteness and the support property are proved along with the same argument of Lemma 3.7, and we leave the readers to check the detail. □

We define the subspaces ${}^p\mathcal{V}_{X/Y} \subset {}^p\mathcal{U}_{X/Y} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ as follows.

Definition 3.11. We define ${}^p\mathcal{U}_{X/Y}$, ${}^p\mathcal{V}_{X/Y}$ to be

$$\begin{aligned} &{}^p\mathcal{U}_{X/Y} = \{\sigma_\xi : \sigma_\xi \text{ is given by } (61)\}, \\ &{}^p\mathcal{V}_{X/Y} = \{\sigma_\xi \in {}^p\mathcal{U}_{X/Y} : \xi \text{ is given by } (60) \text{ with } z_0 = -1\}. \end{aligned}$$

By Lemma 2.15, the subspaces ${}^p\mathcal{U}_{X/Y}$ and ${}^p\mathcal{V}_{X/Y}$ are connected. The map (41) restricts to the homeomorphism,

$$\Pi \colon {}^p\mathcal{V}_{X/Y} \xrightarrow{\sim} \{1\} \times \{-{}^pV(X/Y)\} \times \{-iA(Y)\} \times \mathfrak{H}'. \tag{62}$$

3.4 Flops and weak stability conditions

Let $\phi\colon X^+ \dashrightarrow X$ be a flop as in the diagram (21), and Φ a standard equivalence given in (29). Since the kernel of Φ is supported on the fiber product $X \times_Y X^+$, (cf. [6, Proposition 4.2],) Φ restricts to the equivalence

$$\Phi\colon \mathcal{D}_{X^+} \xrightarrow{\sim} \mathcal{D}_X. \tag{63}$$

Lemma 3.12. *The standard equivalence $\Phi\colon \mathcal{D}_{X^+} \to \mathcal{D}_X$ restricts to the equivalence between ${}^{-1}\mathcal{B}_{X^+/Y}$ and ${}^{0}\mathcal{B}_{X/Y}$.*

Proof. Note that Φ induces the equivalences between ${}^{-1}\mathrm{Per}(X^+/Y)$ and ${}^{0}\mathrm{Per}(X/Y)$, and the equivalence between ${}^{-1}\mathrm{Per}_{\le 1}(X^+/Y)$ and ${}^{0}\mathrm{Per}_{\le 1}(X/Y)$. Hence Φ induces the equivalence,

$$\Phi\colon\ {}^{-1}\mathrm{Per}^{\dagger}(X^+/Y)[-1] \xrightarrow{\sim} {}^{0}\mathrm{Per}^{\dagger}(X/Y)[-1], \tag{64}$$

where ${}^{p}\mathrm{Per}^{\dagger}(X/Y)$ is given in Definition 2.24. Since we have

$${}^{p}\mathcal{B}_{X/Y} = \mathcal{D}_X \cap {}^{p}\mathrm{Per}^{\dagger}(X/Y)[-1],$$

by Proposition 3.3, we obtain the result by restricting (64) to $\mathcal{D}_{X^+}$ and $\mathcal{D}_X$. ∎

Similar to $\Gamma_{\bullet}$, we set

$$\begin{aligned} \Gamma_0^+ &= \mathbb{Z} \oplus N_1(X^+/Y), \\ \Gamma_1^+ &= \mathbb{Z} \oplus N_1(X^+), \\ \Gamma^+ = \Gamma_2^+ &= \mathbb{Z} \oplus N_1(X^+) \oplus \mathbb{Z}. \end{aligned}$$

The associated subquotient is denoted by $\mathbb{H}_i^+$.

Lemma 3.13. *There is a filtration preserving isomorphism $\Phi_{\Gamma}\colon \Gamma_{\bullet}^+ \to \Gamma_{\bullet}$, which satisfies the following.*

- *The following diagram commutes,*

$$\begin{array}{ccc} \mathcal{D}_{X^+} & \xrightarrow{\Phi} & \mathcal{D}_X \\ \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ \Gamma^+ & \xrightarrow{\Phi_{\Gamma}} & \Gamma. \end{array} \tag{65}$$

- *The induced morphism $\mathrm{gr}_{\bullet}\Phi_{\Gamma}$ satisfies*

$$\begin{aligned} \mathrm{gr}_0\Phi_{\Gamma}\colon \mathbb{Z}\oplus N_1(X^+/Y) \ni (z, C) & \\ \mapsto (z, \phi_* C) &\in \mathbb{Z}\oplus N_1(X/Y) \\ \mathrm{gr}_1\Phi_{\Gamma}\colon N_1(Y) \ni D' \mapsto D' &\in N_1(Y), \\ \mathrm{gr}_2\Phi_{\Gamma}\colon \mathbb{Z} \ni z' \mapsto z' &\in \mathbb{Z}. \end{aligned}$$

Proof. First the following diagram is commutative by [31, Proposition 5.2],

$$\begin{array}{ccc} D^b(\mathrm{Coh}_{\le 1}(X^+)) & \xrightarrow{\Phi} & D^b(\mathrm{Coh}_{\le 1}(X)) \\ \downarrow{\scriptstyle(\mathrm{ch}_3,\mathrm{ch}_2)} & & \downarrow{\scriptstyle(\mathrm{ch}_3,\mathrm{ch}_2)} \\ \mathbb{Z}\oplus N_1(X^+) & \xrightarrow{(\mathrm{id},\phi_*)} & \mathbb{Z}\oplus N_1(X). \end{array} \tag{66}$$

Let $v = \mathrm{cl}\,\Phi(\mathcal{O}_{X^+}) \in \Gamma$ and set Φ_Γ as

$$\Phi_\Gamma(s,l,r) = (s, \phi_* l, 0) + rv.$$

The commutativity of (66) implies that Φ_Γ fits into the commutative diagram (65). Since v is of the form $(*,*,1)$, the map Φ_Γ is isomorphism. The induced isomorphism $\mathrm{gr}_\bullet \Phi_\Gamma$ is of the desired form by the construction. □

Remark 3.14. *In fact one can show that* $\Phi(\mathcal{O}_{X^+}) \cong \mathcal{O}_X$*, hence* $\mathrm{cl}\,\Phi(\mathcal{O}_{X^+}) = (0,0,1)$*. However we do not use this fact.*

By Lemma 2.9 and Lemma 3.13, we have the commutative diagram,

$$\begin{array}{ccc} \mathrm{Stab}_{\Gamma^+_\bullet}(\mathcal{D}_{X^+}) & \xrightarrow{\Phi_*} & \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X) \\ \downarrow{\scriptstyle\Pi^+} & & \downarrow{\scriptstyle\Pi} \\ \prod_{i=0}^2 \mathbb{H}_i^{+\vee} & \xrightarrow{(\mathrm{gr}\,\Phi_\Gamma^{-1})^\vee} & \prod_{i=0}^2 \mathbb{H}_i^\vee. \end{array} \tag{67}$$

Proposition 3.15. *(i) We have*

$$\Phi_*({}^{-1}\mathcal{V}_{X^+/Y}) = {}^{0}\mathcal{V}_{X/Y}\,. \tag{68}$$

(ii) We have ${}^{p}\mathcal{V}_{X/Y} \subset \overline{\mathcal{U}}_X$*. In particular, we have the inclusion*

$${}^{0}\mathcal{V}_{X/Y} \subset \overline{\mathcal{U}}_X \cap \Phi_* \overline{\mathcal{U}}_{X^+}, \tag{69}$$

and the following subset $\mathcal{U} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ *is connected,*

$$\mathcal{U} := \mathcal{U}_X \cup \Phi_* \mathcal{U}_{X^+} \cup {}^{0}\mathcal{U}_{X/Y} \cup {}^{-1}\mathcal{U}_{X/Y}\,. \tag{70}$$

Proof. (i) First note that the strict transform $\phi_* \colon N^1(X^+/Y) \to N^1(X/Y)$ induces the homeomorphism,

$$\phi_* \colon\ {}^{-1}V(X^+/Y) \xrightarrow{\sim} {}^{0}V(X/Y).$$

Hence by the homeomorphism (62), the map $(\mathrm{gr}\,\Phi_\Gamma^{-1})^\vee$ in the diagram (67) induces the homeomorphism,

$$\Pi^+\left({}^{-1}\mathcal{V}_{X^+/Y}\right) \xrightarrow{\sim} \Pi\left({}^{0}\mathcal{V}_{X/Y}\right).$$

Combined with Lemma 3.12, we obtain (68).

(ii) By (57) and (62), we have

$$\Pi\left({}^{p}\mathcal{V}_{X/Y}\right) \subset \overline{\Pi\left(\mathcal{U}_X\right)}.$$

By Lemma 2.8 and Lemma 3.6, we obtain ${}^{p}\mathcal{V}_{X/Y} \subset \overline{\mathcal{U}}_X$. Combined with (i), we conclude (69) and the connectedness of (70). □

Remark 3.16. *The subspace*

$$\overline{\mathcal{U}}_X \cup \Phi_* \overline{\mathcal{U}}_{X^+} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X),$$

consists of two chambers $\mathcal{U}_X$ *and* $\Phi_* \mathcal{U}_{X^+}$*, which is an analogue of the chamber structure on the space of stability conditions on* $D^b(\mathrm{Coh}_{\le 1}(X))$*. (cf. [32, Theorem 4.11].) The chamber* $\mathcal{U}_X$ *(resp.* $\Phi_* \mathcal{U}_{X^+}$*) corresponds to the neighborhood of the large volume limit at* X*, (resp.* X^+*,) and the wall* ${}^0\mathcal{V}_{X/Y}$ *corresponds to the locus of non-commutative points. See [27, Figure 8].*

4 Wall-crossing formula

In this section, we review the main results of [30, Section 8]. As in the previous section, $f \colon X \to Y$ is a flopping contraction from a smooth projective Calabi-Yau 3-fold X.

4.1 Assumption

Here we recall the wall-crossing formula of generating series of Donaldson-Thomas type invariants under change of weak stability conditions, given in [30, Section 8]. The formula is established under some conditions given in Assumption 4.1 below. Let us recall that, by the result of Lieblich [19], there is an algebraic stack $\mathcal{M}$ locally of finite type over $\mathbb{C}$ which parameterizes $E \in D^b(\mathrm{Coh}(X))$ satisfying

$$\mathrm{Ext}^i(E, E) = 0, \quad \text{for any } i < 0. \tag{71}$$

Let $\mathcal{M}_0$ be the fiber at $[0] \in \mathrm{Pic}(X)$ of the following morphism,

$$\det \colon \mathcal{M} \ni E \longmapsto \det E \in \mathrm{Pic}(X).$$

For any object $E \in \mathcal{D}_X$, the corresponding $\mathbb{C}$-valued point $[E] \in \mathcal{M}$ is contained in $\mathcal{M}_0$. Let $\mathcal{A} \subset \mathcal{D}_X$ be the heart of a bounded t-structure on $\mathcal{D}_X$. We can consider the following (abstract) substack,

$$\mathcal{O}bj(\mathcal{A}) \subset \mathcal{M}_0,$$

which parameterizes objects $E \in \mathcal{A}$. The above stack decomposes as

$$\mathcal{O}bj(\mathcal{A}) = \coprod_{v \in \Gamma} \mathcal{O}bj^v(\mathcal{A}),$$

where $\mathcal{O}bj^v(\mathcal{A})$ is the stack of objects $E \in \mathcal{A}$ with $\mathrm{cl}(E) = v$.

Let $\Gamma_\bullet$ be the filtration (39). The wall-crossing formula [30, Section 8] is applied for a certain connected subset

$$\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X),$$

satisfying the following assumption.

Assumption 4.1. [30, Assumption 4.1] *For any* $\sigma = (Z = \{Z_i\}_{i=0}^2, \mathcal{P}) \in \mathcal{V}$ *with* $\mathcal{A} = \mathcal{P}((0,1])$*, the following conditions are satisfied.*

- *We have*

$$\mathcal{O}_X \in \mathcal{P}(\psi), \quad \frac{1}{2} < \psi < 1, \tag{72}$$

and $\mathcal{O}_X$ *is the only object* $E \in \mathcal{P}(\psi)$ *with* $\mathrm{cl}(E) = (0,0,1)$.

- *We have*

$$Z_1(\mathbb{H}_1) \subset \mathbb{R} \cdot i. \tag{73}$$

- *For any* $v, v' \in \Gamma_0$ *and any other point* $\tau = (W, \mathcal{Q}) \in \mathcal{V}$, *we have*

$$Z(v) \in \mathbb{R}_{>0} Z(v') \quad \text{if and only if} \quad W(v) \in \mathbb{R}_{>0} W(v'). \tag{74}$$

- *For any* $v \in \Gamma$ *with* $\mathrm{rk}(v) = 1$ *or* $v \in \Gamma_0$, *the stack of objects*

$$\mathcal{O}bj^v(\mathcal{A}) \subset \mathcal{M}_0,$$

is an open substack of $\mathcal{M}_0$. *In particular,* $\mathcal{O}bj^v(\mathcal{A})$ *is an algebraic stack locally of finite type over* $\mathbb{C}$.

- *For any* $v \in \Gamma$ *with* $\mathrm{rk}(v) = 1$ *or* $v \in \Gamma_0$, *the stack of* σ*-semistable objects* $E \in \mathcal{A}$ *with* $\mathrm{cl}(E) = v$,

$$\mathcal{M}^v(\sigma) \subset \mathcal{O}bj^v(\mathcal{A}),$$

is an open substack of finite type over $\mathbb{C}$.

- *There are subsets* $0 \in T \subset S \subset \mathbb{Z} \oplus N_1(X)$, *which satisfy Assumption 4.8 below.*

- *For any other point* $\tau \in \mathcal{V}$, *there is a good path (see Definition 4.2 below) in* $\mathcal{V}$ *which connects* σ *and* τ.

As for the last condition of Assumption 4.1, the notion of good path is defined as follows.

Definition 4.2. A path $[0,1] \ni t \mapsto \sigma_t \in \mathcal{V}$ is *good* if for any $t \in (0,1)$ and $v \in \Gamma_0$ satisfying $Z_t(v) \in \mathbb{R}_{>0} Z_t(\mathcal{O}_X)$, we have either

$$\arg Z_{t+\varepsilon}(v) < \arg Z_{t+\varepsilon}(\mathcal{O}_X), \quad \arg Z_{t-\varepsilon}(v) > \arg Z_{t-\varepsilon}(\mathcal{O}_X), \text{ or} \tag{75}$$

$$\arg Z_{t+\varepsilon}(v) > \arg Z_{t+\varepsilon}(\mathcal{O}_X), \quad \arg Z_{t-\varepsilon}(v) < \arg Z_{t-\varepsilon}(\mathcal{O}_X), \tag{76}$$

for $0 < \varepsilon \ll 1$.

4.2 Wall-crossing formula of the generating series

Let $\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ be a connected subset satisfying Assumption 4.1. We introduce the following notion.

Definition 4.3. We say $\sigma = (Z, \mathcal{P}) \in \mathcal{V}$ is *general* if there is no $v \in \Gamma_0$ which satisfies $Z(v) \in \mathbb{R}_{>0} Z(\mathcal{O}_X)$.

For general $\sigma, \tau \in \mathcal{V}$, take a good path, (cf. Definition 4.2,)

$$[0,1] \ni t \mapsto \sigma_t = (Z_t, \mathcal{P}_t) \in \mathcal{V},$$

which satisfies $\sigma_0 = \sigma$ and $\sigma_1 = \tau$. For $t \in [0,1]$, let W_t be the set,

$$W_t = \{ v \in \Gamma_0 : Z_t(v) \in \mathbb{R}_{>0} Z_t(\mathcal{O}_X) \}.$$

For $t \in [0,1]$ with $W_t \neq \emptyset$, we set

$$\epsilon(t) = 1, \quad (\text{resp. } \epsilon(t) = -1,)$$

if (75) (resp. (76)) happens at t for $v \in W_t$. . By the condition (74), the value $\epsilon(t)$ does not depend on a choice of $v \in W_t$. The main result in [30, Section 8] is summarized as follows.

Theorem 4.4. [30, Theorem 8.9, Corollary 8.10] *Let $\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ be a connected subset satisfying Assumption 4.1. We have the following.*

- *For $\sigma = (Z, \mathcal{A}) \in \mathcal{V}$ and $v = (-n, -\beta, 1) \in \Gamma$, (resp. $v = (-n, -\beta, 0) \in \Gamma_0$,) there is a counting invariant of σ-semistable objects of numerical type v,*

$$\mathrm{DT}_{n,\beta}(\sigma) \in \mathbb{Q}, \quad (\textit{resp. } N_{n,\beta} \in \mathbb{Q},)$$

such that if $\mathcal{M}^v(\sigma)$ is written as $[M/\mathbb{G}_m]$ for a $\mathbb{C}$-scheme M with a trivial $\mathbb{G}_m$-action, we have (cf. Remark 4.5,)

$$\mathrm{DT}_{n,\beta}(\sigma) = \int_{[M^{\mathrm{vir}}]} 1, \quad \left(\textit{resp. } N_{n,\beta} = \int_{[M^{\mathrm{vir}}]} 1. \right) \tag{77}$$

- *Let $\mathrm{DT}(\sigma)$ and $\mathrm{DT}_0(\sigma)$ be the series,*

$$\mathrm{DT}(\sigma) = \sum_{n,\beta} \mathrm{DT}_{n,\beta}(\sigma) x^n y^\beta, \tag{78}$$

$$\mathrm{DT}_0(\sigma) = \sum_{(n,\beta)\in\Gamma_0} \mathrm{DT}_{n,\beta}(\sigma) x^n y^\beta. \tag{79}$$

Then we have the following equalities of the generating series,

$$\mathrm{DT}(\tau) = \mathrm{DT}(\sigma) \cdot \prod_{\substack{-(n,\beta)\in W_t, \\ t\in(0,1).}} \exp((-1)^{n-1} n N_{n,\beta} x^n y^\beta)^{\epsilon(t)}, \tag{80}$$

$$\mathrm{DT}_0(\tau) = \mathrm{DT}_0(\sigma) \cdot \prod_{\substack{-(n,\beta)\in W_t, \\ t\in(0,1).}} \exp((-1)^{n-1} n N_{n,\beta} x^n y^\beta)^{\epsilon(t)}. \tag{81}$$

In particular the quotient series

$$\mathrm{DT}'(\sigma) := \frac{\mathrm{DT}(\sigma)}{\mathrm{DT}_0(\sigma)},$$

is well-defined and does not depend on a general point $\sigma \in \mathcal{V}$.

Remark 4.5. *Suppose that $\mathcal{M}^v(\sigma) = [M/\mathbb{G}_m]$ where M is a scheme with a trivial $\mathbb{G}_m$-action. Then there is a perfect symmetric obstruction theory on M by [10], and hence the associated virtual cycle also exists.*

Remark 4.6. *If $\mathcal{M}^v(\sigma)$ is not written as $[M/\mathbb{G}_m]$ as in Remark 4.5, then the invariant $\mathrm{DT}_{n,\beta}(\sigma)$, (resp. $N_{n,\beta}$,) is defined by the integration of the logarithm of the relevant moduli stacks in the Hall-algebra. Their precise definitions are given in [30, Definition 8.5].*

Remark 4.7. *As in [30, Remark 8.7], the invariant $N_{n,\beta}$ does not depend on $\sigma \in \mathcal{V}$. However it may depend on $\mathcal{V}$, so we may write it as $N_{n,\beta}(\mathcal{V})$. Let $\mathcal{V}_1, \mathcal{V}_2 \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ be connected subsets satisfying Assumption 4.1. If $\mathcal{V}_1 \cup \mathcal{V}_2$ is connected, then the same proof of [30, Proposition-Definition 5.7] and [30, Remark 8.7] show that*

$$N_{n,\beta}(\mathcal{V}_1) = N_{n,\beta}(\mathcal{V}_2),$$

i.e. the invariant $N_{n,\beta}$ does not depend on $\sigma \in \mathcal{V}_1 \cup \mathcal{V}_2$.

4.3 Completions of $\mathbb{C}[N_{\le 1}(X)]$

In this subsection, we discuss certain completions of the group ring $\mathbb{C}[N_{\le 1}(X)]$, in which the generating series $\mathrm{DT}(\sigma)$ and $\mathrm{DT}_0(\sigma)$ are defined. For subsets $S_1, S_2 \subset N_{\le 1}(X) = \mathbb{Z} \oplus N_1(X)$, we set

$$S_1 + S_2 := \{s_1 + s_2 : s_i \in S_i\} \subset N_{\le 1}(X).$$

The sixth condition of Assumption 4.1 is stated as follows.

Assumption 4.8. [30, Assumption 4.4] *In the situation of Assumption 4.1, the subsets $0 \in T \subset S \subset N_{\le 1}(X)$ satisfy the following conditions.*

- *We have*

$$T + T \subset T, \quad S + T \subset S. \tag{82}$$

- *For any $x \in N_{\le 1}(X)$, there are only finitely many ways to write $x = y + z$ for $y, z \in S$.*

- *Let $\psi \in \mathbb{R}$ be as in (72) for $\sigma \in \mathcal{V}$. For $I = (\psi - \varepsilon, \psi + \varepsilon)$ with $0 < \varepsilon \ll 1$, we have*

$$\{(n,\beta) \in N_{\le 1}(X) : (-n, -\beta, 1) \in C_\sigma(I)\} \subset S, \tag{83}$$
$$\{(n,\beta) \in \Gamma_0 : (-n, -\beta, r) \in C_\sigma(I),\ r = 0 \text{ or } 1\} \subset T. \tag{84}$$

Here $C_\sigma(I) \subset \Gamma$ is defined in (16).

- *There is a family of sets $\{S_\lambda\}_{\lambda \in \Lambda}$ with $S_\lambda \subset S$ such that $S \setminus S_\lambda$ is a finite set and*

$$S_\lambda + T \subset S_\lambda, \quad S = \bigcup_{\lambda \in \Lambda} (S \setminus S_\lambda).$$

The existence of such S, T are required to give completions of the group ring $\mathbb{C}[N_{\le 1}(X)]$. We have the following $\mathbb{C}$-vector space,

$$\mathbb{C}\llbracket S \rrbracket := \left\{ f = \sum_{(n,\beta)\in S} a_{n,\beta} x^n y^\beta : a_{n,\beta} \in \mathbb{C} \right\}.$$

The vector spaces $\mathbb{C}\llbracket T \rrbracket$, $\mathbb{C}\llbracket S_\lambda \rrbracket$ are similarly defined. The product on $\mathbb{C}[N_{\le 1}(X)]$ generalizes naturally to products on $\mathbb{C}\llbracket T \rrbracket$, and $\mathbb{C}\llbracket S \rrbracket$, $\mathbb{C}\llbracket S_\lambda \rrbracket$ are $\mathbb{C}\llbracket T \rrbracket$-modules with $\mathbb{C}\llbracket S_\lambda \rrbracket \subset \mathbb{C}\llbracket S \rrbracket$. There is a topology on $\mathbb{C}\llbracket S \rrbracket$, induced by the isomorphism,

$$\mathbb{C}\llbracket S \rrbracket \cong \varprojlim_{\lambda \in \Lambda} \mathbb{C}\llbracket S \rrbracket / \mathbb{C}\llbracket S_\lambda \rrbracket,$$

and the Euclid topology on the finite dimensional vector spaces $\mathbb{C}\llbracket S \rrbracket / \mathbb{C}\llbracket S_\lambda \rrbracket$. (cf. [30, Section 4].) For $\sigma \in \mathcal{V}$ satisfying Assumption 4.1, the third condition of Assumption 4.8 yields,

$$\mathrm{DT}(\sigma) \in \mathbb{C}\llbracket S \rrbracket, \quad \mathrm{DT}_0(\sigma) \in \mathbb{C}\llbracket T \rrbracket,$$

where $\mathrm{DT}(\sigma)$, $\mathrm{DT}_0(\sigma)$ are given in (78), (79).

4.4 Checking assumptions

Let $f \colon X \to Y$ be a flopping contraction. Here we state that the connected subsets $\mathcal{U}_{X,B}$ and ${}^p\mathcal{U}_{X/Y}$ (cf. Definition 3.8, Definition 3.11,) satisfy Assumption 4.1. For $\beta \in \mathrm{NE}(X)$, we set $m(\beta)$ as follows,

$$m(\beta) = \inf\{\chi(\mathcal{O}_C) : \dim C = 1 \text{ with } [C] \le \beta\}. \tag{85}$$

It is well-known that $m(\beta) > -\infty$. (cf. [34, Lemma 3.10].) We set S_X and T_X as

$$S_X := \{(n,\beta) \in N_{\le 1}(X) : \beta \ge 0,\ n \ge m(\beta)\}, \tag{86}$$
$$T_X := \{(n,\beta) \in \Gamma_0 : \beta \ge 0,\ n \ge 0\}. \tag{87}$$

Proposition 4.9. *For $B \in {}^pV(X/Y)$, (cf. (58),) the subset $\mathcal{U}_{X,B} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ satisfies Assumption 4.1 with $S = S_X$ and $T = T_X$.*

Proof. The proof will be given in Section 6. □

For $p = 0, -1$, let ${}^p\mathcal{E} = \mathcal{O}_X \oplus {}^p\mathcal{E}'$ be the vector bundle on X constructed in the proof of Theorem 2.22. We denote by $r(p)$ the rank of ${}^p\mathcal{E}'$. For $v = (n,\beta) \in N_{\le 1}(X)$, we set ${}^p\chi(v)$ as

$$\begin{aligned} {}^p\chi(v) &= \int_X v \cdot \mathrm{ch}\, {}^p\mathcal{E}'^\vee, \\ &= r(p)n + (-1)^p c_1(\mathcal{L}_X) \cdot \beta, \end{aligned}$$

where $\mathcal{L}_X$ is a globally generated ample line bundle which defines ${}^p\mathcal{E}$. (See Theorem 2.22.)

For an effective class $\beta \in N_1(Y)$, we set ${}^p m(\beta)$ as follows,

$$ {}^p m(\beta) = \inf \left\{ \chi(F) : \begin{array}{l} F \in \mathrm{Coh}_{\le 1}(Y),\ [F] \le \beta, \text{ there is a} \\ \text{surjection of sheaves } f_* {}^p\mathcal{E}'^{\vee} \twoheadrightarrow F. \end{array} \right\}, $$

as an analogue of (85). The same proof of [30, Lemma 3.10] shows that ${}^p m(\beta) > -\infty$. We set ${}^p S_{X/Y}$ and ${}^p T_{X/Y}$ to be

$$ {}^p S_{X/Y} = \left\{ v = (n, \beta) \in N_{\le 1}(X) : \begin{array}{l} f_*\beta \ge 0,\ n \ge m(f_*\beta), \\ {}^p\chi(v) \ge {}^p m(r(p) f_*\beta) \end{array} \right\}, $$

$$ {}^p T_{X/Y} = \left\{ v = (n, \beta) \in N_{\le 1}(X) : \begin{array}{l} f_*\beta \ge 0,\ n \ge 0, \\ {}^p\chi(v) \ge 0 \end{array} \right\}, $$

Proposition 4.10. *(i) For $B \in {}^p V(X/Y)$, the subset $\mathcal{U}_{X,B} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ satisfies Assumption 4.1 with $S = {}^p S_{X/Y}$ and $T = {}^p T_{X/Y}$.*

(ii) The subset ${}^p\mathcal{U}_{X/Y} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ satisfies Assumption 4.1 with $S = {}^p S_{X/Y}$ and $T = {}^p T_{X/Y}$.

Proof. The proof will be given in Section 6. □

5 Proof of the main theorem

In this section, we give a proof of Theorem 1.2. Again $f \colon X \to Y$ is a flopping contraction from a smooth projective Calabi-Yau 3-fold X, and $\phi \colon X^+ \dashrightarrow X$ its flop.

5.1 Counting invariants of rank zero objects

Let $\mathcal{U} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ be the connected subset given by (70), and take

$$ v = (-n, -\beta, 0) \in \Gamma_0. $$

By Proposition 3.15 (ii), Proposition 4.10, Theorem 4.4 and Remark 4.7, there is a counting invariant of σ-semistable objects of numerical type v,

$$ N_{n,\beta} \in \mathbb{Q}, \tag{88} $$

which does not depend on $\sigma \in \mathcal{U}$.

Lemma 5.1. *We have the following equality.*

$$ N_{n,\beta} = N_{-n,-\beta} = N_{-n,\beta}. $$

Proof. The first equality is just the definition of $N_{-n,-\beta}$ in [30, Definition 8.1, Definition 8.5]. In ordet to show $N_{n,\beta} = N_{-n,\beta}$, take a data $\xi = (1, -i\omega, -i\omega', z)$ as in (55) with $B = 0$. Then it is easy to see that an object $E \in \mathcal{A}_X$ is Z_ξ-semistable if and only if $E[1] \in \mathrm{Coh}_{\le 1}(X)$ is a ω-Gieseker semistable sheaf. Therefore the dualizing functor

$$ \mathcal{M}^{(-n,-\beta,0)}(\sigma_\xi) \ni E \mapsto \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X) \in \mathcal{M}^{(-n,\beta,0)}(\sigma_\xi), $$

is an isomorphism, hence $N_{n,\beta} = N_{-n,\beta}$ holds. (See [33, Lemma 4.3].) □

Remark 5.2. *By the proof of Lemma 5.1, the invariant (88) coincides with Joyce-Song's generalized DT-invariant [11], which counts ω-Gieseker semistable sheaves $F \in \mathrm{Coh}_{\le 1}(X)$, satisfying $[F] = \beta, \chi(F) = n$.*

5.2 Semistable objects of rank one

Let $\sigma = (Z, \mathcal{P}) \in \mathcal{U} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ be as in the previous subsection, and take

$$v = (-n, -\beta, 1) \in \Gamma.$$

In this subsection, we investigate the moduli stack of σ-semistable objects,

$$\mathcal{M}^v(\sigma) \subset \mathcal{O}bj(\mathcal{A}),$$

which is algebraic by Proposition 4.9 and Proposition 4.10. We first note the following lemma, which follows from (42) and (43) immediately.

Lemma 5.3. *For $E \in \mathcal{A}_X$ (resp. $E \in {}^p\mathcal{B}_{X/Y}$) satisfying $\mathrm{rk}(E) = 1$, there is a filtration in $\mathcal{A}_X$, (resp. ${}^p\mathcal{B}_{X/Y}$),)*

$$0 = E_{-1} \subset E_0 \subset E_1 \subset E_2 = E, \tag{89}$$

such that each subquotient $F_i = E_i/E_{i-1}$ satisfies

$$F_0, F_2 \in \mathrm{Coh}_{\le 1}(X)[-1], (\textit{resp. } {}^p\mathrm{Per}_{\le 1}(X/Y)[-1],) \quad F_1 = \mathcal{O}_X.$$

In particular if $E \in \mathcal{A}_X$, there is an exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow I_C \longrightarrow E \longrightarrow F[-1] \longrightarrow 0, \tag{90}$$

where $C \subset X$ is a 1-dimensional subscheme with the defining ideal $I_C \subset \mathcal{O}_X$, and $F \in \mathrm{Coh}_{\le 1}(X)$.

Let us fix

$$B \in {}^pV(X/Y), \quad \omega' \in A(Y), \quad z \in \mathfrak{H} \text{ with } \arg z \in (\pi/2, \pi), \tag{91}$$

and deform $\omega = tH$ with $t \in \mathbb{R}_{>0}$. Here $H \in A(X/Y)$ is an ample generator given in (20). We obtain a 1-parameter family of weak stability conditions

$$\sigma_{\xi(t)} = (Z_{\xi(t)}, \mathcal{A}_X) \in \mathcal{U}_{X,B}, \tag{92}$$

(cf. Definition 3.8,) where $\xi(t)$ is

$$\xi(t) = (1, -(B + itH), -i\omega', z), \tag{93}$$

which is a family of data (55).

Proposition 5.4. *For a fixed $v = (-n, -\beta, 1) \in \Gamma$ and the data (91), there is $t_0 \in \mathbb{R}$ such that for $t > t_0$, we have*

$$\mathcal{M}^v(\sigma_{\xi(t)}) = [P_n(X, \beta)/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts on $P_n(X, \beta)$ trivially. In particular, the following holds in $\mathbb{C}[\![S_X]\!]$,

$$\lim_{t \to \infty} \mathrm{DT}(\sigma_{\xi(t)}) = \mathrm{PT}(X).$$

Proof. Take a $\sigma_{\xi(t)}$-semistable object $E \in \mathcal{A}_X$ with $\mathrm{cl}(E) = (-n, -\beta, 1)$. Let

$$0 \to I_C \to E \to F[-1] \to 0,$$

be an exact sequence as in (90). We have

$$\mathrm{ch}_3(F) \le n - m(\beta), \tag{94}$$

where $m(\beta)$ is defined by (85). Suppose that the support of F is 1-dimensional. The $\sigma_{\xi(t)}$-semistability of E yields,

$$\arg Z_{\xi(t)}(F[-1]) \ge \arg Z_{\xi(t)}(E) = \arg z > \pi/2.$$

Hence we have $f_* \mathrm{ch}_2(F) = 0$ and

$$\frac{-\mathrm{ch}_3(F) + B \,\mathrm{ch}_2(F)}{tH \cdot \mathrm{ch}_2(F)} \le C < 0, \tag{95}$$

for $c = \mathrm{Re}\, z / \mathrm{Im}\, z$. The inequalities (94) and (95) imply

$$t \le \frac{-n + m(\beta)}{cH \cdot \mathrm{ch}_2(F)} + \frac{b}{c}, \tag{96}$$

where $B = bH$ for $b \in \mathbb{R}$. Since $0 < H \cdot \mathrm{ch}_2(F) \le H \cdot \beta$, there is $t_0 > 0$ (depending only on v, B and z,) such that (96) implies $t \le t_0$. Therefore if we take $t > t_0$, the sheaf F must be 0-dimensional. Also we have $\mathrm{Hom}(\mathcal{O}_x[-1], E) = 0$ for any closed point $x \in X$, since $\mathcal{O}_x[-1]$ is $\sigma_{\xi(t)}$-stable with

$$\pi = \arg Z_{\xi(t)}(\mathcal{O}_x[-1]) > \arg Z_{\xi(t)}(E) = \arg z.$$

Then we apply [30, Lemma 3.11] and conclude that E is a stable pair (35).

Conversely take a stable pair $E = (\mathcal{O}_X \to F) \in \mathcal{A}_X$ with $[F] = \beta$, $\chi(F) = n$, and an exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0.$$

Since there is a surjection of sheaves $\mathcal{H}^1(E) \twoheadrightarrow \mathcal{H}^1(B)$ and $\mathcal{H}^1(E)$ is 0-dimensional, the sheaf $\mathcal{H}^1(B)$ is also 0-dimensional. If $\mathrm{rk}(B) = 0$, then $B = Q[-1]$ for a 0-dimensional sheaf Q. Hence the inequality

$$\arg z = \arg Z_{\xi(t)}(E) < \arg Z_{\xi(t)}(B) = \pi$$

is satisfied. If $\mathrm{rk}(B) \neq 0$, we have $\mathrm{rk}(B) = 1$ and $\mathrm{rk}(A) = 0$. By the exact sequence (90) applied for $B \in \mathcal{A}_X$, we see that $\mathrm{ch}_3(B) \leq -m(\beta)$. Hence we have $A = G[-1]$ for $G \in \mathrm{Coh}_{\leq 1}(X)$ with

$$\mathrm{ch}_3(G) \leq n - m(\beta).$$

Therefore we have

$$\frac{-\mathrm{ch}_3(G) + B\,\mathrm{ch}_2(G)}{tH \cdot \mathrm{ch}_2(G)} \geq \frac{-n + m(\beta)}{tH \cdot \mathrm{ch}_2(F)} + \frac{b}{t}, \tag{97}$$

where $B = bH$ with $b \in \mathbb{R}$. Hence there is $t_0 > 0$ such that the RHS of (97) is bigger than $c = \mathrm{Re}\, z / \mathrm{Im}\, z < 0$ for $t > t_0$, i.e.

$$\arg Z_{\xi(t)}(G[-1]) < \arg Z_{\xi(t)}(E) = \arg z,$$

for $t > t_0$. □

Next let us take $\xi = (-z_0, z_0 B, -i\omega', z_1)$ as in (59), and the associated weak stability condition $\sigma_\xi = (Z_\xi, {}^p\mathcal{B}_{X/Y}) \in {}^p\mathcal{U}_{X/Y}$. (cf. Definition 3.11.) We have the following proposition.

Proposition 5.5. *(i) Suppose that* $\arg z_0 > \arg z_1$. *Then for* $v = (-n, -\beta, 1) \in \Gamma$ *with* $(n, \beta) \in \Gamma_0$, *we have*

$$\mathcal{M}^v(\sigma_\xi) = \begin{cases} [\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m], & n = \beta = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In particular, we have

$$\mathrm{DT}_0(\sigma_\xi) = 1.$$

(ii) Suppose that $\arg z_0 < \arg z_1$. *Then for* $v = (-n, -\beta, 1) \in \Gamma$, *we have*

$$\mathcal{M}^v(\sigma_\xi) = [I_n({}^pA_Y, \beta)/\mathbb{G}_m],$$

where $\mathbb{G}_m$ *acts on* $I_n({}^pA_Y, \beta)$ *trivially. In particular, we have*

$$\mathrm{DT}(\sigma_\xi) = \mathrm{DT}({}^pA_Y).$$

Proof. (i) Take a σ_ξ-semistable object $E \in {}^p\mathcal{B}_{X/Y}$ with $\mathrm{cl}(E) = v$, and a filtration

$$0 = E_{-1} \subset E_0 \subset E_1 \subset E_2 = E,$$

in ${}^p\mathcal{B}_{X/Y}$ as in (89). For each subquotient F_i, the condition $(n, \beta) \in \Gamma_0$ implies

$$F_0, F_2 \in {}^p\mathrm{Per}_0(X/Y)[-1], \quad F_1 = \mathcal{O}_X.$$

(cf. Definition 2.18.) Suppose that $F_0 \neq 0$. Then we have

$$\arg Z_\xi(F_0) = \arg z_0 > \arg z_1 = \arg Z_\xi(E),$$

which contradicts to the σ_ξ-semistability of E. Hence $F_0 = 0$ and we have the exact sequence in ${}^p\mathcal{B}_{X/Y}$,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow F_2 \longrightarrow 0. \tag{98}$$

Since

$$\begin{aligned}\mathrm{Hom}(F_2, \mathcal{O}_X[1]) &\cong \mathcal{H}^2\mathbf{R}\Gamma(X, F_2)^\vee, \\ &= 0,\end{aligned}$$

by the Serre duality and the definition of ${}^p\mathrm{Per}_0(X/Y)$, the sequence (98) splits. Hence E is σ_ξ-semistable if and only if $E \cong \mathcal{O}_X$.

(ii) Let

$${}^p\mathcal{H}^i \colon D^b(\mathrm{Coh}(X)) \longrightarrow {}^p\mathrm{Per}(X/Y)$$

be the i-th cohomology functor with respect to the t-structure on $D^b(\mathrm{Coh}(X))$ with heart ${}^p\mathrm{Per}(X/Y)$. Take a σ_ξ-semistable object $E \in {}^p\mathcal{B}_{X/Y}$ with $\mathrm{cl}(E) = v$, and suppose that ${}^p\mathcal{H}^1(E)$ is non-zero. We have the surjection in ${}^p\mathcal{B}_{X/Y}$,

$$E \twoheadrightarrow {}^p\mathcal{H}^1(E)[-1],$$

and the inequality,

$$\arg Z_\xi(E) = \arg z_1 > \arg z_0 = \arg Z_\xi({}^p\mathcal{H}^1(E)[-1]),$$

by our choice of ξ. This contradicts to the σ_ξ-semistablility of E, hence we have ${}^p\mathcal{H}^1(E) = 0$. Combined with Lemma 5.3, the object E fits into the exact sequence in ${}^p\mathrm{Per}(X/Y)$,

$$0 \to E \to \mathcal{O}_X \to F \to 0, \tag{99}$$

with $F \in {}^p\mathrm{Per}_{\le 1}(X/Y)$.

On the other hand, take $E \in {}^p\mathrm{Per}(X/Y)$ which fits into an exact sequence (99) in ${}^p\mathrm{Per}(X/Y)$. Note that we have $E \in {}^p\mathcal{B}_{X/Y}$ with ${}^p\mathcal{H}^1(E) = 0$. In order to show that E is σ_ξ-stable, let us take an exact sequence in ${}^p\mathcal{B}_{X/Y}$,

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0,$$

such that $E_i \neq 0$ for $i = 1, 2$. Suppose that $\mathrm{rk}(E_1) = 1$ and $\mathrm{rk}(E_2) = 0$, hence $E_2 \in {}^p\mathrm{Per}_{\le 1}(X/Y)[-1]$. The long exact sequence associated to ${}^p\mathcal{H}^\bullet(*)$ together with ${}^p\mathcal{H}^1(E) = 0$ show that $E_2 = 0$. This is a contradiction, hence $\mathrm{rk}(E_1) = 0$ and $\mathrm{rk}(E_2) = 1$ holds. In this case, our choice of ξ yields,

$$\arg Z_\xi(E_1) = \arg z_0 < \arg z_1 = \arg Z_\xi(E),$$

which implies that E is σ_ξ-stable. $\square$

5.3 Local transformation formula of the generating series

In this subsection, we show the transformation formula of $\mathrm{DT}(X/Y)$ and $\mathrm{DT}_0({}^pA_Y)$.

Theorem 5.6. *We have the formula,*

$$\mathrm{DT}(X/Y) = \prod_{\substack{n>0,\beta\ge 0,\\ f_*\beta=0}} \exp((-1)^{n-1} n N_{n,\beta} x^n y^\beta). \tag{100}$$

In particular, we have

$$\mathrm{DT}(X/Y) = i \circ \phi_* \mathrm{DT}(X^+/Y). \tag{101}$$

Here the variable change is $\phi_*(n,\beta) = (n, \phi_*\beta)$ and $i(n,\beta) = (n, -\beta)$.

Proof. Let us take

$$\xi(t) = (1, -(B+itH), -i\omega', z),$$

as in (91), (93) and a 1-parameter family of weak stability conditions $\sigma_{\xi(t)} \in \mathcal{U}_{X,B}$ as in (92). By Proposition 5.4, we have

$$\lim_{t\to\infty} \mathrm{DT}_0(\sigma_{\xi(t)}) = \mathrm{PT}(X/Y),$$

in the topological ring $\mathbb{C}\llbracket T_X \rrbracket$. On the other hand, let us consider the element,

$$\xi(0) = (1, -B, -i\omega', z),$$

which gives data (60) with $z_0 = -1$ in the notation of (60). Note that $\mathrm{DT}_0(\sigma_{\xi(t)})$ and $\mathrm{DT}_0(\sigma_{\xi(0)})$ are contained in $\mathbb{C}\llbracket {}^pT_{X/Y} \rrbracket$ by Proposition 4.10. Since $\sigma_{\xi(0)}$ is a general point, we have

$$\begin{aligned} \lim_{t\to 0} \mathrm{DT}_0(\sigma_{\xi(t)}) &= \mathrm{DT}_0(\sigma_{\xi(0)}) \\ &= 1, \end{aligned}$$

in $\mathbb{C}\llbracket {}^pT_{X/Y} \rrbracket$. Here the second equality is due to Proposition 5.5 (i). Therefore applying (81) in Theorem 4.4, we have

$$\begin{aligned} \mathrm{PT}(X/Y) &= \prod_{\substack{-(n,\beta)\in W_t,\\ 0<t<\infty}} \exp((-1)^{n-1} N_{n,\beta} x^n y^\beta) \\ &= \prod_{\substack{n-B\beta>0,\beta>0,\\ f_*\beta=0}} \exp((-1)^{n-1} n N_{n,\beta} x^n y^\beta) \\ &= \prod_{\substack{n>0,\beta>0,\\ f_*\beta=0}} \exp((-1)^{n-1} n N_{n,\beta} x^n y^\beta). \end{aligned} \tag{102}$$

Here the second equality follows from,

$$\bigcup_{0<t<\infty} W_t = \{-(n,\beta) \in \Gamma_0 : n - B\beta > 0, \beta > 0\}$$

and the last equality follows from taking the limit $B \to 0$ in ${}^{p}V(X/Y)$. Then (100) follows from (102), Theorem 2.31 and the following formula, (cf. [30, Remark 8.13],)

$$\prod_{n>0} \exp((-1)^{n-1} n N_{n,0} x^n) = M(-x)^{\chi(X)}. \tag{103}$$

The formula (101) follows from (100), Lemma 5.1 and the equality $\chi(X) = \chi(X^+)$. □

Next we give the formula for $\mathrm{DT}_0({}^{p}A_Y)$.

Theorem 5.7. *We have the following equality,*

$$\mathrm{DT}_0({}^{p}A_Y) = \prod_{n>0, f_*\beta=0} \exp((-1)^{n-1} n N_{n,\beta} x^n y^\beta). \tag{104}$$

In particular, we have

$$\mathrm{DT}_0({}^{p}A_Y) = \mathrm{DT}(X/Y) \cdot \phi_* \mathrm{DT}'(X^+/Y). \tag{105}$$

Proof. Take data (60),

$$\xi = (-z_0, z_0 B, -i\omega', z_1), \quad \arg z_0 < \arg z_1,$$
$$\xi' = (-z_0', z_0' B, -i\omega', z_1), \quad \arg z_0' > \arg z_1,$$

and the associated weak stability conditions $\sigma_\xi, \sigma_{\xi'} \in {}^{p}\mathcal{U}_{X/Y}$. We consider a family of weak stability conditions

$$\sigma_{\xi(t)} = (Z_{\xi(t)}, {}^{p}\mathcal{B}_{X/Y}) \in {}^{p}\mathcal{U}_{X/Y},$$

which connects σ_ξ and $\sigma_{\xi'}$, where $\xi(t)$ is given by

$$\xi(t) = t\xi + (1-t)\xi'.$$

By Proposition 5.5, we have

$$\mathrm{DT}_0(\sigma_{\xi(0)}) = 1, \quad \mathrm{DT}_0(\sigma_{\xi(1)}) = \mathrm{DT}_0({}^{p}A_Y).$$

Take $t_0 \in (0,1)$ which satisfies

$$t_0 z_0 + (1-t_0) z_0' \in \mathbb{R}_{>0} z_1.$$

We have

$$\begin{aligned} W_{t_0} &= \{v \in N_{\le 1}(X) : Z_{\xi(t_0)}(v) \in \mathbb{R}_{>0} z_1\}, \\ &= \{-(n,\beta) \in N_{\le 1}(X) : n - B\beta > 0\}. \end{aligned}$$

Hence applying (81) in Theorem 4.4, we obtain

$$\begin{aligned} \mathrm{DT}_0({}^{p}A_Y) &= \prod_{n-B\beta>0, f_*\beta=0} \exp((-1)^{n-1} n N_{n,\beta} x^n y^\beta), \\ &= \prod_{n>0, f_*\beta=0} \exp((-1)^{n-1} n N_{n,\beta} x^n y^\beta). \end{aligned}$$

Here the second equality follows from taking the limit $B \to 0$ in ${}^{p}V(X/Y)$. Hence (104) holds. The formula (105) follows from (103), (104) and Theorem 5.6. □

5.4 Global transformation formula

Finally we show the global transformation formula of our generating functions.

Theorem 5.8. *We have the following formula,*

$$\frac{\mathrm{DT}(X)}{\mathrm{DT}(X/Y)} = \frac{\mathrm{DT}({}^{p}A_Y)}{\mathrm{DT}_0({}^{p}A_Y)} = \phi_* \frac{\mathrm{DT}(X^+)}{\mathrm{DT}(X^+/Y)}. \tag{106}$$

Proof. Let us take

$$\xi(t) = (1, -(B+itH), -i\omega', z), \ t \in \mathbb{R}_{>0}$$

as in (91), (93) and a 1-parameter family of weak stability conditions $\sigma_{\xi(t)} \in \mathcal{U}_{X,B}$ as in (92). By Proposition 5.4 and Theorem 2.31, we have

$$\lim_{t\to\infty} \frac{\mathrm{DT}(\sigma_{\xi(t)})}{\mathrm{DT}_0(\sigma_{\xi(t)})} = \frac{\mathrm{PT}(X)}{\mathrm{PT}(X/Y)} = \frac{\mathrm{DT}(X)}{\mathrm{DT}(X/Y)}, \tag{107}$$

in $\mathbb{C}[\![S_X]\!]$. On the other hand, we have

$$\lim_{t\to 0} \frac{\mathrm{DT}(\sigma_{\xi(t)})}{\mathrm{DT}_0(\sigma_{\xi(t)})} = \frac{\mathrm{DT}(\sigma_{\xi(0)})}{\mathrm{DT}_0(\sigma_{\xi(0)})}, \tag{108}$$

since $\sigma_{\xi(0)}$ is a general point and $\lim_{t\to 0}\sigma_{\xi(t)} = \sigma_{\xi(0)}$ by Proposition 3.15. Next let us take a data (60),

$$\xi = (-z_0, z_0 B, -i\omega', z_1), \quad \arg z_0 < \arg z_1.$$

By Theorem 4.4 and Proposition 5.5, we have

$$\frac{\mathrm{DT}(\sigma_{\xi(0)})}{\mathrm{DT}_0(\sigma_{\xi(0)})} = \frac{\mathrm{DT}(\sigma_{\xi})}{\mathrm{DT}_0(\sigma_{\xi})} = \frac{\mathrm{DT}({}^{p}A_Y)}{\mathrm{DT}_0({}^{p}A_Y)}. \tag{109}$$

Finally suppose that $p=0$, i.e. $B \in {}^{0}V(X/Y)$. Note that we have $\phi_*^{-1}(B) \in {}^{-1}V(X^+/Y)$. For $t<0$, we set

$$\xi(t) = (1, -\phi_*^{-1}(B+tiH), -i\omega', z),$$

which gives data (55) for X^+. We have the associated 1-parameter family of weak stability conditions $\sigma^+_{\xi(t)} \in \mathcal{U}_{X^+,\phi_*^{-1}B}$, and we set

$$\sigma_{\xi(t)} := \Phi_* \sigma^+_{\xi(t)} \in \Phi_*(\mathcal{U}_{X^+,\phi_*^{-1}B}) \text{ for } t<0.$$

By Proposition 3.15, the family $\sigma_{\xi(t)}$ is a continuous family for $t \in (-\infty, \infty)$. By Proposition 5.4 and Theorem 2.31, we have

$$\lim_{t\to-\infty} \frac{\mathrm{DT}(\sigma_{\xi(t)})}{\mathrm{DT}_0(\sigma_{\xi(t)})} = \phi_* \frac{\mathrm{PT}(X^+)}{\mathrm{PT}(X^+/Y)} = \phi_* \frac{\mathrm{DT}(X^+)}{\mathrm{DT}(X^+/Y)}. \tag{110}$$

Then the formula (106) follows from (107), (108), (109), (110) and Theorem 4.4. □

6 Some technical lemmas

6.1 Proof of Lemma 3.5

Proof. (i) The noetherian property of $\mathcal{A}_X$ is proved in [30, Lemma 6.2]. Let us show that ${}^p\mathcal{B}_{X/Y}$ is noetherian. For simplicity, we show the case of $p = 0$. Take a chain of surjections in ${}^0\mathcal{B}_{X/Y}$,

$$E_0 \twoheadrightarrow E_1 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots . \tag{111}$$

The description (45) shows that ${}^0\mathcal{B}_{X/Y}$ is concentrated on $[0, 1]$ with respect to the standard t-structure on $D^b(\mathrm{Coh}(X))$, and (111) induces a chain of surjections in $\mathrm{Coh}(X)$,

$$\mathcal{H}^1(E_0) \twoheadrightarrow \mathcal{H}^1(E_1) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{H}^1(E_j) \twoheadrightarrow \mathcal{H}^1(E_{j+1}) \twoheadrightarrow \cdots .$$

Hence we may assume that $\mathcal{H}^1(E_i) \stackrel{\cong}{\to} \mathcal{H}^1(E_{i+1})$ for all i. By (43) and the definition of ${}^0\mathrm{Per}(X/Y)$, we have the exact functor,

$$\mathbf{R}f_* \colon {}^0\mathcal{B}_{X/Y} \longrightarrow \langle \mathcal{O}_Y, \mathrm{Coh}_{\le 1}(Y)[-1] \rangle_{\mathrm{ex}}. \tag{112}$$

The proof that $\mathcal{A}_X$ is noetherian (cf. [30, Lemma 6.2]) is also applied for the singular variety Y, hence the category $\langle \mathcal{O}_Y, \mathrm{Coh}_{\le 1}(Y)[-1] \rangle_{\mathrm{ex}}$ is also noetherian. Therefore we may assume that $\mathbf{R}f_*E_i \stackrel{\cong}{\to} \mathbf{R}f_*E_{i+1}$ for all i. Consider the exact sequence in ${}^0\mathcal{B}_{X/Y}$,

$$0 \longrightarrow F_i \longrightarrow E_0 \longrightarrow E_i \longrightarrow 0.$$

Noting (45) and $\mathbf{R}f_*F_i = 0$, the object F_i is written as $F_i'[-1]$, where F_i' is given by successive extensions of sheaves $\mathcal{O}_{C_k}(-1)$ with $1 \le k \le N$. Hence we obtain the exact sequence of sheaves,

$$0 \longrightarrow \mathcal{H}^0(E_0) \longrightarrow \mathcal{H}^0(E_i) \longrightarrow F_i' \longrightarrow 0.$$

Since $\dim C_k \le 1$, we obtain the sequence of coherent sheaves,

$$\mathcal{H}^0(E_0) \subset \mathcal{H}^0(E_1) \subset \cdots \subset \mathcal{H}^0(E_j) \subset \cdots \subset \mathcal{H}^0(E_0)^{\vee\vee}. \tag{113}$$

Since $\mathrm{Coh}(X)$ is noetherian, the above sequence terminates.

(ii) First we show the termination of (47) in $\mathcal{A}_X$. Take a sequence (47) in $\mathcal{A}_X$, and an ample divisor ω on X. By (42), we have $\mathrm{ch}_0(E) \ge 0$ and $-\mathrm{ch}_2(E) \cdot \omega \ge 0$ for any object $E \in \mathcal{A}_X$. Therefore we may assume that $\mathrm{ch}_0(E_i)$ and $\mathrm{ch}_2(E_i) \cdot \omega$ are constant for all i. Then E_j/E_{j+1} is 0-dimensional, hence it must be zero by the assumption.

Next we show the termination of (47) in ${}^p\mathcal{B}_{X/Y}$. For simplicity we show the case of $p = 0$. By the same argument as in (i), we may assume that $G_j = E_0/E_{j+1}$ in ${}^0\mathcal{B}_{X/Y}$ is written as $G_j'[-1]$, where G_j' is given by successive extensions of sheaves $\mathcal{O}_{C_k}(-1)$ with $1 \le k \le N$. We have the surjections of sheaves

$$\mathcal{H}^1(E_0) \twoheadrightarrow \cdots \twoheadrightarrow \cdots \twoheadrightarrow G_2' \twoheadrightarrow G_1'.$$

Since $\mathcal{H}^1(E_0) \in \mathrm{Coh}_{\le 1}(X)$, the above sequence must terminate, and hence (47) also terminates. □

6.2 Proof of Lemma 3.7

Step 1. *The pair* $\sigma_\xi = (Z_\xi, \mathcal{A}_X)$ *satisfies the Harder-Narasimhan property.*

Proof. It is enough to check (a) and (b) in Proposition 2.5. The condition (b) follows from Lemma 3.5 (i). In order to check (a), take a chain of monomorphisms in $\mathcal{A}_X$,

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

with $\arg Z(E_{j+1}) > \arg Z(E_j/E_{j+1})$ for all j. By Lemma 3.5 (ii), we have $E_j/E_{j+1} \in \mathrm{Coh}_0(X)$ for some j. Then we have $\arg Z(E_{j+1}) > \arg Z(E_j/E_{j+1}) = \pi$, which contradicts to (17). $\square$

Step 2. *Let* $\{\mathcal{P}_\xi(\phi)\}_{\phi\in\mathbb{R}}$ *be the slicing corresponding to the pair* $\sigma_\xi = (Z_\xi, \mathcal{A}_X)$ *via Proposition 2.6. Then* $\{\mathcal{P}_\xi(\phi)\}_{\phi\in\mathbb{R}}$ *is of locally finite.*

Proof. We set $\phi = \frac{1}{\pi}\arg z \in (1/2, 1)$ and take $0 < \eta \ll 1$ satisfying $\phi \pm \eta \in (0,1)$. By Lemma 3.5, it is enough to check that $\mathcal{P}_\xi((\theta-\eta, \theta+\eta))$ is of finite length for any $\theta \in (1-\eta, 1+\eta)$. Let us consider the pair,

$$(Z_{0,\xi}, \mathrm{Coh}(X/Y)), \tag{114}$$

where

$$\mathrm{Coh}(X/Y) = \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}\, f_*E = 0\}.$$

Then the pair (114) determines a locally finite stability condition on $D^b(\mathrm{Coh}(X/Y))$ in the sense of Bridgeland [5]. (cf. [32, Lemma 4.1].) We write the corresponding slicing on $D^b(\mathrm{Coh}(X/Y))$ by $\{\mathcal{Q}(\phi)\}_{\phi\in\mathbb{R}}$. By our choice of η, we have

$$\mathcal{P}_\xi((\theta-\eta, \theta+\eta)) = \mathcal{Q}((\theta-\eta, \theta+\eta)),$$

for any $\theta \in (1-\eta, 1+\eta)$. Therefore $\mathcal{P}_\xi((\theta-\eta, \theta+\eta))$ is of finite length. $\square$

Step 3. *The pair* $\sigma_\xi = (Z_\xi, \mathcal{P}_\xi)$ *satisfies the support property (15).*

Proof. Let $E \in \mathcal{A}_X$ be a non-zero object with $\mathrm{cl}(E) = (-n, -\beta, r)$. We introduce an usual Euclid norm on $\mathbb{H}_2 \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}$. We have

$$\frac{\|E\|}{|Z(E)|} = \begin{cases} |z|, & r>0, \\ \frac{\|f_*\beta\|}{\omega'\cdot f_*\beta}, & r=0,\ f_*\beta \neq 0, \\ \frac{n^2+\|\beta\|^2}{(n-B\beta)^2+(\omega\beta)^2}, & r=f_*\beta=0, n>0. \end{cases}$$

Since β is effective or zero, the above description easily implies the support property. $\square$

6.3 Proof of Proposition 4.9

The conditions of Assumption 4.1 are obviously satisfied except the fourth, fifth and sixth conditions. As for the fourth condition, this is proved in [30, Lemma 3.15]. It is enough to check the fifth and the sixth conditions.

Step 1. *Take $\sigma_\xi \in \mathcal{U}_{X,B}$ and $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$. Then the stack*

$$\mathcal{M}^v(\sigma_\xi) \subset \mathcal{O}bj^v(\mathcal{A}_X),$$

is an open substack of finite type over $\mathbb{C}$.

Proof. As in the proof of [30, Lemma 3.15], it is enough to check the boundedness of σ_ξ-semistable objects of numerical type v. This is proved along with the same argument of [34, Section 3], and we leave the readers to check the detail. □

Step 2. *The sets S_X and T_X satisfy Assumption 4.8.*

Proof. The first and the second conditions of Assumption 4.8 are obviously satisfied. In order to prove the third condition, take an object $E \in C_{\sigma_\xi}(I)$ with $\sigma_\xi = (Z_\xi, \mathcal{A}_X) \in \mathcal{U}_{X,B}$ and $\mathrm{cl}(E) = (-n, -\beta, 1)$. We are going to check that $(n, \beta) \in S_X$. We have the exact sequence in $\mathcal{A}_X$

$$0 \longrightarrow I_C \longrightarrow E \longrightarrow F[-1] \longrightarrow 0, \tag{115}$$

as in (90). Also we have the exact sequence of sheaves,

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0,$$

with $F_1 \in {}^p\mathcal{T}$ and $F_2 \in {}^p\mathcal{F}$, where $({}^p\mathcal{T}, {}^p\mathcal{F})$ is a torsion pair on $\mathrm{Coh}_{\le 1}(X)$ given by (48). Assume that $F_2 \neq 0$. Then we have the surjection $E \twoheadrightarrow F_2[-1]$ in $\mathcal{A}_X$. Then it is easy to see that

$$\arg Z_\xi(E) > \pi/2 > \arg Z_\xi(F_2[-1]),$$

which contradicts to $E \in C_{\sigma_\xi}(I)$. Therefore $F_2 = 0$, and $F \in {}^p\mathcal{T} \subset {}^p\mathrm{Per}_{\le 1}(X/Y)$ follows. Also if $\mathbf{R}f_*F$ is not 0-dimensional, we have

$$\arg Z_\xi(E) > \arg Z_\xi(F) = \pi/2,$$

which contradicts to the σ_ξ-semistability of E. Therefore $\mathbf{R}f_*F$ is 0-dimensional, which means $F \in {}^p\mathrm{Per}_0(X/Y)$. This implies that

$$\mathrm{ch}_3(F) = \mathrm{length}\, \mathbf{R}f_*F \ge 0. \tag{116}$$

On the other hand, the definition of $m(\beta)$ implies $\mathrm{ch}_3(\mathcal{O}_C) \ge m(\beta)$. Hence by (115) and (116), the inequality $n \ge m(\beta)$ holds, i.e. $(n, \beta) \in S_X$.

Also if $(n, \beta) \in \Gamma_0$, then the curve C in the sequence (115) satisfies $f_*[C] = 0$, hence we have $H^1(\mathcal{O}_C) = 0$. This implies that $\mathrm{ch}_3(\mathcal{O}_C) = \chi(\mathcal{O}_C) \ge 0$, hence $(n, \beta) \in T_X$ holds. If $\mathrm{cl}(E) = (-n, -\beta, 0)$, then the same argument shows that $E = F[-1]$ for $F \in {}^p\mathcal{T}$ and $\mathrm{Supp}(F) \subset \mathrm{Ex}(f)$. Hence $\mathrm{ch}_3(F) \ge 0$, and $(n, \beta) \in T_X$ follows.

Finally we check the last condition of Assumption 4.8. Let Λ be the set of pairs (k, β') of $k \in \mathbb{Z}$ and an effective class $\beta' \in N_1(X)$. For $\lambda = (k, \beta')$, we set

$$S_\lambda = \{(n, \beta) \in S : n \ge k \text{ if } \beta \le \beta'\}.$$

Then $\{S_\lambda\}_{\lambda \in \Lambda}$ gives a desired family. □

6.4 Proof of Proposition 4.10

Proof. (i) By Proposition 4.9, it is enough to check the third and the fourth condition of Assumption 4.8. For $\sigma_\xi \in \mathcal{U}_{X,B}$, let us take an object $E \in C_{\sigma_\xi}(I)$ with $\mathrm{cl}(E) = (-n, -\beta, 1)$. We check that $(n, \beta) \in {}^p S_{X/Y}$. We have an exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow E' \longrightarrow E \longrightarrow F[-1] \longrightarrow 0, \tag{117}$$

for $F \in {}^p\mathcal{F}$ and $E' \in {}^p\mathcal{T}'$ by the existence of the torsion pair (49). By the condition $E \in C_\sigma(I)$, we have $F[1] \in {}^p\mathrm{Per}_0(X/Y)$, hence $v = (\mathrm{ch}_3(F), \mathrm{ch}_2(F))$ satisfies

$$\mathrm{ch}_3(F) \geq 0, \quad {}^p\chi(v) \geq 0, \tag{118}$$

by Lemma 6.1 below. On the other hand, since $E' \in {}^p\mathcal{T}' \subset {}^p\mathcal{B}_{X/Y}$, we have the exact sequence in ${}^p\mathcal{B}_{X/Y}$,

$$0 \longrightarrow A \longrightarrow E' \longrightarrow A'[-1] \longrightarrow 0, \tag{119}$$

with $A \in {}^p\mathrm{Per}_{\geq 2}(X/Y)$ and $A' \in {}^p\mathrm{Per}_{\leq 1}(X/Y)$. Taking the long exact sequence of cohomology with respect to the t-structure with heart $\mathcal{A}_X$, we see that $A \in \mathcal{A}_X$ and obtain the following exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow \mathcal{H}^{-1}(B)[-1] \longrightarrow A \longrightarrow E' \longrightarrow \mathcal{H}^0(B)[-1] \longrightarrow 0.$$

By (117) and the condition $E \in C_{\sigma_\xi}(I)$, we conclude that $\mathcal{H}^i(A')$ are supported on fibers of f, hence $A' \in {}^p\mathrm{Per}_0(X/Y)$ follows. Therefore $v' = (\mathrm{ch}_3(A'), \mathrm{ch}_2(A'))$ satisfies

$$\mathrm{ch}_3(A') \geq 0, \quad {}^p\chi(v') \geq 0, \tag{120}$$

by Lemma 6.1. By Lemma 5.3, the object $A \in {}^p\mathrm{Per}(X/Y)$ fits into the exact sequence in ${}^p\mathrm{Per}(X/Y)$,

$$0 \longrightarrow A \longrightarrow \mathcal{O}_X \longrightarrow A'' \longrightarrow 0.$$

Applying $\mathbf{R}f_*$, we obtain surjections in $\mathrm{Coh}(Y)$,

$$\mathcal{O}_Y \twoheadrightarrow \mathbf{R}f_* A'', \quad f_* {}^p\mathcal{E}'^{\vee} \twoheadrightarrow \mathbf{R}f_*(A'' \otimes {}^p\mathcal{E}'^{\vee}).$$

Combining (118) and (120), $v = (n, \beta)$ satisfies

$$\begin{aligned} n &\geq \chi(\mathbf{R}f_* A'') \geq m(f_*\beta), \\ {}^p\chi(v) &\geq \chi(\mathbf{R}f_*(A'' \otimes {}^p\mathcal{E}'^{\vee})) \geq {}^p m(r(p) f_*\beta), \end{aligned}$$

which implies $(n, \beta) \in {}^p S_{X/Y}$. A similar proof shows that if $(n, \beta) \in \Gamma_0$, (or $\mathrm{cl}(E) = (-n, -\beta, 0)$,) then $(n, \beta) \in T_{X/Y}$.

(ii) As for the sixth condition of Assumption 4.8, a similar (and easier) proof to (i) works, and we omit the detail. Here we check the forth and the fifth conditions of Assumption 4.1.

Step 1. *For $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$, the stack of objects*

$$\mathcal{O}bj^v({}^p\mathcal{B}_{X/Y}) \subset \mathcal{M}_0,$$

is an open substack of $\mathcal{M}_0$.

Proof. Note that the category ${}^p\mathcal{B}_{X/Y}$ is equivalent to the category,

$$\langle {}^pA'_Y, \mathrm{Coh}_{\le 1}({}^pA_Y) \rangle_{\mathrm{ex}},$$

via the equivalence (30). Then we can apply the same argument of [30, Lemma 3.15] for the non-commutative scheme $(Y, {}^pA_Y)$, and obtain the result. □

Step 2. *Take $\sigma_\xi = (Z_\xi, {}^p\mathcal{B}_{X/Y}) \in {}^p\mathcal{U}_{X/Y}$ and $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$. Then the substack*

$$\mathcal{M}^v(\sigma_\xi) \subset \mathcal{O}bj^v({}^p\mathcal{B}_{X/Y}),$$

is an open substack and it is of finite type over $\mathbb{C}$.

Proof. As in the proof of [30, Lemma 3.15], it is enough to show the boundedness of σ_ξ-semistable objects of numerical type v. This follows by the same argument as in [34, Section 3], applied for the non-commutative scheme $(Y, {}^pA_Y)$. We leave the readers to check the detail. □

□

We have used the following lemma.

Lemma 6.1. *For $F \in {}^p\mathrm{Per}_0(X/Y)$, set $v = (\mathrm{ch}_3(F), \mathrm{ch}_2(F)) \in N_{\le 1}(X)$. Then we have*

$$\mathrm{ch}_3(F) \ge 0, \quad {}^p\chi(v) \ge 0.$$

Proof. For $F \in {}^p\mathrm{Per}_0(X/Y)$, we have

$$\mathbf{R}\,\mathrm{Hom}(\mathcal{O}_X \oplus {}^p\mathcal{E}', F) \in \mathrm{Coh}_0(Y),$$

by the equivalence (30). Therefore by Riemann-Roch theorem, we have

$$\begin{aligned} \mathrm{ch}_3(F) &= \mathrm{length}\,\mathbf{R}\,\mathrm{Hom}(\mathcal{O}_X, F) \ge 0, \\ {}^p\chi(v) &= \mathrm{length}\,\mathbf{R}\,\mathrm{Hom}({}^p\mathcal{E}', F) \ge 0. \end{aligned}$$

□

7 Appendix

7.1 The formula for the Euler characteristic version

Applying the result of [30, Section 4] and the method in this paper, we can also show the Euler characteristic version of our main result.

Definition 7.1. We define $\widehat{\mathrm{DT}}(X)$, $\widehat{\mathrm{DT}}(X/Y)$, $\widehat{\mathrm{DT}}({}^{p}A_Y)$, $\widehat{\mathrm{DT}}_0({}^{p}A_Y)$ as follows.

$$\widehat{\mathrm{DT}}(X) = \sum_{n,\beta} \chi(I_n(X,\beta))x^n y^\beta,$$
$$\widehat{\mathrm{DT}}(X/Y) = \sum_{n, f_*\beta=0} \chi(I_n(X,\beta))x^n y^\beta,$$
$$\widehat{\mathrm{DT}}({}^{p}A_Y) = \sum_{n,\beta} \chi(I_n({}^{p}A_Y,\beta))x^n y^\beta,$$
$$\widehat{\mathrm{DT}}_0({}^{p}A_Y) = \sum_{n, f_*\beta=0} \chi(I_n({}^{p}A_Y,\beta))x^n y^\beta.$$

The following theorem can be proved along with the same proof of Theorem 5.6, Theorem 5.7 and Theorem 5.8, using [30, Theorem 3.13] instead of Theorem 4.4. [2]

Theorem 7.2. *We have the following formula,*

$$\widehat{\mathrm{DT}}(X/Y) = i \circ \phi_* \widehat{\mathrm{DT}}(X^+/Y),$$
$$\widehat{\mathrm{DT}}_0(A_Y) = \widehat{\mathrm{DT}}(X/Y) \cdot \phi_* \widehat{\mathrm{DT}}'(X^+/Y),$$
$$\frac{\widehat{\mathrm{DT}}(X)}{\widehat{\mathrm{DT}}(X/Y)} = \frac{\widehat{\mathrm{DT}}({}^{p}A_Y)}{\widehat{\mathrm{DT}}_0({}^{p}A_Y)} = \phi_* \frac{\widehat{\mathrm{DT}}(X^+)}{\widehat{\mathrm{DT}}(X^+/Y)}.$$

7.2 Generalization of global ncDT-invariants

The non-commutative Donaldson-Thomas invariant can be defined in a slightly generalized context. Let $f\colon X \to Y$ be a projective birational morphism from a smooth projective Calabi-Yau 3-fold X, satisfying $\dim f^{-1}(y) \le 1$ for any closed point $y \in Y$. Here we do not assume that f is isomorphic in codimension one, so there may be a divisor $E \subset X$ which contracts to a curve on Y. The result of Van den Bergh [7] can be applied in this situation, i.e. there are vector bundles ${}^{p}\mathcal{E}$ on X for $p = 0, -1$, which admit derived equivalences,

$$ {}^{p}\Phi = \mathbf{R}f_*\mathbf{R}\mathcal{H}om({}^{p}\mathcal{E}, *)\colon D^b(\mathrm{Coh}(X)) \xrightarrow{\cong} D^b(\mathrm{Coh}({}^{p}A_Y)). \tag{121}$$

The abelian subcategories ${}^{p}\mathrm{Per}(X/Y) \subset D^b(\mathrm{Coh}(X))$ is also similarly defined, and ${}^{p}\Phi$ restrict to equivalences between ${}^{p}\mathrm{Per}(X/Y)$ and $\mathrm{Coh}({}^{p}A_Y)$. As in subsection 2.9, we can construct the moduli space of perverse ideal sheaves $I_n({}^{p}A_Y, \beta)$, and the counting invariant,

$${}^{p}A_{n,\beta} = \int_{[I_n({}^{p}A_Y,\beta)^{\mathrm{vir}}]} 1 \in \mathbb{Z}.$$

[2]The result of [30, Theorem 3.13], hence Theorem 7.2, does not rely on [3].

The generating series $\mathrm{DT}({}^{p}A_Y)$, $\mathrm{DT}_0({}^{p}A_Y)$, $\widehat{\mathrm{DT}}({}^{p}A_Y)$ and $\widehat{\mathrm{DT}}_0({}^{p}A_Y)$ are similarly defined as in Definition 2.37, Definition 7.1. The following theorem can be proved along with the same proof of Theorem 5.8 and Theorem 7.2.

Theorem 7.3. *We have the following formula.*

$$\frac{\mathrm{DT}(X)}{\mathrm{DT}(X/Y)} = \frac{\mathrm{DT}({}^{p}A_Y)}{\mathrm{DT}_0({}^{p}A_Y)}, \quad \frac{\widehat{\mathrm{DT}}(X)}{\widehat{\mathrm{DT}}(X/Y)} = \frac{\widehat{\mathrm{DT}}({}^{p}A_Y)}{\widehat{\mathrm{DT}}_0({}^{p}A_Y)}.$$

References

[1] K. Behrend. Donaldson-Thomas invariants via microlocal geometry. *Ann. of Math (to appear).* math.AG/0507523.

[2] K. Behrend and B. Fantechi. Symmetric obstruction theories and Hilbert schemes of points on threefolds. *Algebra Number Theory*, Vol. 2, pp. 313–345, 2008.

[3] K. Behrend and E. Getzler. Chern-Simons functional. *in preparation.*

[4] T. Bridgeland. Flops and derived categories. *Invent. Math*, Vol. 147, pp. 613–632, 2002.

[5] T. Bridgeland. Stability conditions on triangulated categories. *Ann. of Math*, Vol. 166, pp. 317–345, 2007.

[6] J-C. Chen. Flops and equivalences of derived categories for three-folds with only Gorenstein singularities. *J. Differential. Geom*, Vol. 61, pp. 227–261, 2002.

[7] M. Van den Bergh. Three dimensional flops and noncommutative rings. *Duke Math. J.* , Vol. 122, pp. 423–455, 2004.

[8] D. Happel, I. Reiten, and S. O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc*, Vol. 120, , 1996.

[9] J. Hu and W. P. Li. The Donaldson-Thomas invariants under blowups and flops. *preprint.* math.AG/0505542.

[10] D. Huybrechts and R. P. Thomas. Deformation-obstruction theory for complexes via Atiyah-Kodaira-Spencer classes. *preprint.* math.AG/0805.3527.

[11] D. Joyce and Y. Song. A theory of generalized Donaldson-Thomas invariants. *preprint.* math.AG/0810.5645.

[12] Y. Kawamata. On the cone of divisors of Calabi-Yau fiber spaces. *Internat. J. Math*, Vol. 5, pp. 665–687, 1997.

[13] Y. Kawamata. Flops connect minimal models. *Publ. Res. Inst. Math. Sci.* , Vol. 44, pp. 419–423, 2008.

[14] Y. Konishi and S. Minabe. Flop invariance of the topological vertex. *Internat. J. Math*, Vol. 19, pp. 27–45, 2008.

[15] M. Kontsevich and Y. Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. *preprint.* math.AG/0811.2435.

[16] M. Levine and R. Pandharipande. Algebraic cobordism revisited. *preprint.* math.AG/0605196.

[17] A.-M. Li and Y. Ruan. Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds. *Invent. Math*, Vol. 145, pp. 151–218, 2001.

[18] J. Li. Zero dimensional Donaldson-Thomas invariants of threefolds. *Geom. Topol.* , Vol. 10, pp. 2117–2171, 2006.

[19] M. Lieblich. Moduli of complexes on a proper morphism. *J. Algebraic Geom.* , Vol. 15, pp. 175–206, 2006.

[20] C. H. Liu and S. T. Yau. Transformations of algebraic Gromov-Witten invariants of three-folds under flops and small extremal transitions, with an appendix from the stringy and the symplectic viewpoint. *preprint.* math.AG/0505084.

[21] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. I. *Compositio. Math*, Vol. 142, pp. 1263–1285, 2006.

[22] D. R. Morrison. Beyond the Kähler cone. *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan,1993)*, pp. 361–376.

[23] S. Mozgovoy and M. Reineke. On the noncommutative Donaldson-Thomas invariants arising from brane tilings. *preprint.* math.AG/0809.0117.

[24] K. Nagao. Derived categories of small toric calabi-yau 3-folds and counting invariants. *preprint.* math.AG/0809.2994.

[25] K. Nagao and H. Nakajima. Counting invariant of perverse coherent sheaves and its wall-crossing. *preprint.* math.AG/0809.2992.

[26] R. Pandharipande and R. P. Thomas. Curve counting via stable pairs in the derived category. *preprint.* math.AG/0707.2348.

[27] B. Szendrői. Non-commutative Donaldson-Thomas theory and the conifold. *Geom. Topol.* , Vol. 12, pp. 1171–1202, 2008.

[28] R. P. Thomas. A holomorphic casson invariant for Calabi-Yau 3-folds and bundles on $K3$-fibrations. *J. Differential. Geom*, Vol. 54, pp. 367–438, 2000.

[29] R. P. Thomas. Stability conditions and the braid groups. *Comm. Anal. Geom.* , Vol. 14, pp. 135–161, 2006.

[30] Y. Toda. Curve counting theories via stable objects I: DT/PT correspondence. *preprint*. math.AG/0902.4371.

[31] Y. Toda. Birational Calabi-Yau 3-folds and BPS state counting. *Communications in Number Theory and Physics*, Vol. 2, pp. 63–112, 2008.

[32] Y. Toda. Stability conditions and crepant small resolutions. *Trans. Amer. Math. Soc.*, Vol. 360, pp. 6149–6178, 2008.

[33] Y. Toda. Generating functions of stable pair invariants via wall-crossings in derived categories. *preprint*. math.AG/0806.0062.

[34] Y. Toda. Limit stable objects on Calabi-Yau 3-folds. *Duke Math. J.* , Vol. 149, pp. 157–208, 2009.

[35] B. Young with an appndix by J. Bryan. Generating functions for colored 3D Young diagrams and the Donaldson-Thomas invariants of orbifolds. *preprint*. math.AG/0802.3948.

[36] B. Young. Computing a pyramid partition generating function with dimer shuffling. *J. Combin. Theory Ser.* , pp. 334–350, 2009.

Institute for the Physics and Mathematics of the Universe, University of Tokyo
E-mail address:toda-914@pj9.so-net.ne.jp