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Bounds for state degeneracies in 2D conformal field theory

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Abstract: In this note we explore the application of modular invariance in 2-dimensional CFT to derive universal bounds for quantities describing certain state degeneracies, such as the thermodynamic entropy, or the number of marginal operators. We show that the entropy at inverse temperature 2π satisfies a universal *lower* bound, and we enumerate the principal obstacles to deriving *upper* bounds on entropies or quantum mechanical degeneracies for completely general CFT. We then restrict our attention to infrared-stable CFT with moderately low central charge, in addition to the usual assumptions of modular invariance, unitarity and discrete operator spectrum. For CFT in the range $c_{\mathrm{L}} + c_{\mathrm{R}} < 48$ with no relevant operators, we are able to prove an upper bound on the thermodynamic entropy at inverse temperature 2π. Under the same conditions we also prove that a CFT can have no more than $\left(\frac{c_{\mathrm{L}}+c_{\mathrm{R}}}{48-c_{\mathrm{L}}-c_{\mathrm{R}}}\right) \cdot \exp\{+4\pi\} - 2$ marginal deformations.

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1 Introduction

Conformal field theory plays several distinct roles in our understanding of nature, including quantum gravity. First, 2D CFT defines the classical solutions of the mysterious system whose perturbative amplitudes are given by the dynamics of relativistic strings. Second of all, conformal field theory defines the Hamiltonian of quantum gravity in spaces with negative cosmological constant and asymptotically anti-de Sitter boundary conditions. Third of all, conformal field theory describes fixed points of the renormalization group flow that organizes the behavior of all local quantum systems, including string theory in the low-energy approximation. Beyond these important roles, CFT appears in many other guises in various aspects of fundamental physics.

It is therefore of interest to understand as clearly as possible the gross features of the "landscape" of conformal field theories [3, 20]. Even the case that is by far best explored, that of two-dimensional CFT, is far from being understood in a systematic way.

Recently multiple independent lines of development have converged to probe the consistency conditions on the space of conformal field theories in extreme limits. In particular, various papers over recent years have investigated the question of how big a gap can ever be opened in the spectrum of operator dimensions, above a universal sector defined by products of stress tensors.

This question is explored under various simplifying assumptions in different articles: in [4, 8, 9], the two-dimensional case is explored under the assumption of holomorphic factorization of the Hilbert space; in [18] the two-dimensional case is explored again and extended superconformal symmetry is assumed; in [5, 6] the authors examine the maximum possible dimension of the lowest-dimension operator appearing in the operator product expansion of two scalar operators whose dimensions are taken as given; in [2] the two-dimensional case is probed once again and the dimension of the lowest primary operator is bounded in a completely general CFT satisfying the minimal properties of unitarity and discrete operator spectrum.

In all cases the result is qualitatively the same: if one fixes the input parameters — in two dimensions, the central charge, and in the case of [5, 6] the dimensions of the external scalar operators in the OPE — the lowest-dimension operator that appears can never have a dimension higher than some universal bound.[1] Strengthening the assumptions — adding special conditions such as supersymmetry or holomorphic factorization — improves the bound numerically but does not qualitatively change the result.

In each case discussed above, the constraining principle is either modular invariance, or else associativity of the OPE, which is closely related. Both are consistency conditions expressing the constraint that the theory must make sense when quantized in two different, inequivalent channels — that is, foliating the space with two different time-slicings whose leaves may be orthogonal to one another. It would appear that the condition of covariance among channels, or democracy among foliations, is a fundamental principle that makes it impossible to pick and choose the spectrum of a CFT at will. In particular, the condition of modular invariance, or channel covariance of the OPE, is incompatible with an attempt to deform the spectrum in an extreme way.

In all cases discussed above, the gap in operator dimensions is the quantity under examination, that is bounded by the principle of modular invariance, or channel covariance more generally. There are other "extreme directions" in the space of possible spectra, in which we suspect it may be impossible to engineer the spectrum of a consistent CFT. In particular, one expects that a CFT of fixed central charge ought not to have a number of marginal deformations greater than some universal number depending on the central charge. That is, for a CFT of central charge $c_{\rm tot}$ and discrete spectrum, it would appear likely that there may be a fundamental limit on the dimension of moduli space.

This idea is correlated with commonly held beliefs in mathematics and physics. In physics, the holographic principle of 't Hooft and Susskind [26, 27, 29] could be vitiated by limits in which the number of massless species is pushed infinitely large [28, 29]. In mathematics, it is thought that there is likely an upper bound on the Euler number of compact Calabi-Yau threefolds [30–32], which would follow immediately from a fully general bound on the number of marginal operators of a superconformal field theory of central charge $c_{\rm L} = c_{\rm R} = c = \frac{3}{2}\hat{c} = 9$.

In this note we derive rigorous bounds on certain state degeneracies: First, a lower bound on the thermodynamic entropy at the inverse temperature $\beta = 2\pi$ that maps to itself under the modular S-transformation. Second, we prove an upper bound on the number of marginal operators of a two-dimensional CFT of a given central charge, under certain conditions. Under the same conditions we also derive an upper bound on the thermodynamic entropy at inverse temperature $\beta = 2\pi$.

[1]In [7] the converse is shown: By making the central charge sufficiently large, it is possible to push the gap to infinity and obtain a consistent limit where the theory contains only a sector of low-dimension operators.

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The bounds proven herein, though requiring certain conditions to hold that are less than fully general, provide an illustration of a principle bounding state degeneracies from above, that the authors hope may apply in a broader set of circumstances.

2 Energy and entropy bounds from modular invariance

2.1 Review of modular invariance

The principle underlying our bounds is modular invariance, in particular invariance under the modular S-transformation $\tau \to -\frac{1}{\tau}$ of the partition function of the CFT on a two-torus with complex structure τ. The partition function of a two-dimensional CFT on such a torus can be written as

$$Z[\tau] = \mathrm{tr}\left(\exp\left\{2\pi i\tau\left(L_0 - \tfrac{c_{\mathrm{R}}}{24}\right) - 2\pi i\bar{\tau}\left(\tilde{L}_0 - \tfrac{c_{\mathrm{L}}}{24}\right)\right\}\right) , \tag{2.1}$$

where L_0 and $\tilde{L}_0$ are the zeroth right- and left-moving Virasoro generators, c_{R} and c_{L} are the right- and left-moving central charges, and the complex structure τ lies in the upper half plane. The torus can be thought of as a quotient of the complex plane $\mathbb{C} = \{\sigma \equiv \sigma^1 + i\sigma^2\}$ by the identifications $\sigma \sim \sigma + 2\pi \sim \sigma + 2\pi\tau$. The generators L_0 and $\tilde{L}_0$ are related to energy H and momentum P_1 by

$$\begin{aligned} L_0 &= \tfrac{1}{2}(H + P_1) + \tfrac{c_{\mathrm{R}}}{24} , \\ \tilde{L}_0 &= \tfrac{1}{2}(H - P_1) + \tfrac{c_{\mathrm{L}}}{24} . \end{aligned}$$

We can represent this partition function as a path integral on a torus with metric

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \frac{1}{\mathrm{Im}\ \tau} \begin{pmatrix} 1 & \mathrm{Re}\ \tau \\ \mathrm{Re}\ \tau & |\tau|^2 \end{pmatrix}$$

Modular invariance of the partition function is the statement that a local conformal field theory has nothing to distinguish the various cycles of the two-torus other than the background metric itself. Therefore, partition functions on two distinct background metrics differing by a large coordinate transformation should have the same partition function. A large coordinate transformation acts on the cycles of the torus as

$$\mathrm{SL}(2, \mathbb{Z}) \ni \Gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

which induces an action $\tau \mapsto \frac{a\tau + b}{c\tau + d}$ on the complex structure of the torus.

The modular group is generated by two elements, $S \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T \equiv \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which satisfy the relation $(ST)^3 = -1$. For modular invariance to be a good symmetry of the CFT at the quantum level, it suffices to check that the partition function transforms under both S and T without anomalous phases.

For purposes of this note, as in [2], we shall not require invariance under the T-transformation, but only under the S-transformation. Failure of the partition function to be invariant under the T-transformation is easily understood in the Hamiltonian framework as a failure of the momentum P_1 to be quantized in integer units; the expression (2.1) is invariant under $\tau \to \tau + 1$ if and only if every state in the spectrum has $P_1 \in \mathbb{Z}$.

The S-transformation is both more obscure and more robust.

It is obscure in the sense that it is completely non-manifest in the Hamiltonian formalism. That is to say, a Hamiltonian formulation of a quantum theory depends on a foliation of spacetime, with Hilbert spaces associated to the individual leaves and Hamiltonian flow implementing linear transformations between the Hilbert spaces on different leaves. But the S-transformation is not a canonical transformation, and does not preserve even a single leaf of any smooth foliation of the torus, even up to Hamiltonian flow. The S-transformation, even when an exact symmetry of the system, is *not* realized as an action on the Hilbert space of quantum states.

On the other hand, the S-transformation is a good symmetry of the quantum theory under very broad conditions: if the theory has a Poincaré-invariant path integral formulation in terms of local variables, then the path integral automatically respects the S-transformation at the quantum level. There are many known theories that satisfy all axioms of CFT except symmetry under the S-transformation; however all such examples involve imposing projections directly on a Hilbert space and do not have partition functions defined by a path integral over local variables. In this paper we shall restrict our attention to modular-invariant CFT, though only invariance under the S-transformation is necessary for our results to hold.

2.2 Review of previous work

The medium-temperature expansion.

Going forward let us assume that the CFT is described by a unitary quantum mechanics (i.e., a Hermitean Hamiltonian with a positive definite norm on the Hilbert space), and that the spectrum of the Hamiltonian in finite volume is discrete. Thus the partition function can be written as

$$Z[\beta] = \mathrm{tr}\,(\exp\{-\beta H\}) = \textstyle\sum_n \exp\{-\beta E_n\}\,, \tag{2.2}$$

where E_n are the discrete, real eigenvalues of the Hamiltonian H of the theory on a circle of length 2π.

By virtue of Cardy's formula, the partition function and all of its derivatives are convergent for any positive β. The $p^{\underline{\mathrm{th}}}$ derivative is equal to a sum of derivatives of exponentials, that is

$$\left(\frac{\partial}{\partial\beta}\right)^p Z[\beta] = (-1)^p\,\mathrm{tr}\,(H^p \exp\{-\beta H\}) = (-1)^p \textstyle\sum_n E_n^p \exp\{-\beta E_n\}\ . \tag{2.3}$$

For purely imaginary complex structure $\tau = \frac{i\beta}{2\pi}$ expression (2.1) reduces to the usual thermodynamic partition function (2.2) at inverse temperature β. Then the S-transformation acts on the partition function as

$$Z[\beta] \to Z[\tfrac{4\pi^2}{\beta}]\ . \tag{2.4}$$

A modular invariant partition function is invariant under this transformation:

$$Z[\beta] = Z[\tfrac{4\pi^2}{\beta}] \ . \tag{2.5}$$

It follows immediately [2] that

$$(\beta\partial_\beta)^p Z[\beta]|_{\beta=2\pi} = 0 \ . \tag{2.6}$$

for any positive odd p. Written in terms of the energies E_n, this can be written

$$\textstyle\sum_n \exp\{-2\pi E_n\}\, f_p(E_n) = 0\,, \tag{2.7}$$

where $f_p(E)$ is a $p^{\underline{\text{th}}}$ order polynomial defined by

$$f_p(E) \equiv \exp\{+2\pi E\}\ \left(\beta\tfrac{\partial}{\partial\beta}\right)^p\ \exp\{-\beta E\}\Big|_{\beta=2\pi} \ . \tag{2.8}$$

Explicit expressions for low p are:

$$\begin{aligned} f_1(E) &= -2\pi E \\ f_3(E) &= -(2\pi E)^3 + 3(2\pi E)^2 - (2\pi E) \end{aligned} \tag{2.9}$$

More generally, if $F(x)$ is any odd function of x, then we define a derived function

$$f_F(E) \equiv \exp\{+2\pi E\}\ F\left(\beta\tfrac{\partial}{\partial\beta}\right)\cdot\exp\{-\beta E\}\Big|_{\beta=2\pi} \ . \tag{2.10}$$

The polynomials f_p are just the derived functions corresponding to $F(x) = x^p$.

For any odd F the derived function $f_F(E)$ satisfies

$$\textstyle\sum_n \exp\{-2\pi E_n\} f_F(E_n) = 0\,, \tag{2.11}$$

where $\{E_n\}$ is the spectrum of a modular invariant CFT. For some purposes it is convenient to think of this condition in terms of a density

$$\rho(E) \equiv \exp\{-2\pi E\}\ \textstyle\sum_n \delta(E - E_n)\,, \tag{2.12}$$

which is $\exp\{-2\pi E\}$ times the usual spectral density. Then we can write the condition (2.11) as

$$\textstyle\int dE\, \rho(E)\, f_F(E) = 0 \tag{2.13}$$

for an f derived from any odd F. By virtue of Cardy's formula, the partition function is real analytic as a function of β for any β. Therefore the conditions (2.7) for all odd p are not only a consequence of invariance under the S-transformation, but when taken together are sufficient to imply it as well.

The condition (2.7) follows directly from writing

$$\beta \equiv 2\pi \exp\{s\}\,,$$

noting that the S-transformation acts as $s \mapsto -s$, and expanding the equation (2.5) to $p^{\underline{\text{th}}}$ order in s. Since we are expanding the partition function in the neighborhood of $\beta = 2\pi$, which is intermediate between the high-temperature $\beta \to 0$ and low-temperature $\beta \to \infty$ régimes, we refer to the expansion in s as the *medium-temperature* expansion [2]. The medium-temperature expansion has proven to be useful in deriving general constraints on the spectrum of a modular invariant conformal field theory [2, 10, 11].

Review of the upper bound on Δ_1.

In [2] we used the medium-temperature expansion to prove that any conformal field theory satisfying unitarity and modular invariance, with a discrete spectrum, satisfies the bound

$$\Delta_1 \leq \tfrac{c_{\text{tot}}}{12} + 0.48\,, \tag{2.14}$$

where

$$c_{\text{tot}} \equiv c_{\text{L}} + c_{\text{R}} \tag{2.15}$$

and Δ_1 is the dimension of the primary operator of lowest dimension other than the identity itself. For $c_{\text{tot}} < 24 - \frac{18}{\pi} \simeq 18.270$, the proof is completely elementary and does not depend on any use of representation theory of the Virasoro algebra. We recall the proof here.

Fix a value of the central charge c_{tot} less than $24 - \frac{18}{\pi}$. Now consider the cubic polynomial

$$F(x) \equiv x\,(x^2 - 4\pi^2 E_0^2 + 6\pi E_0 - 1)\,, \tag{2.16}$$

where E_0 is the ground state energy

$$E_0 = -\tfrac{c_{\text{tot}}}{24}. \tag{2.17}$$

Using expressions (2.9), the derived polynomial is

$$\begin{aligned} f_F(E) &= f_3(E) - (4\pi^2 E_0^2 - 6\pi E_0 + 1) f_1(E) \\ &= -8\pi^3\, E\,(E - E_0)\,(E - E_+)\,, \end{aligned} \tag{2.18}$$

where $E_+ \equiv \frac{3}{2\pi} - E_0$. By equation (2.13), the quantity

$$\textstyle\int_{E_0}^{\infty} f_F(E)\rho(E)dE \tag{2.19}$$

must vanish. The measure $\rho(E)$ is positive and the derived polynomial $f_F(E)$ vanishes at E_0 and is negative for $E > E_+$. If all excited energy levels were to be E_+ or higher, then the quantity (2.19) could not vanish, as it would receive only negative contributions. The lowest excited level E_1 must therefore be lower than $E_+ = \frac{3}{2\pi} - E_0$. Translating into operator dimensions via $\Delta = E - E_0$, we find that the lowest-dimension operator other than the identity can have dimension no higher than $\Delta_+ \equiv E_+ - E_0 = \frac{3}{2\pi} - 2E_0$. Using (2.17) we write the bound as

$$\Delta_1 < \Delta_+ \equiv \tfrac{3}{2\pi} + \tfrac{c_{\text{tot}}}{12}\ . \tag{2.20}$$

For $c_{\text{tot}} < 24 - \frac{18}{\pi}$ the value of Δ_+ is less than 2, and it follows that the operator of dimension Δ_1 cannot be a descendant of the identity; it must therefore be primary or else the descendant of a primary operator of even lower dimension. So in this range of central charge, we learn that equation (2.20) can be taken to apply specifically to the dimension of the lowest *primary* operator above the identity.

Medium-temperature equations with characters.

For higher central charge a similar bound can be proven using roughly the same argument, with the contributions to the partition function organized according to full representations of the Virasoro algebra rather than individual energy levels. The proof for $c_{\text{tot}} > 24 - \frac{18}{\pi}$ uses some elementary facts about the representation theory of the Virasoro algebra, in particular formulae for the characters of the Virasoro algebra for various representations [12–15].

It is natural to generalize the derived polynomials $f_p(E)$ to derived polynomials $f_{p|\chi}$ with respect to characters χ of the Virasoro algebra. In this context, we will only consider characters restricted to the imaginary axis of their argument, $\chi(\beta)$ with $\tau = \frac{i\beta}{2\pi}$.

Our simplifying assumptions, that $c_{\text{L}}, c_{\text{R}} > 1$ and that there are no holomorphic or antiholomorphic operators, guarantee that there are only two types of representations of the Virasoro algebra for a unitary CFT with discrete spectrum [12, 13, 15]:

- the conformal family of the vacuum, generated by the independent states

$$\prod_{m,n\geq 2} (L_{-m})^{N_m} (\tilde{L}_{-n})^{\tilde{N}_n} \, |0\rangle \ , \tag{2.21}$$

  and

- the conformal family of a generic primary with dimension Δ, generated by the independent states

$$\prod_{m,n\geq 1} (L_{-m})^{N_m} (\tilde{L}_{-n})^{\tilde{N}_n} \, |\Delta\rangle \ , \tag{2.22}$$

with the $N_m, \tilde{N}_n$ running over all possible $N_m, \tilde{N}_n \geq 0$ in each case.

Each type of conformal family is then spanned by different monomials in Virasoro raising operators acting on the primary state, with each monomial raising the energy of the vacuum by an amount $\sum_n n\, N_n + n\tilde{N}_n$. The contributions of the two families to the partition function are then given by

$$\chi_v(\beta) \exp\{-E_0\beta\} \tag{2.23}$$

for the vacuum family, and

$$\chi_g(\beta) \exp\{-(\Delta + E_0)\beta\} \tag{2.24}$$

for the family of the generic primary, with the vacuum and generic characters given by

$$\begin{aligned} \chi_v(\beta) &\equiv \textstyle\prod_{n\geq 2} (1 - \exp\{-n\beta\})^{-2} \, , \\ \chi_g(\beta) &\equiv \textstyle\prod_{n\geq 1} (1 - \exp\{-n\beta\})^{-2} \, , \end{aligned} \tag{2.25}$$

respectively. Then the full partition function can be written as

$$Z[\beta] = \chi_v(\beta)\exp\{-\beta E_0\} + \chi_g(\beta)(\textstyle\sum_{n=1}^{\infty} \exp\{-\beta E_n\})\,, \tag{2.26}$$

where the sum in the second term runs over energies $E_n \equiv \Delta_n + E_0$ of *primary states* alone. Then invariance under the modular S-transformation $\beta \to 4\pi^2/\beta$ can be expressed in terms of energies of primary states. For a given function $F(x)$ and any character χ, define the derived functions

$$f_{F\,|\,\chi}(E) \equiv \tfrac{\exp\{+2\pi E\}}{\chi(2\pi)}\; F\left(\beta\partial_\beta\right)\cdot \left[\chi(\beta)\,\exp\{-\beta E\}\right]\Big|_{\beta=2\pi}\;. \tag{2.27}$$

Then the invariance under the modular S-transformation can be expressed by the condition that

$$\chi_v(2\pi)\exp\{-2\pi E_0\} f_{F\,|\,\chi_v}(E_0) + \textstyle\sum_{n=1}^{\infty}\chi_g(2\pi)\exp\{-2\pi E_n\}\, f_{F\,|\,\chi_g}(E_n) = 0\,, \tag{2.28}$$

for any odd F. To express this condition in terms of measures, define

$$\rho_v(E) \equiv \chi_v(2\pi)\exp\{-2\pi E\}\delta(E - E_0) \tag{2.29}$$

and

$$\rho_g(E) \equiv \chi_g(2\pi)\exp\{-2\pi E\}\textstyle\sum_{n=1}^{\infty}\delta(E - E_n)\,, \tag{2.30}$$

where n runs over non-vacuum primary states. Then the condition for modular invariance is

$$\textstyle\int dE\, \rho_v(E) f_{F\,|\,\chi_v}(E) + \int dE\, \rho_g(E) f_{F\,|\,\chi_g}(E) = 0 \tag{2.31}$$

for any odd F. For an arbitrary character $\chi(\beta)$, the derived polynomials $f_{p\,|\,\chi}$ corresponding to low-order monomials $F(x) = x^p$ are

$$\begin{aligned}
f_{1\,|\,\chi}(E) &= -(2\pi E) + 2\pi\frac{\chi'(2\pi)}{\chi(2\pi)}\\
f_{3\,|\,\chi}(E) &= -(2\pi E)^3 + (2\pi E)^2\left(6\pi\frac{\chi'(2\pi)}{\chi(2\pi)} + 3\right)\\
&\quad -(2\pi E)\left(12\pi^2\frac{\chi''(2\pi)}{\chi(2\pi)} + 12\pi\frac{\chi'(2\pi)}{\chi(2\pi)} + 1\right) + \left(8\pi^3\frac{\chi'''(2\pi)}{\chi(2\pi)} + 12\pi^2\frac{\chi''(2\pi)}{\chi(2\pi)} + 2\pi\frac{\chi'(2\pi)}{\chi(2\pi)}\right)
\end{aligned} \tag{2.32}$$

In [2] we use this structure to derive a bound on the weight of the lowest non-vacuum primary dimension Δ_1 that applies for arbitrarily high values of the total central charge. We refer the reader to [2] for details. We want to emphasize, however, that the generalized proof is not much more complicated than the elementary proof for low central charge that we have reviewed in the previous subsection.

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2.3 A lower bound for the entropy at medium temperature

The most straightforward consequence of modular invariance is to give a *lower*, rather than an upper bound for the thermodynamic entropy of the canonical ensemble, particularly at inverse temperature $\beta = 2\pi$.

From the first order of the medium-temperature expansion, it follows that a universal lower bound holds for the thermodynamic entropy at medium temperature $\beta^{-1} = \frac{1}{2\pi}$. The entropy σ is related to the partition function via

$$\sigma = \ln(Z) + \beta \langle E \rangle \ . \tag{2.33}$$

Using the derived function $f_1(E) = -2\pi E$ in equation (2.7), we have

$$\langle E \rangle|_{\beta=2\pi} = 0 \ . \tag{2.34}$$

By unitarity, every contribution to the partition function is positive, so the value of Z is bounded below by its vacuum contribution $\exp\{-2\pi E_0\}$, so we have

$$\sigma|_{\beta=2\pi} \geq -2\pi E_0 = \tfrac{\pi c_{\text{tot}}}{12} \ . \tag{2.35}$$

We have given an elementary proof of a universal lower bound for the thermodynamic entropy in a modular invariant 2D CFT at a particular temperature. On the other hand we shall see in the next section that there can be no fully general upper bound on the thermodynamic entropy or on the microstate degeneracies that would hold without imposing additional assumptions on the CFT. Understanding the issues involved will help us to formulate a useful set of additional assumptions on the CFT as we go forward.

3 Meta-problem: why are upper bounds for state degeneracies hard?

In the previous section we reviewed the derivation of an upper bound on the energy of the first excited energy level. Should it not be possible, then, to prove an upper bound on entropy — thermodynamic entropy or quantum mechanical degeneracies — using similar techniques? Let us examine, briefly, a few reasons why the type of argument in the previous section cannot be generalized very easily to give a fully general upper bound for entropy or quantum mechanical degeneracies, without imposing additional assumptions on the CFT as inputs.

(a) *The homogeneity problem.*

The homogeneity problem is a meta-problem with any candidate for a method to bound the entropy above, using invariance under the S-transformation. Suppose we had some equation of type (2.13) that would *always be violated* if the entropy — the thermodynamic entropy or the quantum mechanical degeneracy, in some energy range, according to whatever definition — were to be sufficiently high. We can argue by contradiction that no such equation can ever exist.

Suppose such an equation did exist, that ruled out the possibility of a modular invariant spectrum with effective state degeneracy greater than n_{max}, by whatever definition.

But we could always take k copies of a modular invariant spectrum with effective state degeneracy $n < n_{\max}$, such that $k\, n > n_{\max}$. Taking k copies of a modular invariant spectrum automatically yields a modular invariant spectrum, so it is clear that there can be no direct constraint from modular invariance bounding the effective state degeneracy above, without using additional inputs.

(b) *The continuum problem at the vacuum.*

One way around the homogeneity problem is to use the fact that the spectrum under consideration is not arbitrary, but corresponds to the spectrum of a good CFT, satisfying all the usual CFT axioms including cluster decomposition. Together with unitarity and the state-operator correspondence, cluster decomposition implies that there is a unique lowest state, with energy $E_0 = -\frac{c_{\rm tot}}{24}$.

Then it may be possible in principle to find an upper bound on the entropy (or the degeneracy of some states in some range of energy) that evades the homogeneity problem by using the fact that the degeneracy of the vacuum is never greater than 1.

To exploit this fact, it would be necessary to find an odd function F such that the derived function $f_F(E)$ gets a positive contribution from the vacuum, a negative contribution from the states of interest, and a sum of contributions from all other states that is bounded above independently. (In particular, it would not do to pick an F such that $f_F(E_0) = 0$, as we did to prove the universal upper bound on E_1.)

This type of approach to bounding the entropy suffers from a separate meta-problem that we shall call the *continuum problem*. By the continuum problem, we mean that such a proof could never apply in cases where the spectrum develops an approximate continuum of states with energies an arbitrarily small amount above the vacuum. In such a case the continuum would contribute with the same sign as the vacuum (by continuity of $f(E)$) with an unboundedly large coefficient, due to the presence of an arbitrarily numerous set of levels in the range between E_0 and $E_0 + \epsilon$.

Of course, no CFT under our consideration ever has a strict continuum; we are always assuming that our CFT have a discrete operator spectrum, or equivalently discrete spectrum of the Hamiltonian in finite volume. But many CFT are known to come in families with singular limits where the limiting spectrum has a continuum of some kind. In particular, many familiar moduli spaces of CFT have limits in which the CFT can be thought of as a sigma model with volume approaching infinity. So the continuum problem is a general no-go principle for upper bounds on CFT degeneracies of any sort that do not use additional consistency conditions of the CFT, or assume a minimum gap in the spectrum above the vacuum.

(c) *The hyperfine structure problem of character corrections.*

To evade the continuum problem without assuming a minimum gap in the spectrum as an input, we could try using other consistency conditions of the CFT; in particular, we could try using the organization of the spectrum into representations of the Virasoro algebra. However this approach immediately runs into the problem that the differences between different characters of the Virasoro algebra are numerically very tiny.

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As in [2], simplify the discussion by assuming that c_{L} and c_{R} are both greater than 1, and that there is no chiral algebra of the theory other than the Virasoro algebra. The condition (2.31) then treats the vacuum conformal family, which contributes to ρ_v, differently from the conformal families of the other primaries, which contribute to ρ_g. Therefore it is possible in principle to find odd F such that the derived polynomials $f_{F\,|\,\chi_v}(E_0)$ and $f_{F\,|\,\chi_g}(E)$ contribute with different sign, even when E is arbitrarily close to E_0. In practice, however, it seems difficult to generate a useful bound this way.

An approach to overcoming the continuum problem based on the difference in Virasoro representations between that of the vacuum and that of the generic primaries lying in the continuum just above the vacuum, would need to exploit the differences between the derived functions $f_{F\,|\,\chi_v}$ and $f_{F\,|\,\chi_g}$, evaluated at E_0. But the absolute difference between the derived functions for the two different types of conformal family is numerically small, being proportional to $\exp\{-2\pi\} \simeq \frac{1}{535}$.

For instance, suppose we want to take the lowest-order polynomial possible, $F(x) = x^1$. Then the two derived polynomials are

$$\begin{aligned} f_{1\,|\,\chi_v}(E) &= -2\pi E + 4\pi \textstyle\sum_{n=2}^{\infty} \frac{n \exp\{-2\pi n\}}{1-\exp\{-2\pi n\}} \\ f_{1\,|\,\chi_g}(E) &= -2\pi E + 4\pi \textstyle\sum_{n=1}^{\infty} \frac{n \exp\{-2\pi n\}}{1-\exp\{-2\pi n\}} \end{aligned} \tag{3.1}$$

The two differ only by

$$f_{1\,|\,\chi_g}(E) - f_{1\,|\,\chi_v}(E) = 4\pi \frac{\exp\{-2\pi\}}{1-\exp\{-2\pi\}} \simeq 2.35 \times 10^{-2}\ . \tag{3.2}$$

Furthermore, a minimal criterion for overcoming the continuum problem is that the value of $\lim\limits_{E\to E_0} f_{F\,|\,\chi_g}(E)$ should have a different sign from $f_{F\,|\,\chi_v}(E_0)$. In the case $F(x) = x$, it is impossible for the two to differ in sign unless $E_0 > 0$, which would be inconsistent with positivity of the central charge, and thus with unitarity.

(d) *The fine structure problem at large central charge*

There is a separate problem in attempting to bound the entropy from above by using invariance of the partition function under the modular S-transformation, that is particularly acute when the central charge becomes large. To see the nature of the problem, consider in particular an attempt to bound the degeneracy of marginal operators in the limit of large central charge. When the central charge is large, it becomes increasingly difficult to find derived functions $f_F(E)$ that are positive for $E = E_0$, and negative for all $E \geq E_0 + 2$. In the limit $E_0 \to -\infty$ with E/E_0 held fixed, the derived function $f_F(E)$ is to good approximation equal to $F(-2\pi E)$, which is an *odd* function. For $F(x) = x^p$, we have

$$f_p(E) = (-2\pi E)^p + O(E^{p-1}). \tag{3.3}$$

Thus if we take an odd polynomial $F(x)$ such that $f_F(E)$ is positive at $E = E_0$ but negative at $E_0 + 2$, then it will tend to be positive again by the time it reaches $E = -(E_0 + 2)$, if $|E_0|$ is large.

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The even part of the function $f_F(E)$ is a kind of "fine structure" of subleading order at large central charge. But it is only by tuning the form of the function $F(x)$ to enhance the contribution of the even part of f_F relative to the odd part, that one has any hope of obtaining a derived function with properties that could imply a bound.

(e) *The resolutional problem.*

The *resolutional problem* is the difficulty in formulating a suitable definition of the entropy of marginal operators that is robust against small perturbations of the spectrum. Certainly it seems quite unlikely that there should be any unitary conformal field theory, with discrete spectrum and central charges $c_L = c_R = 2$, say, with $10^{10^{100}}$ marginal operators, for instance. And yet, if we alter the question slightly to ask whether there might be a unitary conformal field theory with discrete spectrum and central charges $c_L = c_R = 2$, and $10^{10^{100}}$ operators between dimensions $2 - \epsilon$ and $2 + \epsilon$, for arbitrarily small values of ϵ, the answer is that, *yes*, there certainly do exist such CFT. Simply consider a sigma model with target space T^2, with cycles of length $2\pi R$, and the kinetic term for the target space coordinates X^a normalized as $\mathcal{L} = \frac{1}{4\pi\alpha'} g^{ij}(\partial i X^a)(\partial j X^a)$.

The theory contains scalar primary operators of the form $\mathcal{O}_{n_1,n_2} \equiv : \exp\{i n_a X_a/R\} :$, which have dimension $\Delta = \frac{\alpha'}{2R^2} n_a^2$. For R much larger than $\sqrt{\alpha'}$, the set of operators with $2-\epsilon \leq \Delta \leq 2+\epsilon$ corresponds to the set of integer pairs (n_1, n_2) lying in the Euclidean plane between two spheres centered at the origin, with radii $\sqrt{\frac{2R^2(2-\epsilon)}{\alpha'}}$ and $\sqrt{\frac{2R^2(2+\epsilon)}{\alpha'}}$. The region of interest is an annular region of radius $2\frac{R}{\sqrt{\alpha'}}$, circumference $4\pi\frac{R}{\sqrt{\alpha'}}$ and thickness $\frac{R\epsilon}{\alpha'}$. The lattice points are distributed with unit density in the plane, so the annular region contains of order $\frac{4\pi R^2\epsilon}{\alpha'}$ lattice points. Thus, no matter how small ϵ is chosen to be, the radius R can always be made sufficiently large that the number of almost-marginal operators — scalar operators with dimensions between $2 - \epsilon$ and $2 + \epsilon$ can be made as large as desired.

The five problems described above are not entirely logically independent from one another. The strict version of problem (a) can be solved trivially by assuming cluster decomposition, but even then this solution is "unstable" against turning into problem (b) under a small perturbation of the spectrum. Problem (b) can in principle be solved by organizing the partition function using characters of the Virasoro algebra to separate the vacuum from the continuum. But in practice one runs into problem (c), that the effect of the distinct characters at $\beta = 2\pi$ is so small that it is not easy to exploit the separateness of these contributions to derive a bound.

Problems (a)-(c) all concern the difficulty of controlling unwanted contributions to (2.7)–(2.13) from energy levels near the vacuum E_0. Problem (d) concerns the difficulty of controlling unwanted contributions from energy levels much higher than the marginal operators at $E_0 + 2$, and problem (e) concerns the difficulty of controlling unwanted contributions from levels arbitrarily close to $E_0 + 2$.

The purpose of discussing these meta-problems is not only to warn ourselves away from too-naive attempts to prove a bound using modular invariance, but also to guide ourselves in choosing a favorable set of additional assumptions on the CFT that will allow us to avoid these persistent difficulties.

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4 Upper bounds for state degeneracies under certain conditions

In this section we will prove a bound on the number of marginal operators in an infrared stable CFT with $c_{\rm L} + c_{\rm R}$ less than 48. We will also derive an upper bound on the thermodynamic entropy at inverse temperature 2π, under the same conditions. We conclude with a discussion of the possibility of bounding the number of marginal operators under other sets of assumptions, such as extended superconformal symmetry.

4.1 A bound on the number of marginal operators

Keeping in mind the meta problems discussed in the section above, we will now choose some simple assumptions that will allow us to avoid problems (a) through (e). We can avoid problems (a)-(c) by assuming cluster decomposition, and the absence of a continuum of states just above E_0: in fact the bound is simplest if we restrict our considerations to infrared stable fixed points of the renormalization group — CFT with no relevant operators at all other than the identity. We will avoid problem (d) by restricting our considerations to CFT with moderately low central charge — say, less than 24 on the left and on the right, so $c_{\rm tot} < 48$. We need not make any additional assumption in order to evade problem (e) — in fact we will see that the assumptions of infrared stability and $c_{\rm tot} < 48$ are enough to prove a bound; problem (e) is avoided automatically.

Under these assumptions it is possible to prove a bound using only the first order in the medium-temperature expansion. Let N be the number of primary operators of dimension 2. The degeneracy at energy $E_0 + 2$ is then $N + 2$, with the 2 extra operators coming from the left- and right-moving stress tensor. The number N includes both scalar primaries of weight $(1, 1)$ as well as spin-1 operators of weights $(3/2, 1/2)$ and $(1/2, 3/2)$. The simple method we are using is not sufficiently refined to distinguish these, but an upper bound on N necessarily bounds the number of scalar primary operators, so it will not matter too much that we do not distinguish them.

Taking $F(x) = -(x/(2\pi))$ gives $f_F(E) = E$, and we have the equation

$$\begin{aligned}0 \;=\;& E_0 \exp\{-2\pi E_0\} \\ &+(E_0+2)(N+2)\exp\{-2\pi(E_0+2)\} \\ &+\sum (E_0+\Delta)\exp\{-2\pi(E_0+\Delta)\}\end{aligned} \tag{4.1}$$

where the sum in the third term runs over all operators with dimension $\Delta > 2$. For $c_{\rm tot} < 48$, the quantity $E_0 + 2 = \frac{48 - c_{\rm tot}}{24}$ is positive, so the only negative contribution in equation (4.1) is the first. Multiplying (4.1) through by $\exp\{+2\pi(E_0 + 2)\}$ we obtain

$$\begin{aligned}0 \;&<\; (E_0+2)(N+2) \\ &<\; (E_0+2)(N+2) + \sum (E_0+\Delta)\exp\{-2\pi(\Delta-2)\} \\ &=\; -E_0 \exp\{+4\pi\}\,,\end{aligned} \tag{4.2}$$

giving us a bound

$$N < \left(\frac{c_{\rm L} + c_{\rm R}}{48 - c_{\rm L} - c_{\rm R}}\right) \cdot \exp\{+4\pi\} - 2\ . \tag{4.3}$$

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c_{tot}	$N^{\max}$
19	187,869
20	204,820
21	223,026
22	242,633
23	263,809
24	286,749
25	311,684
26	338,885
27	368,678
28	401,449
29	437,671
30	477,916
31	522,897
32	573,500
33	630,850

c_{tot}	$N^{\max}$
34	696,394
35	772,020
36	860,251
37	964,525
38	1,089,652
39	1,242,587
40	1,433,754
41	1,679,541
42	2,007,257
43	2,466,059
44	3,154,262
45	4,301,267
46	6,595,278
47	13,477,309
48	∞

Table 1. Maximal number (4.3) of marginal operators in a stable unitary CFT for integer values of the central charge in the range (4.4).

This result depends on the assumption that there are no operators with dimensions Δ between 0 and 2. Due to the earlier result [2], reviewed in the second section, this can only be the case if $c_{\text{tot}} > 24 - \frac{18}{\pi} \simeq 18.27$. So the interesting range of central charge, where a useful bound may be proven from modular invariance at first order in the medium-temperature expansion, is

$$\sim 18.27 < c_{\text{tot}} < 48\ . \tag{4.4}$$

For integer values of c_{tot} in this range, we give the maximum possible number of marginal operators in a CFT with no relevant operators in table 1. As an immediate corollary of our bound, note that the resolutional problem, problem (e) of the previous section, has resolved itself automatically: through modular invariance as expressed at first order in the medium-temperature expansion, the assumption of no relevant operators other than the identity implies the absence of a continuum of states with dimensions near 2. If the latter did exist, then equation (4.1) would receive an unboundedly large number of positive contributions going as $(E_0 + 2 + \epsilon)\exp\{-2\pi(E_0 + 2 + \epsilon)\}$, for ϵ made arbitrarily small, without offsetting negative contributions; if there were a (near)-continuum close to $\Delta = 2$ but no states between $\Delta = 0$ and $\Delta = 2$, then the first-order medium-temperature condition for modular invariance could not be satisfied.

4.2 An upper bound on the entropy at medium temperature

To complement our discussion of a lower bound for thermodynamic entropy in the second section, we would like to establish an upper bound on the thermodynamic entropy of the canonical ensemble at medium temperature, $\beta = 2\pi$, under certain conditions. We impose

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the same assumptions on our CFT as we did to derive the upper bound on the degeneracy of marginal operators in the previous subsection: That is, in addition to unitarity, cluster decomposition, discrete operator spectrum and invariance under the modular S-transformation, we also assume that $c_{\rm tot} < 48$ and that there are no relevant operators other than the identity.

As in the previous subsection, we need consider only the leading nontrivial order in the medium-temperature expansion in order to derive an interesting bound. Using the fact that $E_0 < 0$, we can write $E_0 = -|E_0|$, and equation (4.1) can be written as

$$1 = \textstyle\sum_{n\geq 1} \frac{E_n}{|E_0|} \exp\{-2\pi(E_n + |E_0|)\} \ . \tag{4.5}$$

We are assuming $0 < c_{\rm tot} < 48$, and that $2 \leq \Delta_1 < \Delta_2 < \cdots$. Then using $E_n = \Delta_n + E_0 = \Delta_n - \frac{c_{\rm tot}}{24}$, we have

$$0 \leq E_1 \leq E_2 \cdots \tag{4.6}$$

So the $n^{\underline{\rm th}}$ term on the right hand side of (4.5) is no smaller than $\frac{E_1}{|E_0|} \exp\{-2\pi(E_n + |E_0|)\}$. Summing terms, we find

$$\tfrac{E_1}{|E_0|} \textstyle\sum_{n\geq 1} \exp\{-2\pi(E_n + |E_0|)\} \leq \sum_{n\geq 1} \frac{E_n}{|E_0|} \exp\{-2\pi(E_n + |E_0|)\} = 1 \ . \tag{4.7}$$

Multiply each side by $\frac{|E_0|}{E_1} \exp\{+2\pi|E_0|\}$ to derive the inequality

$$\textstyle\sum_{n\geq 1} \exp\{-2\pi E_n\} \leq \frac{|E_0|}{E_1} \exp\{+2\pi|E_0|\} \ . \tag{4.8}$$

We may have no information about the specific value of E_1, but under our assumptions we do know that $E_1 \geq 2 - |E_0| > 0$, so

$$\textstyle\sum_{n\geq 1} \exp\{-2\pi E_n\} \leq \frac{|E_0|}{2-|E_0|} \exp\{+2\pi|E_0|\} \ . \tag{4.9}$$

Adding the vacuum contribution $\exp\{-2\pi E_0\} = \exp\{+2\pi|E_0|\}$ yields an upper bound for the full partition function at $\beta = 2\pi$:

$$\begin{aligned} Z[2\pi] &= \sum_{n\geq 0} \exp\{-2\pi E_n\} \\ &\leq \left(1 + \frac{|E_0|}{2 - |E_0|}\right) \exp\{+2\pi|E_0|\} = \frac{48}{48 - c_{\rm tot}} \exp\left\{+\frac{\pi c_{\rm tot}}{12}\right\} \ . \end{aligned} \tag{4.10}$$

The bound (4.10) on the partition function in turn gives us an upper bound for the thermodynamic entropy of the canonical ensemble at medium temperature. Using the fact that $< E >= 0$ at $\beta = 2\pi$ and combining with the lower bound (2.35), we have:

$$\tfrac{\pi c_{\rm tot}}{12} \leq \sigma|_{\beta=2\pi} \leq \tfrac{\pi c_{\rm tot}}{12} + \ln\left(\tfrac{48}{48-c_{\rm tot}}\right) \ . \tag{4.11}$$

4.3 Discussion

We have proven a bound on the number of marginal operators for a general unitary CFT with discrete spectrum, no relevant operators other than the identity, and central charge in a certain range, $c_{\rm L} + c_{\rm R} < 48$. We have also proven upper and lower bounds for the thermodynamic entropy at inverse temperature 2π, under the same assumptions. Ultimately, we would like to improve the bounds, both by extracting more refined information, and by proving similar bounds under weakened hypotheses.

In terms of more refined information, it would be good to be able to bound the number of scalar marginal operators alone, without including spin-1 operators of weight 2 into the same count. It seems possible that this could be done by using refined medium-temperature equations that consider the partition function as a function of τ and $\bar{\tau}$ instead of $\mathrm{Im}(\tau)$ alone, and using the stronger condition $(\bar{\tau}\partial_{\bar{\tau}})^{p_1}(\tau\partial_\tau)^{p_2} Z[\tau]|_{\tau=i} = 0$ for $p_1 + p_2$ odd.

As far as weakening the assumptions is concerned, it would be good to be able to bound the thermodynamic entropy and microscopic state degeneracies for arbitrarily high central charge. Also, the condition that there be no relevant operators is a strong one, and it is certainly desirable to derive limits on state degeneracies without this assumption. Attempts to weaken either of these two conditions will necessarily meet some of the difficulties enumerated in the third section. New ideas may be required to circumvent those.

In one particular circumstance, the restriction to theories without relevant operators seems particularly natural. Two-dimensional CFT with extended supersymmetry and integrally quantized U(1) charges play an important role as theories representing string propagation in spaces with unbroken spacetime SUSY, for instance on Calabi-Yau threefolds.

These CFT may be a particularly tractable special class in which the number of marginal operators may be bounded. First of all, these theories possess a large chiral algebra — at least $\mathcal{N} = 2$ superconformal symmetry, together with spectral flow generators [25]. They have low central charge, circumventing the fine-structure problem discussed earlier. As for the continuum problem, conformal sigma models on Calabi-Yau spaces admit non-thermal boundary conditions under which the partition function can be evaluated, which project out the contribution from the near-continuum of operators that may be present with dimension close to 0. Each non-thermal boundary condition for the partition function nonetheless has simple modular transformation properties. Among the possible boundary conditions for the partition function are those describing the elliptic genus [24, 25]. Interesting work has been done recently [18, 21–23] on the derivation of general consistency conditions for elliptic genera. Perhaps the medium-temperature techniques discussed here combined with results such as [18, 21] may yield useful information about the still mostly-uncharted landscape of Calabi-Yau manifolds. We hope the present note will provide clues for further progress in that direction and others.

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