Monoids in the fundamental groups of the complement of logarithmic free divisors in $\mathbb{C}^3$

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**Abstract**

We study monoids generated by certain Zariski-van Kampen generators in the 17 fundamental groups of the complement of logarithmic free divisors in $\mathbb{C}^3$ listed by Sekiguchi. They admit positive homogeneous presentations (Theorem 1). Five of them are Artin monoids and eight of them are free abelian monoids. *The remaining four monoids are not Gaußian and, hence, are neither Garside nor Artin* (Theorem 2). However, *we introduce the concept of fundamental elements for positive homogenously presented monoids, and show that all 17 monoids posses fundamental elements* (Theorem 3).

# 1 Introduction

A hypersurface $D$ in $\mathbb{C}^l$ ($l \in \mathbb{Z}_{\geq 0}$) is called a *logarithmic free divisor* ([S1]), if the associated module $Der_{\mathbb{C}^l}(-\log(D))$ of logarithmic vector fields is a free $\mathcal{O}_{\mathbb{C}^l}$-module. Classical example of logarithmic free divisors is the discriminant loci of a finite reflection group ([S1,2,3,4]). The fundamental group of the complement of the discriminant loci is presented (Brieskorn [B]) by certain positive homogeneous relations, called Artin braid relations. The group (resp. monoid) defined by that presentation is called an *Artin group* (resp. *Artin monoid*) of finite type [B-S], for which the word problem and other problems are solved using a particular element $\Delta$, the *fundamental elements*, in the corresponding monoid ([B-S],[D],[G]).

In [Se1], Sekiguchi listed up 17 weighted homogeneous polynomials, defining logarithmic free divisors in $\mathbb{C}^3$, whose weights coincide with those of the discriminant of types $\mathrm{A}_3$, $\mathrm{B}_3$ or $\mathrm{H}_3$. Then, the fundamental groups of the complements of the divisors are presented by Zariski-van Kampen method by [I1] (we recall the result in §3). It turns out that the defining relations can be reformulated by a system of positive homogeneous relations in the sense explained in §4 of the present paper, so that we can introduce monoids defined by them. We show that, among 17 monoids, five are Artin monoids, and eight are free abelian monoids. However, four remaining monoids are not Gaussian, and hence are neither Garside nor Artin (§5). Nevertheless, we show that they carry certain particular elements similar to the fundamental elements in Artin monoids (§6).

Let us explain more details of the contents. The 17 Sekiguchi-polynomials $\Delta_X(x,y,z)$ are labeled by the type $X \in \{\mathrm{A_i},\mathrm{A_{ii}},\mathrm{B_i},\mathrm{B_{ii}},\mathrm{B_{iii}},\mathrm{B_{iv}},\mathrm{B_v},\mathrm{B_{vi}},\mathrm{B_{vii}},\mathrm{H_i},$ $\mathrm{H_{ii}},\mathrm{H_{iii}},\ \mathrm{H_{iv}},\mathrm{H_v},\ \mathrm{H_{vi}},\mathrm{H_{vii}},\mathrm{H_{viii}}\}$ (§2). They are monic polynomials of degree 3 in the variable $z$. We calculate the fundamental group of the complement of the divisor $D_X := \{\Delta_X(x,y,z)=0\}$ in $\mathbb{C}^3$ by choosing Zariski-pencils $l$ in $z$-coordinate direction, which intersect the divisor $D_X$ at 3 points. Zariski-van Kampen method gives a presentation of the fundamental group $\pi_1(\mathbb{C}^3\setminus D_X,*)$ with respect to three generators $a$, $b$ and $c$ presented by a suitable choice of paths in the pencil counterclockwise turning once around each of three intersection points.

We rewrite the Zariski-van Kampen relations into a system of positive homogeneous relations (not unique, §4 Theorem 1), and study the group $G_X$ and the monoid $G_X^+$ defined by the relations as well as the localization homomorphism $\pi: G_X^+ \to G_X$, where $G_X$ is naturally isomorphic to $\pi_1(\mathbb{C}^3\setminus D_X,*)$. We denote by $\pi G_X^+$ the image of $G_X^+$ in $G_X$, that is, the monoid generated by the Zariski-van Kampen generators $\{a,b,c\}$ in $\pi_1(\mathbb{C}^3\setminus D_X,*)$.[1] The monoid $\pi G_X^+$ depends on the choice of generators but not on the choice of homogeneous relations, whereas the monoid $G_X^+$ does. It turns out that $G_X^+$ are Artin monoids for the types $X \in \{\mathrm{A_i},\mathrm{B_i},\mathrm{H_i},\mathrm{A_{ii}},\mathrm{B_{iv}}\}$, and are free abelian monoids for the types $X \in \{\mathrm{B_{iii}},\mathrm{B_v},\mathrm{B_{vii}},\mathrm{H_{iv}},\mathrm{H_v},\ \mathrm{H_{vi}},\mathrm{H_{vii}},\mathrm{H_{viii}}\}$ so that one has the injectivities: $G_X^+ \to G_X$ . However, for all the remaining four types $\mathrm{B_{ii}},\mathrm{B_{vi}},\mathrm{H_{ii}},\mathrm{H_{iii}}$, the monoids $\pi G_X^+$ does not admit the divisibility theory (see [B-S, §5], or §5 Theorem 2 of present paper). That is, their associated groups are not Gaussian groups [D-P, §2], and, hence, they are neither Artin nor Garside groups (actually, we have an isomorphism $G_{\mathrm{B_{vi}}}^+ \simeq G_{\mathrm{H_{iii}}}^+$ and hence $\pi G_{\mathrm{B_{vi}}}^+ \simeq \pi G_{\mathrm{H_{iii}}}^+$).

On the other hand, as one main result of the present paper, we show that the monoid $G_X^+$ carries some distinguished elements, which we call *fundamental* (§6 Theorem 3). Namely, we call an element $\Delta \in G_X^+$ fundamental (see §6 ) if there exists a permutation $\sigma_\Delta$ of the set $\{a,b,c\}/\sim$ (:=the image of the set $\{a,b,c\}$ in $G_X^+$) such that for any $d \in \{a,b,c\}/\sim$, there exists $\Delta_d \in G_X^+$ such

[1] We changed the notation from [S-I1]. Namely, $G_X^+$ and $\pi G_X^+$ in the present paper are denoted by $M_X$ and $G_X^+$, respectively, in [S-I1].

that the following relation holds:

$$\Delta \; = \; d\cdot\Delta_d \; = \; \Delta_d\cdot\sigma_\Delta(d).$$

The set $\mathcal{F}(G_X^+)$ of fundamental elements in $G_X^+$ form a subsemigroup of $G_X^+$ such that $\mathcal{QZ}(G_X^+)\mathcal{F}(G_X^+)=\mathcal{F}(G_X^+)\mathcal{QZ}(G_X^+)=\mathcal{F}(G_X^+)$ (see §6 Fact 3.) where $\mathcal{QZ}(G_X^+)$ is the quasi-center of $G_X^+$.[2]

Since the localization homomorphism induces a map $\mathcal{F}(G_X^+)\to\mathcal{F}(\pi G_X^+)$, the fact $\mathcal{F}(G_X^+)\neq\emptyset$ for all 17 monoids (§6 Theorem3) implies $\mathcal{F}(\pi G_X^+)\neq\emptyset$. [3]

In §7, we discuss the cancellation condition on the monoid $G_X^+$. In fact, this condition together with the existence of fundamental elements (shown in §6), imply that the localization homomorphism $\pi: G_X^+\to G_X$ is injective. An Artin monoid or a free abelian monoid satisfies already the cancellation condition ([B-S]). We show that *the monoid* $G_{\mathrm{B_{ii}}}^+$ *satisfies the cancellation condition* (Theorem 4). For the remaining three types $\mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}$, we do not know whether the localization homomorphism $\pi$ is injective or not. That is, we don't know whether we have sufficiently many defining relations to assert the cancellation condition or not.

Finally in §8, we construct non-abelian representations of the groups $G_{\mathrm{B_{ii}}}, G_{\mathrm{B_{vi}}}$, $G_{\mathrm{H_{ii}}}$ and $G_{\mathrm{H_{iii}}}$ into $\mathrm{GL}_2(\mathbb{C})$ (Theorem 5). Actually, this result is independent of §5, 6 and 7, and is used in the proof of Theorem 2 in §5.

[2] An element $\Delta\in G_X^+$ is called quasi-central ([B-S, 7.1]) if $d\cdot\Delta=\Delta\cdot\sigma_\Delta(d)$ for $d\in\{a,b,c\}$.

[3] We ask, more generally, *whether the monoid generated by Zariski-van Kampen generators in the local fundamental group of the complement of a free divisor has always a fundamental element* (see §6 Remark 6.4). In the 4 types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}$, we observe that $\mathcal{F}(\pi G_X^+)$ is not singly generated. Therefore, we ask, also, *whether the set of fundamental elements* $\mathcal{F}(\pi G_X^+)$ *is finitely generated over* $\mathcal{QZ}(\pi G_X^+)$ *or not.* For an Artin monoid of finite type, $\mathcal{F}(G_X^+)$ is generated by a single element $\Delta$ and $\mathcal{F}(G_X^+)=\Delta^{\mathbb{Z}_{\geq 1}}$ ([B-S]).

## 2 Sekiguchi's Polynomial

J. Sekiguchi [Se1,2] listed the following 17 weighted homogeneous polynomials $\Delta$ in three variables $(x, y, z)$ satisfying freeness criterion by K.Saito [S1].

$$
\begin{aligned}
\Delta_{\mathrm{A_i}}(x,y,z) &:= -4x^3y^2 - 27y^4 + 16x^4z + 144xy^2z - 128x^2z^2 + 256z^3 \\
\Delta_{\mathrm{A_{ii}}}(x,y,z) &:= 2x^6 - 3x^4z + 18x^3y^2 - 18xy^2z + 27y^4 + z^3 \\
\Delta_{\mathrm{B_i}}(x,y,z) &:= z(x^2y^2 - 4y^3 - 4x^3z + 18xyz - 27z^2) \\
\Delta_{\mathrm{B_{ii}}}(x,y,z) &:= z(-2y^3 + 4x^3z + 18xyz + 27z^2) \\
\Delta_{\mathrm{B_{iii}}}(x,y,z) &:= z(-2y^3 + 9xyz + 45z^2) \\
\Delta_{\mathrm{B_{iv}}}(x,y,z) &:= z(9x^2y^2 - 4y^3 + 18xyz + 9z^2) \\
\Delta_{\mathrm{B_v}}(x,y,z) &:= xy^4 + y^3z + z^3 \\
\Delta_{\mathrm{B_{vi}}}(x,y,z) &:= 9xy^4 + 6x^2y^2z - 4y^3z + x^3z^2 - 12xyz^2 + 4z^3 \\
\Delta_{\mathrm{B_{vii}}}(x,y,z) &:= (1/2)xy^4 - 2x^2y^2z - y^3z + 2x^3z^2 + 2xyz^2 + z^3 \\
\Delta_{\mathrm{H_i}}(x,y,z) &:= -50z^3 + (4x^5 - 50x^2y)z^2 + (4x^7 + 60x^4y^2 + 225xy^3)z \\
&\quad\; -(135/2)y^5 - 115x^3y^4 - 10x^6y^3 - 4x^9y^2 \\
\Delta_{\mathrm{H_{ii}}}(x,y,z) &:= 100x^3y^4 + y^5 + 40x^4y^2z - 10xy^3z + 4x^5z^2 - 15x^2yz^2 + z^3 \\
\Delta_{\mathrm{H_{iii}}}(x,y,z) &:= 8x^3y^4 + 108y^5 - 36xy^3z - x^2yz^2 + 4z^3 \\
\Delta_{\mathrm{H_{iv}}}(x,y,z) &:= y^5 - 2xy^3z + x^2yz^2 + z^3 \\
\Delta_{\mathrm{H_v}}(x,y,z) &:= x^3y^4 - y^5 + 3xy^3z + z^3 \\
\Delta_{\mathrm{H_{vi}}}(x,y,z) &:= x^3y^4 + y^5 - 2x^4y^2z - 4xy^3z + x^5z^2 + 3x^2yz^2 + z^3 \\
\Delta_{\mathrm{H_{vii}}}(x,y,z) &:= xy^3z + y^5 + z^3 \\
\Delta_{\mathrm{H_{viii}}}(x,y,z) &:= x^3y^4 + y^5 - 8x^4y^2z - 7xy^3z + 16x^5z^2 + 12x^2yz^2 + z^3.
\end{aligned}
$$

Here, the polynomials are classified into three types A, B and H according to whether the numerical data $(\deg(x), \deg(y), \deg(z); \deg(\Delta))$ is equal to $(2,3,4;12)$, $(2,4,6;18)$ or $(2,6,10;30)$, respectively. In each type, the polynomials are numbered by small Roman numerals i, ii,... etc. We remark that, in all cases, the polynomial is a monic polynomial of degree 3 in the variable $z$.

## 3 Zariski-van Kampen Presentation

Let $X$ be one of the 17 types $\mathrm{A_i}$,$\mathrm{A_{ii}}$,$\mathrm{B_i}$,..., $\mathrm{B_{vii}}$, $\mathrm{H_i}$,...,$\mathrm{H_{viii}}$. In the present section, we recall in Table 1 from [I1] [S-I1] the result of the calculation of the fundamental group $\pi_1(S_X \setminus D_X, *_X)$ of the complement of the free divisor $D_X$ in the space $S_X$ by Zarisik-van Kampen method (see [Ch],[T-S] for instance), where we put $S_X := \mathbb{C}^3$ and

$$D_X := \{(x,y,z) \in \mathbb{C}^3 \mid \Delta_X(x,y,z) = 0\}. \tag{3.1}$$

**Table 1.**

$$\pi_1(S_{\mathrm{A_i}} \setminus D_{\mathrm{A_i}}, *_{\mathrm{A_i}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} ab = ba, \\ bcb = cbc, \\ aca = cac \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{A_{ii}}} \setminus D_{\mathrm{A_{ii}}}, *_{\mathrm{A_{ii}}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} ababab = bababa, \\ aba = bab, \\ b = c \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_i}} \setminus D_{\mathrm{B_i}}, *_{\mathrm{B_i}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} abab = baba, \\ bc = cb, \\ aca = cac, \\ cbac = baca \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{ii}}} \setminus D_{\mathrm{B_{ii}}}, *_{\mathrm{B_{ii}}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} ababab = bababa, \\ bc = ab, \\ ac = ca \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{iii}}} \setminus D_{\mathrm{B_{iii}}}, *_{\mathrm{B_{iii}}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} a = b, \\ a = cbab^{-1}c^{-1}, \\ b = cbacbc^{-1}a^{-1}b^{-1}c^{-1}, \\ c = cbacbcb^{-1}c^{-1}a^{-1}b^{-1}c^{-1} \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{iv}}} \setminus D_{\mathrm{B_{iv}}}, *_{\mathrm{B_{iv}}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} acb = cba, \\ bcba = cbac, \\ cbac = bacb, \\ ab = ba \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_v}} \setminus D_{\mathrm{B_v}}, *_{\mathrm{B_v}}) \cong \left\langle a,b,c \,\middle|\, a = b = c \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{vi}}} \setminus D_{\mathrm{B_{vi}}}, *_{\mathrm{B_{vi}}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} aba = bab, \\ aca = bac, \\ acaca = cacac \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{vii}}} \setminus D_{\mathrm{B_{vii}}}, *_{\mathrm{B_{vii}}})$$

$$\cong \left\langle a,b,c \,\middle|\, \begin{array}{c} a = b^{-1}cbab^{-1}cbab^{-1}cbab^{-1}cba^{-1}b^{-1}c^{-1}ba^{-1}b^{-1}c^{-1}ba^{-1}b^{-1}c^{-1}b, \\ c = bab^{-1}cbab^{-1}cbab^{-1}cbab^{-1}c^{-1}ba^{-1}b^{-1}c^{-1}ba^{-1}b^{-1}c^{-1}ba^{-1}b^{-1}, \\ a = ba^{-1}b^{-1}c^{-1}bab^{-1}cbab^{-1}, \\ cba = bab, cba = bcb, cba = bab^{-1}c^{-1}b^{-1}cbcb \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_i}} \setminus D_{\mathrm{H_i}}, *_{\mathrm{H_i}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} ababa = babab, \\ bc = cb, \\ aca = cac \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{ii}}} \setminus D_{\mathrm{H_{ii}}}, *_{\mathrm{H_{ii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} abab = baba, \\ aca = bac, \\ acaca = cacac \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{iii}}} \setminus D_{\mathrm{H_{iii}}}, *_{\mathrm{H_{iii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} aba = bab, \\ bcba = cbac, \\ cba = acb \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{iv}}} \setminus D_{\mathrm{H_{iv}}}, *_{\mathrm{H_{iv}}}) \cong \left\langle a,b,c \;\middle|\; a = b = c \right\rangle.$$

$$\pi_1(S_{\mathrm{H_v}} \setminus D_{\mathrm{H_v}}, *_{\mathrm{H_v}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} acba = cbac, \\ bcbac = cbacb, \\ bacb = cbac, \\ bc = cb \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{vi}}} \setminus D_{\mathrm{H_{vi}}}, *_{\mathrm{H_{vi}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} abababab = babababa, \\ ba = cb, \\ ac = ba \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{vii}}} \setminus D_{\mathrm{H_{vii}}}, *_{\mathrm{H_{vii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} a = cbaca^{-1}b^{-1}c^{-1}, \\ b = cbacbc^{-1}a^{-1}b^{-1}c^{-1}, \\ c = cbacbab^{-1}c^{-1}a^{-1}b^{-1}c^{-1}, \\ b = c \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{viii}}} \setminus D_{\mathrm{H_{viii}}}, *_{\mathrm{H_{viii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} abababa = bababab, \\ ab = bc, \\ ac = ca \end{array} \right\rangle.$$

# 4 Positive Homogeneous Presentation

In the present section, we rewrite the presentations of the fundamental groups in section 3 to a positive homogeneous form. We, first, prepare some terminology.

**Definition.** 1. Let $G = \langle L \mid R \rangle$ be a presentation of a group $G$, where $L$ is the set of generators (called letters) and $R$ is the set of relations. We say that the presentation is *positive homogeneous*, if $R$ consists of relations of the form $R_i = S_i$ where $R_i$ and $S_i$ are positive words in $L$ (i.e. words consisting of only non-negative powers of the letters in $L$) of the same length.

2. If a positive homogeneous presentation $\langle L \mid R \rangle$ of a group $G$ is given, then we associate a monoid $G^+$ defined as the quotient of free monoid $L^*$ generated by $L$ by the equivalence relation $\simeq$ defined as follows:

1) two words $U$ and $V$ in $L^*$ are called *elementarily equivalent* if either $U = V$ or $V$ is obtained from $U$ by substituting a substring $R_i$ of $U$ by $S_i$ where $R_i = S_i$ is a relation of $R$ ($S_i = R_i$ is also a relation if $R_i = S_i$ is a relation),

2) two words $U$ and $V$ in $L^*$ are called *equivalent*, denoted by $U \simeq V$, if there exists a sequence $U = W_0, W_1, \cdots, W_n = V$ of words in $L^*$ for $n \in \mathbb{Z}_{\geq 0}$ such that $W_i$ is elementarily equivalent to $W_{i-1}$ for $i = 1, \cdots, n$.

3. The natural homomorphism $\pi : G^+ \to G$ will be called the *localization homomorphism*. The image of the localization homomorphism is denoted by $\pi G^+$.

*Note.* 1. The monoid $\pi G^+$ depends on the choice of the generators for the group $G$. Even if we choose the same generators for the same group $G$, the monoid $G^+$ depends on the choice of the relations $R$.

2. Due to the homogeneity of the relations, one defines a homomorphism:

$$l \ : \ G \ \longrightarrow \ \mathbb{Z}$$

by associating 1 to each letter in $L$. The restriction of the homomorphism on $\pi G^+$ and its pull-back to $G^+$ by the composition with the localization homomorphism are called *length functions*. Length functions have the additivity: $l(UV) = l(U) + l(V)$ and the conicity: $l(U) = 0$ implies $U = 1$ in the monoids. The existence of such length functions implies that the monoids $G^+$ and $\pi G^+$ are *atomic* ([D-P, §2]) and that $\pi G^+$ is also a positive homogeneously presented monoid.

**Theorem 1.** *The fundamental group in* **Table 1.** *of type $X$ is naturally isomorphic to the following positive homogeneously presented group $G_X$ by identifying the generators $\{a, b, c\}$ in both groups.*

$$\mathrm{A_i} \quad : G_{\mathrm{A_i}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} ab=ba, \\ bcb=cbc, \\ aca=cac \end{array} \right\rangle .$$

$$\mathrm{A_{ii}} \quad : G_{\mathrm{A_{ii}}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} aba=bab, \\ b=c \end{array} \right\rangle .$$

$$\mathrm{B_i} \quad : G_{\mathrm{B_i}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} abab=baba, \\ bc=cb, \\ aca=cac \end{array} \right\rangle .$$

$$\mathrm{B_{ii}} \quad : G_{\mathrm{B_{ii}}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} cbb=bba, \\ bc=ab, \\ ac=ca \end{array} \right\rangle .$$

$$\mathrm{B_{iii}} \quad : G_{\mathrm{B_{iii}}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} a=b, \\ ac=ca \end{array} \right\rangle .$$

$$\mathrm{B_{iv}} \quad : G_{\mathrm{B_{iv}}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} ab=ba, \\ bcb=cbc, \\ ac=ca \end{array} \right\rangle .$$

$$\mathrm{B_v} \quad : G_{\mathrm{B_v}} := \left\langle a,b,c \,\middle|\, a=b=c \right\rangle .$$

$$\mathrm{B_{vi}} \quad : G_{\mathrm{B_{vi}}} := \left\langle a,b,c \,\middle|\, \begin{array}{l} aba=bab,\ bcb=cbc,\ aca=bac,\ cab=bca,\ acb=cac,\ abb=bbc, \\ bcca=ccac,\ bbac=caab,\ cbbb=bbba,\ acbcb=bccca, \\ accbb=bccba, accaa=ccaac, caacc=aacca,\ baaccba=cbaaccb, \\ acccc=bcccb,\ bbaac=cbaab,\ bbaab=caaaa,\ caaab=abaac, \\ a^5=b^5=c^5,\ baaab=aaaac,\ cccca=bccbb,\ ccbaac=accbaa \end{array} \right\rangle .$$

$$\mathrm{B_{vii}} \quad : G_{\mathrm{B_{vii}}} := \left\langle a,b,c \,\middle|\, a=b=c \right\rangle .$$

$$\mathrm{H_i} \quad : G_{\mathrm{H_i}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} ababa=babab, \\ bc=cb, \\ aca=cac \end{array} \right\rangle .$$

$\mathrm{H_{ii}} \quad : G_{\mathrm{H_{ii}}} := \left\langle a,b,c \,\middle|\, R_{\mathrm{H_{ii}}} \right\rangle$ ($R_{\mathrm{H_{ii}}}$ is given at the end of present Table).

$$\mathrm{H_{iii}} \quad : G_{\mathrm{H_{iii}}} := \left\langle a,b,c \,\middle|\, \begin{array}{l} aba=bab,\ aca=cac,\ bcb=abc,\ cba=acb,\ bca=cbc,\ baa=aac, \\ accb=ccbc,\ aabc=cbba,\ caaa=aaab,\ bcaca=accca, \\ bccaa=accab, bccbb=ccbbc, cbbcc=bbccb,\ abbccab=cabbcca, \\ bcccc=accca,\ aabbc=cabba,\ aabba=cbbbb,\ cbbba=babbc, \\ a^5=b^5=c^5,\ abbba=bbbbc,\ ccccb=accaa,\ ccabbc=bccabb \end{array} \right\rangle .$$

$$\mathrm{H_{iv}} \quad : G_{\mathrm{H_{iv}}} := \left\langle a,b,c \,\middle|\, a=b=c \right\rangle .$$

$$\mathrm{H_v} \quad : G_{\mathrm{H_v}} := \left\langle a,b,c \,\middle|\, a=b=c \right\rangle .$$

$$\mathrm{H_{vi}} \quad : G_{\mathrm{H_{vi}}} := \left\langle a,b,c \,\middle|\, a=b=c \right\rangle .$$

$$\mathrm{H_{vii}} \quad : G_{\mathrm{H_{vii}}} := \left\langle a,b,c \,\middle|\, a=b=c \right\rangle .$$

$$\mathrm{H_{viii}} \quad : G_{\mathrm{H_{viii}}} := \left\langle a,b,c \,\middle|\, a=b=c \right\rangle .$$

$$
R_{\mathrm{H_{ii}}} := \left\{ \begin{array}{l}
abab = baba, aca = bac, bcbc = cbcb, acb = cac, bbcaba = abccac, \\
abbbca = baaaac, bbbabb = abbaaa, baaaaba = abbbbab, \\
baabbb = aaabaa, abccc = cccab, bbcbab = cccaac, \\
cccbcaa = bbbccab, bccbbb = cccbcc, bbccab = caaccc, \\
ccaac = bccaa, ccaab = accaa, ccabaac = accbcaa, \\
caaccab = bcaacca, aabaaa = bbbaab, bbbaaa = aaabbb, \\
abaaaab = babbbba, aaabba = bbabbb, baabbaa = aabbaab, \\
baabaabaa = abbabbabb, aabbaac = babbcaa, aaabc = bcaaa, \\
abbaabaac = babbabbca, cccaaa = aaaccc, cccbbb = bbbccc, \\
caacaac = aabccba, bbbcbb = cbbccc, abacbc = cbcaba, \\
cbbbbcb = bccccbc, cabbbc = accccb, bcccccaa = cbbbcaac, \\
ccbccc = bbbccb, cbcaaab = bcccaba, caabcb = baccca, \\
bcbaab = aaccba, baaccbbc = caccabcb, bccabb = accaaa, \\
babcbab = cabcaca, caabbbbcb = bacccccca, cbaacc = bccbab, \\
abcbaa = ccbabb, bcbbaa = ccbbab, caacac = babcca, \\
cbbaaaacc = acacbbcba, caaaacc = aabccca, bcabbcc = aabbcbb, \\
bbcaabc = cccaabb, cbbcaab = bccabba, bbaabba = abbaabb, \\
abaabcc = bbabbcb, bacbcab = cabcaba, cbcabca = bcaacab, \\
caaccbba = bcabcaab, babbcbb = aabccbc, bbcbbb = cccbbc, \\
bcbbbbc = cbccccb, bccbbabbc = abcabccba, bbabcbbab = cbbabbccc, \\
cabaaccc = abbcbbab, bacabc = cbcabb, bcabaab = abcabaa, \\
aaccbcab = bcabaacc, cbaabcc = baccbca, cccbaabc = baaccaba, \\
bccbaabc = cabacbca, abaabcaba = bbaabcabb, ccbbaaa = aaaccbb, \\
ccbbaabca = abcabbaac, baabcabba = abacabaab, bcaabb = aaacca, \\
accbbcc = ccabbcb, bbcabbccc = abbabbccb, bcaaccbc = abcabcca, \\
cabaabcc = babccbca, babccba = cbbabcb, aabccbca = cabaabcb, \\
abcbbaaccb = ccbaaacaba, baaccbca = abacbaac, ccbcabaa = abccbaab, \\
bbcbbabba = ccbccbaab, cbabbcba = accbaacb, aacccbab = cbcaaaba, \\
cbcaaac = aacccca, baaccca = cccaaab, caaaccb = bcccaaa, \\
bbaaccc = cccbbaa, acabbabc = cbbabccb, ccabaaa = aaaccab
\end{array} \right\}
$$

*Proof.* Except for the types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}, \mathrm{H_{viii}}$, the relations are obtained by elementary reductions of the Zariski-van Kampen relations, and we omit details.

Some new relations for the cases of types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}$ are obtained by cancelling common factors from the left or from the right of equivalent expressions of the same fundamental elements (introduced in §6 6.1. See §7 Definition 7.1), where these equivalent expressions of a fundamental element are obtained with the help of Hayashi's computer program (see http://www.kurims.kyoto-u.ac.jp/ saito/SI/). In the following, we sketch how some of them are obtained by hand calculations. In the proof, "the first relation, the second relation, . . . ", mean "the relation which is at the first place, the second place, . . . in Table 1. of Zariski-van Kampen relations in §3".

The case for the type $\mathrm{H_{viii}}$ needs to be treated separately because its calculations are non-trivial. Detailed verifications are left to the reader.

$\mathrm{B_{ii}}$: Using $ab = bc$, rewrite the LHS $ababab$ (resp. RHS $bababa$) of the first relation to $bcabbc$ (resp. $babbca$). Then, using the commutativity of $a$ and $c$, we cancel $ba$ from left and $c$ from right so that we obtain a new relation $cbb \simeq bba$.

$\mathrm{B}_{\mathrm{vi}}$: Using the defining relation $aca = bac$, rewrite the LHS $acaca$ of the third relation to $acbac$ so that the relation turns to $acbac \simeq cacac$. We cancel $ac$ from right and obtain a new relation $acb \simeq cac$. Using this, one has $bcbac \simeq bcaca \simeq bacba \simeq acaba \simeq acbab \simeq cacab \simeq cbacb \simeq cbcac$.

We cancel $ac$ from right and obtain $bcb \simeq cbc$. Using this, one has $acabc \simeq bacbc \simeq babcb \simeq abacb \simeq abcac$. Cancelling $a$ and $c$ for left and right, we obtain a new relation $cab \simeq bca$. Using this, one has $cabba \simeq bcaba \simeq bcbab \simeq cbcab \simeq cbbca$. Cancelling $c$ and $a$ for left and right, we obtain a new relation $abb \simeq bbc$. The second relation of length 4 is obtained by cancelling $a$ from left of the equality: $abbac \simeq bbcac \simeq bbacb \simeq bacab \simeq acaab$.

$\mathrm{H}_{\mathrm{ii}}$: Using the defining relation $aca = bac$, rewrite the LHS $acaca$ of the third relation to $acbac$ so that the relation turns to $acbac \simeq cacac$. We cancel $ac$ from right so that we obtain a new relation $acb \simeq cac$.

$\mathrm{H}_{\mathrm{iii}}$: Multiply $b$ to the second relation from the right, and rewire the LHS to $bcaba$ (by a use of the defining relation $bab = aba$ and rewrite the RHS to $cbcba$ (by a use of the defining relation $acb = cba$). Cancelling by $ba$ from right, we obtain a new relation $bca \simeq cbc$.

Using the length 3 relations, one has $acabc \simeq acbcb \simeq cbacb \simeq cbcba \simeq cabca \simeq cacbc$. Cancelling by $bc$ from right, we obtain a new relation $aca \simeq cac$.

Using the length 3 relations, one has $bcaac \simeq cbcac \simeq cbaca \simeq acbca \simeq abcaa \simeq bcbaa$. Cancelling by $bc$ from left, we obtain a new relation $aac \simeq baa$.

In the above sequence, the middle term $acbca$ is also equivalent to $accbc$. Thus, cancelling $c$ from right, we obtain a new relation $accb \simeq cbca (\simeq bcaa)$.

$\mathrm{H}_{\mathrm{viii}}$: From the defining relations, we have $abababa \simeq bcbcbca$, $bababab \simeq bbcbcbc$, and, hence, $bcbcbca \simeq bbcbcbc$ (1). Multiplying $b$ from the right, we get $bcbcbcbc \simeq bcbcbcab \simeq bbcbcbcb$ (2). In the equality (2), dividing by $b$ from the left, we get $cbcbcbc \simeq bcbcbcb$. Dividing (1) by $b$ from the left, we get $cbcbca \simeq bcbcbc$. The left hand side of this equality is equivalent to $cabbca \simeq acbbca$, and the right hand side of the equality is equivalent to $abbcbc$ so that $acbbca \simeq abbcbc$. Dividing by $a$ from the left, we get $bbcbc \simeq cbbca \simeq cbbac$. Dividing by $c$ from the light, we get $cbba \simeq bbcb (\simeq babb)$. Multiplying $cbcb$ from the right, we get $cbbacbcb \simeq bbcbcbcb$. The right hand side is equivalent to $bbcbcbcb \simeq bcbcbcbc \simeq cbcbcbcc \simeq cbcbcabc \simeq cbcbacbc$. The left hand side is equivalent to $cbbacbcb \simeq cbbcabcb \simeq cbababcb$, and hence $cbababcb \simeq cbcbacbc$. Dividing by $cb$ from the left, we get $cbacbc \simeq ababcb$. The left hand side is equivalent to $cbacab \simeq cbaacb$. Dividing by $cb$ from the right, we get $abab \simeq cbaa$ (3). Mutiplying $b$ from the right, the left hand side is equivalent to $ababb \simeq acbba \simeq cabba \simeq cbcba$ so that $cbcba \simeq cbaab$. Dividing by $cb$ from the left, we get $cba \simeq aab$ (4). Applying (4) to the equality (3), we get $abab \simeq cbaa \simeq aaba$. Dividing by $a$ from the left, we get $bab \simeq aba \simeq bca$. Dividing by $b$ from the left, we get $ab \simeq ca = ac$, and hence $b \simeq c$.

This completes a proof of Theorem 1. □

**Notation.** *For each type* $X \in \{\mathrm{A}_{\mathrm{i}}, \mathrm{A}_{\mathrm{ii}}, \mathrm{B}_{\mathrm{i}}, \mathrm{B}_{\mathrm{ii}}, \mathrm{B}_{\mathrm{iii}}, \mathrm{B}_{\mathrm{iv}}, \mathrm{B}_{\mathrm{v}}, \mathrm{B}_{\mathrm{vi}}, \mathrm{B}_{\mathrm{vii}}, \mathrm{H}_{\mathrm{i}}, \mathrm{H}_{\mathrm{ii}}, \mathrm{H}_{\mathrm{iii}}, \mathrm{H}_{\mathrm{iv}}, \mathrm{H}_{\mathrm{v}}, \mathrm{H}_{\mathrm{vi}}, \mathrm{H}_{\mathrm{vii}}, \mathrm{H}_{\mathrm{viii}}\}$*, we denote by* $G_X$*,* $G_X^+$ *and* $\pi G_X^+$ *the* **group***, the* **monoid** *and the* **image of localization** $\pi : G_X^+ \to G_X$*, respectively, associated with the positive homogeneous relations of type* $X$ *given in Theorem 1.*

From the presentations, we immediately observe the followings.

**Corollary 1.** i) *For the type* $X \in \{\mathrm{A}_{\mathrm{i}}, \mathrm{A}_{\mathrm{ii}}, \mathrm{B}_{\mathrm{i}}, \mathrm{B}_{\mathrm{iv}}, \mathrm{H}_{\mathrm{i}}\}$*, the monoid* $G_X^+$ *and the group* $G_X$ *is an Artin monoid and an Artin group of type* $\mathrm{A}_3$, $\mathrm{A}_2$, $\mathrm{B}_3$, $\mathrm{A}_1 \times \mathrm{A}_2$ *and* $\mathrm{H}_3$*, respectively. As a consequence, we have the injectivity:* $G_X^+ \to G_X$*.*

ii) *For the type* $X \in \{\mathrm{B_v}, \mathrm{B_{vii}}, \mathrm{H_{iv}}, \mathrm{H_v}, \mathrm{H_{vi}}, \mathrm{H_{vii}}, \mathrm{H_{viii}}\}$*, the monoid* $G_X^+$ *and the group* $G_X$ *is the infinite cyclic monoid* $\mathbb{Z}_{\geq 0}$ *and group* $\mathbb{Z}$*, respectively. The monoid* $G_{\mathrm{B_{iii}}}^+$ *and the group* $G_{\mathrm{B_{iii}}}$ *is a free abelian monoid* $(\mathbb{Z}_{\geq 0})^2$ *and group* $\mathbb{Z}^2$ *of rank 2. As a consequence, we have the injectivity:* $G_X^+ \to G_X$*.*

iii) *The correspondence:* $\{a \mapsto b, b \mapsto a, c \mapsto c\}$ *induces an isomorphism:*

$$G_{\mathrm{B_{vi}}}^+ \quad \simeq \quad G_{\mathrm{H_{iii}}}^+$$

*and, hence, also the isomorphisms:* $G_{\mathrm{B_{vi}}} \simeq G_{\mathrm{H_{iii}}}$ *and* $\pi G_{\mathrm{B_{vi}}}^+ \simeq \pi G_{\mathrm{H_{iii}}}^+$*. Note that the isomorphism does not identify the Coxeter elements (c.f. Proposition 6.5).*

*Proof.* We can show that the Zariski-van Kampen relations of one of the two types can be deduced, up to the transposition of $a$ and $b$, from that of the other type. □

As the consequence of Corollary 1, in the rest of the present paper, we shall focus our attention to the remaining 4 types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}$ and $\mathrm{H_{iii}}$ together with the "constraint $\mathrm{B_{vi}} \simeq \mathrm{H_{iii}}$".

**Corollary 2.** *The groups* $G_{\mathrm{B_{vi}}}$ *and* $G_{\mathrm{H_{iii}}}$ *do not admit Artin group presentation with respect to any Zariski-van Kampen type generator system.*

*Proof.* Due to Theorem 1., both groups have the relations: $a^5 = b^5 = c^5$, which are invariant by the change of generator system by the braid group $B(3)$. □

**Remark 4.1.** The group $G_X$ is naturally isomorphic to the fundamental group, which does not depend on the choice of Zariski-van Kampen generators $\{a, b, c\}$, but the monoid $\pi G_X^+$ depends on that choice (see next Remark 4.2).

Furthermore, the monoid $G_X^+$, a priori, depends on the choice of relations in Theorem 1. The injectivity in the above corollary follows from cancellation conditions on $G_X^+$ (see [B-S]). We shall show that, also for $G_{\mathrm{B_{ii}}}^+$ in §7, the cancellation condition holds, implying the injectivity $\pi : G_{\mathrm{B_{ii}}}^+ \to G_{\mathrm{B_{ii}}}$. Thus, for these cases as a consequence of the cancellation condition, $G_X^+$ does not depend on the choice of relations in Theorem 1. However, for the remaining types $\mathrm{B_{vi}}$, $\mathrm{H_{ii}}$ and $\mathrm{H_{iii}}$, it may be still possible that we need more relations in order to obtain the injectivity of the localization homomorphism.

**Remark 4.2.** Recall that we have chosen Zariski pencils for the calculation of the fundamental group of $\mathbb{C}^3 \setminus D_X$ in the direction of the $z$-axis, where $z$ is the weighted homogeneous coordinate of the highest weight so that the pencils intersects the divisor $D_X$ at three points and, for a generic choice of a pencil, we get three generators $\{a, b, c\}$ of the fundamental group(see §1 Introduction). However, this does not determine $\{a, b, c\}$ uniquely. It is wellknown that the ambiguity of the choices of the generators is described by the action of the braid group $B(3)$ with three strings on the free group $F_3$ generated by $\{a, b, c\}$. Here is a remarkable observation for the type $\mathrm{B_{ii}}$.

**Assertion.** *For any choice of Zariski-van Kampen generator system* $\{a, b, c\}$ *(up to a permutation), the fundamental group admits only one of the following* *two positive homogeneous presentations I. and II.*

$$\mathrm{I}: \quad \left\langle a,b,c \;\middle|\; \begin{array}{c} cbb = bba, \\ bc = ab, \\ ac = ca \end{array} \right\rangle .$$

$$\mathrm{II}: \quad \left\langle a,b,c \;\middle|\; \begin{array}{c} ababab = bababa, \\ b = c, \\ aabab = baaba \end{array} \right\rangle .$$

# 5 Non-division property of the monoid $\pi G_X^+$

In the present section, we show that none of the monoids $\pi G_X^+$ of the four types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}$ and $\mathrm{H_{iii}}$ does admit the divisibility theory ([B-S, §4]), and therefore the monoid is neither Gaussian, Garside nor Artin.

We first recall some terminologies and concepts on the monoid $\pi G^+$.
An element $U \in \pi G^+$ is said to *divide* $V \in \pi G^+$ from the left (resp. right), denoted by $U|_l V$ (resp. $U|_r V$), if there exists $W \in \pi G^+$ such that $V = UW$ (resp. $V = WU$). We also say $V$ is *left-divisible* by $U$, or $V$ is a *right-multiple* of $U$.

We say that $\pi G^+$ *admits the left* (resp. *right*) *divisibility theory*, if for any two elements $U, V$ of $\pi G_X^+$, there always exists a left (resp. right) least common multiple, i.e. a left (resp. right) common multiple which divides any other left (resp. right) common multiple. Since $\pi G_X^+$ can be positive homogeneously presented, the only invertible elements in the monoid is the unit element, so that we have the unique left (resp. right) least common multiple, denoted by $\mathrm{lcm}_l(U,V)$ (resp. $\mathrm{lcm}_r(U,V)$).

**Theorem 2.** *The monoids $\pi G^+_{\mathrm{B_{ii}}}, \pi G^+_{\mathrm{B_{vi}}}, \pi G^+_{\mathrm{H_{ii}}}, \pi G^+_{\mathrm{H_{iii}}}$ admits neither the left-divisibility theory nor the right divisibility theory.*

*Proof.* We claim a fact, which shall be proven in §8 Theorem 5 ii) independent of the results of §5, 6 and 7.

**Fact 5.1.** None of the groups $G_{\mathrm{B_{ii}}}, G_{\mathrm{B_{vi}}}, G_{\mathrm{H_{ii}}}$ and $G_{\mathrm{H_{iii}}}$ is abelian.

Assuming that the monoid $\pi G_X^+$ admits the right divisibility theory, we show that $G_X$ becomes an abelian group: a contradiction! to Fact 5.1. The case for the left divisibility theory can be shown similarly.

1) $\pi G^+_{\mathrm{B_{ii}}}$: It is immediate to see $l(\mathrm{lcm}_r(b,c)) > 2$ from the defining relations in Theorem 1. Then, $bba = cbb$ is a common multiple of $b$ and $c$ of the shortest length 3, and, hence, should be equal to $\mathrm{lcm}_r(b,c)$. On the other hand, we have the following sequence of elementary equivalent words: $bcba$, $abba$, $acbb$, $cabb$. That is, $bcba = cabb$ in $\pi G^+_{\mathrm{B_{ii}}}$ is another common right-multiple of $b$ and $c$. If $bba = cbb$ divides $bcba = cabb$ from the left, there exists $d \in \{a,b,c\}$ such that $bcba = bbad$. So, in $\pi G^+_{\mathrm{B_{ii}}}$, we have $cba = bad$ which is again a common right-multiple of $b$ and $c$. Thus, we have the equality: $cba = cbb$ in $\pi G^+_{\mathrm{B_{ii}}}$. That is, $a = b$ in $\pi G^+_{\mathrm{B_{ii}}}$. By adding this relation $a = b$ to the set of the defining relations of the group $G_{\mathrm{B_{ii}}}$, we get $G_{\mathrm{B_{ii}}} \simeq \mathbb{Z}$. A contradiction!

2) $\pi G^+_{\mathrm{B_{vi}}}$: Due to the first defining relation in Theorem 1., we have $l(\mathrm{lcm}_r(a,b)) \leq 3$. Let us consider 3 cases:

i) $l(\mathrm{lcm}_r(a,b)) = 1$. This means $l(\mathrm{lcm}_r(a,b)) = a = b$. By adding this relation to the defining relation of the group $G_{\mathrm{B_{vi}}}$, we get $G_{\mathrm{B_{vi}}} \simeq \mathbb{Z}$. A contradiction!

ii) $l(\mathrm{lcm}_r(a,b)) = 2$. This means that there exists $u, v \in \{a,b,c\}$ such that $l(\mathrm{lcm}_r(a,b)) = au = bv$. Depending on each choice of $u$ and $v$, one can show that this assumption leads to a contradictory conclusion $G_{\mathrm{B_{vi}}} \simeq \mathbb{Z}$. Details are left to the reader.

iii) $l(\mathrm{lcm}_r(a,b)) = 3$. In view of the first two defining relations in Theorem 1., one has $aba = bab = aca = bac$. By adding this relation to the set of the defining relations of the group $G_{\mathrm{B_{vi}}}$, we get $G_{\mathrm{B_{vi}}} \simeq \mathbb{Z}$. A contradiction!.

3) $\pi G^+_{\mathrm{H_{ii}}}$: Due to the second defining relation in Theorem 1., we have $l(\mathrm{lcm}_r(a,b)) \leq 3$. Let us consider 3 cases:

i) $l(\mathrm{lcm}_(a,b)) = 1$. This means $l(\mathrm{lcm}_r(a,b)) = a = b$. By adding this relation to the defining relation of the group $G_{\mathrm{H_{ii}}}$, we get a contradiction $G_{\mathrm{H_{ii}}} \simeq \mathbb{Z}$.

ii) $l(\mathrm{lcm}_r(a,b)) = 2$. This means that there exists $u, v \in \{a,b,c\}$ such that $l(\mathrm{lcm}_r(a,b)) = au = bv$. Depending on each choice of $u$ and $v$, one can show that this assumption leads to a contradictory conclusion $G_{\mathrm{H_{ii}}} \simeq \mathbb{Z}$. Details are left to the reader.

iii) $l(\mathrm{lcm}_r(a,b)) = 3$. In view of the first two defining relations, one has $\mathrm{lcm}_r(a,b) = aca = bac$, and it divides $abab = baba$ (from left). This means that there exists $d \in \{a,b,c\}$ such that $cd = ba$ in $G_{\mathrm{H_{ii}}}$. For each case $d = a, b$ or $c$ separately, one can show that $G_{\mathrm{H_{ii}}} \simeq \mathbb{Z}$. A contradiction!.

4) $\pi G^+_{\mathrm{H_{iii}}}$: Due to the isomorphism $\pi G_{\mathrm{B_{vi}}} \simeq \pi G^+_{\mathrm{H_{iii}}}$ (Corollary 1,iii) of Theorem 1), we can reduce this case to the case 2).

These complete the proof of Theorem 2. $\square$

**Corollary 5.2.** *The monoids $\pi G^+_{\mathrm{B_{ii}}}, \pi G^+_{\mathrm{B_{vi}}}, \pi G^+_{\mathrm{H_{ii}}}, \pi G^+_{\mathrm{H_{iii}}}$ are not Gaussian and hence are niether Gaussian nor Garside (a monoid is Gaussian ([D-P, §2]) if it is atomic, satisfies the cancellation condition and admits divisibility theory).*

# 6 Fundamental elements of the monoid $G^+_X$

An Artin monoid of finite type has a particular element, denoted by $\Delta$ and called the *fundamental element* ([B-S] §6). In this section, we generalize the concept for positive homogeneously presented monoids.

In view of Theorem 2, we do not naively employ the original definition: the left and right least common multiple of the generators. Instead of that, analyzing equivalent defining properties of the fundamental element for Artin monoid case, we consider two classes of elements in the monoid $G^+$: quasi-central elements and fundamental elements, forming subsemigroups $\mathcal{QZ}(G^+)$ and $\mathcal{F}(G^+)$ in $G^+$, respectively, with $\mathcal{F}(G^+) \subset \mathcal{QZ}(G^+)$. The goal of the present section is to show $\mathcal{F}(G^+_X) \neq \emptyset$ for all types $X$, implying also $\mathcal{F}(\pi G^+_X) \neq \emptyset$ for all types $X$.

Let $G^+$ be a monoid given in §4, i.e. defined by a positive homogeneous relations on a generator set $L$. Let us denote by $L/\sim$ the quotient set of $L$

divided by the equivalence relation generated by the equalities between two letters (in the relation set $R$). An element $\Delta \in G^+$ is called *quasi-central* ([B-S] 7.1), if there exists a permutation $\sigma_\Delta$ of $L/\!\sim$ such that

$$a \cdot \Delta \simeq \Delta \cdot \sigma_\Delta(a)$$

holds for all generators $a \in L/\sim$. The set of all quasi-central elements is denoted by $\mathcal{QZ}(G^+)$. The following is an immediate consequence of the definition.

**Fact 2.** *The $\mathcal{QZ}(G^+)$ is closed under the product. For two elements $\Delta_1, \Delta_2 \in \mathcal{QZ}(G^+)$, we have $\sigma_{\Delta_1\cdot\Delta_2} = \sigma_{\Delta_2} \cdot \sigma_{\Delta_1}$.*

According to Fact 2, we introduce an anti-homomorphism:

$$\sigma \ : \ \ \mathcal{QZ}(G^+) \ \longrightarrow \ \mathfrak{S}(L/\!\sim), \quad \Delta \mapsto \sigma_\Delta.$$

The kernel of $\sigma$ is the center $\mathcal{Z}(G^+)$ of the monoid $G^+$.

Next, we introduce the concept of a fundamental element.

**Definition 6.1.** An element $\Delta \in G^+$ is called *fundamental* if there exists a permutation $\sigma_\Delta$ of $L/\sim$ such that, for any $a \in L/\sim$, there exists $\Delta_a \in \pi G_X^+$ satisfying the following relation:

$$\Delta \ \simeq \ a \cdot \Delta_a \ \simeq \ \Delta_a \cdot \sigma_\Delta(a).$$

We denote by $\mathcal{F}(G^+)$ the set of all fundamental elements of $G^+$. Note that $1 \in \mathcal{QZ}(G^+)$ but $1 \notin \mathcal{F}(G^+)$

**Fact 3.** *The $\mathcal{F}(G^+)$ has the following two properties.*

i) *A fundamental element is a quasi-central element: $\mathcal{F}(G^+) \subset \mathcal{QZ}(G^+)$. The associated permutation of $L/\sim$ as a fundamental element coincides with that as a quasi-central element.*

ii) *Products $\Delta\cdot\Delta'$ and $\Delta'\cdot\Delta$ of a fundamental element $\Delta$ and a quasi-central element $\Delta'$ are again fundamental elements whose permutation of $L/\sim$ is given in Fact 2. We have $(\Delta\Delta')_a = \Delta_a\Delta'$, and $(\Delta'\Delta)_a = \Delta'\Delta_{\sigma_{\Delta'}(a)}$.*

$$\mathcal{F}(G^+)\mathcal{QZ}(G^+) = \mathcal{QZ}(G^+)\mathcal{F}(G^+) = \mathcal{F}(G^+).$$

*Proof.* i) We have $a \cdot \Delta \simeq a{\cdot}\Delta_a{\cdot}\sigma_\Delta(a) \simeq \Delta{\cdot}\sigma_\Delta(a)$ for all $a{\in}L/\!\sim$.

ii) We prove only the case $\Delta \cdot \Delta'$.

On one side, one has:

$$\Delta \cdot \Delta' \simeq (a \cdot \Delta_a) \cdot \Delta' \simeq a \cdot (\Delta_a \cdot \Delta').$$

On the other side, one has:

$$\begin{aligned} \Delta \cdot \Delta' &\simeq (\Delta_a \cdot \sigma_\Delta(a)) \cdot \Delta' \simeq \Delta_a \cdot (\sigma_\Delta(a) \cdot \Delta') \simeq \Delta_a \cdot (\Delta' \cdot \sigma_{\Delta'}(\sigma_\Delta(a))) \\ &\simeq (\Delta_a \cdot \Delta') \cdot \sigma_{\Delta'}(\sigma_\Delta(a)) \simeq (\Delta_a \cdot \Delta') \cdot \sigma_{\Delta\Delta'}(a)). \end{aligned}$$ □

One basic property of a fundamental element is that it can be a universal denominator for the localization homomorphism (c.f. §7 Lemma 7.2.2).

**Fact 4.** *Let $\Delta$ be a fundamental element of $G^+$. Then, for any $U \in G^+$, $U$ divides $\Delta^{l(U)}$ from left and from right.*

*Proof.* We prove only for the left division. Right division can be shown similarly. We show the statement by induction on $l(U)$, where the case $l(U)=1$ follows from the definition of a fundamental element. Let $l(U)>1$ and $U \simeq U' \cdot a$. By induction hypothesis, we have $\Delta^{l(U)-1} \simeq U' \cdot V$ for some $V$. Then, multiplying $\Delta$ from right, we have $\Delta^{l(U)} \simeq U' \cdot V \cdot \Delta \simeq U' \cdot \Delta \cdot \sigma_\Delta(V) \simeq U' \cdot a \cdot \Delta_a \cdot \sigma_\Delta(V)$. Here, if $V$ is the word $v_1 \cdots v_n$ then $\sigma_\Delta(V)$ is a word $\sigma_\Delta(v_1) \cdots \sigma_\Delta(v_n)$ $\square$

**Remark 6.2.** If $G^+$ is an indecomposable Artin monoid (of finite type), then any non-trivial quasi-central element is fundamental ([B-S] 5.2 and 7.1). That is, one has the "opposite" inclusion: $(\mathcal{QZ}(G^+) \setminus \{1\}) \subset \mathcal{F}(G^+)$.

**Remark 6.3.** By definition, any fundamental element is divisible from both left and right by all generators in $L$. However, a (non-trivial) quasi-central element in general may not have this property.

(i) $b^3 \in \mathcal{QZ}(G^+_{\mathrm{B_{ii}}})$ is central. However, it is not divisible by $a$ and $c$ from the left and right.

(ii) $ababa \in G^+_{\mathrm{B_{ii}}}$ is divisible by all generators from both sides, but it does not belong to $\mathcal{QZ}(G^+_{\mathrm{B_{ii}}})$.

**Definition 6.4** A fundamental element $\Delta$ is called a *minimal fundamental element* if any fundamental element dividing $\Delta$ from right or left coincides with $\Delta$ itself.

**Remark 6.5** A fundamental element is called *prime*, if it does not decompose into a product of two nontrivial quasi-central elements. In general, a minimal fundamental element may not be prime (see [I2]).

We state the second main result of the present paper.

**Theorem 3.** *The following elements are minimal fundamental elements in $G^+_{\mathrm{X}}$ for any type $X$. Except for the types $\mathrm{B_{vi}}$, $\mathrm{H_{ii}}$ and $\mathrm{H_{iii}}$, they are the complete list of minimal fundamental elements.*

| | | | | |
|---|---|---|---|---|
| $\mathrm{A_i}$ : | $\Delta_{\mathrm{A_i}}$ | $:=$ | $(cba)^2$ | $\sigma : \binom{a,\ b,\ c}{c,\ b,\ a}$ |
| $\mathrm{A_{ii}}$ : | $\Delta_{\mathrm{A_{ii}}}$ | $:=$ | $aba$ | $\sigma : \binom{a,\ b{=}c}{b{=}c,\ a}$ |
| $\mathrm{B_i}$ : | $\Delta_{\mathrm{B_i}}$ | $:=$ | $(cba)^3$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| $\mathrm{B_{ii}}$ : | $\Delta_{\mathrm{B_{ii}},k}$ | $:=$ | $(a^kb)^3 \quad (k \geq 1)$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| $\mathrm{B_{iii}}$ : | $\Delta_{\mathrm{B_{iii}}}$ | $:=$ | $ac$ | $\sigma : \binom{a{=}b,\ c}{a{=}b,\ c}$ |
| $\mathrm{B_{iv}}$ : | $\Delta_{\mathrm{B_{iv}}}$ | $:=$ | $abcb$ | $\sigma : \binom{a,\ b,\ c}{a,\ c,\ b}$ |
| $\mathrm{B_v}$ : | $\Delta_{\mathrm{B_v}}$ | $:=$ | $a$ | $\sigma : \binom{a{=}b{=}c}{a{=}b{=}c}$ |
| $\mathrm{B_{vi}}$ : | $\Delta_{\mathrm{B_{vi}}1}$ | $:=$ | $a^5 \simeq b^5 \simeq c^5$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{B_{vi}}2}$ | $:=$ | $(aba)^2$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{B_{vi}}3}$ | $:=$ | $bccabcb$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{B_{vi}}4}$ | $:=$ | $(bbac)^2$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{B_{vi}}5}$ | $:=$ | $(acaca)^2$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{B_{vi}}6}$ | $:=$ | $(cba)^3$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |

| | | | |
|---|---|---|---|
| $\mathrm{B_{vii}}$ : | $\Delta_{\mathrm{B_{vii}}}$ | $:= \ a$ | $\sigma : \binom{a=b=c}{a=b=c}$ |
| $\mathrm{H_{i}}$ : | $\Delta_{\mathrm{H_{i}}}$ | $:= \ (cba)^5$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| $\mathrm{H_{ii}}$ : | $\Delta_{\mathrm{H_{ii}}1}$ | $:= \ (acaca)^2 \simeq (ac)^5$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{H_{ii}}2}$ | $:= \ (babac)^3 \simeq (cba)^5$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| $\mathrm{H_{iii}}$ : | $\Delta_{\mathrm{H_{iii}}1}$ | $:= a^5 \simeq b^5 \simeq c^5$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{H_{iii}}2}$ | $:= (aba)^2$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{H_{iii}}3}$ | $:= accbaca$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{H_{iii}}4}$ | $:= \ (bcba)^2$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{H_{iii}}5}$ | $:= \ (bcbcb)^2 \simeq (bc)^5$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| | $\Delta_{\mathrm{H_{iii}}6}$ | $:= (abc)^3$ | $\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$ |
| $\mathrm{H_{iv}}$ : | $\Delta_{\mathrm{H_{iv}}}$ | $:= \ a$ | $\sigma : \binom{a=b=c}{a=b=c}$ |
| $\mathrm{H_{v}}$ : | $\Delta_{\mathrm{H_{v}}}$ | $:= \ a$ | $\sigma : \binom{a=b=c}{a=b=c}$ |
| $\mathrm{H_{vi}}$ : | $\Delta_{\mathrm{H_{vi}}}$ | $:= \ a$ | $\sigma : \binom{a=b=c}{a=b=c}$ |
| $\mathrm{H_{vii}}$ : | $\Delta_{\mathrm{H_{vii}}}$ | $:= \ a$ | $\sigma : \binom{a=b=c}{a=b=c}$ |
| $\mathrm{H_{viii}}$ : | $\Delta_{\mathrm{H_{viii}}}$ | $:= \ a$ | $\sigma : \binom{a=b=c}{a=b=c}$ |

*Proof.* Since the cases for an Artin monoid or a free abelian monoid are classical, we show only the 4 exceptional cases.

$\mathrm{B_{ii}}$ : For the proof of this case , it is sufficient to show that $\Delta_{\mathrm{B_{ii}},k}$ are quasi central elements which are divisible by the generators $a, b$ and $c$ (see Proposition 7.4). Actually, it is easy to show the following:

$$(a^k b)^3 \simeq (ba^k)^3 \simeq (bc^k)^3 \simeq (c^k b)^3.$$

For the proof of the facts that they are quasi-central and they form the complete list of minimal fundamental elements, one is refered to [I2].

$\mathrm{B_{vi}}$ : Since the monoids of types $\mathrm{B_{vi}}$ and $\mathrm{H_{iii}}$ are isomorphic to each other (see Remark after Theorem 1 in §4), we may reduce the proof to the case $\mathrm{H_{iii}}$.

$\mathrm{H_{ii}}$ : First, let us show a relation: $acaca \simeq cacac$ ($acaca \simeq acbac \simeq cacac$), which shall be used in the sequel.

$\Delta_{\mathrm{H_{ii}}1} := acacaacaca.$
$\Delta_{\mathrm{H_{ii}}1} = a(cacaacaca) \simeq cacacacaca \simeq (cacaacaca)a.$
$\Delta_{\mathrm{H_{ii}}1} \simeq c(acacacaca) \simeq acacaacaca \simeq (acacacaca)c.$

$\Delta_{\mathrm{H_{ii}}1} \simeq acacaacaca \simeq b(accaacaca) \simeq acacaacaca \simeq acacacacac$
$\quad \simeq accacaccac \simeq accaacbcac \simeq accaacbacb \simeq (accaacaca)b,$
$\Delta_{\mathrm{H_{ii}}2} := babacbabacbabac \simeq ababcbabacbabac.$

$\Delta_{\mathrm{H_{ii}}2} = a(babcbabacbabac) \simeq bababcbabacbabac \simeq babcacabacbabac$
$\quad \simeq babcbacbacbabac \simeq babcbacacababac \simeq babcbaacbababac$
$\quad \simeq babcbaacababbac \simeq babcbabacbabbac \simeq (babcbabacbabac)a.$
$\Delta_{\mathrm{H_{ii}}2} = b(abacbabacbabac) \simeq ababcbabacbabac \simeq ababcababcbabac$
$\quad \simeq ababcababcbaaca \simeq ababcbabacbaaca \simeq ababacbaacabaaca$
$\quad \simeq ababcbaacababac \simeq ababcbaacbabaac \simeq ababcbacacabaac$

$\simeq ababcacaacabaac \simeq ababacbaacabaac \simeq abaacabaacabaac$
$\simeq abaacababacbaac \simeq abaacbabaacbaac \simeq abacacabaacbaac$
$\simeq abacbacbaacbaac \simeq abacbacbacacaac \simeq abacbacacaacaac$
$\simeq abacbaacbaacac \simeq abacbaacbabacac \simeq abacbaacababcac$
$\simeq abacbabacbabcac \simeq (abacbabacbabac)b.$

$\Delta_{\mathrm{H_{ii}}2} = abacbabacbabacb \simeq aacababacbabacb \simeq aacbabaacbabacb$
$\simeq acacabaacbabacb \simeq acbacbaacbabacb \simeq c(acacbaacbabacb)$
$\simeq acbacbaacbabacb \simeq acacabaacababcb \simeq acacababacbabcb$
$\simeq acacbabaacbabcb \simeq acacbabacacabcb \simeq acacbaacaacabcb$
$\simeq acacbaacabacbcb \simeq acacbaacababcbc \simeq (acacbaacbabacb)c.$

$\mathrm{H_{iii}}$ :

$\Delta_{\mathrm{H_{iii}}2} := (aba)^2.$
$\Delta_{\mathrm{H_{iii}}2} = a(baaba) \simeq bababa \simeq (baaba)a.$
$\Delta_{\mathrm{H_{iii}}2} = b(ababa) \simeq abaaba \simeq (ababa)b.$
$\Delta_{\mathrm{H_{iii}}2} = abaaba \simeq aaacba \simeq aacbaa \simeq aacaac$

$\Delta_{\mathrm{H_{iii}}3} := accbaca.$
$\Delta_{\mathrm{H_{iii}}3} = a(ccbaca) \simeq cbcaaca \simeq ccbcaca \simeq (ccbaca)a.$
$\Delta_{\mathrm{H_{iii}}3} = accbaca \simeq cbcaaca \simeq b(caaaca).$
$\Delta_{\mathrm{H_{iii}}3} = accbaca \simeq cbcaaca \simeq ccbcaca \simeq caccbca \simeq cacbcaa$
$\simeq caaccba \simeq caacacb \simeq (caaaca)b.$
$\Delta_{\mathrm{H_{iii}}3} = accbaca \simeq c(bcaaca).$
$\Delta_{\mathrm{H_{iii}}3} = accbaca \simeq cbcaaca \simeq bcaaaca \simeq (bcaaca)c.$

$\Delta_{\mathrm{H_{iii}}4} := bcbabcba.$
$\Delta_{\mathrm{H_{iii}}4} \simeq a(bcabcba) \simeq bcabacba \simeq (bcabcba)a.$

$\Delta_{\mathrm{H_{iii}}4} = b(cbabcba) \simeq bcabacba \simeq cbcbacba \simeq cbacbcba$
$\simeq cbabcaba \simeq (cbabcba)b.$

$\Delta_{\mathrm{H_{iii}}4} \simeq bcabacba \simeq c(bcbacba) \simeq bcbaabca \simeq bcbaacbc \simeq bcbacbac.$

$\Delta_{\mathrm{H_{iii}}5} := bcbcbbcbcb.$
$\Delta_{\mathrm{H_{iii}}5} \simeq abccbbcbcb \simeq abccbcbcbc \simeq a(bcbcbcbbc),$
$\Delta_{\mathrm{H_{iii}}5} \simeq bcbcbcbcbc \simeq (bcbcbcbbc)a.$

$\Delta_{\mathrm{H_{iii}}5} = b(cbcbbcbcb) \simeq cbcbcbcbcb \simeq (cbcbbcbcb)b.$

$\Delta_{\mathrm{H_{iii}}5} \simeq c(bcbcbcbcb) \simeq (bcbcbcbcb)c.$

$\Delta_{\mathrm{H_{iii}}6} := (abc)^3$
$\Delta_{\mathrm{H_{iii}}6} = a(bcabcabc) \simeq bcbabcab \simeq bcabacabc \simeq bcabcacbc \simeq (bcabcabc)a$
$\Delta_{\mathrm{H_{iii}}6} \simeq (abcabcab)c \simeq abcabcbcb \simeq abcabbcab \simeq acbcbbcab$
$\simeq acabcbcab \simeq cacbcbcab \simeq c(abcabcab).$

These complete the proof of Theorem 3. $\square$

Let us state some observations related to the fundamental elements.

Let $G^+$ be a monoid defined by positive homogeneous relations. Recall (§4 Definition) that $\pi G^+$ is the image of $G^+$ in the group $G$ by the localization homomorphism $\pi$. We define quasi-central elements and fundamental elements of $\pi G^+$ exactly by the same defining relations for $G^+$. Let us denote by $\mathcal{QZ}(\pi G^+)$ and $\mathcal{F}(\pi G^+)$ the set of quasi-central elements and fundamental elements in $\pi G^+$, respectively. Then, the localization homomorphism induces homomorphisms: $\mathcal{QZ}(G^+) \to \mathcal{QZ}(\pi G^+)$ and $\mathcal{F}(G^+) \to \mathcal{F}(\pi G^+)$, which may be neither injective nor surjective. However, Theorem 3 implies the following fact.

**Corollary 6.4** *For any type $X$, the set of fundamental elements $\mathcal{F}(\pi G_X^+)$ is non-empty.*

We note that $\mathcal{F}(\pi G_X^+)$ may not be singly generated. Evenmore, it is infinitely generated for the type $\mathrm{B_{ii}}$ (see details [I2]).

Next, we state an observation that a power of the Coxeter element yields a fundamental element.

**Proposition 6.5** *Except for the types when the monoid decomposes into direct products or when we have a nontrivial relation $\sim$ on $L$ (explicitly, except for types $\mathrm{A_{ii}}, \mathrm{B_{iii}}, \mathrm{B_{iv}}, \mathrm{B_v}, \mathrm{B_{vii}}, \mathrm{H_{iv}}, \mathrm{H_v}, \mathrm{H_{vi}}, \mathrm{H_{vii}}, \mathrm{H_{viii}}$), $\deg(z)$-th power of the Coxeter element $C := cba$ (= a homotopy class which turns once around all the three points $C_X \cap l_{*1,\mathbb{C}}$ counterclockwise) is a fundamental element.*

*Proof.* Except for the type $\mathrm{H_{iii}}$, the statement is true due to Theorem 3. In the case of type $\mathrm{H_{iii}}$, we have:

$$(cba)^5 \simeq \Delta_{\mathrm{H_{iii}}1}\Delta_{\mathrm{H_{iii}}5} \simeq \Delta_{\mathrm{H_{iii}}2}\Delta_{\mathrm{H_{iii}}6} \simeq \Delta_{\mathrm{H_{iii}}3}\Delta_{\mathrm{H_{iii}}4}.$$

□

Let us give further exmaples of local fundamental groups, where the Coxeter element plays a similar role as in the 17 cases treated in the present paper. In order to state the result, we introduce a property:

(P): The local fundamental group of the complement of a logarithmic-free *indecomposable*[4] local divisor admits a positive homogeneous presentation by a suitable choice of Zariski-van Kampen generators such that a power of the Coxeter element, defined as a suitable product of the generators whose realizing path has no self-intersecting point, gives a fundamental element of the monoid generated by them in the fundamental group.

1. The discriminant of a finite irreducible reflection group satisfies the property (P) ([B-S, S2, S3]).
2. The discriminant of a finite irreducible well-generated complex reflection group ([B-M-R, Be]) satisfies the property (P) if their generators are identified with certain Zariski-van Kampen generators.
3. The zero-loci of Sekiguchi polynomials define divisors satisfying (P) (Theorems 1. and 3. of the present paper).
4. A plane curve is locally logarithmic free ([S1]), and, conjecturally, satisfies (P) (c.f. [K]).
5. The discriminant of elliptic Weyl group is a free divisor ([S4]II), which satisfies, conjecturally, the property (P), where the hyperbolic Coxeter element in the elliptic Weyl group ([S4]I,III) can be lifted in the fundamental group to an element whose power of order $m_\Gamma$ is a fundamental element.

**Question.** We ask whether the property (P) holds for any indecomposable logarithmic free local divisor or not.

[4] A local divisor $D$ in $(\mathbb{C}^n, O)$ at the origin is called *decomposable* if there exist local divisors $D_i$ in $(\mathbb{C}^{n_i}, O)$ $(i = 1, 2)$ and a local analytic isomorphism $(\mathbb{C}^n, O) \simeq (\mathbb{C}^{n_1}, O)\times(\mathbb{C}^{n_2}, O)$ which induces a local isomorphism $D \simeq (D_1 \times \mathbb{C}^{n_2}) \cup (\mathbb{C}^{n_1} \times D_2)$. A local divisor $D$ in $(\mathbb{C}^n, O)$ is called *indecomposable* if it is not decomposable.

# 7 Cancellation conditions on the monoid $G_X^+$

In the present section, we study the *cancellation condition* on a monoid $G^+$. In the first half, we show some general consequences on the monoid $G^+$ under the cancellation condition, or under its weaker version: a *weak cancellation condition*. In the latter half, we prove that the monoid $G^+_{\mathrm{B_{ii}}}$ satisfies the cancellation condition, however, we do not know whether the monoids $G^+_{\mathrm{B_{vi}}}$, $G^+_{\mathrm{H_{ii}}}$ and $G^+_{\mathrm{H_{iii}}}$ satisfy it or not.

**Definition 7.1.** A monoid $G^+$ is said to *satisfy the cancellation condition*, if an equality $AXB=AYB$ for $A,B,X,Y\in M$ implies $X=Y$.

It is well-known that an Artin monoid satisfies the cancellation condition [B-S, Prop.2.3]. Let us state some important consequences of the cancellation condition on a monoid defined by positive homogeneous relations.

**Lemma 7.2.** *Let $G^+$ be a monoid defined by positive homogeneous relations. Suppose it satisfies the cancellation condition. Then, we have the following.*
**1.** *For any $\Delta \in \mathcal{QZ}(G^+)$, the associated permutation $\sigma_\Delta$ of $L/\sim$ extends to an isomorphism, denoted also $\sigma_\Delta$, of $G^+$. The correspondence: $\Delta \mapsto \sigma_\Delta$ induces an anti-homomorphism:*

$$\mathcal{QZ}(G^+) \longrightarrow Aut(G^+).$$

**2.** *If $\mathcal{F}(G^+) \neq \emptyset$, then the localization homomorphism $\pi$ is injective.*
**3.** *For any element $A\in G$ and any $\Delta\in\mathcal{F}(G^+)$, there exist $B\in\pi G^+$ and $n\in\mathbb{Z}_{\geq 0}$ such that, in $G$, one has equalities:*

$$A = B\cdot(\Delta)^{-n} = (\Delta^{-n})\cdot\sigma_\Delta^{-n}(B).$$

*Proof.* **1.** First, we note that the permutation $\sigma_\Delta$ induces an isomorphism of the free monoid $(L/\sim)^*$, denoted also $\sigma_\Delta$. Let $U$ and $V$ be words in $(L/\sim)^*$ which are equivalent by the relations $R$ (i.e. give the same element in $G^+$). Then, by definition, $U\Delta \simeq \Delta\sigma_\Delta(U)$ and $V\Delta \simeq \Delta\sigma_\Delta(V)$ are equivalent. That is, $\Delta\sigma_\Delta(U)$ and $\Delta\sigma_\Delta(V)$ give the same element in $G^+$. Then, cancelling $\Delta$ from the left, we see that $\sigma_\Delta(U)$ and $\sigma_\Delta(V)$ give the same element in $G^+$. Thus $\sigma_\Delta$ induces a homomorphism from $G^+$ to $G^+$. The homomorphism is invertible, since a finite power of it is an identity. By definition, for any $U\in G^+$ and $\Delta_1,\Delta_2 \in \mathcal{QZ}(G^+)$, one has:

$$U\cdot\Delta_1\Delta_2 \ \simeq\ \Delta_1\cdot\sigma_{\Delta_1}(U)\cdot\Delta_2 \simeq \Delta_1\Delta_2\cdot\sigma_{\Delta_2}(\sigma_{\Delta_1}(U)).$$

**2.** For a localization homomorphism to be injective, it is sufficient to show that the monoid satisfies the cancellation condition and that any two elements of the monoid have (at least) one (left and right) common multiple (Öre's condition, see [C-P]). In view of Fact 4. in §6, for any two elements $U,V\in G^+$ and $\Delta\in\mathcal{F}(G^+)$, $\Delta^{\max\{l(U),l(V)\}}$ is a common multiple of $U$ and $V$ from both sides.

**3.** Owing to the previous **2.**, it is sufficient to show that, for any element $A\in G$ and any $\Delta\in\mathcal{F}(G^+)$, there exists $k\in\mathbb{Z}_{\geq 0}$ such that $\Delta^k\cdot A\in\pi G^+$. This can be easily shown by an induction on $k(A)\in\mathbb{Z}_{\geq 0}$ where $k(A)$ is the (minimal) number of letters of negative power in a word expression of $A$ in $(L\cup L^{-1})^*$. Details are left to the reader. $\square$

Next, we formulate a weak cancellation condition and its consequences.

**Definition 7.2.** An element $\Delta \in G^+$ is called left (resp. right) *weakly cancellative*, if an equality $\Delta = U \cdot V = U \cdot W$ (resp. $\Delta = V \cdot U = W \cdot U$) in $G^+$ for some $U, V, W \in G^+$, implies $V = W$ in $G^+$.

Using the concept of weakly cancellativity, we give a proposition characterizing fundamental elements.

**Proposition 7.4.** *Let $G^+$ be a monoid defined by positive homogeneous relations. A quasi-central element $\Delta$ is a fundamental element if, for any $s \in L$, $s\Delta$ is left weakly cancellative and $s$ divides $\Delta$ from the left.*

*Proof.* Since $\Delta$ is divisible by any $s \in L/\sim$ from the left, we put $\Delta = s\Delta_s$ for a suitable $\Delta_s$. Multiply, $\sigma_\Delta(s)$ from the right so that we obtain $\Delta\sigma_\Delta(s) = s\Delta_s\sigma_\Delta(s)$, where the left hand side is equal to $s\Delta = ss\Delta_s$. Therefore, using the weakly cancellativity of $s\Delta$, dividing by $s$ from the left, we obtain $s\Delta_s = \Delta_s\sigma(s)$. This implies the statemnt. □

*Notation.* For an element $\Delta \in G^+$, we put
$Div_l(\Delta) := \{U \in G^+ : U |_l \Delta\}$ and $Div_r(\Delta) := \{U \in G^+ : U |_r \Delta\}$.

**Proposition 7.5.** *Let a fundamental element $\Delta \in \mathcal{F}(G^+)$ be left weakly cancellative. Then the following* i), ii), iii) *and* iv) *hold.*

i) *For any element $U \in Div_l(\Delta)$, let $\tilde{U} \in (L/\sim)^*$ be a lifting to a word. Then, the class of $\sigma_\Delta(\tilde{U})$ in $G^+$ depends only on the class $U$ but not on the lifting $\tilde{U}$. Let us denote the class in $G^+$ by $\sigma_\Delta(U)$.*

ii) *The divisor set $Div_l(\Delta)$ is invariant under the action of $\sigma_\Delta$. In particular, the unique longest element $\Delta$ is fixed by $\sigma_\Delta$.*

iii) *The fundamental element $\Delta$ is right weakly cancellative.*

iv) *We have the equality:* $Div_l(\Delta) = Div_r(\Delta)$.

*Proof.* i) Suppose one has a decomposition $\Delta \simeq U \cdot V$ for $U, V \in G^+$, and let $\tilde{U}$ be a lifting of $U$ into a word in $(L/\sim)^*$. Then, $\sigma_\Delta(\tilde{U})$ is well-defined as a word and hence induce an element in $G^+$, which we denote by the same $\sigma_\Delta(\tilde{U})$. We claim that $\Delta$ is equivalent to $V \cdot \sigma_\Delta(\tilde{U})$. This is shown by induction on $l(U)$. If $l(U) = 1$, this is the definition of fundamental elements. Let $l(U) > 1$, $\tilde{U} = \tilde{U}' \cdot a$ and $\Delta \simeq \tilde{U}' \cdot a \cdot V$. By induction hypothesis, we have $\Delta \simeq a \cdot V \cdot \sigma_\Delta(\tilde{U}')$. Due to the weak cancellativity, $V \cdot \sigma_\Delta(\tilde{U}')$ is equivalent to $\Delta_a$. Then, by definition of fundamental elements, $\Delta$ is equivalent to $V \cdot \sigma_\Delta(\tilde{U}') \cdot \sigma_\Delta(a) \simeq V \cdot \sigma_\Delta(\tilde{U})$.

Let $\tilde{U}_1$ and $\tilde{U}_2$ be liftings of $U$. Then, applying the above result, we see that $\Delta$ is equal to $V \cdot \sigma_\Delta(\tilde{U}_1)$ and $V \cdot \sigma_\Delta(\tilde{U}_2)$. Then, applying the weak cancellativity of $\Delta$, we see that $\sigma_\Delta(\tilde{U}_1)$ and $\sigma_\Delta(\tilde{U}_2)$ define the same element in $G^+$, which we shall denote by $\sigma_\Delta(U)$.

ii) In the proof of i), taking $U = \Delta$ and $V = 1$, we obtain $\Delta = \sigma_\Delta(\Delta)$. Then, since $\sigma_\Delta$ is of finite order, we obtain $\sigma_\Delta(Div_l(\Delta)) = Div_l(\sigma_\Delta(\Delta)) = Div_l(\Delta)$.

iii) Suppose $\Delta = V \cdot U = W \cdot U$. Then according to i), we have $\Delta = U \cdot \sigma_\Delta(V) = U \cdot \sigma_\Delta(W)$. Then the left cancellation condition implies $\sigma_\Delta(V) =$

$\sigma_\Delta(W)$. On the other hand, according to ii), $\sigma_\Delta(V) = \sigma_\Delta(W)$ are again elements of $Div_l(\Delta)$ so that we can apply $\sigma_\Delta$ to the equality. Since $\sigma_\Delta$ is of finite order, after repeating this several times, we obtain the equality $V = W$.

iv) $\Delta$ is left divisible by $U$ if and only if $\Delta$ is right divisible by $\sigma_\Delta(U)$. That is, the set $Div_r(\Delta)$ of right divisors of $\Delta$ is equal to $\sigma_\Delta(Div_l(\Delta)) = Div_l(\Delta)$. $\square$

**Conjecture.** Let $C^k$ of the element in §6 Proposition 6.5. If $C^{k\cdot\mathrm{ord}(\sigma_{C^k})}$ is weakly cancellative, then $G^+$ satisfies the cancellation condition.

The following theorem shows that we have already enough relations for type $\mathrm{B_{ii}}$.

**Theorem 4.** *The monoid $G^+_{\mathrm{B_{ii}}}$ satisfies the cancellation condition.*

*Proof.* We, first, remark the following.

**Proposition 7.6.** *The left cancellation condition on $\mathrm{G}^+_{\mathrm{B_{ii}}}$ implies the right cancellation condition.*

*Proof.* Consider a map $\varphi : \mathrm{G}^+_{\mathrm{B_{ii}}} \to \mathrm{G}^+_{\mathrm{B_{ii}}}$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where $\sigma$ is a permutation $\binom{a\ b\ c}{c\ b\ a}$ and $rev(W)$ is the reverse of the word $W = x_1x_2\cdots x_t$ ($x_i$ is a letter or an inverse of a letter) given by the word $x_t\cdots x_2x_1$. In view of the defining relation of $\mathrm{G}^+_{\mathrm{B_{ii}}}$ in Theorem 1., $\varphi$ is well defined and is an anti-isomorphism. If $\beta\alpha \simeq \gamma\alpha$, then $\varphi(\beta\alpha) \simeq \varphi(\gamma\alpha)$, i.e., $\varphi(\alpha)\varphi(\beta) \simeq \varphi(\alpha)\varphi(\gamma)$. Using left cancellation condition, we obtain $\varphi(\beta) = \varphi(\gamma)$ and, hence, $\beta \simeq \gamma$. $\square$

The following is sufficient to show the left cancellation condition on $\mathrm{G}^+_{\mathrm{B_{ii}}}$.

**Proposition 7.7.** *Let $X$ and $Y$ be positive words in $\mathrm{G}^+_{\mathrm{B_{ii}}}$ of length $r \in \mathbb{Z}_{\geq 0}$.*

(i) *If $uX \simeq uY$ for some $u \in \{a,b,c\}$, then $X \simeq Y$.*
(ii) *If $aX \simeq bY$, then $X \simeq bZ$, $Y \simeq cZ$ for some positive word $Z$.*
(iii) *If $aX \simeq cY$, then $X \simeq cZ$, $Y \simeq aZ$ for some positive word $Z$.*
(iv) *If $bX \simeq cY$, then there exist an integer $k$ $(0 \leq k < r-1)$ and a word $Z$ such that $X \simeq c^kbaZ$ and $Y \simeq a^kbbZ$.*

*Proof.* Let us denote by $H(r,t)$ the statement in Proposition 7.7 for all pairs of words $X$ and $Y$ such that their word-lengths are $r$ and for all $u, v \in \{a,b,c\}$ such that $uX \simeq vY$ and the number of elementary transformations to bring $uX$ to $vY$ is less or equal than $t$. It is easy to see that $H(r,t)$ is true if $r \leq 1$ or $t \leq 1$.

For $r, t \in \mathbb{Z}_{>1}$, we prove $H(r,t)$ under the induction hypothesis that $H(s,u)$ holds for $(s,u)$ such that either $s < r$ and arbitrary $u$ or $s = r$ and $u < t$.

Let $X, Y$ be of word-length $r$, and let $u_1X \simeq u_2W_2 \simeq \cdots \simeq u_tW_t \simeq u_{t+1}Y$ be a sequence of elementary transformations of $t$ steps, where $u_1, \cdots, u_{t+1} \in \{a,b,c\}$ and $W_2, \cdots, W_t$ are positive words of length $r$. By assumption $t > 1$, there exists an index $i \in \{2, ..., t\}$ so that we decompose the sequence into two steps $u_1X \simeq u_iW_i \simeq u_{t+1}Y$, where each step satisfies the induction hypothesis.

If there exists $i$ such that $u_i$ is equal either to $u_1$ or $u_{t+1}$, then by induction hypothesis, $W_i$ is equivalent either to $X$ or to $Y$. Then, again, applying the induction hypothesis to the remaining step, we obtain the statement for the $u_1X \simeq u_{t+1}Y$. Thus, we assume from now on $u_i \neq u_1, u_{t+1}$ for $1 < i \leq t$.

Suppose $u_1 = u_{t+1}$. If there exists $i$ such that $\{u_1 = u_{t+1}, u_i\} \neq \{b, c\}$, then each of the equivalence says the existence of $\alpha, \beta \in \{a, b, c\}$ and words $Z_1, Z_2$ such that $X \simeq \alpha Z_1$, $W_i \simeq \beta Z_1 \simeq \beta Z_2$ and $Y \simeq \alpha Z_2$. Applying the induction hypothesis for $r$ to $\beta Z_1 \simeq \beta Z_2$, we get $Z_1 \simeq Z_2$ and, hence, we obtained the statement $X \simeq \alpha Z_1 \simeq \alpha Z_2 \simeq Y$. Thus, we exclude these cases from our considerations. Next, we consider the case $\{u_1 = u_{t+1}, u_i\} = \{b, c\}$. However, due to the above consideration, we have only the case $u_2 = u_3 = \cdots = u_t$. Then, by induction hypothesis, we have $W_2 \simeq \cdots \simeq W_t$. On the other hand, since the equivalences $u_1 X \simeq u_2 W_2$ and $u_{t+1} Y \simeq u_t W_t$ are the elementary transformations at the beginning of the words, there exist again $\alpha, \beta \in \{bb, ba\}$ and words $Z_1, Z_2$ with the similar descriptions as above hold, implying again $X \simeq Y$.

To complete the proof, we have to examine three more cases $(u_1, u_2, u_3) = (a, b, c), (a, c, b)$ and $(b, a, c)$ for $t = 2$, where we shall put $W := W_2$.

(I) Case $(a, b, c)$. We have $aX \simeq bW \simeq cY$.
Since the equivalences are single elementary transformations, there exist words $Z_1$ and $Z_2$ such that $X \simeq bZ_1$, $W \simeq cZ_1 \simeq baZ_2$ and $Y \simeq bbZ_2$. Applying the induction hypothesis for $r$ to the two equivalent expressions of $W$, we see that there exist $k$ and a word $Z_3$ such that $0 \leq k < r - 2$, $Z_1 \simeq a^k bbZ_3$ and $aZ_2 \simeq c^k baZ_3$. We can apply $k$-times the induction hypothesis to the last two equivalent expressions and we see that there exists a word $Z_4$ such that $Z_2 \simeq c^k Z_4$ and $baZ_3 \simeq aZ_4$. Applying again the induction hypothesis to the last equivalence relation, there exists a word $Z_5$ such that $Z_4 \simeq bZ_5$ and $aZ_3 \simeq cZ_5$. Once again applying the induction hypothesis to the last equivalence relation, we finally obtain $Z_3 \simeq cZ_6$ and $Z_5 \simeq aZ_6$ for a word $Z_6$. Reversing the procedure, obtain the descriptions:

$$\begin{array}{lcl} X & \simeq & bZ_1 \simeq ba^k bbZ_3 \simeq ba^k bbcZ_6, \\ Y & \simeq & bbZ_2 \simeq bbc^k Z_4 \simeq bbc^k bZ_5 \simeq bbc^k baZ_6. \end{array}$$

By using the relations of $G^+_{B_{ii}}$, we can show $ba^k bbc \simeq cbbc^k b$ and $bbc^k ba \simeq abbc^k b$. So, we conclude that $X \simeq cZ, Y \simeq aZ$ for $Z \simeq bbc^k bZ_6$.

(II) Case $(a, c, b)$. We have $aX \simeq cW \simeq bY$.
Since the equivalences are single elementary transformations, there exist words $Z_1$ and $Z_2$ such that $X \simeq cZ_1$, $W \simeq aZ_1 \simeq bbZ_2$ and $Y \simeq baZ_2$. Applying the induction hypothesis for $r$ to the two equivalent expressions of $W$, we see that there exists a word $Z_3$ such that $Z_1 \simeq bZ_3$ and $bZ_2 \simeq cZ_3$. Again applying the induction hypothesis to the last two equivalent expressions, we see that there exist an integer $k$ with $0 \leq k < r - 3$ and a word $Z_4$ such that $Z_2 \simeq c^k baZ_4$ and $Z_3 \simeq a^k bbZ_4$. Reversing the procedure, obtain the descriptions:

$$X \simeq cZ_1 \simeq cbZ_3 \simeq cba^k bbZ_4 \quad \text{and} \quad Y \simeq baZ_2 \simeq bac^k baZ_4.$$

It is not hard to show the equivalences $cba^k bb \simeq bbac^k b$ and $bac^k ba \simeq cbac^k b$. Thus, we obtain $X \simeq bZ, Y \simeq cZ$ for $Z := bac^k bZ_4$.

(III) Case $(b, a, c)$. We have $bX \simeq aW \simeq cY$.
By induction hypothesis, there exist words $Z_1$ and $Z_2$ such that $X \simeq cZ_1$, $W \simeq bZ_1 \simeq cZ_2$ and $Y \simeq aZ_2$. Applying the induction hypothesis for $r$ to the two equivalent expressions of $W$, we see that there exist $k$ and a word $Z_3$ such that $0 \leq k < r - 2$, $Z_1 \simeq c^k baZ_3$ and $Z_2 \simeq a^k bbZ_3$. Thus, we obtain the descriptions:

$$X \simeq cZ_1 \simeq cc^k baZ_3 \quad \text{and} \quad Y \simeq aZ_2 \simeq aa^k bbZ_3.$$

This is the conclusion in Proposition 7.7 (iv) with $0 \leq k+1 < r-1$, which we looked for.

This completes the proof of Proposition. $\square$

This completes the proof of Theorem 4. $\square$

**Remark 7.3.** The sufficient criterion for the cancellation condition given in [D-P, Prop. 3.6] is not satisfied by the monoid $G^+_{\mathrm{B_{ii}}}$.

# 8 $2 \times 2$-matrix representation of the group $G_X$

We construct non-abelian representations of the groups $G_{\mathrm{B_{ii}}}, G_{\mathrm{B_{vi}}}, G_{\mathrm{H_{ii}}}, G_{\mathrm{H_{iii}}}$.

**Theorem 5.** *For each type $X \in \{\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}\}$, consider matrices $A, B, C \in \mathrm{GL}(2, \mathbb{C})$ listed below. Then we have the following i) and ii).*

*i) The correspondence $a \mapsto A, b \mapsto B, c \mapsto C$ induces representations*

$$\rho \;:\; G_X \longrightarrow \mathrm{GL}(2, \mathbb{C}).$$

*ii) The image $\rho(G_X)$ is not an abelian group if $l^2 \neq 1$.*

*1. Type* $\mathrm{B_{ii}}$*:*

$$A = u\begin{pmatrix} 1 & l^2 \\ 0 & 1 \end{pmatrix}, \quad B = v\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \quad C = u\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

*where $l^6 = 1$ and $u, v \in \mathbb{C}^\times$.*

*2. Type* $\mathrm{B_{vi}}$*:*

$$A = u\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \quad B = u\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad C = u\begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

$$A = u\begin{pmatrix} 1 & l^2 \\ 0 & 1 \end{pmatrix}, \quad B = v\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \quad C = u\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

*where $l^{10} = 1$ $(l^2 \neq 1)$ and $u \in \mathbb{C}^\times$*

$$a = -\frac{1}{l(l^2-1)}, \quad bc = \frac{-l^4 + l^2 - 1}{(1-l^2)^2}, \quad d = \frac{l^3}{l^2-1}$$

$$p = -l^4 a, \quad q = -\frac{b}{l^4}, \quad r = -l^4 c, \quad s = -\frac{d}{l^4}$$

*3. Type* $\mathrm{H_{ii}}$*:*

$$A = u\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \quad B = u\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad C = u\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

*where $u \in \mathbb{C}^\times$ and one of the following two cases holds.*

*i) $l^2 + l + 1 = 0$ and $3p^2 + 3p + 2 = 0$*

$$a = \frac{l-1}{3}, \; d = \frac{-l-2}{3}, \; bc = -\frac{2}{3}, \; q = \frac{-b(l+2)}{3p}, \; r = \frac{p(1-l)}{3b}, \; s = \frac{2}{3p}.$$

*ii)* $l^2 - l + 1 = 0$ *and* $3p^2 - 3p + 2 = 0$.

$$a = \frac{l+1}{3},\ d = \frac{-l+2}{3}, bc = -\frac{2}{3},\ q = \frac{b(-l+2)}{3p},\ r = \frac{-p(l+1)}{3b},\ s = \frac{2}{3p}.$$

*4. Type* $\mathrm{H_{iii}}$*:*

$$A = u\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix},\quad B = u\begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad C = u\begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

*where* $l^{10} = 1$ $(l^2 \neq 1)$ *and* $u \in \mathbb{C}^\times$

$$a = -\frac{1}{l(l^2-1)},\quad bc = \frac{-l^4+l^2-1}{(l^2-1)^2},\quad d = \frac{l^3}{l^2-1}$$

$$p = a,\quad q = \frac{b}{l^4},\quad r = l^4 c,\quad s = d$$

*Proof.* It is sufficient to prove only for the case $u = v = 1$.

We present the matrices $A, B$ and $C$ by the indeterminates $a, b, c, d, p, q, r, s$ and $l$ as in the statement, and then solve the polynomial equation on them defined by the relations listed in Theorem 1. It is unnecessary to check all relations, since some relations are included in the list because of the cancellation condition, whereas $\mathrm{GL}(2,\mathbb{C})$ is a group where the cancellation condition is automatically satisfied. However, as we shall see, it is sometimes convenient to take these "superfluous" relations in account. Detailed calculations are left to the reader.

1.Type $\mathrm{B_{ii}}$: We need to show $CBB = BBA, BC = AB, AC = CA$, whose verifications are left to the reader. We have $\det(A) = \det(C) = u^2 \neq 0, \det(B) = v^2 \neq 0$. Since $ABA^{-1}B^{-1} = \begin{pmatrix} 1 & l^2(1-l^2) \\ 0 & 1 \end{pmatrix}$ and $BCB^{-1}C^{-1} = \begin{pmatrix} 1 & l^2-1 \\ 0 & 1 \end{pmatrix}$, $\rho(G_{\mathrm{B_{ii}}})$ is abelian if and only if $l^2 = 1$.

2. Type $\mathrm{B_{vi}}$: We need to show $ABA = BAB, ACA = BAC, ACB = CAC$. Actually, solving the (1,1) entry of the equation $ABA = BAB$, $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(B) = 1$, w obtain the expressions for $a, b, c, d$. Then, using the relation $C = ABA^{-2}B$, we obtain the expressions for $p, q, r, s$. Furthermore, comparing the $(1,1)$-entry of $A^5 = B^5$, we get $l^8 + l^6 + l^4 + l^2 + 1 = 0$.

3. Type $\mathrm{H_{ii}}$: We need to show $ABAB = BABA, ACA = BAC, ACB = CAC$.

$$ABAB = \begin{pmatrix} bc + a^2l^2 & bd + abl^2 \\ ac + cd/l^2 & bc + d^2/l^2 \end{pmatrix}, BABA = \begin{pmatrix} bc + a^2l^2 & ab + bd/l^2 \\ cd + acl^2 & bc + d^2/l^2 \end{pmatrix}$$

By these calculations, we have $d + al^2 = 0$. By $\mathrm{Tr}A = \mathrm{Tr}B = \mathrm{Tr}C$ and $\det A = \det B = \det C$, we have $a + d = l + l^{-1} = p + s$, $ad - bc = ps - qr = 1$.

$$a = \frac{l^2+1}{l(1-l^2)}, d = \frac{l(l^2+1)}{l^2-1}, bc = \frac{-2(l^4+1)}{(l^2-1)^2}$$

$$ACA = \begin{pmatrix} l^2p & q \\ r & s/l^2 \end{pmatrix}, BAC = \begin{pmatrix} alp + br/l & alq + bs/l \\ clp + dr/l & clq + ds/l \end{pmatrix}$$

$$q=\frac{b}{lp(1-l^2)}, r=\frac{l(l^4+1)p}{b(l^2-1)}, s=\frac{-2l^2}{(1-l^2)^2p}$$

$ACB=CAC \Leftrightarrow (1-l+l^2)=0$ and $3p^2-3p+2=0$, or , $(1+l+l^2)=0$ and $3p^2+3p+2=0$

We calculate each cases separately and obtain the result.

4. Type $\mathrm{H_{iii}}$: We need to show $ABA=BAB, CBA=ACB, BCA=CBC$. As in case of $\mathrm{B_{vi}}$, already the first relation $ABA=BAB$ (in particular $\mathrm{tr}(A)=\mathrm{tr}(B)$ and $\det(B)=1$) implies the expressions for $a,b,c,d$. Using further the relation $ACA=CAC$, we obtain $a=p$, $d=s$ and $bc=qr$. Then applying the relation $A^2C=BA^2$, we get $q=l^4b$ and $r=l^{-4}c$. Further, using the relation $CA^3=A^3B$, we obtain $l^{10}=1$. □

**Corollary.** For $X\in\{\mathrm{B_{ii}},\ \mathrm{B_{vi}},\ \mathrm{H_{ii}}, \mathrm{H_{iii}}\}$, $\sigma(\mathcal{QZ}(\pi G_X^+))$ consists only of the identity.

*Sketch of Proof.* For $\sigma\in\mathfrak{S}(L)$, we consider a matrix $X\in\mathrm{Mat}(2,\mathbb{C})$ satisfying the equations: $AX=X\sigma(A)$, $BX=X\sigma(B)$, $CX=X\sigma(C)$. If $\sigma=1$, then the solutions are *constant* $\times$ id. If $\sigma\neq 1$, then $X=0$.

**Remark.** J.Sekiguchi constructed the following $3\times 3$-matrices representation: $a\mapsto A, b\mapsto B, c\mapsto C$ of the group of type $\mathrm{B_{ii}}$.

$$A=\begin{pmatrix} a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & 0 & a_3\end{pmatrix},\quad B=\begin{pmatrix} 0 & 0 & b_1\\ b_2 & 0 & 0\\ 0 & b_3 & 0\end{pmatrix},\quad C=\begin{pmatrix} a_2 & 0 & 0\\ 0 & a_3 & 0\\ 0 & 0 & a_1\end{pmatrix},$$

for $a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{C}^\times$.

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