PRODUCT STRUCTURES IN MOTIVIC COHOMOLOGY AND HIGHER CHOW GROUPS

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ABSTRACT. It is shown that the product structures of motivic cohomology groups and of higher Chow groups are compatible under the comparison isomorphism of Voevodsky [Vo2]. This extends the result of Weibel [We1], where he used the comparison isomorphism which assumed that the base field admits resolution of singularities.

The mod n motivic cohomology groups and product structures in motivic homotopy theory are defined, and it is shown that the product structures are compatible under the comparison isomorphisms.

1. INTRODUCTION

Let k be a perfect field. Let X be a scheme smooth over Spec k. For nonnegative integers $i, j \geq 0$, we let $H^i_{\mathcal{M},Ho}(X,\mathbb{Z}(j))$ denote the motivic cohomology group defined using motivic homotopy theory (see Section 2.2.1 for the precise definition). We let $H^i_{\mathcal{M},DM}(X,\mathbb{Z}(j))$ denote the motivic cohomology defined using the motivic complex (see Section 2.1.1 for the precise definition). We also have its higher Chow group $\mathrm{CH}^{2j-i}(X,j)$. It is known that the three groups are isomorphic. The result (Proposition 2.2 and Theorem 3.1) is that the isomorphisms are compatible with the product structures. The case of mod n coefficients is also proved.

We note that such compatibility result between the motivic cohomology groups with Z-coefficients has been obtained by Weibel ([We2], Lemma 2, p. 391), and the compatibility with higher Chow groups is proved under the assumption that the base field admits resolution of singularities by Weibel ([We1]).

2. Compatibility in motivic cohomology

2.1. Motivic cohomology in $DM_{-}^{eff}(k)$.

2.1.1. Throughout this section we fix a perfect field k. We use the symbol \times to denote the fiber product over Spec k of two schemes over Spec k.

Let $Shv_{Nis}(SmCor(k))$ denote the category of Nisnevich sheaves with transfers in the sense of [Vo-Su-Fr], Definition 3.1.1, p. 199. Recall that an object in $Shv_{Nis}(SmCor(k))$ is a presheaf of abelian groups on the category SmCor(k) introduced in [Vo-Su-Fr], Chapter 5, p. 190, whose restriction to the category Sm/k is a sheaf with respect to the Nisnevich topology. Let $DM_{-}^{eff}(k)$ be the triangulated

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category introduced in [Vo-Su-Fr], Chapter 5, p. 205. By [Vo-Su-Fr], Proposition 3.2.3, p. 208, there exists a covariant functor $RC : D^{-}(Shv_{Nis}(SmCor(k))) \rightarrow DM_{-}^{eff}(k)$ from the derived category of complexes in $Shv_{Nis}(SmCor(k))$ bounded above to the category $DM_{-}^{eff}(k)$. For an object M in $Shv_{Nis}(SmCor(k))$ or $D^{-}(Shv_{Nis}(SmCor(k)))$, we denote by the same symbol M its image RC(M) under RC if there is no risk of confusion.

We say that a morphism between bounded above complexes of $Shv_{Nis}(SmCor(k))$ is an isomorphism in $DM_{-}^{eff}(k)$ if it becomes an isomorphism in $DM_{-}^{eff}(k)$ when we apply RC to it. In [Vo-Su-Fr], p. 206 (resp. [Vo-Su-Fr], p. 210), tensor products in the category $Shv_{Nis}(SmCor(k))$ and $D^{-}(Shv_{Nis}(SmCor(k)))$ (resp. the category $DM_{-}^{eff}(k)$) are defined. The functor RC preserves the tensor products. We denote the tensor product by the symbol \otimes .

For a separated scheme X of finite type (not necessarily smooth) over Spec k, let $\mathbb{Z}_{tr}(X)$ (resp. $z_{equi}(X,0)$) denote the presheaf with transfers L(X) (resp. $L^c(X)$) defined in [Vo-Su-Fr], p. 223. The presheaf $\mathbb{Z}_{tr}(X)$ is a Nisnevich sheaf with transfers (see [Ma-Vo-We, p.15, Exercise 2.11], [Ma-Vo-We, p.37, Lemma 6.2]). By definition, for a scheme U smooth over Spec k, the abelian group $\mathbb{Z}_{tr}(X)(U)$ (resp. $z_{equi}(X,0)(U)$) is the free abelian group generated by closed integral subschemes Z of $X \times U$ which are finite and surjective (resp. quasi-finite and dominant) over an irreducible component of U.

Let $q \geq 0$ be a non-negative integer. Following [Su-Vo], we define the object $\mathbb{Z}(q)$ to be $C_*\mathbb{Z}'(q)$ where $\mathbb{Z}'(q)$ is the q-fold tensor product of the object $\mathbb{Z}'(1) = [\mathbb{Z}_{tr}(\mathbb{G}_m) \to \mathbb{Z}_{tr}(\operatorname{Spec} k)]$, where the object $\mathbb{Z}_{tr}(\mathbb{G}_m)$ is placed in degree zero, in the category of (bounded above) complexes of presheaves with transfers. (See [Vo-Su-Fr, p.207] or [Ma-Vo-We, p.16, Definition 2.14] for the definition of C_* .) We regard them as objects in the category of complexes of Nisnevich sheaves with transfers and in $DM_{-f}^{eff}(k)$.

2.1.2. Mod n motivic cohomology in $DM_{-}^{eff}(k)$. We define motivic cohomology groups in $DM_{-}^{eff}(k)$ as follows. Let $i, j \ge 0$ be integers. For a scheme X smooth over Spec k, we let

$$H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}(j)) = \operatorname{Hom}_{DM^{eff}(k)}(\mathbb{Z}_{tr}(X),\mathbb{Z}(j)[i]).$$

For an integer $n \geq 2$, we let

$$H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j)) = \operatorname{Hom}_{DM^{eff}_{-}(k)}(\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}/n,\mathbb{Z}(j)[i+1]).$$

The subscript DM is added to distinguish these from the motivic cohomology groups defined in another manner later.

2.1.3. Product structures. Let j, j' be a pair of non-negative integers. We use the product map $\mu' : \mathbb{Z}'(j) \otimes \mathbb{Z}'(j') \to \mathbb{Z}'(j+j')$ of [Ma-Vo-We, p.24, Construction 3.11], and the product map $\mu : \mathbb{Z}(j) \otimes \mathbb{Z}(j') \to \mathbb{Z}(j+j')$ of [Ma-Vo-We, p.23, Construction 3.10].

The map μ defines a product map

$$H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}(j))\otimes H^{i'}_{\mathcal{M},DM}(X,\mathbb{Z}(j'))\to H^{i+i'}_{\mathcal{M},DM}(X,\mathbb{Z}(j+j')).$$

We define a product map

$$H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j))\otimes H^{i'}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j'))\to H^{i+i'}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j+j'))$$

for mod n coefficients as follows. Let $n \ge 2$ and i, i' be integers. We have a canonical quasi-isomorphism

$$\begin{split} \mathbb{Z}/n[-i-1] \otimes^{\mathbb{L}} \mathbb{Z}/n[-i'-1] \\ &= (\mathbb{Z} \xrightarrow{n} \mathbb{Z})[-i-1] \otimes (\mathbb{Z} \xrightarrow{n} \mathbb{Z})[-i'-1] \\ &= (\mathbb{Z} \xrightarrow{n,-n} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{n,n} \mathbb{Z})[-i-i'-2] \\ &\cong (\mathbb{Z}/n \xrightarrow{0} \mathbb{Z}/n)[-i-i'-2] \\ &\cong \mathbb{Z}/n[-i-i'-1] \oplus \mathbb{Z}/n[-i-i'-2]. \end{split}$$

Here, the right most object in a complex (such as $\mathbb{Z} \to \mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$) is placed in degree zero. We let $\Delta_{i,i'} : \mathbb{Z}/n[-i-i'-1] \to \mathbb{Z}/n[-i-1] \otimes^{\mathbb{L}} \mathbb{Z}/n[-i'-1]$ denote the identity map to the direct summand. We also let $\Delta_{i,i'} : \mathbb{Z}/n[-i-i'-1] \to \mathbb{Z}/n[-i-1] \otimes^{\mathbb{L}} \mathbb{Z}[-i']$ denote the canonical isomorphism.

Let $f \in H^i_{\mathcal{M},DM}(X,\mathbb{Z}/n(j))$ and $g \in H^{i'}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j'))$. We define the image of $f \otimes g$ to be the class of the map

$$\frac{\mathbb{Z}/n[-i-i'-1]}{\overset{\Delta_{i,i'}\otimes\Delta_X}{\longrightarrow}} \mathbb{Z}/n[-i-1] \otimes^{\mathbb{L}} \mathbb{Z}/n[-i'-1] \otimes \mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(X)$$

$$\frac{f^{\otimes g}}{\overset{\Delta_{i,i'}\otimes\Delta_X}{\longrightarrow}} \mathbb{Z}(j) \otimes^{\mathbb{L}} \mathbb{Z}(j') \xrightarrow{\mu} \mathbb{Z}(j+j')$$

We also define a product map

$$H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j))\otimes H^{i'}_{\mathcal{M},DM}(X,\mathbb{Z}(j'))\to H^{i+i'}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j+j'))$$

in a similar manner.

2.2. Motivic cohomology in motivic homotopy theory.

2.2.1. Let k be a field. Let Sm/k denote the category of schemes which is smooth over Spec k. The (unstable) homotopy category of pointed simplicial Nisnevich sheaves on Sm/k ([Mo-Vo], p. 109) is denoted by $\text{H}_{\bullet}(k)$. (We do not use the homotopy category of Nisnevich sheaves (spaces) of [Vo1], Definition 3.5, p. 585.) We regard a Nisnevich sheaf (of sets) on Sm/k as a simplicial Nisnevich sheaf by regarding a set as a 0-dimensional simplicial set.

For $n \geq 0$, let $K(\mathbb{Z}(n), 2n)$ denote the Eilenberg-MacLane space, which is a Nisnevich sheaf on Sm/k, as in [Vo1], DEFINITION 6.1, p. 597. We regard these objects as simplicial Nisnevich sheaves on Sm/k by the procedure above. The product map $m_{m,n} : K(\mathbb{Z}(m), 2m) \wedge K(\mathbb{Z}(n), 2n) \to K(\mathbb{Z}(m+n), 2(m+n))$ of [Vo1], p. 597, bottom, for $m, n \geq 0$, of Nisnevich sheaves induces a product map as simplicial sheaves. We use the same notation $m_{m,n}$ for the latter. For n < 0, we denote by $K(\mathbb{Z}(n), 2n)$ the zero object in the category of pointed simplicial Nisnevich sheaves. For $m, n \in \mathbb{Z}$ with m < 0 or n < 0, we denote by $m_{m,n}$ the unique map $m_{m,n} : K(\mathbb{Z}(m), 2m) \wedge K(\mathbb{Z}(n), 2n) \to K(\mathbb{Z}(m+n), 2(m+n))$.

From now on we assume that the base field k is perfect. For $i, j \geq 0$ and a pointed simplicial Nisnevich sheaf X, we put $H^{2j-i}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)) = \operatorname{Hom}_{H_{\bullet}(k)}(S^{i}_{s} \wedge X, K(\mathbb{Z}(j), 2j))$. For $i, j \in \mathbb{Z}$ with i < 0 or j < 0, we set $H^{2j-i}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)) = 0$. The subscript Ho is added to distinguish these from the motivic cohomology groups defined in Section 2.1.2.

2.2.2. Mod n motivic cohomology groups. Let $m \ge 2$ and $n \ge 1$ be integers. We define the mod n Moore space $P^m(n)$ of dimension m to be the topological space $S^{m-1} \cup_{\alpha_n} e^m$ where e^m is the m-cell and $\alpha_n : S^{m-1} \to S^{m-1}$ is a map of degree n. We have a sequence of cofibrations

$$S^{i-1} \xrightarrow{\alpha_n} S^{i-1} \to P^i(n) \to S^i \xrightarrow{\alpha_n} S^i$$

for $i \geq 2$. (By this, we mean that each of the sequences $S^{i-1} \xrightarrow{\alpha_n} S^{i-1} \to P^i(n)$, $S^{i-1} \to P^i(n) \to S^i$, and $P^i(n) \to S^i \xrightarrow{\alpha_n} S^i$ can be identified up to weak equivalence with a cofibration sequence.)

Suppose n is odd or 4 divides n. For $m, m' \ge 2$, there exists a continuous map

$$\Delta_{m,m'}: P^{m+m'}(n) \to P^m(n) \land P^{m'}(n)$$

called coproduct map (see [N], Lemma 8.2, p. 40). Let us recall a property of the coproduct map to be used later. For a topological space X, there is a canonical map

$$[P^m(n), X] \to H^m(X, \mathbb{Z}/n)$$

called the mod *n* Hurewicz map (see [N], Definition 3.1, p.10), where the bracket denotes the homotopy classes of maps. Then the class of the coproduct map $\Delta_{m,m'}$ is sent via the Hurewicz map to the class of $e^m \wedge e^{m'}$ in $H^{m+m'}(P^m(n) \wedge P^{m'}(n), \mathbb{Z}/n)$.

Let m, m' be as above and let $n \geq 1$ be an integer. Note that the two maps $S^{m-1} \wedge S^{m'} \xrightarrow{\alpha_n \wedge \text{id}} S^{m-1} \wedge S^{m'}$ and $S^{m+m'-1} \xrightarrow{\alpha_n} S^{m+m'-1}$ belong to the same homotopy class. Hence there exists a canonical isomorphism

$$P^{m+m'}(n) \to P^m(n) \wedge S^m$$

(in the homotopy category of pointed topological spaces). By abuse of notation, we denote this map by $\Delta_{m,m'}$.

We recall that there is a singular functor from the category of topological spaces to the category of simplicial sets. By abuse, we let $P^m(n)$ denote the image of the topological space $P^m(n)$ by the singular functor. We also have a functor from the category of simplicial sets to the category of simplicial presheaves which sends a simplicial set to the constant presheaf. We also let $P^m(n)$ denote the image of the Moore space by this functor.

Let X be a smooth scheme over k, and X_+ denote the pointed object corresponding to X. (This is an object in the category of pointed simplicial Nisnevich sheaves on Sm/k.) Let $n \ge 2$ be an integer. Let $i, j \ge 0$ be integers. When $2j - i \ge 2$, we let

$$H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j)) = \operatorname{Hom}_{H_{\bullet}(k)}(P^{2j-i}(n) \wedge X_{+}, K(\mathbb{Z}(j), 2j)).$$

If 2j - i = 1, we let

$$H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j)) = \operatorname{Hom}_{H_{\bullet}(k)}(P^{2}(n) \wedge X_{+} \wedge S^{1}_{t}, K(\mathbb{Z}(j+1), 2(j+1))).$$

From the sequence of cofibrations above, we obtain a long exact sequence

$$\cdots \to H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)) \xrightarrow{n} H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)) \to H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j)) \\ \to H^{i+1}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)) \xrightarrow{n} \cdots \xrightarrow{n} H^{2j}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)).$$

2.2.3. Product structures. Let $n \ge 1$ be an integer. We assume either n is odd or 4 divides n. We define a product map

(2.1)
$$H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j))\otimes H^{i'}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j'))\to H^{i+i'}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j+j'))$$

as follows.

When $2j - i \ge 2$ and $2j' - i' \ge 2$, the element $f \otimes g$ of the source is sent to the class of the map

$$\begin{array}{l} P^{2(j+j')-(i+i')}(n) \wedge X_{+} \\ \xrightarrow{\Delta_{2j-i,2j'-i'} \wedge \Delta_{X}} & P^{2j-i}(n) \wedge P^{2j'-i'}(n) \wedge X_{+} \wedge X_{+} \\ \xrightarrow{f \wedge g} & K(\mathbb{Z}(j),2j) \wedge K(\mathbb{Z}(j'),2j') \\ \xrightarrow{\mu} & K(\mathbb{Z}(j+j'),2(j+j')). \end{array}$$

Suppose 2j - i = 1 and $2j' - i' \ge 2$. Given an element $f \otimes g$ of the source, consider the composite map

$$\begin{array}{l} P^{2(j+j')-(i+i')}(n) \wedge X_{+} \wedge S_{s}^{1} \wedge S_{t}^{1} \\ \xrightarrow{=} P^{2(j+j')-(i+i')+1}(n) \wedge X_{+} \wedge S_{t}^{1} \\ \xrightarrow{\Delta_{2,2j'-i'} \wedge \Delta_{X}} \\ \xrightarrow{f \wedge g} K(\mathbb{Z}(j+1), 2(j+1)) \wedge K(\mathbb{Z}(j'), 2j') \\ \xrightarrow{\mu} K(\mathbb{Z}(j+j'+1), 2(j'+j+1)). \end{array}$$

We need a lemma. The similar statement for the homotopy category of spaces is [Vo1], p. 598, Theorem 6.2.

Lemma 2.1. Let Y be a smooth scheme over k. Let $m \ge 1$ and $t \ge 2$ be integers. Let $n \ge 1$. There is a canonical isomorphism

$$\begin{split} &\operatorname{Hom}_{H_{\bullet}(k)}(P^{t}(n) \wedge Y \wedge (S^{1}_{t} \wedge S^{1}_{s})^{\wedge m}, K(\mathbb{Z}(j^{\prime\prime}), 2j^{\prime\prime})) \\ &\xrightarrow{\cong} \operatorname{Hom}_{H_{\bullet}(k)}(P^{t}(n) \wedge Y \wedge (S^{1}_{t} \wedge S^{1}_{s})^{\wedge (m+1)}, K(\mathbb{Z}(j^{\prime\prime}+1), 2(j^{\prime\prime}+1))) \end{split}$$

for any j''.

Proof. By [Mo-Vo], Proposition 2.17, p.112, we know that $S_t^1 \wedge S_s^1$ is canonically isomorphic to (\mathbb{P}^1, ∞) . By the Dold-Kan correspondence (recalled in Section 2.3; we use the notations below), we have

$$\begin{split} &\operatorname{Hom}_{H_{\bullet}(k)}(P^{t}(n) \wedge Y \wedge (S_{t}^{1} \wedge S_{s}^{1})^{\wedge m}, K(\mathbb{Z}(j''+1), 2(j''+1))) \\ &= \operatorname{Hom}_{DM_{-}^{eff}(k)}(\mathbb{Z}/n[t-1] \otimes \mathbb{Z}_{tr}(Y) \otimes (\mathbb{Z}_{tr}(\mathbb{P}^{1})/\mathbb{Z}_{tr}(\infty))^{\otimes m}, \mathbb{Z}'(1)^{\otimes (j''+1)}) \\ &= \operatorname{Hom}_{DM_{-}^{eff}(k)}(\mathbb{Z}/n[t-1] \otimes \mathbb{Z}_{tr}(Y) \otimes \mathbb{Z}'(m)[2m], \mathbb{Z}'(1)^{\otimes (j''+1)}[2(j''+1)]). \end{split}$$

Hence the endofunctor $- \otimes \mathbb{Z}(1)$ of $DM_{-}^{eff}(k)$ gives the map in the statement of the lemma. The fact that it is an isomorphism is the cancellation theorem, which is known to hold for k perfect ([Vo3]).

Using this lemma, the composite map above defines an element of $H^{i+i'}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j+j'))$. This then defines the product map in this case.

Suppose 2j - i = 1 and 2j' - i' = 1. Given an element $f \otimes g$ of the source, consider the composite map

$$\begin{split} & P^2(n) \wedge X_+ \wedge S_s^1 \wedge S_s^1 \wedge S_t^1 \wedge S_t^1 \\ & \xrightarrow{\equiv} P^4(n) \wedge X_+ \wedge S_t^1 \wedge S_t^1 \\ & \xrightarrow{\Delta_{2,2} \wedge \Delta_X} P^2(n) \wedge X_+ \wedge P^2(n) \wedge X_+ \wedge S_t^1 \wedge S_t^1 \\ & \xrightarrow{f \wedge g} K(\mathbb{Z}(j+1), 2(j+1)) \wedge K(\mathbb{Z}(j'+1), 2(j'+1)) \\ & \xrightarrow{\mu} K(\mathbb{Z}(j+j'+2), 2(j+j'+2)). \end{split}$$

Again using the canonical isomorphism in Lemma 2.1 we obtain a map

$$\begin{split} &\operatorname{Hom}_{H_{\bullet}(k)}(Y \wedge S^{1}_{s} \wedge S^{1}_{s} \wedge S^{1}_{t} \wedge S^{1}_{t}, K(\mathbb{Z}(j''+2), 2(j''+2)) \\ &\cong \operatorname{Hom}_{H_{\bullet}(k)}(Y, K(\mathbb{Z}(j''), 2j'')). \end{split}$$

Thus the composite map above defines an element of $H^{i+i'}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j+j'))$. This then defines the product map in this case.

We also define product maps

$$H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)) \otimes H^{i'}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j')) \to H^{i+i'}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j+j'))$$
$$H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)) \otimes H^{i'}_{\mathcal{M},Ho}(X,\mathbb{Z}(j')) \to H^{i+i'}_{\mathcal{M},Ho}(X,\mathbb{Z}(j+j'))$$

for $2j' - i' \ge 1$ and for $n \ge 1$ in a similar manner using the coproduct map $\Delta_{i,i'}: P^{i+i'}(n) \to P^i(n) \land S_s^{i'}$ and the map $\Delta_{i,i'}: S_s^{i+i'} \cong S_s^i \land S_s^{i'}$ respectively.

2.3. The Dold-Kan correspondence. Much of the material in this section is taken from the notes by Riou ([Ri]).

Let $n \ge 1$ be an integer. Using the Dold-Kan correspondence, we have isomorphisms

(2.2)
$$H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}(j)) \cong H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}(j))$$

(2.3)
$$H^{i}_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j)) \cong H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j))$$

(the second line holds in the range $2j - i \ge 1$).

The isomorphisms are given essentially by a series of adjunctions. Let us recall briefly the mod n case. We assume $2j - i \ge 2$ below.

$$\begin{aligned} &H_{\mathcal{M},Ho}(X,\mathbb{Z}/n(j)) = \operatorname{Hom}_{H_{\bullet}(k)}(P^{2j-i}(n) \wedge X_{+}, K(\mathbb{Z}(j), 2j)) \\ \stackrel{(1)}{=} \operatorname{Hom}_{H_{s,\bullet}(k)}(P^{2j-i}(n) \wedge X_{+}, \operatorname{Sing}^{\mathbb{A}^{1}}K(\mathbb{Z}(j), 2j)) \\ \stackrel{(2)}{=} \operatorname{Hom}_{D^{-}(AbShv(\operatorname{Sm}/k))}(N\mathbb{Z}P^{2j-i}(n) \otimes N\mathbb{Z}X_{+}, N\operatorname{Sing}^{\mathbb{A}^{1}}K(\mathbb{Z}(j), 2j)) \\ \stackrel{(3)}{=} \operatorname{Hom}_{D^{-}(AbShv(\operatorname{Sm}/k))}(\mathbb{Z}/n[2j-i-1] \otimes N\mathbb{Z}X_{+}, N\operatorname{Sing}^{\mathbb{A}^{1}}K(\mathbb{Z}(j), 2j)) \\ \stackrel{(4)}{=} \operatorname{Hom}_{DM^{eff}(k)}(\mathbb{Z}/n[2j-i-1] \otimes \mathbb{Z}_{tr}(X), \mathbb{Z}(j)[2j]) = H^{i}_{\mathcal{M},DM}(X, \mathbb{Z}/n(j)). \end{aligned}$$

By $H_{s,\bullet}(k)$, we mean the homotopy category of pointed simplicial Nisnevich sheaves on Sm/k before A¹-localization as defined in [Mo-Vo], p. 82. For the definition of the functor Sing^{A¹}, we refer to [Mo-Vo], p. 87. The equality (1) follows from the definition of A¹-localization and the fact that Sing^{A¹} $K(\mathbb{Z}(j), 2j)$ is A¹-local. The equality (2) is the Dold-Kan correspondence. We refer to [Mo-Vo], p. 56, Proposition 1.24, and the discussions thereof. We wrote N for the functor induced by the usual functor of normalized complexes, and Z for the functor induced by the left adjoint of the forgetful functor from the category of abelian groups to the

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category of sets. We also used the fact that N commutes with products up to quasiisomorphism (by the Eilenberg-Zilber theorem). We let $D^{-}(AbShv(Sm/k))$ denote the derived category of chain complexes of abelian sheaves bounded above on Sm/k with respect to the Nisnevich topology (see [Mo-Vo], p. 95 and also [Mo-Vo], p. 56). The equality (3) follows by using the canonical quasi-isomorphism $\mathbb{Z}/n[2j - i - 1] \cong N\mathbb{Z}P^{2j-i}(n)$, which follows from the fact that $N\mathbb{Z}P^{2j-i}(n)$ is the (classical) Eilenberg-MacLane space. The equality (4) follows by using the canonical quasiisomorphism $N\mathrm{Sing}^{\mathbb{A}^{1}} K(\mathbb{Z}(j), 2j) \cong C_*\mathbb{Z}'(j)[2j]$. This follows from the definitions of the functor $\mathrm{Sing}^{\mathbb{A}^{1}}$, of the Eilenberg-MacLane space $K(\mathbb{Z}(j), 2j)$, and of the motivic complex $\mathbb{Z}(j)$, using the Eilenberg-Zilber theorem.

Proposition 2.2. The isomorphisms (2.2) and (2.3) are compatible with the product structures. For the compatibility of (2.3), we assume that n is odd or 4 divides n.

Remark 2.3. The assumption on the integer n is used in the definition of the product map (2.1).

As a consequence of Proposition 2.2, we see that the product defined in Section 2.2.3 does not depend on the choice of the coproduct map $\Delta_{m,m'}$ of Section 2.2.2.

Proof of Proposition 2.2. We need to go through the Dold-Kan correspondence step-by-step. Let X be a smooth k-scheme and let X_+ denote the corresponding pointed object in the category of simplicial Nisnevich sheaves. Let us write $K_j = K(\mathbb{Z}(j), 2j)$ for short.

For the equality (1) above, we consider the following diagram:



The horizontal arrows are by adjunction (1) and are isomorphisms. There is a canonical map $\operatorname{Sing}^{\mathbb{A}^1} K_j \wedge \operatorname{Sing}^{\mathbb{A}^1} K_{j'} \to \operatorname{Sing}^{\mathbb{A}^1} K_{j+j'}$ constructed from the product map $K_j \wedge K_{j'} \to K_{j+j'}$ of Nisnevich sheaves (of sets). The map (a) is the map induced from this map. The diagram is commutative.

Let us write $D = D^{-}(AbShv(Sm/k))$ for short. For the equality (2), we consider the following commutative diagram:

$$\begin{split} \operatorname{Hom}_{\operatorname{H}_{\mathrm{S},\bullet}(k)}(X_{+},\operatorname{Sing}^{\mathbb{A}^{1}}K_{j})\otimes \operatorname{Hom}_{\operatorname{H}_{\mathrm{S},\bullet}(k)}(X_{+},\operatorname{Sing}^{\mathbb{A}^{1}}K_{j'}) & \overset{\cong}{\longrightarrow} \operatorname{Hom}_{D}(N\mathbb{Z}X,N\operatorname{Sing}^{\mathbb{A}^{1}}K_{j})\otimes \operatorname{Hom}_{D}(N\mathbb{Z}X,N\operatorname{Sing}^{\mathbb{A}^{1}}K_{j}) \\ & \downarrow \\ \operatorname{Hom}_{\operatorname{H}_{\mathrm{S},\bullet}(k)}(X_{+},\operatorname{Sing}^{\mathbb{A}^{1}}K_{j} \wedge \operatorname{Sing}^{\mathbb{A}^{1}}K_{j'}) & \overset{\cong}{\longrightarrow} \operatorname{Hom}_{D}(N\mathbb{Z}X,N\operatorname{Sing}^{\mathbb{A}^{1}}K_{j} \otimes N\operatorname{Sing}^{\mathbb{A}^{1}}K_{j'}) \\ & (a) \\ \operatorname{Hom}_{\operatorname{H}_{\mathrm{S},\bullet}(k)}(X_{+},\operatorname{Sing}^{\mathbb{A}^{1}}K_{j+j'}) & \overset{\cong}{\longrightarrow} \operatorname{Hom}_{D}(N\mathbb{Z}X,N\operatorname{Sing}^{\mathbb{A}^{1}}K_{j+j'}) \end{split}$$

The horizontal arrows are the equality (2) above. The commutativity follows essentially from the Eilenberg-Zilber theorem.

For the equality (3), one needs the compatibility of the coproduct maps (for n odd or for 4 dividing n) $\mathbb{Z}/n[i+i'-1] \to \mathbb{Z}/n[i-1] \otimes^{\mathbb{L}} \mathbb{Z}/n[i'-1]$ and $P^{i+i'}(n) \to P^{i}(n) \wedge P^{i'}(n)$. This follows from the commutativity up to homotopy of the following diagram in the derived category of complexes of abelian groups:

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The commutativity follows from the property of $\Delta_{i,i'}$ recalled in Section 2.2.2.

For the equality (4), one needs the compatibility of the product maps $\mathbb{Z}(j) \otimes \mathbb{Z}(j') \to \mathbb{Z}(j+j')$ and $K(\mathbb{Z}(j), 2j) \wedge K(\mathbb{Z}(j'), 2j') \to K(\mathbb{Z}(j+j'), 2(j+j'))$ under the sequence of equalities above. By this, we mean the commutativity of the following diagram

 $N\mathrm{Sing}^{\mathbb{A}^1}K(\mathbb{Z}(j),2j) \otimes N\mathrm{Sing}^{\mathbb{A}^1}K(\mathbb{Z}(j),2j) \longrightarrow N\mathrm{Sing}^{\mathbb{A}^1}K(\mathbb{Z}(j+j'),2(j+j'))$ in $D^-(AbShv(\mathrm{Sm}/k))$, where the vertical arrows are the isomorphisms mentioned above and the bottom horizontal arrow is the map induced from the product map. This follows essentially from [We2], Lemma 2, p. 391. \Box

3. Compatibility with higher Chow groups

By the main result of [Vo2] we have a canonical isomorphism

(3.1)
$$H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}(j)) \cong CH^{j}(X,2j-i)$$

for any pair (i, j) of integers and for any scheme X smooth over Spec k. By the same method we obtain a canonical isomorphism

(3.2)
$$H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j)) \cong CH^{j}(X,\mathbb{Z}/n,2j-i)$$

for each integer $n \geq 1$, where the right hand side is Bloch's higher Chow group with coefficients in $\mathbb{Z}/n\mathbb{Z}$, which is introduced and is denoted by $H^i(X, \mathbb{Z}/n(j))$ in [Ge-Le], Section 2.5. These isomorphisms are functorial in the sense that they are compatible with the pullback homomorphisms.

In [B1] Bloch defines a product structure $CH^{j}(X,i) \otimes_{\mathbb{Z}} CH^{j'}(X,i') \to CH^{j+j'}(X,i+i')$ of higher Chow groups. This is extended by Geisser and Levine [Ge-Le], Section 2.10 to the product structure

$$CH^{j}(X, \mathbb{Z}/n, i) \otimes_{\mathbb{Z}/n\mathbb{Z}} CH^{j'}(X, \mathbb{Z}/n, i') \to CH^{j+j'}(X, \mathbb{Z}/n, i+i')$$

in the case of $\mathbb{Z}/n\mathbb{Z}$ -coefficients.

The main statement of this section is as follows.

Theorem 3.1. Let X be a scheme which is smooth over Spec k. Then the isomorphisms (3.1) and (3.2) are compatible with the product structures on both sides.

We remark that compatibility of a similar kind for (3.1) has already been established in [We1] if the field k admits resolution of singularities. **3.1.** Let X be a scheme smooth over Spec k. By definition, the isomorphisms (3.1) and (3.2) for X are equal to the composite

(3.3)
$$\begin{aligned} H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}(j)) &= \operatorname{Hom}_{DM^{eff}_{-}(k)}(\mathbb{Z}_{tr}(X),\mathbb{Z}(j)[i]) \\ &\cong \operatorname{Hom}_{DM^{-ff}_{-}(k)}(\mathbb{Z}_{tr}(X),z_{equi}(\mathbb{A}^{j},0)[i-2j]) \\ &\cong CH^{j}(X,2j-i) \end{aligned}$$

and

$$(3.4) \begin{array}{l} H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}/n(j)) &= \operatorname{Hom}_{DM_{-}^{eff}(k)}(\mathbb{Z}_{tr}(X)\otimes\mathbb{Z}/n,\mathbb{Z}(j)[i]) \\ &\cong \operatorname{Hom}_{DM_{-}^{eff}(k)}(\mathbb{Z}_{tr}(X)\otimes\mathbb{Z}/n,z_{equi}(\mathbb{A}^{j},0)[i-2j]) \\ &\cong CH^{j}(X,\mathbb{Z}/n,2j-i) \end{array}$$

respectively, where, in each of (3.3) and (3.4), the first isomorphism follows from the isomorphism

(3.5)
$$\mathbb{Z}(q) \cong z_{equi}(\mathbb{A}^q, 0)[-2q]$$

in $DM_{-}^{eff}(k)$ constructed by Voevodsky in [Vo2], and the second isomorphism is constructed by Friedlander ans Suslin in [Fr-Su], Proposition 12.1, p. 831 (in Section 8 of [Fr-Su] they assume that the base field is infinite, and some of the arguments in [Fr-Su] rely on the unpublished preprint [Bl-Li] by Bloch and Lichtenbaum whose validity is not widely accepted, however they do not use the assumption that the base field is infinite or the result in [Bl-Li] for the construction of the morphism and for the verification that it is an isomorphism).

Remark 3.2. We can prove that the isomorphism (3.1) can also be described as the composite

(3.6)

$$\begin{aligned}
H^{i}_{\mathcal{M},DM}(X,\mathbb{Z}(j)) &= \operatorname{Hom}_{DM_{-}^{eff}(k)}(\mathbb{Z}_{tr}(X),(\mathbb{Z}_{tr}(\mathbb{P}^{1})/\mathbb{Z}_{tr}(\{\infty\}))^{\otimes j}) \\
&\to \operatorname{Hom}_{DM_{-}^{eff}(k)}(\mathbb{Z}_{tr}(X),z_{equi}(\mathbb{A}^{j},0)[i-2j]) \\
&\cong CH^{j}(X,2j-i)
\end{aligned}$$

where the map is induced by the restriction of the cycles on $(\mathbb{P}^1)^j \times U$ to $\mathbb{A}^j \times U$ for each scheme U smooth over Spec k, and the isomorphism is the same as that in (3.3). If we admit this, then we do not need to use Proposition 3.3 below for the proof of Theorem 3.1 since the map in (3.6) is clearly compatible with the product structures on both sides. However we do not take this route since the second author feels that to give a proof of the decomposition (3.6) requires many more pages than to give a proof of Proposition 3.3.

Each of the three groups in each of (3.3) and (3.4) has a product structure. We have already explained the product structure for the first one and the last one. The product structure for the second group is supplied by the canonical morphism $z_{equi}(\mathbb{A}^{j}, 0) \otimes z_{equi}(\mathbb{A}^{j'}, 0) \rightarrow z_{equi}(\mathbb{A}^{j+j'}, 0).$

It follows from the argument in the proof of [We1], Corollary 2.4, p. 308 that the second isomorphisms in (3.3) and (3.4) preserve the product structures. Therefore, to prove Theorem 3.1, it suffices to show that the first isomorphisms in (3.3) and (3.4) are compatible with the product structures on both sides, which is a consequence of the following proposition.

Proposition 3.3. The diagram

in the category $DM_{-}^{eff}(k)$ is commutative. Here the vertical maps in the diagram are the isomorphisms supplied by the isomorphism (3.5) and the horizontal maps are the product maps.

3.2. We recall in this paragraph some the basic properties of the object $\mathbb{Z}_{tr}(X)$ necessary for the argument used in this section.

3.2.1. We defined a presheaf with transfers $\mathbb{Z}_{tr}(X)$ for a (not necessarily smooth) separated scheme X of finite type over Spec k in Section 2.1.1.

For a morphism $X \to Y$ of separated of finite type k-schemes, there is induced a morphism $\mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(Y)$ in the category of presheaves with transfers. This can be seen from [Vo-Su-Fr, p.51, Corollary 3.6.3].

3.2.2. Let X and Y be smooth schemes over $\operatorname{Spec} k$. By definition of the product in the category of presheaves with transfers, we have an isomorphism

$$(3.7) \qquad \qquad \mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y) \to \mathbb{Z}_{tr}(X \times Y)$$

(see [Ma-Vo-We], p.57, Example 8.10).

3.2.3. Let X be a separated scheme of finite type over Spec k. Write X as the union of its irreducible components: $X = \bigcup_{i \in I} X_i$. For our application, the intersection $\bigcap_{j \in J} X_j$ will be smooth for all subsets $J \subset I$. Let us regard I as a totally ordered set.

Lemma 3.4. The augmented Čech complex

(3.8)
$$\cdots \to \bigoplus_{i_1 < i_2} \mathbb{Z}_{tr}(X_{i_1} \cap X_{i_2}) \to \bigoplus_{i \in I} \mathbb{Z}_{tr}(X_i) \to \mathbb{Z}_{tr}(X) \to 0,$$

associated with the (closed) cover $(X_i \to X)_{i \in I}$, is exact in the category of presheaves with transfers.

Proof. Let U be a smooth k-scheme. Let $Z \subset X \times U$ be an integral closed subscheme. Then, by the integrality, there exists an $i \in I$ such that $Z \subset X_i \times U$.

Let $\operatorname{CIS}(X, U)$ denote the set of closed integral subschemes which is finite and surjective over a connected component of U. We may regard $\operatorname{CIS}(X_i, U)$ as a subset of $\operatorname{CIS}(X, U)$ (see Section 3.2.1). It is easy to check that $\operatorname{CIS}(X, U) = \bigcup_{i \in I} \operatorname{CIS}(X_i, U)$, and that for any subset $J \subset I$, the equality $\bigcap_{j \in J} \operatorname{CIS}(X_j, U) = \operatorname{CIS}(\bigcap_{i \in J} X_j, U)$ holds. Now the exactness is obvious.

3.3. Let us introduce an intermediary object B which will be used in the proof.

3.3.1. We introduce some notation. For any integer $n \ge 1$, the *n*-dimensional projective space \mathbb{P}^n over Spec *k* contains the *n*-dimensional affine space \mathbb{A}^n over Spec *k* as an open dense subscheme. We identify the complement of \mathbb{A}^n in \mathbb{P}^n (with the reduced scheme structure) with \mathbb{P}^{n-1} . We denote by 0 the origin of \mathbb{A}^n .

Let j and j' be positive integers. We let $B = B_{j,j'}$ be the blowup of $\mathbb{P}^j \times \mathbb{P}^{j'}$ with center $\mathbb{P}^{j-1} \times \mathbb{P}^{j'-1}$. There is a canonical morphism $\mathbb{P}^j \times \mathbb{P}^{j'} \leftarrow B$.

We construct a morphism $B \to \mathbb{P}^{j+j'}$. Note that $\mathbb{A}^j \times \mathbb{A}^{j'} \subset \mathbb{P}^j \times \mathbb{P}^{j'}$ embeds as a dense open subscheme in B, which does not intersect the exceptional divisor. Then the morphism $B \to \mathbb{P}^{j+j'}$ is the unique map which extends the isomorphism of open dense subschemes $\mathbb{A}^j \times \mathbb{A}^{j'} \to \mathbb{A}^{j+j'}$. Let us give the explicit construction of the morphism using toric geometry.

Let $M = \mathbb{Z}^{j}$ and $M' = \mathbb{Z}^{j'}$. We let $e_1 = (1, 0, ..., 0), ..., e_j = (0, ..., 0, 1)$ be the standard basis of M, and $e'_1 = (1, 0, ..., 0), ..., e'_{j'} = (0, ..., 0, 1)$ be the standard basis of M'. We put $e_0 = -(e_1 + \cdots + e_j) = (-1, ..., -1)$ and $e'_0 = -(e'_1 + \cdots + e'_{j'}) = (-1, ..., -1)$.

Let $N = M \oplus M'$. We let $E = \{(e_i, 0) | 0 \le i \le j\}$. For $0 \le i \le j$, we let $E_i = E \setminus \{(e_i, 0)\}$. Similarly, we let $E' = \{(0, e'_{i'}) | 0 \le i' \le j'\}$ and for $0 \le i' \le j'$ let $E'_{i'} = E' \setminus \{(0, e_{i'})\}$.

For $0 \leq i \leq j$, $0 \leq i' \leq j'$, we let C(i,i') denote the cone spanned by $E_i \cup E'_{i'}$. For $1 \leq i \leq j$, $0 \leq i' \leq j'$, let D(i,i') denote the cone spanned by $(E_i \setminus \{(e_0,0)\}) \cup \{(e_0,e'_0)\} \cup E'_{i'}$. For $0 \leq i \leq j$, $1 \leq i' \leq j'$, let D'(i,i') denote the cone spanned by $E_i \cup \{(e_0,e'_0)\} \cup (E'_{i'} \setminus \{(0,e'_0)\})$.

Then B defined above is the toric variety constructed from the fan σ'' which consists of the cones C(i,0) $(0 \le i \le j)$, C(0,i') $(1 \le i' \le j')$, D(i,i') $(1 \le i \le j, 1 \le i' \le j')$, and D'(i,i') $(1 \le i \le j, 1 \le i' \le j')$ and their faces. The toric variety constructed from the fan σ which consists of the cones C(i,i') $(0 \le i \le j, 0 \le i' \le j')$ and their faces is $\mathbb{P}^j \times \mathbb{P}^{j'}$. The toric variety constructed from the fan σ' which consists of the cones C(0,0), D(i,0) $(1 \le i \le j)$, D'(0,i') $(1 \le i' \le j')$ and their faces is $\mathbb{P}^{j+j'}$. Then since the fan σ'' is a refinement of σ and of σ' , we obtain two morphisms

$$(3.9) \qquad \qquad \mathbb{P}^{j} \times \mathbb{P}^{j'} \longleftarrow B \longrightarrow \mathbb{P}^{j+j'}$$

The left arrow is the canonical map mentioned above, and the right arrow is the morphism with the uniqueness property mentioned above.

3.4. We construct some diagrams in this section.

3.4.1. Consider the following diagram in the category of complexes of presheaves with transfers:

where the map (1) is as in [Ma-Vo-We, p.24, Construction 3.11], the map (1)' is as in [Ma-Vo-We, p.23, Construction 3.10], and the vertical arrows are the canonical maps. One can check that it is commutative. (See [Vo-Su-Fr, p.207] or [Ma-Vo-We, p.16, Definition 2.14] for the definition of C_* .) **3.4.2.** Let $M_j = [\mathbb{Z}_{tr}(\mathbb{A}^j \setminus \{0\}) \to \mathbb{Z}_{tr}(\mathbb{A}^j)]$ denote the complex with $\mathbb{Z}_{tr}(\mathbb{A}^j)$ in degree zero. Consider the following commutative diagram in the category of complexes of presheaves with transfers:

$$\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge j})[-j] \otimes \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge j'})[-j'] \xrightarrow{(1)'} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge (j+j')})[-j-j'] \xrightarrow{(2)} M_j \otimes M_{j'} \xrightarrow{(3)} M_{j+j'}.$$

Here the maps are defined as follows. The maps (2) and (3) are supplied by the canonical morphism $S(\mathcal{U}) \to \mathbb{Z}_{tr}(X)$ in [Vo2], Lemma 3, p. 352, with respect to the covering denoted by \mathcal{V}_n in the last line of [Vo2], p. 352, of $X = \mathbb{A}^n \setminus \{0\}$, where n is j, j', or j + j'. The map (4) is supplied by the canonical map (3.7) and by the canonical morphism $S(\mathcal{U}) \to \mathbb{Z}_{tr}(X)$ in [Vo2], Lemma 3, p. 352, with respect to the covering of $X = \mathbb{A}^{j+j'} \setminus \{0\}$ by $(\mathbb{A}^j \setminus \{0\}) \times \mathbb{A}^{j'}$ and $\mathbb{A}^j \times (\mathbb{A}^{j'} \setminus \{0\})$.

3.4.3. We consider the following commutative diagrams in the category of presheaves with transfers.



The map (4)' is defined in a manner similar to that of (4). The map (5) is induced by the canonical inclusion $\mathbb{A}^n \subset \mathbb{P}^n$ where n is j or j'. The map (6) is supplied by the canonical map (3.7) and by the canonical morphism $S(\mathcal{U}) \to \mathbb{Z}_{tr}(X)$ in [Vo2], Lemma 3, p. 352, with respect to the covering of $X = \mathbb{P}^j \times \mathbb{P}^{j'} \setminus \{(0,0)\}$ by $(\mathbb{P}^j \setminus \{0\}) \times \mathbb{P}^{j'}$ and $\mathbb{P}^j \times (\mathbb{P}^{j'} \setminus \{0\})$. The maps (7), (9) and (11) are induced by the canonical inclusion $\mathbb{A}^{j+j'} \subset X$ where X is $\mathbb{P}^j \times \mathbb{P}^{j'}$, B, or $\mathbb{P}^{j+j'}$. The maps (8), (10), (15), and (17) are induced by the canonical morphisms in (3.9). The map (13) is supplied by the canonical map (3.7) and by the sequence (3.8) for the normal crossing variety $X = (\mathbb{P}^{j-1} \times \mathbb{P}^{j'}) \cup (\mathbb{P}^j \times \mathbb{P}^{j'-1})$. The maps (12) and (18) are induced by the canonical inclusion $\mathbb{P}^{n-1} \subset \mathbb{P}^n \setminus \{0\}$ for n = j, j' or j+j'. The maps (14) and (16) are induced by the canonical inclusions $(\mathbb{P}^{j-1} \times \mathbb{P}^{j'}) \cup (\mathbb{P}^j \times \mathbb{P}^{j'-1}) \subset$ $\mathbb{P}^j \times \mathbb{P}^{j'} \setminus \{(0,0)\}$ and $B \setminus (\mathbb{A}^j \times \mathbb{A}^{j'}) \subset B \setminus \{(0,0)\}$ respectively.

3.5. We claim that all the maps (i) for $i \geq 2$ and (4)' in the diagrams above are isomorphisms in $DM_{-}^{eff}(k)$, that is, after sheafification and application of RC. The maps (2), (3), (4), (4)', and (6) are isomorphisms by [Vo2], Lemma 3, p. 352. We can easily see that the maps (5) and (11) are isomorphisms by applying [Vo2], Lemma 3, p. 352, to the covering $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n \setminus \{0\})$ where n = j, j', or j + j'. Similarly, we can see that the maps (7) and (9) are isomorphisms by applying [Vo2], Lemma 3, p. 352, to the covering $X = \mathbb{A}^n \cup (X \setminus \{(0,0)\})$ where X is $\mathbb{P}^j \times \mathbb{P}^{j'}$ or B. Hence all the maps in the first diagram are isomorphisms. It follows from the commutativity of the diagram that the maps (8) and (10) are isomorphisms. The maps (12) and (18) are isomorphisms in $DM_{-}^{eff}(k)$ since the canonical inclusion $\mathbb{P}^{n-1} \subset \mathbb{P}^n \setminus \{0\}$ can be regarded as the zero section of a line bundle over \mathbb{P}^{n-1} . It follows from the exact sequence (3.8) that the map (13) is an isomorphism in $DM_{-}^{eff}(k)$. Below we prove that the map (16) is an isomorphism. Then from the commutativity of the diagram, it follows that the maps (15) and (17) are isomorphisms, thus proving the claim.

Lemma 3.5. The map (16) is an isomorphism.

Proof. Let E denote the exceptional divisor of the blowup $B \to \mathbb{P}^j \times \mathbb{P}^{j'}$. The scheme $B \setminus \{(0,0)\}$ is the union $B \setminus \{(0,0)\} = U_0 \cup U_1 \cup U_2$ of three open subschemes

where $U_0 = B \setminus (\{0\} \times \mathbb{P}^{j'} \cup \mathbb{P}^{j'} \times \{0\}), U_1 = B \setminus (\{0\} \times \mathbb{P}^{j'} \cup E \cup (\mathbb{P}^j \times \mathbb{P}^{j'-1}))$ and $U_2 = B \setminus (\mathbb{P}^j \times \{0\} \cup E \cup (\mathbb{P}^{j-1} \times \mathbb{P}^{j'}))$. We put $D = B \setminus (\mathbb{A}^j \times \mathbb{A}^{j'})$. Let I be a subset of $\{0, 1, 2\}$. We claim that the canonical map

$$\mathbb{Z}_{tr}(\bigcap_{i\in I}U_i\cap D)\to\mathbb{Z}_{tr}(\bigcap_{i\in I}U_i)$$

is an isomorphism in $DM_{-}^{eff}(k)$ unless $I = \emptyset$, $\{1, 2\}$, or $\{0, 1, 2\}$. In the next paragraph we give a proof of the claim only in the case where $I = \{0\}$, since in the other cases the proof is much simpler.

The scheme U_0 is the blowup of $(\mathbb{P}^j \setminus \{0\}) \times (\mathbb{P}^{j'} \setminus \{0\})$ at the center $\mathbb{P}^{j-1} \times \mathbb{P}^{j'-1}$. Let Y_1 (resp. Y_2) denote the proper transform of $(\mathbb{P}^j \setminus \{0\}) \times \mathbb{P}^{j'-1}$ (resp. $\mathbb{P}^{j-1} \times (\mathbb{P}^{j'-1} \setminus \{0\}))$ in U_0 . Then the scheme $U_0 \cap D$ is a normal crossing variety whose decomposition into irreducible components is described as $U_0 \cap D = E \cup Y_1 \cup Y_2$. Moreover the intersection $Y_1 \cap Y_2$ is empty. Hence by (3.8) we have a canonical isomorphism from the complex

$$(3.10) \qquad \mathbb{Z}_{tr}(E \cap Y_1) \oplus \mathbb{Z}_{tr}(E \cap Y_2) \to \mathbb{Z}_{tr}(E) \oplus \mathbb{Z}_{tr}(Y_1) \oplus \mathbb{Z}_{tr}(Y_2)$$

(where $\mathbb{Z}_{tr}(E) \oplus \mathbb{Z}_{tr}(Y_1) \oplus \mathbb{Z}_{tr}(Y_2)$ is placed in degree zero) to $\mathbb{Z}_{tr}(U_0 \cap D)$ in $DM_-^{eff}(k)$. We have a canonical isomorphism $Y_1 \cong (\mathbb{P}^j \setminus \{0\}) \times \mathbb{P}^{j'-1}$ (resp. $Y_2 \cong \mathbb{P}^{j-1} \times (\mathbb{P}^{j'-1} \setminus \{0\}))$ whose restriction to $Y_1 \cap E$ (resp. $Y_2 \cap E$) gives an isomorphism $Y_1 \cap E \cong \mathbb{P}^{j-1} \times \mathbb{P}^{j'-1}$ (resp. $Y_2 \cap E \cong \mathbb{P}^{j-1} \times \mathbb{P}^{j'-1}$). Thus the inclusion $Y_i \cap E \subset Y_i$ for i = 1, 2 can be regarded as the zero section of a line bundle over $Y_i \cap E$, which implies that the canonical map $\mathbb{Z}_{tr}(Y_i \cap E) \to \mathbb{Z}_{tr}(Y_i)$ is an isomorphism in $DM_-^{eff}(k)$. This shows that the canonical map from $\mathbb{Z}_{tr}(E)$ to the complex (3.10) is an isomorphism in $DM_-^{eff}(k)$. Hence the map $\mathbb{Z}_{tr}(E) \to \mathbb{Z}_{tr}(U_0 \cap D)$ induced by the inclusion $E \subset U_0 \cap D$ is an isomorphism in $DM_-^{eff}(k)$. Since $(\mathbb{P}^j \setminus \{0\}) \times (\mathbb{P}^{j'} \setminus \{0\})$ is canonically isomorphic to a vector bundle of rank two over $\mathbb{P}^{j-1} \times \mathbb{P}^{j'-1}$, the canonical map $\mathbb{Z}_{tr}(E) \to \mathbb{Z}_{tr}(U_0)$ induced by the inclusion $E \hookrightarrow U_0$ is an isomorphism in $DM_-^{eff}(k)$ by Lemma 3.6 below. This proves the claim in the previous paragraph in the case where $I = \{0\}$.

Since $U_0 \cap U_1 \cap U_2$ is equal to $U_1 \cap U_2$, it follows from [Vo2], Lemma 3, p. 352, that the canonical map from the complex

$$(3.11) \qquad \mathbb{Z}_{tr}(U_0 \cap U_1) \oplus \mathbb{Z}_{tr}(U_0 \cap U_2) \to \mathbb{Z}_{tr}(U_0) \oplus \mathbb{Z}_{tr}(U_1) \oplus \mathbb{Z}_{tr}(U_2)$$

(where $\mathbb{Z}_{tr}(U_0) \oplus \mathbb{Z}_{tr}(U_1) \oplus \mathbb{Z}_{tr}(U_2)$ is placed in degree zero) to $\mathbb{Z}_{tr}(B \setminus \{(0,0)\})$ is an isomorphism in $DM_{-}^{eff}(k)$. The claim in the previous paragraph shows that the canonical map from the complex

$$\mathbb{Z}_{tr}(U_0 \cap U_1 \cap D) \oplus \mathbb{Z}_{tr}(U_0 \cap U_2 \cap D) \to \mathbb{Z}_{tr}(U_0 \cap D) \oplus \mathbb{Z}_{tr}(U_1 \cap D) \oplus \mathbb{Z}_{tr}(U_2 \cap D)$$

to the complex (3.11) is an isomorphism in $DM_{-}^{eff}(k)$. Again by [Vo2], Lemma 3, p. 352, the canonical map from the latter complex to $\mathbb{Z}_{tr}(D)$ is an isomorphism in $DM_{-}^{eff}(k)$. This proves that the map (16) is an isomorphism in $DM_{-}^{eff}(k)$.

Lemma 3.6. Let X be a scheme which is smooth over Spec k. Let $V \to X$ be a vector bundle over X. We regard X as a closed subscheme of V via the zero section $X \hookrightarrow V$. Let B denote the blowup of V with center X. Let $E \subset B$ be the exceptional divisor. Then the canonical map $\mathbb{Z}_{tr}(E) \to \mathbb{Z}_{tr}(B)$ is an isomorphism in $DM_{-}^{eff}(k)$. *Proof.* We give a proof which was suggested by the referee.

Let the notations be as above. Consider the cartesian square:



where s is the zero section and p is the blowup. Then it follows from [Vo-Su-Fr], Proposition 3.5.2, p.219, that the following square

$$\mathbb{Z}_{tr}(E) \xrightarrow{t_*} \mathbb{Z}_{tr}(B)$$

$$-q_* \downarrow \qquad \qquad \downarrow p_*$$

$$\mathbb{Z}_{tr}(X) \xrightarrow{s_*} \mathbb{Z}_{tr}(V)$$

is homotopy cartesian. In other words, the morphism $\operatorname{Cone}(t_*) \to \operatorname{Cone}(s_*)$, induced by taking the cones of the horizontal maps in the diagram above, is an isomorphism in $DM_{-}^{eff}(k)$. As $\operatorname{Cone}(s_*) = 0$, we have the claim.

3.6.

Proof of Proposition 3.3. By the definition of the isomorphism (3.5) in $DM_{-}^{eff}(k)$, the commutativity of the diagram in the statement of Proposition 3.3 follows from the commutativity of the following diagram

where the maps (19) and (23) are the maps induced by the restriction of the cycles on $\mathbb{P}^n \times U$ to $\mathbb{A}^n \times U$ for each scheme U smooth over Spec k and for n = j, j' or j+j', the map (20) is the canonical map (see the paragraphs preceding Proposition 3.3) and the maps (21) and (22) are the maps induced by the restriction of the cycles on $X \times U$ to $\mathbb{A}^j \times \mathbb{A}^{j'} \times U$ for each scheme U smooth over Spec k and for $X = \mathbb{P}^j \times \mathbb{P}^{j'}$ or B. The commutativity of the top square follows since the product appearing in the vertical maps is given by an exterior product of cycles. The commutativity of the rest of the diagram is easy to see. This completes the proof of Proposition 3.3.

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