LOCAL L AND EPSILON FACTORS IN HECKE EIGENVALUES

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Abstract. Formulas (Theorems 4.2 and 5.1) which express the local L-factor and the local epsilon factor of an irreducible admissible representation of GL_d over a non-archimedean local field in terms of the eigenvalues of some explicitly given Hecke operators are derived.

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1. Introduction

The aim of this article is to derive formulas (Theorems 4.2 and 5.1) which express the local L-factor and the local epsilon factor of an irreducible admissible representation of GL_d over a non-archimedean local field in terms of the eigenvalues of some explicitly given Hecke operators.

We need some notations to state our result. Let K be a nonarchimedean local field. Let $\mathcal{O}$ be the ring of integers and fix a uniformizer $\varpi \in \mathcal{O}$. Let $d \geq 1$. Let (π, V) be an irreducible admissible representation of $G = \mathrm{GL}_d(K)$ where V is a complex vector space. We let ω_π denote the central character. By the classification theorem ([18], see [15, Théorème 3, p.211] also for the notations used below) of admissible representations of G, the representation π is of the form $L(\Delta_1, \dots, \Delta_m)$ where $\Delta_1, \dots, \Delta_m$ are segments such that Δ_i does not precede Δ_j if $m \geq i > j \geq 1$. Each $L(\Delta_i)$ is an essentially square integrable representation of $\mathrm{GL}_{d_i}(K)$. Then $\mathbf{d} = (d_1, \dots, d_m)$ is a partition of $d = d_1 + \cdots + d_m$. We set $\pi_i = |\det|^{s_i} \otimes L(\Delta_i)$ for each i where $s_i = (\sum_{j<i} d_j - \sum_{i<j} d_j)/2$. Then π is the unique irreducible quotient of the (unnormalized) induced representation $\xi = \mathrm{Ind}(G, P_{\mathbf{d}}; \mathrm{Inf}(\pi_1 \otimes \cdots \otimes \pi_m))$ where $P_{\mathbf{d}}$ is the parabolic subgroup corresponding to $\mathbf{d}$. We note that we use different normalization for the induced representation from [15], and the s_i's as above compensate the difference. Let $c = \sum_{i=1}^m \mathrm{cond}\, \pi_i$ be the sum of (the exponent of) conductors of π_i and c' be the conductor of ω_π.

Let $\mathbb{K}_c \subset \mathrm{GL}_d(\mathcal{O})$ denote the subgroup of elements $(x_{ij})_{1 \leq i,j \leq d} \in \mathrm{GL}_d(\mathcal{O})$ such that $(x_{id})_{1 \leq i \leq d}$ is congruent to $(0, \dots, 0, 1)$ modulo (ϖ^c). We let $\mathcal{H} = \mathcal{H}(G, \mathbb{K}_c)$ denote the Hecke algebra of $\mathbb{K}_c$-biinvariant ($\mathbb{C}$-valued) functions on G. For $0 \leq i \leq d$, we let $T_i \in \mathcal{H}$ denote the characteristic function of the double coset $\mathbb{K}_c \mathrm{diag}(\varpi, \cdots, \varpi, 1, \cdots, 1)\mathbb{K}_c$ where ϖ appears i times and 1 appears $d-i$ times. (We write $\mathrm{diag}(a_1, \dots, a_d)$ for the diagonal matrix with diagonal entries $a_1, \dots, a_d$.) We let $T(c') \in \mathcal{H}$ denote the characteristic function of the double coset $\mathbb{K}_c x_{\varpi^{-c'}} \mathbb{K}_c$

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(see Section 5.1 for the definition of $x_{\varpi^{-c'}}$). Let W denote the representation space of ξ. Then using the theory of new vectors ([8]), we know (Lemma 4.1) that $\dim_{\mathbb{C}} W^{\mathbb{K}_c} = 1$, and the action of Hecke operators on this space gives a representation $\chi_W : \mathcal{H} \to \mathbb{C}^\times$.

We fix an additive character $\psi : K \to \mathbb{C}^\times$ of conductor 0. (The conductor of an additive character ψ is the largest integer a such that $\psi(\varpi^{-a}\mathcal{O}) = 1$.) We let $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ denote the L-factor and the epsilon factor of π as defined in [7].

Let q denote the cardinality of the residue field $\mathcal{O}/\varpi\mathcal{O}$. We are ready to state our main result (Theorems 4.2 and 5.1).

Theorem 1.1. *Let the notations be as above. If $c \geq 1$, then*

$$L(s, \pi) = \left(\sum_{r=0}^{d-1} (-1)^r \chi_W(T_{c,r}) q^{\frac{r(r-1)}{2} - r(\frac{d-1}{2}+s)} \right)^{-1}.$$

If $c \geq 1$ and $c' \geq 1$, then

$$\varepsilon(s, \pi, \psi) = q^{(-c+c')s} \varepsilon(s, \omega_\pi, \psi) \omega_\pi(\varpi^{c'}) \chi_W(T(c')).$$

If $c \geq 1$ and $c' = 0$, then

$$\varepsilon(s, \pi, \psi) = \frac{q^{-cs+1}}{q-1} \chi_W(T(0)).$$

The formula of the L-factor of an unramified representation, i.e., the case $c = 0$, is well-known (for example, see [3, Lecture 7]), and its prototype is found in [16, p.394, THEOREM 3]. In the generic case, there is a formula, established in [8, p. 208, Théorème], which expresses such local L-factor in terms of a certain integral of the Whittaker function associated with a new vector. However, it does not seem to the second author that Theorem 4.2 is an immediate consequence of [8, p. 208, Théorème]. For the epsilon factor, we did not find a reference.

The result on the L-factor (for generic, not necessarily unramified representations) will be used in our other paper [9] where we compute a certain zeta integral. We do not have any application of the result on the epsilon factor.

Let us remark on the proof. Even though the definition of the L-factor and the epsilon factor is given in a uniform manner using zeta integrals (see Section 1 of [7]), we use the classification theorem of admissible representations and the known computations of L-factors and epsilon factors (see Section 3 of [7]). For the L-factor, the key computation is that for induced representations (see Section 2 of [7]) and appears as Lemma 4.7 in our paper. This lemma depends on Corollary 3.4, to which most of Sections 2 and 3 are devoted. For the epsilon factor, we use a result from our other paper [10] on Euler system relations. This key fact enables us to make an explicit choice of zeta integral which appears in the definition of the epsilon factor.

The paper is organized as follows. In Sections 2 and 3, we develop a sheaf theory for Hecke algebra. The aim is to introduce a category such that the category of sheaves (which differs slightly from the usual notion) on the category becomes equivalent to the category of smooth representations. Operations such as restriction, inflation, and parabolic induction of representations have a natural interpretation as operations on the category of sheaves. The main result of these two sections is Corollary 3.4, and will be used in the proof of Lemma 4.7. We mention that in our other paper [10], a more general setup for this sheaf theory is developed, in order to prove the Euler system relation of certain elements in algebraic K-groups of Drinfeld modular varieties. We believe that some computations of Hecke operators are clarified in this language of sheaves. In Sections 4 and 5, we give a computation of the L-factor and the epsilon factor respectively. See their introduction for more details.

2. A Sheaf theory for Hecke algebra

We introduce the categories $\mathcal{C}^d$ and $\mathcal{FC}^d$, and develop sheaf theory on them with respect to (an analogue of) a Grothendieck topology. Actually, since the categories are not closed under fiber products, they do not convey topology in the sense of Verdier [19, Expose II]. However we have a cofinality lemma (Lemma 2.2) to circumvent the technical difficulty caused by this shortcoming, and many of the useful notions of sheaf theory become available in our setting.

The connection with the representations of $GL_d(K)$ is given in Section 2.2. We have a functor ω (Section 2.2.1) which gives an equivalence of categories between the category of sheaves on $\mathcal{FC}^d$ with values in complex vector spaces and the category of admissible representations of G (Proposition 2.5).

In Section 2.3, we define the notion of transfers. This is called norm, trace, or pushforward depending on the context to which it applies. There appears a pushforward map in the definition of a Hecke operator (Section 3.2.2), and that is exactly the place where it will be used.

2.1. Let $d \geq 1$ be a positive integer. Let K be a local field and $\mathcal{O}$ be its ring of integers.

2.1.1. We define the category $\mathcal{C}^d$ as follows. An object in $\mathcal{C}^d$ is an $\mathcal{O}$-module of finite length which admits a surjection from $\mathcal{O}^{\oplus d}$. For two objects N and N' in $\mathcal{C}^d$, the set $\mathrm{Hom}_{\mathcal{C}^d}(N, N')$ of morphisms from N to N' is the set of isomorphism classes of diagrams

$$N' \twoheadleftarrow N'' \hookrightarrow N$$

in the category of finitely generated $\mathcal{O}$-modules where the left arrow is a surjection and the right arrow is an injection. Here two diagrams $N' \twoheadleftarrow N'' \hookrightarrow N$ and $N' \twoheadleftarrow N''' \hookrightarrow N$ are considered to be isomorphic if there exists an isomorphism $N'' \xrightarrow{\cong} N'''$ of $\mathcal{O}$-modules such that the diagram

$$\begin{array}{ccccc} N' & \twoheadleftarrow & N'' & \hookrightarrow & N \\ \| & & \downarrow \cong & & \| \\ N' & \twoheadleftarrow & N''' & \hookrightarrow & N \end{array}$$

is commutative. The composition of two morphisms $N' \twoheadleftarrow M \hookrightarrow N$ and $N'' \twoheadleftarrow M' \hookrightarrow N'$ is seen in the following diagram:

$$\begin{array}{ccccc} & & & & N \\ & & & & \uparrow \\ & & N' & \twoheadleftarrow & M \\ & & \uparrow & \circ & \uparrow \\ N'' & \twoheadleftarrow & M' & \twoheadleftarrow & M \times_{N'} M' \end{array}$$

where the circle means that the square containing it is cartesian. This definition of morphisms is taken from [14] except that here we take morphisms in the opposite direction.

We often consider the following two types of morphisms in $\mathcal{C}^d$. Let N be an object in $\mathcal{C}^d$. For a sub $\mathcal{O}$-module N' of N, the morphism $N' = N' \hookrightarrow N$ in $\mathcal{C}^d$ is denoted by $r_{N,N'} : N \to N'$. For a quotient $\mathcal{O}$-module N'' of N, the morphism $N'' \twoheadleftarrow N = N$ in $\mathcal{C}^d$ is denoted by $m_{N,N''} : N \to N''$.

2.1.2. Let $\mathcal{FC}^d$ denote the category of finite families of objects in $\mathcal{C}^d$. An object in $\mathcal{FC}^d$ is a pair $(J, (N_j)_{j\in J})$ where J is a (possibly empty) finite set and $(N_j)_{j\in J}$ is a family of objects in $\mathcal{C}^d$ indexed by J. We denote the object $(J, (N_j)_{j\in J})$ by $\coprod_{j\in J} N_j$. For two objects $\coprod_{i\in I} M_i$ and $\coprod_{j\in J} N_j$, the set $\mathrm{Hom}_{\mathcal{FC}^d}(\coprod_{i\in I} M_i, \coprod_{j\in J} N_j)$ is, by definition, the set $\prod_{i\in I}\coprod_{j\in J} \mathrm{Hom}_{\mathcal{C}^d}(M_i, N_j)$. We regard $\mathcal{C}^d$ as a full subcategory of $\mathcal{FC}^d$. We define $\pi_0(\coprod_{j\in J} N_j)$ to be the set J. A morphism $f : M \to M'$ in the category $\mathcal{FC}^d$ is said to be a *covering* if the underlying morphism $\pi_0(M) \to \pi_0(M')$ is surjective. We note that every covering in $\mathcal{FC}^d$ is an epimorphism.

2.1.3. Let $f : N' \to N$ be a covering in $\mathcal{FC}^d$. We let $\mathrm{Aut}_N(N')$ denote the group of automorphisms σ in $\mathcal{FC}^d$ of N' such that $f \circ \sigma = f$. Let $f : N' \to N$ be a morphism in $\mathcal{FC}^d$, and let G be a subgroup of $\mathrm{Aut}_N(N')$. We say that f is a *Galois covering* of Galois group G if the fiber product $N' \times_N N'$ exists and if the morphism $\coprod_{g\in G}(g, \mathrm{id}) : \coprod_{g\in G} N' \to N' \times_N N'$ is an isomorphism. We note that if $f : N' \to N$ is a Galois covering with Galois group G, then the induced morphism $\pi_0(N')/G \to \pi_0(N)$ is an isomorphism. If $f : N' \to N$ is a Galois covering with N' and N in $\mathcal{C}^d$, then the standard argument in the theory of Galois categories shows that its Galois group equals $\mathrm{Aut}_N(N')$.

2.1.4. We prove that there are enough Galois coverings in the category $\mathcal{FC}^d$, and give a sheaf criterion in terms of Galois coverings.

Lemma 2.1. *Let $f : N' \to N$ be a morphism in $\mathcal{C}^d$ given by the diagram $N \overset{p}{\twoheadleftarrow} N'' \overset{i}{\hookrightarrow} N'$. Suppose there exists a sub $\mathcal{O}$-module N_1 of N such that $p^{-1}(N_1) \cong M_1^{\oplus d}$ and $N'/i(p^{-1}(N_1)) \cong M_2^{\oplus d}$ for some M_1, M_2 in $\mathcal{C}^1$. Then f is a Galois covering.*

Proof. Let M be an object in $\mathcal{C}^d$. It suffices to show the map $\alpha_M : \mathrm{Hom}_{\mathcal{FC}^d}(M, N') \to \mathrm{Hom}_{\mathcal{FC}^d}(M, N)$ induced by f is an $\mathrm{Aut}_N(N')$-torsor over the set $\mathrm{Hom}_{\mathcal{FC}^d}(M, N)$.

Since M_1 and M_2 are generated by one element, there exist ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{O}$ such that $M_1 \cong \mathcal{O}/\mathcal{I}_1$ and $M_2 \cong \mathcal{O}/\mathcal{I}_2$.

Take an element $x \in \mathrm{Hom}_{\mathcal{FC}^d}(M, N)$ and let us consider the set $\alpha_M^{-1}(x)$. Suppose $y \in \alpha_M^{-1}(x)$ is given by the diagram $N' \overset{s'}{\twoheadleftarrow} F \overset{s}{\hookrightarrow} M$. We let $F' = s'^{-1}(i(p^{-1}(N_1)))$ and $F'' = \mathrm{Ker}\, s'$.

Since $F'/F'' \cong (\mathcal{O}/\mathcal{I}_1)^{\oplus d}$, $F/F' \cong (\mathcal{O}/\mathcal{I}_2)^{\oplus d}$, and M is generated by d elements, it follows that $F'/F'' = F'/\mathcal{I}_1 F'$ and F/F' is the set of elements z in M/F' such that $\mathcal{I}_2 z = 0$. Hence $F'' = \mathcal{I}_1 F'$ and F is the set of elements z in M such that $\mathcal{I}_2 z \subset F'$. In particular, $s(F)$ and $s(F'')$ as sub $\mathcal{O}$-modules of M are uniquely determined independent of the choice of y. Note that y is the composition of the canonical morphism $s(F)/s(F'') \twoheadleftarrow s(F) \hookrightarrow M$ and an isomorphism $s(F)/s(F'') \cong N'$. Thus the set $\alpha_M^{-1}(x)$ is canonically isomorphic to the subset of the set of isomorphisms

$s(F)/s(F'') \cong N'$ such that the composition $M \to s(F)/s(F'') \cong N' \to N$ equals the morphism x. Hence the set $\alpha_M^{-1}(x)$ is an $\mathrm{Aut}_N(N')$-torsor. □

Lemma 2.2 (cofinality). *Let k be a positive integer. Let N_i $(1 \le i \le k)$ and N be objects in $\mathcal{C}^d$, and $g_i : N_i \to N$ be morphisms. Then there exist an object M in $\mathcal{C}^d$ and morphisms $f_i : M \to N_i$ such that $g_i \circ f_i = g_j \circ f_j$ for any $1 \le i, j \le k$ and $g_1 \circ f_1$ is a Galois covering.*

Proof. We take $\mathcal{O}$-lattices L_{N_i}, L'_{N_i} $(1 \le i \le k)$, L_N, and L'_N in K^d such that

- $L_{N_i} \supset L_N \supset L'_N \supset L'_{N_i}$ for $1 \le i \le k$,
- $L_{N_i}/L'_{N_i} \cong N_i$ for all i and $L_N/L'_N \cong N$ as $\mathcal{O}$-modules,
- $L_N/L'_N \twoheadleftarrow L_N/L'_{N_i} \hookrightarrow L_{N_i}/L'_{N_i}$ is identified with g_i for all i.

There exists an integral ideal $I \subset \mathcal{O}$ such that $I^{-1}L_N \supset L_{N_i}$ and $L'_{N_i} \subset IL'_N$ for all i. Set $M = I^{-1}L_N/IL_N$ and define the morphisms $M \to N_i$ by $N_i \cong L_{N_i}/L'_{N_i} \twoheadleftarrow L_{N_i}/IL_N \hookrightarrow I^{-1}L_N/IL_N$ for all i. Then by Lemma 2.1, the morphism $M \to N$ and the morphisms $M \to N_i$ for all i are Galois. □

Definition 2.3. *A* presheaf *on $\mathcal{FC}^d$ is a contravariant functor from $\mathcal{FC}^d$ to the category of sets. A presheaf F on $\mathcal{FC}^d$ is a* sheaf *if it satisfies the following conditions (1), (2) and (3):*

1. *The image of the empty set $F(\emptyset)$ is the set of one element.*
2. *For two objects N and N' in $\mathcal{FC}^d$, the canonical map $F(N \amalg N') \to F(N) \times F(N')$ is an isomorphism.*
3. *For any Galois covering $N \to N'$ in $\mathcal{C}^d$, the set $F(N')$ is canonically isomorphic to the $\mathrm{Aut}_{N'}(N)$-fixed part $F(N)^{\mathrm{Aut}_{N'}(N)}$ of $F(N)$.*

We note that a representable presheaf is not necessarily a sheaf.

2.1.5. The inclusion of the category of sheaves on $\mathcal{FC}^d$ into the category of presheaves on $\mathcal{FC}^d$ has a left adjoint $(-)^a$. Let us describe the construction. Given a presheaf $F : \mathcal{FC}^d \to (\mathrm{Sets})$, we define the functor $F^a : \mathcal{FC}^d \to (\mathrm{Sets})$. Let N be an object in $\mathcal{C}^d$. Then the section $F^a(N)$ is given by

$$\varinjlim_{M \to N} \mathrm{Ker}\,[F(M) \rightrightarrows F(M \times_N M)] = \varinjlim_{M \to N} F(M)^{\mathrm{Gal}(M/N)}$$

where the limit is taken over (a small skeleton of the category of) all Galois coverings $M \to N$ in $\mathcal{C}^d$. To check that F^a satisfies (1)(2) and (3), one uses Lemma 2.2. The details are omitted. Note that since $F^a(N)$ is expressed as filtered inductive limit, the functor $(-)^a$ commutes with finite (projective) limits ([13] Ch. IX).

2.2. Connection with smooth representations.

2.2.1. Let $\mathrm{Presh}(\mathcal{FC}^d)$ denote the category of presheaves on $\mathcal{FC}^d$. We define the functor $\omega : \mathrm{Presh}(\mathcal{FC}^d) \to (G-\mathrm{Sets})$ from $\mathrm{Presh}(\mathcal{FC}^d)$ to the category of left G-sets as follows. We consider $K^d = K^{\oplus d}$ as the space of row vectors. Given a presheaf $F \in \mathrm{Presh}(\mathcal{FC}^d)$, we define $\omega(F)$ to be

$$\omega(F) = \varinjlim_{L_1 \subset L_2 \subset K^d} F(L_2/L_1)$$

where the inductive limit is taken over the filtered ordered set of the pairs of two $\mathcal{O}$-lattices (L_1, L_2) in K^d with $L_1 \subset L_2$. The order is defined as follows: for two such pairs (L_1, L_2) and (L'_1, L'_2), we say $(L_1, L_2) > (L'_1, L'_2)$ if and only if

$L'_1 \subset L_1 \subset L_2 \subset L'_2$. We note that whenever $(L_1, L_2) > (L'_1, L'_2)$, there is a morphism

$$L_2/L_1 \twoheadleftarrow L_2/L'_1 \hookrightarrow L'_2/L'_1$$

in $\mathcal{FC}^d$. The transition maps in the inductive limit above are given by these morphisms. The action of G on K^d which appears in the index of the limit $\varinjlim_{L_1 \subset L_2 \subset K^d} F(L_2/L_1)$ makes $\omega(F)$ a G-set.

Lemma 2.4. *Let $L_1 \subset L_2 \subset K^d$ be two $\mathcal{O}$-lattices of K^d. Let $\mathbb{K}_{L_1,L_2} \subset G$ denote the compact open subgroup of the elements $g \in G$ such that $L_i g = L_i$ for $i = 1, 2$ and the map induced by g on L_2/L_1 is the identity. Then for a sheaf $F \in \mathrm{Shv}(\mathcal{FC}^d)$, the canonical map $F(L_2/L_1) \to \omega(F)$ induces an isomorphism $F(L_2/L_1) \cong \omega(F)^{\mathbb{K}_{L_1,L_2}}$.*

Proof. By definition, $F(L_2/L_1) = \varinjlim_{M \to L_2/L_1} F(M)^{\mathrm{Gal}(M/(L_2/L_1))}$ where the limit is taken over (a small skeleton of the category of) all Galois coverings of L_2/L_1 in $\mathcal{C}^d$. By the definition of ω, we have $\omega(F)^{\mathbb{K}} = \varinjlim_{L'_1 \subset L'_2 \subset K^d} F(L'_2/L'_1)^{\mathrm{Gal}((L'_2/L'_1)/(L_2/L_1))}$ where the limit is taken over all Galois coverings of the form $L_2/L_1 \twoheadleftarrow L_2/L'_1 \hookrightarrow L'_2/L'_1$. One sees that the two limits are equal using the argument in the proof of Lemma 2.2. □

Let $\mathrm{Rep}(G)$ denote the (abelian) category of smooth representations of G. Let $\mathrm{Shv}(\mathcal{FC}^d) = \mathrm{Shv}_{\mathbb{C}}(\mathcal{FC}^d)$ denote the category of sheaves on $\mathcal{FC}^d$ with values in the category of complex vector spaces.

Proposition 2.5. *The functor ω induces an equivalence*

$$\mathrm{Shv}(\mathcal{FC}^d) \to \mathrm{Rep}(G).$$

Proof. Let us construct a functor $\rho : \mathrm{Rep}(G) \to \mathrm{Shv}(\mathcal{FC}^d)$ in the opposite direction.

For an object N in $\mathcal{FC}^d$, let

$$s(N) = \varinjlim_{L_1 \subset L_2 \subset K^d} \mathrm{Hom}_{\mathcal{FC}^d}(L_2/L_1, N).$$

The limit is taken as in the definition of ω in Section 2.2.1, hence $s(N)$ is equipped with the left G-action, and makes s a functor to the category of left G-sets. For a smooth representation V, we set $\rho(V)(N) = \mathrm{Hom}_G(s(N), V)$. Note that when N is of the form L_2/L_1 for some lattices $L_1 \subset L_2$, we have a canonical isomorphism $s(N) = G/\mathbb{K}_{L_1,L_2}$. Hence

$$\rho(V)(L_1/L_2) = \mathrm{Hom}_G(G/\mathbb{K}_{L_1,L_2}, V) = V^{\mathbb{K}_{L_1,L_2}}$$

and $\rho(V)$ is a sheaf. This defines a functor $\rho : \mathrm{Rep}(G) \to \mathrm{Shv}(\mathcal{FC}^d)$. We see that $\rho \circ \omega = \mathrm{id}_{\mathrm{Shv}_{\mathbb{C}}(\mathcal{FC}^d)}$ since

$$\rho(\omega(F))(L_2/L_1) = \mathrm{Hom}_G(G/\mathbb{K}_{L_1,L_2}, \omega(F)) = \omega(F)^{\mathbb{K}_{L_1,L_2}} \cong F(L_1/L_2)$$

where the last isomorphism follows from Lemma 2.4. We also see that $\omega \circ \rho = \mathrm{id}_{\mathrm{Rep}(G)}$ since, using that V is smooth, $\omega(\rho(V)) = \varinjlim_{L_1 \subset L_2 \subset K^d} V^{\mathbb{K}_{L_1,L_2}}$. The claim is proved. □

2.2.2. Let $f : N_1 \to N_2$ be a morphism in $\mathcal{FC}^d$ with $N_2 \in \mathcal{C}^d$. Choose an object N and morphisms $f_1 : N \to N_1$ and $f_2 : N \to N_2$ such that f_1 and f_2 are Galois using Lemma 2.2. The integer $\#\mathrm{Aut}_{N_1}(N)/\#\mathrm{Aut}_{N_2}(N)$ is independent of the choice of N, and we call it the degree of f. We define $\deg : \pi_0(N_2) \to \mathbb{Z}_{\geq 0}$ for $N_2 \in \mathcal{FC}^d$ extending the degree above.

2.3. Presheaves with transfers.

Definition 2.6. *An* abelian presheaf with transfers *on $\mathcal{FC}^d$ is a presheaf F of abelian groups on $\mathcal{FC}^d$ equipped with, for each morphism $f : N \to N'$ in $\mathcal{FC}^d$ a homomorphism $f_* : F(N) \to F(N')$ satisfying the following properties:*

(1) *For any two composable morphisms (resp. fibrations) f and f', we have $(f \circ f')_* = f_* \circ f'_*$.*
(2) *For any cartesian diagram*

$$\begin{array}{ccc} N_1' & \xrightarrow{g_1} & N_1 \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ N_2' & \xrightarrow{g_2} & N_2 \end{array}$$

in $\mathcal{FC}^d$, we have $g_2^* \circ f_* = f'_* \circ g_1^*$.
(3) *The composite $f_* \circ f^*$ is the multiplication by* $\deg f$.

2.3.1. Any abelian sheaf F on $\mathcal{FC}^d$ has a unique structure of abelian presheaf with transfers on $\mathcal{FC}^d$. Let $f : N \to N'$ be a morphism in $\mathcal{FC}^d$. We assume that N and N' are objects in $\mathcal{C}^d$ and give the construction of the map $f_* : F(N) \to F(N')$ below. For general objects in $\mathcal{FC}^d$, we extend f_* in a canonical way to obtain the structure of transfers on F.

Using Lemma 2.2, take an object M and Galois morphisms $g : M \to N$ and $g' : M \to N'$ such that $f \circ g = g'$. Recall that by the definition of a sheaf we have $F(N) = F(M)^{\mathrm{Gal}(M/N)}$ and $F(N') = F(M)^{\mathrm{Gal}(M/N')}$. Then for an element $x \in F(N)$, we put $f_*(x) = \sum_{\sigma \in \mathrm{Gal}(M/N)/\mathrm{Gal}(M/N')} \sigma x$. This defines the structure of transfers on F.

2.3.2. A homomorphism of abelian presheaves with transfers is a homomorphism of abelian presheaves compatible with f_*. If F is an abelian sheaf, any homomorphism of abelian presheaves from an abelian presheaf with transfers to F is compatible with f_*.

3. Parabolic subgroups and Levi quotients

This section contains our main technical result, namely Corollary 3.4. This is stated in the language of sheaves of the previous section, and it corresponds to Lemma 4.7 via the equivalence of categories of Proposition 2.5.

In Section 3.1, we extend the results of Section 2 for the case of GL_d to the case of parabolic subgroups and their Levi quotients. We interpret operations such as restriction, inflation, and parabolic induction on admissible representations, in terms of operations such as sheafification, pushforward, and pullback on sheaves. In Section 3.2, we give the computation of some transfers on sheaves. Hecke operators in the context of sheaves are introduced in Section 3.2.2, and the main technical result (Corollary 3.4) is proved.

3.1.

3.1.1. For a partition $\mathbf{d} = (d_1, \dots, d_m)$, $d = d_1 + \cdots + d_m$, $d_1, \dots, d_m \geq 1$ of d, let $\mathcal{E}^{\mathbf{d}}$ denote the following category. An object in $\mathcal{E}^{\mathbf{d}}$ is an object M in $\mathcal{C}^d$ endowed with a decreasing filtration

$$M = \mathrm{Fil}^1 M \supset \mathrm{Fil}^2 M \supset \cdots \supset \mathrm{Fil}^{m+1} M = 0$$

of M by sub $\mathcal{O}$-modules such that for each $i = 1, \dots, m$, $\mathrm{Gr}^i M = \mathrm{Fil}^i M/\mathrm{Fil}^{i+1} M$ is an object in $\mathcal{C}^{d_i}$. For two objects $(M, \mathrm{Fil}^\bullet)$ and $(N, \mathrm{Fil}^\bullet)$ in $\mathcal{E}^{\mathbf{d}}$, a morphism from $(M, \mathrm{Fil}^\bullet)$ to $(N, \mathrm{Fil}^\bullet)$ is a morphism from M to N in $\mathcal{C}^d$ such that the filtration $\mathrm{Fil}^\bullet N$ coincides with the filtration on N induced from the filtration $\mathrm{Fil}^\bullet M$ on M.

We have the following diagram of categories

$$\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m} \xleftarrow{\mathrm{gr}} \mathcal{E}^{\mathbf{d}} \xrightarrow{\mathrm{for}} \mathcal{C}^d,$$

where, gr (resp. for) denotes the functor which sends an object $(M, \mathrm{Fil}^\bullet)$ in $\mathcal{E}^{\mathbf{d}}$ to the object $(\mathrm{Gr}^1 M, \dots, \mathrm{Gr}^m M)$ in $\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m}$ (resp. the object M in $\mathcal{C}^d$).

3.1.2. For the category $\mathcal{E}^{\mathbf{d}}$, we can define, in a similar way as we have done for the category $\mathcal{C}^d$, the category $\mathcal{F}\mathcal{E}^{\mathbf{d}}$ of finite families of objects in $\mathcal{E}^{\mathbf{d}}$, the notions of presheaves and sheaves on $\mathcal{F}\mathcal{E}^{\mathbf{d}}$, and of the sheaf associated to a presheaf on $\mathcal{F}\mathcal{E}^{\mathbf{d}}$ in a manner similar to that in Definition 2.3. Let $\mathrm{Fil}^\bullet K^d$ be the decreasing filtration on the vector space K^d characterized by the following properties: $\mathrm{Fil}^i K^d = K^d$ (resp. $\mathrm{Fil}^i K^d = 0$) for $i \leq 1$ (resp. for $i \geq m+1$), and for $2 \leq i \leq m$, the subspace $\mathrm{Fil}^i K^d \subset K^d$ is the space of vectors whose first $d_1 + \cdots + d_{i-1}$ coefficients are zero. (We regard K^d as the space of row vectors.) Given a presheaf F of sets on $\mathcal{F}\mathcal{E}^{\mathbf{d}}$, we define the set $\omega^{\mathbf{d}}(F)$ to be

$$\omega^{\mathbf{d}}(F) = \varinjlim_{L_1 \subset L_2 \subset K^d} F(L_2/L_1, \mathrm{Fil}^\bullet)$$

where the inductive limit is taken over the pairs of two $\mathcal{O}$-lattices (L_1, L_2) in K^d with $L_1 \subset L_2$ and the filtration $\mathrm{Fil}^\bullet$ on L_2/L_1 is the one induced by the filtration $\mathrm{Fil}^\bullet K^d$ on K^d. Then the group $P_{\mathbf{d}}$ (the standard parabolic subgroup associated with $\mathbf{d}$) acts continuously on the set $\omega^{\mathbf{d}}(F)$ from the left where we endow the set $\omega^{\mathbf{d}}(F)$ with the discrete topology. Using the functor associating $\omega^{\mathbf{d}}(F)$ to F, we can check that the category of sheaves on $\mathcal{F}\mathcal{E}^{\mathbf{d}}$ with values in complex vector spaces is canonically equivalent to the category of smooth representation of $P_{\mathbf{d}}$.

One can proceed in a similar manner as above with the category $\mathcal{F}(\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m})$ of finite families of objects in $\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m}$. Then we can check that the category of sheaves on $\mathcal{F}(\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m})$ with values in complex vector spaces is canonically equivalent to the category of smooth representations of the group $\mathrm{GL}_{d_1}(K) \times \cdots \times \mathrm{GL}_{d_m}(K)$.

3.1.3. *Functors between categories of presheaves (cf.* [19, Expose I]*).* We recall here the definitions of the pushforward and pullback.

For two categories $\mathcal{A}$, $\mathcal{C}$, let $\mathrm{Presh}(\mathcal{C}, \mathcal{A})$ denote the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$. In this section, we assume that any category denoted by a letter $\mathcal{C}$ with some subscripts is essentially small.

Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a covariant functor. Then the pullback functor $f^* : \mathrm{Presh}(\mathcal{C}_2, \mathcal{A}) \to \mathrm{Presh}(\mathcal{C}_1, \mathcal{A})$ is canonically defined.

If the category $\mathcal{A}$ has a limit (= projective limit), there is a right adjoint functor of f^*, which we denote by $f_* : \mathrm{Presh}(\mathcal{C}_1, \mathcal{A}) \to \mathrm{Presh}(\mathcal{C}_2, \mathcal{A})$. The functor f_* can be explicitly given as follows. Let F be a presheaf on $\mathcal{C}_1$, and X be an object in $\mathcal{C}_2$. Then $(f_*F)(X)$ is a limit of $F(Y)$. Here the limit is taken over (a small skeleton of) the category of pairs (Y, α) of an object Y in $\mathcal{C}_1$ and a morphism $\alpha : f(Y) \to X$ in $\mathcal{C}_2$. When $g : \mathcal{C}_2 \to \mathcal{C}_3$ is another covariant functor, we have $g_* f_* = (gf)_*$.

3.1.4. *Restriction and inflation.* Let $\mathbf{d}$ be a partition of d and let $P_{\mathbf{d}} \subset G$ be the standard parabolic subgroup associated with $\mathbf{d}$. We let $L_{\mathbf{d}} = \mathrm{GL}_{d_1}(K) \times \cdots \times \mathrm{GL}_{d_m}(K)$ denote the Levi quotient of $P_{\mathbf{d}}$. We have a functor called inflation (resp. restriction) from the category $\mathrm{Rep}(L_{\mathbf{d}})$ of smooth representations of $L_{\mathbf{d}}$ to the category $\mathrm{Rep}(P_{\mathbf{d}})$ of smooth representations of $P_{\mathbf{d}}$ (resp. from the category $\mathrm{Rep}(G)$ of smooth representations of G to the category $\mathrm{Rep}(P_{\mathbf{d}})$).

Let $\mathrm{Shv}(\mathcal{F}(\mathcal{E}^{\mathbf{d}}))$ (resp. $\mathrm{Shv}(\mathcal{F}(\mathcal{C}^{\mathbf{d}}))$ denote the category of sheaves on $\mathcal{F}(\mathcal{E}^{\mathbf{d}})$ (resp. on $\mathcal{F}(\mathcal{C}^{\mathbf{d}})$) with values in the complex vector spaces. The restriction functor and the inflation functor are compatible with the equivalence of categories in the sense that the following two diagrams are commutative:

$$\begin{array}{ccc} \mathrm{Shv}(\mathcal{F}(\mathcal{E}^{\mathbf{d}})) & \longrightarrow & \mathrm{Rep}(P_{\mathbf{d}}) \\ {\scriptstyle \mathrm{gr}^*}\uparrow & & {\scriptstyle \mathrm{Inf}}\uparrow \\ \mathrm{Shv}(\mathcal{F}(\mathcal{C}^{\mathbf{d}})) & \longrightarrow & \mathrm{Rep}(L_{\mathbf{d}}), \end{array}$$

$$\begin{array}{ccc} \mathrm{Shv}(\mathcal{F}(\mathcal{E}^{\mathbf{d}})) & \longrightarrow & \mathrm{Rep}(P_{\mathbf{d}}) \\ {\scriptstyle \mathrm{for}^*}\uparrow & & {\scriptstyle \mathrm{Res}}\uparrow \\ \mathrm{Shv}(\mathcal{F}(\mathcal{C}^{d})) & \longrightarrow & \mathrm{Rep}(G). \end{array}$$

The horizontal arrows are the equivalence of categories induced by the equivalence discussed in Section 3.1.2.

3.1.5. *Parabolic induction.* Note that the construction of the unnormalized induced representation gives an induction functor $\mathrm{Ind} : \mathrm{Rep}(P_{\mathbf{d}}) \to \mathrm{Rep}(G)$. This is right adjoint to the restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(P_{\mathbf{d}})$.

Consider the functor $\mathrm{for}_*^a = a \circ \mathrm{for}_* : \mathrm{Shv}(\mathcal{F}(\mathcal{E}^{\mathbf{d}})) \to \mathrm{Shv}(\mathcal{F}(\mathcal{C}^d))$ which sends an abelian sheaf F to the sheaf associated to the presheaf $\mathrm{for}_* F$. One can check that this is right adjoint to the pullback functor for^*.

By the uniqueness of the adjoint functor and the commutativity of the diagram above, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Shv}(\mathcal{F}(\mathcal{E}^{\mathbf{d}})) & \longrightarrow & \mathrm{Rep}(P_{\mathbf{d}}) \\ {\scriptstyle \mathrm{for}_*^a}\downarrow & & \downarrow{\scriptstyle \mathrm{Ind}} \\ \mathrm{Shv}\mathcal{F}\mathcal{C}^d & \longrightarrow & \mathrm{Rep}(G), \end{array}$$

where the horizontal arrows are the equivalence of categories.

3.1.6. Let $\mathbf{d} = (d_1, \ldots, d_m)$ be a partition of d. We let F_i be a sheaf with values in complex vector spaces on $\mathcal{F}\mathcal{C}^{d_i}$ for each $i = 1, \ldots, m$. We define $F_1 \boxtimes \cdots \boxtimes F_m$ to be the sheaf on $\mathcal{F}\mathcal{C}^d$ with values in complex vector spaces which sends an object $(M_1, \ldots, M_m)$ of $\mathcal{F}\mathcal{C}^{d_1} \times \cdots \times \mathcal{F}\mathcal{C}^{d_m}$ to $F_1(M_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} F_m(M_m)$.

Lemma 3.1. *Let the notations be as above. Then*

$$\mathrm{for}_*\mathrm{gr}^*(F_1 \boxtimes \cdots \boxtimes F_m)$$

is a sheaf.

Proof of Lemma 3.1. Let $F'' = \mathrm{for}_*\mathrm{gr}^*(F_1|_{\mathcal{C}^{d_1}} \boxtimes \cdots \boxtimes F_m|_{\mathcal{C}^{d_m}})$. Let $f : M \to N$ be a Galois covering in $\mathcal{C}^d$ with Galois group G. We set $e(M) = \coprod_{x\in\pi_0(e(M))} e(M)_x$ and $e(N) = \coprod_{y\in\pi_0(e(N))} e(N)_y$. Then for any $y \in \pi_0(e(N))$, the morphism

$$\coprod_{\pi_0(e(f))(x)=y} e(M)_x \to e(N)_y$$

is a Galois covering in $\mathcal{F}(\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m})$ whose Galois group G_y is a quotient of G. Hence $F''(N)$ is isomorphic to the G-invariant part of $F''(M)$, whence the assertion follows. □

3.1.7. We define a covariant functor $h : \mathcal{FC}^d \to \mathcal{F}(\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m})$ in the following way. For an object M in $\mathcal{C}^d$, let $\mathrm{Flag}^{\mathbf{d}}(M)$ denote the set of decreasing filtrations

$$M = \mathrm{Fil}^1 M \supset \mathrm{Fil}^2 M \supset \cdots \supset \mathrm{Fil}^{m+1} M = 0$$

of M by sub $\mathcal{O}$-modules such that for each $i = 1, \ldots, m$, $\mathrm{Gr}^i M = \mathrm{Fil}^i M/\mathrm{Fil}^{i+1} M$ is an object in $\mathcal{C}^{d_i}$. We define the object $h(M)$ in $\mathcal{F}(\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m})$ to be the disjoint sum

$$h(M) = \coprod_{\mathrm{Flag}^{\mathbf{d}}(M)} (\mathrm{Gr}^1 M, \ldots, \mathrm{Gr}^m M).$$

For an object $M = \coprod_j M_j$ in $\mathcal{FC}^d$, we set $h(M) = \coprod_j h(M_j)$. The proof of the following corollary is omitted.

Corollary 3.2. *In the notation of Lemma 3.1, we have an equality of functors*

$$a \circ (\mathrm{for}_*\mathrm{gr}^*) = h^*.$$

from the category of sheaves on $\mathcal{F}(\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m})$ to the category of sheaves on $\mathcal{FC}^d$. Here a denotes the sheafification functor. □

3.2. Computation of the transfers.

3.2.1. For a morphism $f : M \to N$ in $\mathcal{FC}^d$ and for $x \in \pi_0(h(M))$, we define the multiplicity $\mathrm{mult}_x(f)$ of f at x which is a power of q as follows. An element $x \in \pi_0(h(M))$ corresponds to a pair $(M_0, \mathrm{Fil}^\bullet M_0)$ of a connected component M_0 of M and a decreasing filtration

$$M_0 = \mathrm{Fil}^1 M_0 \supset \cdots \supset \mathrm{Fil}^{m+1} M_0 = 0$$

such that $\mathrm{Gr}^i M_0$ is an object in $\mathcal{C}^{d_i}$ for each $i = 1, \ldots, m$. Let $N_0 \twoheadleftarrow M_0' \hookrightarrow M_0$ be the restriction of f to M_0, where N_0 is an appropriate connected component of N. The filtration $\mathrm{Fil}^\bullet M_0$ on M_0 induces a filtration $\mathrm{Fil}^\bullet M_0'$ on M_0' and a filtration $\mathrm{Fil}^\bullet N_0$ on N_0. We define $\mathrm{mult}_x(f)$ to be

$$\mathrm{mult}_x(f) = \sharp(M_0/M_0')^d \prod_{j=1}^m \left(\frac{(\sharp \mathrm{Fil}^{j+1} M_0')^2}{\sharp \mathrm{Fil}^{j+1} M_0 \cdot \sharp \mathrm{Fil}^{j+1} N_0} \right)^{d_j}.$$

Proposition 3.3. *Let the notations be as in Sections 3.1.6 and 3.1.7. Let* $F' = h^*(F_1 \boxtimes \cdots \boxtimes F_m)$*. Then for any morphism* $f : M \to N$ *in* $\mathcal{FC}^d$*, the transfer map* $f_* : F'(M) \to F'(N)$ *is canonically identified with the map*

$$f'_* : [F_1 \boxtimes \cdots \boxtimes F_m](h(M)) \to [F_1 \boxtimes \cdots \boxtimes F_m](h(N))$$

which is defined as follows. We set $h(M) = \coprod_{x \in \pi_0(h(M))} h(M)_x$ *and* $h(N) = \coprod_{y \in \pi_0(h(N))} h(N)_y$*. On each* $x \in \pi_0(h(M))$*, we define* $h(f)_x : h(M)_x \to h(N)_{\pi_0(h(f))(x)}$ *to be the restriction of the morphism* $h(f) : h(M) \to h(N)$ *to the component* $h(M)_x$*. Then* f'_* *is given as the direct sum of the morphisms*

$$\mathrm{mult}_x(f)(h(f)_x)_* : [F_1 \boxtimes \cdots \boxtimes F_m](h(M)_x) \to [F_1 \boxtimes \cdots \boxtimes F_m](h(N)_{\pi_0(h(f))(x)}).$$

Proof. We easily reduce to the case where $f : M \to N$ is a Galois covering in $\mathcal{C}^d$. Moreover we may assume that $M = (\mathcal{O}/\varpi^n)^{\oplus d}$ for some n, and that for the diagram $N \twoheadleftarrow M' \hookrightarrow M$ giving f, M' is equal to either M or N. Let G be the Galois group of M over N. We set $h(M) = \coprod_{x \in \pi_0(h(M))} h(M)_x$ and $h(N) = \coprod_{y \in \pi_0(h(N))} h(N)_y$. For $x \in \pi_0(h(M))$, let G_x denote the Galois group of $h(M)_x$ over $h(N)_{\pi_0(h(f))(x)}$. Then it is easily checked that the cardinality of the kernel of $G \twoheadrightarrow G_x$ is equal to $\mathrm{mult}_x(f)$. Hence the assertion follows. □

3.2.2. Let F be a sheaf on $\mathcal{FC}^d$ with values in a complex vector space. Consider the cyclic $\mathcal{O}$-module $N = \varpi^{-n}\mathcal{O}/\mathcal{O}$ of length n. For $r = 1, \ldots, d-1$, let $r_r = r_r^{(d)}$ and $m_r = m_r^{(d)}$ denote the morphisms $(\mathcal{O}/\varpi)^{\oplus r} \oplus N \to N$ in $\mathcal{C}^d$ given by the canonical inclusion $N = N \hookrightarrow (\mathcal{O}/\varpi)^{\oplus r} \oplus N$ and by the canonical quotient map $N \twoheadleftarrow (\mathcal{O}/\varpi)^{\oplus r} \oplus N = (\mathcal{O}/\varpi)^{\oplus r} \oplus N$ respectively. We put

$$T_{n,r} = \frac{1}{\#\mathrm{GL}_r(\mathcal{O}/\varpi)}(r_r)_*(m_r)^* : F(N) \to F(N).$$

These are the Hecke operators. See Section 4.1.2.

Let F be a sheaf on $\mathcal{FC}^d$ with values in complex vector spaces. When there exists a nonnegative integer n such that $F(\varpi^{-n}\mathcal{O}/\mathcal{O}) \neq 0$, we define the conductor of F to be the smallest such n.

Corollary 3.4. *Let* F_i *be a sheaf on* $\mathcal{FC}^{d_i}$ *with values in complex vector spaces. Let* n_i *be the conductor of* F_i *for each* $i = 1, \ldots, m$ *and assume that* $F_i(\varpi^{-n_i}\mathcal{O}/\mathcal{O})$ *is one-dimensional. Let* $F' = h^*(F_1 \boxtimes \cdots \boxtimes F_m)$ *(see discussion preceding Corollary 3.2 for the definition of* h*). Let* $n' = \sum_{i=1}^m n_i$ *and* $N = \varpi^{-n'}\mathcal{O}/\mathcal{O}$*. Then* $F'(N)$ *is one-dimensional, and the operator* $T_{n,r}$ *on* $F'(N)$ *is equal to the sum*

$$\sum_{\substack{r = r_1 + \cdots + r_m, \\ r_i \leq \max(d_i - n_i, d_i - 1)}} \frac{\prod_{i=1}^m q^{r_i(\sum_{1 \leq j < i} d_j)}}{q^{\sum_{1 \leq i < j \leq m} r_i r_j}} T^{(d_1)}_{n_1, r_1} \otimes \cdots \otimes T^{(d_m)}_{n_m, r_m}.$$

Proof. Let $\mathrm{Fil}^\bullet N$ be the decreasing filtration of N defined by $\mathrm{Fil}^i N = N$ for $i \leq 1$, $\mathrm{Fil}^i N = \varpi^{n_1 + \cdots + n_{i-1}} N$ for $2 \leq i \leq m$, and $\mathrm{Fil}^i N = 0$ for $i \geq m+1$. Let $x \in \pi_0(h(N))$ be the connected component corresponding to this filtration. Then it is easily checked that

$$[F_1 \boxtimes \cdots \boxtimes F_m](h(N)) = [F_1 \boxtimes \cdots \boxtimes F_m](\mathrm{Fil}^1 N/\mathrm{Fil}^2 N, \ldots, \mathrm{Fil}^m N/\mathrm{Fil}^{m+1} N).$$

Hence $F'(N) = F_1(\mathrm{Gr}^1 N) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} F_m(\mathrm{Gr}^m N)$ is one-dimensional. Now let us compute the operator $T_{n,r} = \frac{1}{\sharp \mathrm{GL}_r(\mathcal{O}/\varpi)}(r_r)_* m_r^*$ on $F'(N)$. The only involved connected components $\widetilde{x} \in \pi_0(h((\mathcal{O}/\varpi)^{\oplus r} \oplus N))$ are those which satisfy $\pi_0(h(m_r))(\widetilde{x}) = \pi_0(h(r_r))(\widetilde{x}) = x$. For each such $\widetilde{x}$, the filtration on $(\mathcal{O}/\varpi)^{\oplus r} \oplus N$ corresponding to $\widetilde{x}$ is the direct sum of a filtration on $(\mathcal{O}/\varpi)^{\oplus r}$ and the filtration $\mathrm{Fil}^\bullet N$ on N. Hence $(r_r)_*(m_r)^*$ is equal to

$$\sum_{\substack{r=r_1+\cdots+r_m,\\ r_i\le \max(d_i-n_i,d_i-1)}} \frac{\sharp\mathrm{GL}_r(\mathcal{O}/\varpi)\prod_{i=1}^m q^{d_i(\sum_{i<j\le m} r_j)}}{\prod_{i=1}^m \sharp\mathrm{GL}_{r_i}(\mathcal{O}/\varpi)\cdot q^{r_i(\sum_{i<j\le m} r_j)}} \times((r_{r_1}^{(d_1)})_* m_{r_1}^{(d_1)*})\otimes\cdots\otimes((r_{r_m}^{(d_m)})_* m_{r_m}^{(d_m)*}).$$

The assertion follows. □

4. Local L-factor in Hecke eigenvalues

The aim of this section is to prove Theorem 4.2. In Section 4.1, we give the setup and the precise statement of the theorem on local L-factor. Lemma 4.1 may be ignored if one is interested only in generic representation, in which case the result is due to [8]. In Section 4.2, we give a summary of the classifications of admissible representations of G, along with some known facts on the L-factors. Note that L-factors are computed inductively. Section 4.3 is devoted to the proof of Theorem 4.2. We prove that the same inductive properties hold for the L-factor L_H which is defined using Hecke eigenvalues.

4.1. Local L-factors of GL_d. Let K be a non-archimedean local field, $\mathcal{O}$ be its ring of integers, and ϖ be a uniformizer. Let q be the cardinality of the residue field. Let (π, V) be an irreducible admissible representation of $G = \mathrm{GL}_d(K)$.

4.1.1. For an integer $n \geq 0$, let $\mathbb{K}_n \subset G$ be the open compact subgroup consisting of the elements in $\mathrm{GL}_d(\mathcal{O})$ whose last row is congruent to $(0, \ldots, 0, 1)$ modulo (ϖ^n). Let $\mathcal{H}(G, \mathbb{K}_n)$ be the Hecke algebra consisting of the bi-$\mathbb{K}_n$-invariant functions on G with compact supports. Then $\mathcal{H}(G, \mathbb{K}_n)$ is a convolution algebra with respect to the Haar measure of G satisfying $\mathrm{vol}(\mathbb{K}_n) = 1$, whose unit is the characteristic function of $\mathbb{K}_n$. For $r = 0, \ldots, d$, let $T_{n,r} = T_{n,r}^{(d)} \in \mathcal{H}(G, \mathbb{K}_n)$ denote the characteristic function of the double coset

$$\mathbb{K}_n \begin{pmatrix} \varpi & & & & & \\ & \ddots & & & & \\ & & \varpi & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \mathbb{K}_n$$

where in the above diagonal matrix ϖ appears r times and 1 appears $d-r$ times. We note that if $r \leq d-1$ or $n = 0$ then $T_{n,r}^{(d)}$ does not depend on the choice of the uniformizer.

We also define dual Hecke operators $T_{n,r}^* \in \mathcal{H}(G, \mathbb{K}_n)$ as the characteristic function of the double coset $\mathbb{K}_n \mathrm{diag}(\varpi^{-1} \ldots \varpi^{-1} 1 \ldots 1)\mathbb{K}_n$ where ϖ^{-1} appears r times and 1 appears $d-r$ times.

4.1.2. The Hecke operators defined in Section 3.2.2 and in Section 4.1.1 are compatible in the following sense. Let (π, V) be a smooth representation. Let $0 \le r \le n$ be integers. Let $L_1 = \mathcal{O}^{\oplus d}$ and $L_2 = \mathcal{O}^{\oplus d-1} \oplus \varpi^{-n}\mathcal{O}$ be lattices in K_d. Then $\mathbb{K}_{L_1,L_2} = \mathbb{K}_n$ holds. Let us identify L_2/L_1 and $N = \varpi^{-n}\mathcal{O}/\mathcal{O}$. Then the diagram

$$\begin{array}{ccc} V^{\mathbb{K}_n} & \xrightarrow{T_{n,r}} & V^{\mathbb{K}_n} \\ \rho \downarrow \cong & & \rho \downarrow \cong \\ \rho(V)(N) & \xrightarrow{T_{n,r}} & \rho(V)(N) \end{array}$$

is commutative. Here N and $T_{n,r}$ on the bottom are as in Section 3.2.2, and the $T_{n,r}$ on the top is that of Section 4.1.1. The vertical ρ is that appeared in the proof of Proposition 2.5.

4.1.3. Let $\mathbf{d} = (d_1, \ldots, d_m)$, $d = d_1 + \cdots + d_m$, $d_1, \ldots, d_m \ge 1$ be a partition of d, and let $P_{\mathbf{d}}$ denote the group of K-valued points of the standard parabolic subgroup of GL_d corresponding to the partition $\mathbf{d}$. Let σ be a smooth representation of $P_{\mathbf{d}}$ on a complex vector space V. We consider the unnormalized induced representation $\mathrm{Ind}(G, P_{\mathbf{d}}; \sigma)$ which is defined to be the space of locally constant functions $f : G \to V$ such that $f(pg) = \sigma(p)f(g)$ holds for any $p \in P_{\mathbf{d}}$ and any $g \in G$.

4.1.4. We use the notations from the introduction. Let $d \ge 1$. Let (π, V) be an irreducible admissible representation of $G = \mathrm{GL}_d(K)$ where V is a complex vector space. We let ω_π denote the central character. By the classification theorem ([18], see [15, Théorème 3, p.211], also for the notations used below) of admissible representations of G, the representation π is of the form $L(\Delta_1, \ldots, \Delta_m)$ where $\Delta_1, \ldots, \Delta_m$ are segments such that Δ_i does not precede Δ_j if $m \ge i > j \ge 1$. Each $L(\Delta_i)$ is a square integrable representation of $\mathrm{GL}_{d_i}(K)$. Then $\mathbf{d} = (d_1, \ldots, d_m)$ is a partition of $d = d_1 + \cdots + d_m$. We set $\pi_i = |\det_{d_i}|^{s_i} \otimes L(\Delta_i)$ for each i where $s_i = (\sum_{j<i} d_j - \sum_{i<j} d_j)/2$, and $\det_{d_i}$ is the determinant character on GL_{d_i}. Then π is the unique irreducible quotient of the induced representation $\xi = \mathrm{Ind}(G, P_{\mathbf{d}}; \mathrm{Inf}(\pi_1 \otimes \cdots \otimes \pi_m))$ where $P_{\mathbf{d}}$ is the parabolic subgroup corresponding to $\mathbf{d}$. We write W for the representation space of ξ. We note that we use different normalization (see Section 4.1.3 for the normalization) for the induced representation from [15]. The difference is the square root of the modulus of $P_{\mathbf{d}}$, which is explicitly given as $|\det_{d_1}|^{s_1} \otimes \cdots \otimes |\det_{d_m}|^{s_m}$.

Note that π_i is generic. We call the integer m in the equation (1) of [8, p.211] (the exponent of) the conductor of π_i and denote it by c_i or by $\mathrm{cond}\, \pi_i$. We let $c = \sum_i c_i$.

Lemma 4.1. *Let the notations be as above. Then $W^{\mathbb{K}_c}$ is one-dimensional and $W^{\mathbb{K}_{c-1}} = 0$.*

Proof. Translate Corollary 3.4 using the functor ω of Proposition 2.5. □

The action of $\mathcal{H}(G, \mathbb{K}_c)$ on $W^{\mathbb{K}_c}$ defines an algebra homomorphism $\chi_W : \mathcal{H}(G, \mathbb{K}_c) \to \mathbb{C}$. We define the local L-factor in Hecke eigenvalues of π as follows. If $c = 0$, then we put

$$L_H(s, \pi) = \left(\sum_{r=0}^{d} (-1)^r \chi_W(T_{c,r}) q^{\frac{r(r-1)}{2} - r(\frac{d-1}{2} + s)} \right)^{-1}.$$

If $c \geq 1$, then we put

$$L_H(s,\pi) = \left(\sum_{r=0}^{d-1} (-1)^r \chi_W(T_{c,r}) q^{\frac{r(r-1)}{2} - r(\frac{d-1}{2}+s)}\right)^{-1}.$$

Theorem 4.2. *Let the notations and assumptions be as above. Let $L(s,\pi)$ be the local L-factor of π defined by Godement and Jacquet [4]. Then we have*

$$L(s,\pi) = L_H(s,\pi).$$

4.2. Here we give a summary of what we use in our proof from the classification theory of the irreducible admissible representations of G. Most of the facts needed for this article are contained in the papers [18], [7], and [8].

Lemma 4.3. *Let π be an irreducible admissible representation. Using the notation in Section 4.1.4, we have*

$$L(s,\pi) = \prod_i L(s+s_i, \pi_i)$$

where we put $s_i = (\sum_{j<i} d_j - \sum_{i<j} d_j)/2$ for each i. If moreover π is generic, so that the conductor is defined, then

$$\operatorname{cond} \pi = \sum_i \operatorname{cond} \pi_i$$

holds.

Proof. The first part may be deduced from [18, 9.7. THEOREM, p.199]. The facts on the L-function and the conductor can be found in [7, (3.4) THEOREM, p.72].

For our normalization of the induced representation differs from that in [7] by a square root of the modulus of $P_{\mathbf{d}}$, we need to compute the difference, which appears as the values s_i. □

We also record the following lemma.

Lemma 4.4. *Let π be essentially square integrable. Then for some m dividing d and for some supercuspidal representation σ of $\mathrm{GL}_{d/m}(K)$, we have*

$$\pi \cong Q(\sigma, \ldots, \sigma(m-1)).$$

Here the right hand side denotes the unique irreducible quotient of $\mathrm{Ind}(G, P_{(d/m,\ldots,d/m)}; \mathrm{Inf}(\sigma \otimes \cdots \otimes \sigma(m-1))$, where for an integer i, we denote by $\sigma(i)$ the twist of σ by the quasicharacter $x \mapsto |x|^i$ of $K^\times$. In this case, we have

$$L(s,\pi) = L(s, \sigma(m-1)).$$

Moreover we have $\operatorname{cond}\pi = m \operatorname{cond}\sigma$ unless $d=1$ and σ is unramified. In the latter case we have $\operatorname{cond}\pi = d-1$.

Proof. The first part is [18, 9.3. THEOREM, p.198]. The remaining part follows from the computation of the L-factor and the epsilon factor given in [6, p.153]. □

The reader is referred to the beginning of [11, p.377, 3.1] for a summary of the computation of the L-factor. We note here that any supercuspidal representation π is generic, and for such π we have $c \geq 1$ and $L(s,\pi) = 1$ except for the case where $d = 1$ and $c = 0$.

With these lemmas, one can inductively compute the L-factor of an arbitrary irreducible admissible representation in terms of those of one-dimensional unramified representations.

4.3. Proof of Theorem 4.2. To prove the proposition, we prove the following series of lemmas.

Lemma 4.5. *Let π be unramified and $n = 1$. Then $L_H(s, \pi) = L(s, \pi)$.*

Proof of Lemma 4.5. This is well known. □

Lemma 4.6. *Let (π, V) be supercuspidal and suppose $c \geq 1$. Then $L_H(s, \pi) = 1(= L(s, \pi))$.*

Proof of Lemma 4.6. Let $r \in \{1, \ldots, d-1\}$ and $m \geq 1$. Let $S_{r,m} \subset G$ be the set of $g = (g_{ij}) \in G$ such that the valuation of $\det g$ is mr and g_{dd} belongs to the set $1 + \varpi^c$. Let $W \subset G$ be a subset which is compact modulo center. Then the intersection $S_{r,m} \cap W$ is empty for sufficiently large m, since the set of the determinants of the elements in W such that g_{dd} is congruent to 1 modulo (ϖ^c) is bounded.

Let $f \in V^{\mathbb{K}_c}$ be a new vector ("vecteur essentiel" in [8, p.211, (4.4)]), so that $\chi_V(T_{c,r})f = T_{c,r}f$ for $r \in \{1, \ldots, d-1\}$. We take a nonzero vector $w \in (V^\vee)^{\mathbb{K}_c}$, so that $(f, w) \neq 0$. Consider $\chi_V(T_{c,r})^m(f, w) = (T_{c,r}^m f, w)$ where $(-,-)$ is the canonical pairing $V \times V^\vee \to \mathbb{C}$. Since π is supercuspidal, the matrix coefficients of π are compactly supported modulo center. Observe that the support of $(T_{c,r})^m$ is contained in the set $S_{m,r}$. Then from the argument in the previous paragraph, it follows that $(T_{c,r}^m f, w)$ is zero for sufficiently large m. Hence $\chi_V(T_{c,r}) = 0$. □

Lemma 4.7. *Let the notations and assumptions be as in Lemma 4.3. We have*

$$L_H(s, \pi) = \prod_i L_H(s + s_i, \pi_i).$$

Proof of Lemma 4.7. Let F (resp. F_i) be the sheaf on $\mathcal{FC}^d$ (resp. on $\mathcal{FC}^{d_i}$ for each $i = 1, \ldots, m$) which corresponds to π (resp. π_i) via the equivalence of categories of Proposition 2.5. Then the assumptions of Corollary 3.4 are satisfied. In view of the remark in Section 4.1.2, it suffices to prove that

$$\left[\sum_{\substack{r=r_1+\cdots+r_m,\\ r_i \leq \max(d_i-n_i, d_i-1)}} \frac{\prod_{i=1}^m q^{r_i(\sum_{1\leq j<i} d_j)}}{q^{\sum_{1\leq i<j\leq m} r_i r_j}} \right] q^{\frac{r(r-1)}{2} - \frac{r(d-1)}{2}}$$

equals

$$\sum_{\substack{r=r_1+\cdots+r_m,\\ r_i \leq \max(d_i-n_i, d_i-1)}} q^{\sum_{i=1}^m [r_i(r_i-1)/2 - r_i((d_i-1)/2+s_i)]},$$

or that

$$\frac{r^2 - rd}{2} + \sum_{i=1}^m (r_i \sum_{1\leq j<i} d_j) - \sum_{1\leq i<j\leq m} r_i r_j = \sum_{i=1}^m (r_i^2 - r_i d_i + 2 r_i s_i)/2$$

where $r = \sum_{i=1}^m r_i$ and $d = \sum_{i=1}^m d_i$. This follows easily. □

Lemma 4.8. *Let the notations and assumptions be as in Lemma 4.4. We have*

$$L_H(s, \pi) = L_H(s, \sigma(m-1)).$$

Proof of Lemma 4.8. Suppose we are either in the case $d/m \geq 1$ or in the case $d/m = 1$ and σ is ramified. Applying Lemma 4.4 to the contragradient, we may and will assume that π is the unique irreducible subrepresentation of the induced representation $\mathrm{Ind}(\sigma'_1, \ldots, \sigma'_m)$ for a supercuspidal representation σ' and $\sigma'_k = \sigma'((m-k) + \frac{d}{2m}(m-2k+1))$ for $k = 1, \ldots, m$. Let F_π (resp. F_i) be the sheaf on $\mathcal{FC}^d$ (resp. $\mathcal{FC}^{d/m}$) corresponding to the representation π (resp. σ'_i for each i). Then $(F_\pi)^{\mathbb{K}_{n_\pi}}$, where n_π is the conductor of F_π, is one-dimensional and is a subspace of $(h^*(F_1 \boxtimes \cdots \boxtimes F_m))^{\mathbb{K}_{n_\pi}}$. We then apply Corollary 3.4 and Lemma 4.6 to obtain the claim.

The claim for the case $d/m = 1$ and σ is unramified follows from Lemma 4.9. $\square$

Lemma 4.9. *Let π be the Steinberg representation. Then*
$$L_H(s, \pi) = L(s, \pi).$$

Proof of Lemma 4.9. It is known that $L(s,\pi) = (1 - q^{-s})^{-1}$. We compute the left hand side below.

Let $1^{(i)}$ denote the trivial representation of $\mathrm{GL}_i(K)$ for $i = 1, 2$. As π is the Steinberg representation, it is isomorphic (see [12, p. 193]) to the quotient of the unnormalized parabolic induction $\mathrm{Ind}(G, P_{1,1,\ldots,1}; \mathrm{Inf}(1^{(1)} \times \cdots \times 1^{(1)}))$, by the canonical image of the direct sum
$$\begin{aligned}&\mathrm{Ind}(1^{(2)} \times 1^{(1)} \times \cdots \times 1^{(1)}) \oplus \mathrm{Ind}(1^{(1)} \times 1^{(2)} \times 1^{(1)} \times \cdots \times 1^{(1)})\\ &\oplus \cdots \oplus \mathrm{Ind}(1^{(1)} \times \cdots \times 1^{(1)} \times 1^{(2)}).\end{aligned}$$

Let $\mathbb{C}_{\mathcal{FC}^i}$ denote the constant sheaf on $\mathcal{FC}^i$ with the value $\mathbb{C}$ for $i = 1, 2$. Put $F' = h^*[\mathbb{C}_{\mathcal{FC}^1} \boxtimes \cdots \boxtimes \mathbb{C}_{\mathcal{FC}^1}]$ and $F_2 = h^*[\mathbb{C}_{\mathcal{FC}^2} \boxtimes \mathbb{C}_{\mathcal{FC}^1} \boxtimes \cdots \boxtimes \mathbb{C}_{\mathcal{FC}^1}]$, $F_3 = h^*[\mathbb{C}_{\mathcal{FC}^1} \boxtimes \mathbb{C}_{\mathcal{FC}^2} \boxtimes \mathbb{C}_{\mathcal{FC}^1} \boxtimes \cdots \boxtimes \mathbb{C}_{\mathcal{FC}^1}]$, ..., $F_d = h^*[\mathbb{C}_{\mathcal{FC}^1} \boxtimes \cdots \boxtimes \mathbb{C}_{\mathcal{FC}^1} \boxtimes \mathbb{C}_{\mathcal{FC}^2}]$. Then $F', F_2, \ldots, F_d$ are the sheaves on $\mathcal{FC}^d$ corresponding to $\mathrm{Ind}(1^{(1)} \times \cdots \times 1^{(1)})$, $\mathrm{Ind}(1^{(2)} \times 1^{(1)} \cdots \times 1^{(1)})$, ..., $\mathrm{Ind}(1^{(1)} \times \cdots \times 1^{(1)} \times 1^{(2)})$, respectively. Hence the sheaf on $\mathcal{FC}^d$ corresponding to the Steinberg representation is the cokernel sheaf F of
$$\bigoplus_{i=2}^{d} F_i \to F'.$$

Let $\mathbb{K}$ be a compact open subgroup of G. Then the functor $V \mapsto V^{\mathbb{K}}$ from the category of smooth representations of G with coefficients in $\mathbb{C}$ is exact. This implies that the cokernel presheaf of $\bigoplus_{i=2}^d F_i \to F'$ is a sheaf and hence equals F.

We put $N = \varpi^{-d+1}\mathcal{O}/\mathcal{O}$. By the equivalence of categories of Proposition 2.5, $F(N)$ is isomorphic to $V^{\mathbb{K}_{d-1}}$ where V is the representation space of π, and it is known that it is one-dimensional. Since $F(N)$ is one-dimensional, there exists a non-trivial linear form β. We construct explicitly such β.

We set $S = \{2, \ldots, d\}$. From Corollary 3.2, it follows that $F'(N)$ is canonically identified with the direct sum
$$F'(N) = \bigoplus_{\alpha : S \to \{0,\ldots,d-1\}} \mathbb{C},$$
where α runs over the non-decreasing maps from S to $\{0, \ldots, d-1\}$. Similarly $F_i(N)$ for $i \in S$ is canonically identified with the direct sum
$$F_i(N) = \bigoplus_{\alpha_i : S - \{i\} \to \{0,\ldots,d-1\}} \mathbb{C},$$

where α_i runs over the non-decreasing map from $S - \{i\}$ to $\{0, \ldots, d-1\}$. For a map $\varepsilon : S \to \{0, 1\}$, let $\alpha_\varepsilon : S \to \{0, \ldots, d-1\}$ denote the non-decreasing map defined by $\alpha_\varepsilon(i) = i - 2 + \varepsilon(i)$. We also set $s(\varepsilon) = (-1)^{\sum_i \varepsilon(i)}$. Define the $\mathbb{C}$-linear map $\beta : F(N) \to \mathbb{C}$ by sending $(c_\alpha)_\alpha$ to $\sum_\varepsilon s(\varepsilon) c_{\alpha_\varepsilon}$. Then it is easily checked that for each $i \in S$ the composition $F_i(N) \to F'(N) \xrightarrow{\beta} \mathbb{C}$ is zero.

Let $\varepsilon_0 : S \to \{0\} \subset \{0, 1\}$ be the constant map on S. Let $v \in F'(N)$ the element whose $\alpha = \alpha_{\varepsilon_0}$-component is 1 and whose $\alpha \neq \alpha_{\varepsilon_0}$-component is 0. For $r = 1, \ldots, d-1$, set $T_{d-1,r}(v) = (w_{r,\alpha})_\alpha$. We compute

$$C_r = \beta(T_{d-1,r}(v)) = \sum_\varepsilon s(\varepsilon) w_{r,\alpha_\varepsilon}.$$

From Proposition 3.3, it follows that $w_{r,\alpha}$ is expressed as the linear combination of the α'-component of v where $\alpha' : S \to \{0, \ldots, d-1\}$ is such that $\alpha'(i) \geq \alpha(i)$ for each $i \in S$. Among the functions of the form α_ε, the function α_{ε_0} is the one which takes the minimal value at each point on S. It follows from this that $C_r = w_{r,\alpha_{\varepsilon_0}}$. It is checked easily that $w_{r,\alpha_{\varepsilon_0}} = 0$ for $r \geq 2$ and $w_{1,\alpha_{\varepsilon_0}} = 1$.

This implies that $L_H(s, \pi) = (1 - q^{-s})^{-1}$. This completes the proof of Proposition 4.2. □

Proof of Theorem 4.2. In view of Lemmas 4.3 and 4.4, the series of Lemmas 4.5, 4.7, 4.8, and 4.9 imply Theorem 4.2. □

5. Local epsilon factor in Hecke eigenvalues

The aim of this section is to prove Theorem 5.1. The reader is referred to Section 1 of [7, p.63] for the details on the definition of the epsilon factor. The input from other sections is concentrated in Lemma 5.2 where we use Theorem 4.2. Another input is needed from our other paper [10], and is used in the proof of Lemma 5.3.

In Section 5.1, we give the precise statement of the theorem. We briefly recall the zeta integral and the definition of the epsilon factor in the form we need in Section 5.2. Section 5.3 is where we use a result from our other paper. We make an explicit choice f_ξ of the coefficient of ξ and a Bruhat-Schwartz function Φ_c, and then compute the zeta integral (Lemma 5.3). In Section 5.4, we recall the epsilon factor of a one-dimensional representation. The proof of Theorem 5.1 is given in the last section.

5.1. Let π be a admissible irreducible representation of G. We use the notations from Section 4.1.4. Let c' denote the conductor of the central character ω_π of π. For $a \in K$, we define $x_a = (x_{a,i,j})_{1 \leq i,j \leq d} \in G$ as follows. We let $x_{a,1,d} = \varpi^{-c}$ and $x_{a,d,d} = a$. We let $x_{a,i+1,d-i} = 1$ for $1 \leq i \leq d-1$, and put $x_{a,i,j} = 0$ otherwise. When $d = 1$, we only allow $a = \varpi^{-c} = \varpi^{-c'}$.

We let $T(c') \in \mathcal{H}(G, \mathbb{K}_c)$ denote the Hecke operator corresponding to the double coset $\mathbb{K}_c x_{\varpi^{-c'}} \mathbb{K}_c$. We fix an additive character $\psi : K \to \mathbb{C}^\times$ of conductor 0. (The conductor of an additive character ψ is the largest integer a such that $\psi(\varpi^{-a}\mathcal{O}) = 1$.)

When $c \geq 1$ and $c' \geq 1$, we put

$$\varepsilon_H(s, \pi, \psi) = q^{(-c+c')s} \varepsilon(s, \omega_\pi, \psi) \omega_\pi(\varpi^{c'}) \chi_W(T(c')).$$

When $c \geq 1$ and $c' = 0$, we put

$$\varepsilon_H(s, \pi, \psi) = \frac{q^{-cs+1}}{q-1} \chi_W(T(0))$$

When $c = 0$, we put $\varepsilon_H(s, \pi, \psi) = 1$. We note that since the double coset $\mathbb{K}_c \varpi^{c'} x_{\varpi^{-c'}} \mathbb{K}_c$ is independent of the choice of the uniformizer ϖ, the factor $\varepsilon_H(s, \pi, \psi)$ is also independent of the choice.

Theorem 5.1. *Let the notations be as above. We have* $\varepsilon(s, \pi, \psi) = \varepsilon_H(s, \pi, \psi)$ *where* $\varepsilon(s, \pi, \psi)$ *is the epsilon factor as defined in* [7, p.64, (1.3.6)].

5.2. Let $(\pi^\vee, V^\vee)$ denote the contragradient of (π, V). By definition, $V^\vee$ is the subspace of smooth elements of $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. We write $\langle , \rangle : V \times V^\vee \to \mathbb{C}$ for the canonical pairing.

5.2.1. We fix a Haar measure dg on G. For a Bruhat-Schwartz function Φ (i.e., a locally constant compactly supported function) on $\mathrm{Mat}_d(K)$ and a function h on G, we set

$$Z(\Phi, s, h) = \int_G \Phi(g) |\det g|^s h(g) dg.$$

For a Bruhat-Schwartz function Φ on $\mathrm{Mat}_d(K)$, we write

$$\widehat{\Phi}(x) = \int_{\mathrm{Mat}_d(K)} \Phi(y) \psi(\mathrm{tr}(xy)) dy$$

for its Fourier transform. Here dy is the Haar measure on $\mathrm{Mat}_d(K)$ which is self-dual with respect to the pairing $(x, y) \mapsto \psi(\mathrm{tr}(xy))$.

Lemma 5.2. *Let* Φ *be a Bruhat-Schwartz function on* $\mathrm{Mat}_d(K)$ *and let* f *be a coefficient of* ξ*. Then we have*

$$\varepsilon(s, \pi, \psi) Z(\Phi, s, f) = Z(\widehat{\Phi}, 1 - s + \frac{d-1}{2}, f^\vee) \frac{L_H(s, \pi)}{L_H(1 - s, \pi^\vee)} \tag{5.1}$$

where $f^\vee(g) = f(g^{-1})$ *is a coefficient of* $\xi^\vee$*.*

Proof. By definition ([7, (1.3.6), p.64]), we have $\varepsilon(s, \pi, \psi) = \gamma(s, \pi, \psi) L(s, \pi) L(1 - s, \pi^\vee)^{-1}$, where $\gamma(s, \pi, \psi)$ is as in [7, Proposition 1.2, p.63]. From Theorem 4.2, it follows that $L(s, \pi) = L_H(s, \pi)$ and $L(1 - s, \pi^\vee) = L_H(1 - s, \pi^\vee)$. It is shown in [7, (2.7.3), p.69] that $\gamma(s, \pi, \psi) = \gamma(s, \xi, \psi)$. Hence the claim follows by using the remark preceding [7, Proposition 2.3, p.67] which says that [7, Proposition 1.2, p.63] holds for ξ. □

When π is unramified, $\varepsilon(s, \pi, \psi) = 1$. Since each factor of $\varepsilon_H(s, \pi, \psi)$ is 1, the equality holds in this case. We assume from now on that $c \geq 1$.

5.3. To prove Theorem 5.1, we make a suitable choice f_ξ of a coefficient of ξ and a choice Φ_c of a Bruhat-Schwartz function, and then compute the right hand side of (5.1) of Lemma 5.2.

Let us define f_ξ. From Lemma 4.1, we know that $W^{\mathbb{K}_c}$ is one-dimensional. Let $w \in W^{\mathbb{K}_c}$ be a nonzero element. We write $w^\vee \in (W^\vee)^{\mathbb{K}_c}$ for the element such that $\langle w, w^\vee \rangle = 1$. We then put $f_\xi(g) = \langle gw, w^\vee \rangle$ for $g \in G$. We see that both f_ξ and $f_\xi^\vee$ are biinvariant under $\mathbb{K}_n$, i.e., $f_\xi(kgk') = f_\xi(g)$ for $k, k' \in \mathbb{K}_c$ and $g \in G$, and similarly for $f_\xi^\vee$.

Let us define Φ_c. Let $Y_c = \{(x_{ij}) \in \mathrm{Mat}_d(\mathcal{O}) | (x_{id})_{1\le i\le d} \equiv (0,\cdots,0,1) \bmod (\varpi^c)\}$ denote the subset of elements of $\mathrm{Mat}_d(\mathcal{O})$ such that the g-th row is congruent to $(0,\cdots,0,1)$ modulo (ϖ^c). We let Φ_c be the characteristic function of the set $Y_c \subset \mathrm{Mat}_d(\mathcal{O})$. We have $\mathbb{K}_c = Y_c \cap \mathrm{GL}_d(\mathcal{O})$.

5.3.1.

Lemma 5.3. *Let f_ξ and Φ_c be as above. Then*

$$Z(\Phi_c, s + \frac{d-1}{2}, f_\xi) L_H(s,\pi)^{-1} = \mathrm{vol}(\mathbb{K}_c).$$

Proof. We let G act on $\mathrm{Mat}_d(K)$ by multiplication from the right, and we regard the space $\mathcal{S}(\mathrm{Mat}_d(K))^{\mathbb{K}_c}$ of Bruhat-Schwartz functions on $\mathrm{Mat}_d(K)$ that are $\mathbb{K}_c$-invariant as a left $\mathcal{H}(G,\mathbb{K}_c)$-module.

Let Φ'_c denote the characteristic function of $\mathbb{K}_c$. We regard Φ_c and Φ'_c as elements in $\mathcal{S}(\mathrm{Mat}_d(K))^{\mathbb{K}_c}$. We obtain

$$\Phi'_c = \sum_{i=0}^{d-1} (-1)^i q^{\frac{i(i-1)}{2}} T^*_{c,i} \Phi_c$$

using the Euler system relation. We refer to [10] for a proof. See also the thesis by Grigorov [5, p.25, Theorem 1.4.6] where a relevant portion of the proof is presented.

Since f_ξ is $\mathbb{K}_c$-invariant and $|\det g| = 1$ for $g \in \mathbb{K}_c \subset \mathrm{GL}_d(\mathcal{O})$, we have

$$Z(\Phi'_c, s, f_\xi) = \int_G \Phi'_c(g) |\det g|^s f_\xi(g) dg = \int_{\mathbb{K}_c} |\det g|^s f_\xi(g) dg = \mathrm{vol}(\mathbb{K}_c).$$

On the other hand we have

$$\begin{aligned} Z(\Phi'_c, s, f_\xi) &= \sum_{i=0}^{d-1} (-1)^i q^{\frac{i(i-1)}{2}} \int_G (T^*_i \Phi_c)(g) |\det g|^s f_\xi(g) dg \\ &\overset{(1)}{=} \sum_{i=0}^{d-1} (-1)^i q^{\frac{i(i-1)}{2}} q^{-is} \chi_W(T^*_{c,i}) \int_G \Phi_c(g) |\det g|^s f_\xi(g) dg \\ &\overset{(2)}{=} L_H(s,\pi)^{-1} Z(\Phi_c, s, f_\xi). \end{aligned}$$

For the equality (1), we use that, $|\det g| = q^{-i}$ for g in the support of $T^*_{c,i}$. The equality (2) follows from the definitions. □

5.3.2. Let $a \in \varpi^{-c}\mathcal{O}/\mathcal{O}$. We define a subset $Y_a \subset \mathrm{Mat}_d(\mathcal{O})\mathrm{diag}(1,\cdots,1,\varpi^{-c})$ to be those elements (y_{ij}) such that y_{dd} modulo $\mathcal{O}$ is equal to a. Then a direct computation shows that

$$\widehat{\Phi}_c(x) = \left(\sum_{a \in \varpi^{-c}\mathcal{O}/\mathcal{O}} \psi(a) \mathrm{ch}_{Y_a}(x) \right) q^{-dc} \tag{5.2}$$

where ch_{Y_a} is the characteristic function of Y_a.

We let

$$\widehat{\Phi}'_c = \sum_{i=0}^{d-1} (-1)^i q^{\frac{i(i-1)}{2}} T^*_{c,i} \widehat{\Phi}_c.$$

Then the argument as in the proof of Lemma 5.3 implies

$$Z(\widehat{\Phi}'_c, 1-s+\frac{d-1}{2}, f^\vee_\xi) = L_H(1-s,\pi^\vee)^{-1} Z(\widehat{\Phi}_c, 1-s+\frac{d-1}{2}, f^\vee_\xi). \tag{5.3}$$

5.4. The epsilon factor of the central character ω_π is given explicitly (see [17, p.13, (3.2.6)]). If ω_π is ramified (i.e., $c' \geq 1$), we have

$$\varepsilon(s,\omega_\pi,\psi) = \int_{K^\times} \psi(a)\omega_\pi^{-1}(a)|a|^{-s}da = \sum_{r\in\mathbb{Z}} q^{rs} \int_{\varpi^r\mathcal{O}^\times} \omega_\pi^{-1}(a)\psi(a)da$$

where da is the Haar measure of the additive group K such that the volume of $\mathcal{O}$ is 1.

Suppose $c' \geq 1$. Then $\int_{\mathcal{O}^\times} \omega_\pi^{-1}(\varpi^r a)\psi(\omega^r a)da = 0$ for $r \neq -c$. Hence $\varepsilon(s,\omega_\pi,\psi)$ is of the form $q^{-c's}$ times a constant. (See [17, (3.2.6.2), (3.4.5)].)

5.5. Proof of Theorem 5.1. We give a proof of Theorem 5.1 in this section.

5.5.1. Let $W_c \subset G$ be the subset of elements g such that $|\det g| = q^c$. We have

$$\begin{aligned}
&\mathrm{vol}(\mathbb{K}_c)\varepsilon(s,\pi,\psi) \overset{(1)}{=} Z(\widehat{\Phi}'_c, 1-s+\tfrac{d-1}{2}, f_\xi^\vee) \\
&\overset{(2)}{=} \sum_{i=0}^{d-1}(-1)^i q^{\frac{i(i-1)}{2}} \int_G (T_{c,i}^*\widehat{\Phi}_c)(g)|\det g|^{1-s+\frac{d-1}{2}} f_\xi^\vee(g)dg \\
&\overset{(3)}{=} \sum_{i=0}^{d-1}(-1)^i q^{\frac{i(i-1)}{2}} \int_{W_c} (T_{c,i}^*\widehat{\Phi}_c)(g)|\det g|^{1-s+\frac{d-1}{2}} f_\xi^\vee(g)dg.
\end{aligned}$$

For (1), we used Lemmas 5.2 and 5.3 and the equation (5.3). The equality (2) is by definition. Since $\varepsilon(s,\pi,\psi)$ is of the form q^{-cs} times a constant (If π is generic, this follows from [8, p.211, Théorème]. The general case is reduced to the case where π is essentially square integrable (hence generic), by using Lemma 4.3 and [7, p.67, (2.3) Proposition].), we see that (3) holds. Further, this equals

$$\begin{aligned}
&\overset{(4)}{=} \sum_{i=0}^{d-1}(-1)^i q^{\frac{i(i-1)}{2}} \sum_{a\in\varpi^{-c}\mathcal{O}/\mathcal{O}} \psi(a) \int_{W_c} (T_{c,i}^*\mathrm{ch}_{Y_a})(g) q^{-dc} q^{c(1-s+\frac{d-1}{2})} f_\xi^\vee(g)dg \\
&\overset{(5)}{=} q^{\frac{c}{2}(-d+1)} q^{c(1-s+\frac{d-1}{2})} \sum_{a\in\varpi^{-c}\mathcal{O}/\mathcal{O}} \psi(a) \int_{W_c\cap Y_a} f_\xi^\vee(g)dg.
\end{aligned}$$

The equality (4) follows from (5.2). For (5), we use the following lemma.

Lemma 5.4. *Let $i \geq 1$. Then*

$$(T_i^*\mathrm{ch}_{Y_a})(X) = 0$$

for $X \in Y_a \cap W_c$.

Proof. If $Y \in Y_a$, then $|\det Y| \leq q^c$. If g is in the support of T_i^*, then $|\det g| = q^i$. Hence if X belongs to the support of $T_i^*\mathrm{ch}_{Y_a}$, then $|\det X| \leq q^{c-i}$. This implies in particular that $X \notin W_c$ if $i \geq 1$. This proves the claim. □

5.5.2. Let $\alpha_a = \int_{W_c\cap Y_a} f_\xi^\vee(g)dg$ for short. Note that $\alpha_{au} = \omega_\pi^{-1}(u)\alpha_a$ holds for $u \in \mathcal{O}^\times$. Then

$$\begin{aligned}
&\sum_{a\in\varpi^{-c}\mathcal{O}/\mathcal{O}} \psi(a) \int_{W_c\cap Y_a} f_\xi^\vee(g)dg \\
&= \sum_{r\geq -c} \int_{\varpi^r\mathcal{O}^\times} \psi(a)\alpha_a da \\
&= \sum_{r\geq -c} \left(\int_{\varpi^r\mathcal{O}^\times} \psi(a)\omega_\pi^{-1}(a)da\right) \omega_\pi(\varpi^r)\alpha_{\varpi^r}.
\end{aligned}$$

When $c' \geq 1$, this equals $\varepsilon(s, \omega_\pi, \psi) q^{-c's} \omega_\pi(\varpi^{c'}) \alpha_{\varpi^{c'}}$ (see Section 5.4). When $c' = 0$, this equals $-\alpha_{\varpi^{-1}} + \alpha_1$.

5.5.3.

Lemma 5.5. *We have* $\chi_W(T(c')) = \alpha_{\varpi^{c'}}$.

Proof. Let $a \in \varpi^{-c}\mathcal{O}/\mathcal{O}$. Take a lift $\tilde{a} \in \varpi^{-c}\mathcal{O}$ and regard it as an element in K. The double coset $\mathbb{K}_c x_{\tilde{a}} \mathbb{K}_c$ (see Section 5.1 for the definition of $x_{\tilde{a}}$) is independent of the choice of the lift $\tilde{a}$. We will write $\mathbb{K}_c x_a \mathbb{K}_c$ for this double coset.

We claim that $W_c \cap Y_a = \mathbb{K}_c x_a \mathbb{K}_c$. To prove the claim we consider the union $\mathrm{GL}_d(\mathcal{O})\mathrm{diag}(1, \cdots, 1, \varpi^c) = \coprod_{a \in \varpi^{-c}\mathcal{O}/\mathcal{O}} W_c \cap Y_a$. Let

$$\mathbb{K}'_c = \mathrm{diag}(1, \cdots, 1, \varpi^c)\mathbb{K}_c \mathrm{diag}(1, \cdots, 1, \varpi^c)^{-1}.$$

Then one can check that

$$\begin{aligned} &\textstyle\coprod_{a \in \varpi^{-c}\mathcal{O}/\mathcal{O}} \mathbb{K}_c \backslash (W \cap Y_a)/\mathbb{K}_c = \mathbb{K}_c \backslash \mathrm{GL}_d(\mathcal{O})/\mathbb{K}'_c \\ &\xrightarrow{(1)} \mathbb{K}_c \backslash [(\mathcal{O}/\varpi^c\mathcal{O})^{\oplus d} \setminus (\varpi\mathcal{O}/\varpi^c\mathcal{O})^{\oplus d}] \xrightarrow{(2)} \mathcal{O}/\varpi^c\mathcal{O} \end{aligned}$$

is an isomorphism. Here the map (1) sends the class of $(x_{ij}) \in \mathrm{GL}_d(\mathcal{O})$ to the vector $(x_{id})_{1 \leq i \leq d}$ and the map (2) sends the class of (y_i) to y_d. An element in $W \cap Y_a$ is sent to $a\varpi^c$ via this map. Since x_a is sent to $a\varpi^c$, we obtain the claim. The claim implies in particular that $\chi_W(T(c')) = \alpha_{\varpi^{c'}}$. □

This proves Theorem 5.1 for the case $c' \geq 1$.

5.5.4.

Lemma 5.6. *When* $c' = 0$,

$$(q-1)\chi_W(T(1)) + \chi_W(T(0)) = 0$$

holds.

Proof. Consider the disjoint union

$$\coprod_{a \in \varpi^{-1}\mathcal{O}/\mathcal{O}} Y_a \subset \mathrm{Mat}_d(K).$$

One can check that the set

$$\mathrm{diag}(1, \ldots, 1, \varpi) \coprod_{a \in \varpi^{-1}\mathcal{O}/\mathcal{O}} Y_a$$

is invariant under multiplication from the right by an element of $\mathbb{K}_{c-1}$. Hence the set

$$S = \mathrm{diag}(1, \ldots, 1, \varpi) \coprod_{a \in \varpi^{-1}\mathcal{O}/\mathcal{O}} Y_a \cap W_c \subset G$$

is also invariant under multiplication from the right by an element of $\mathbb{K}_{c-1}$.

By Lemma 4.1, we have $W^{\mathbb{K}_{c-1}} = 0$. Hence if we regard the characteristic function ch_S of S as an element of the Hecke algebra $\mathcal{H}(G, \mathbb{K}_c)$, then $\chi_W(\mathrm{ch}_S) = 0$ holds.

Now for each $a \in \varpi^{-1}\mathcal{O}/\mathcal{O}$, let $S(a)$ denote the characteristic function of $W_c \cap Y_a$. We regard them as elements of the Hecke algebra $\mathcal{H}(G, \mathbb{K}_c)$. Since $\chi_W(\mathrm{ch}_S) = 0$, for an element $w \in W^{\mathbb{K}_c}$, we have

$$\mathrm{diag}(1, \ldots, 1, \varpi) \cdot \sum_{a \in \varpi^{-1}\mathcal{O}/\mathcal{O}} S(a)w = 0$$

Hence

$$\sum_{a \in \varpi^{-1}\mathcal{O}/\mathcal{O}} \chi_W(S(a)) = 0.$$

We write a_0 for the element $\varpi^{-1} \bmod \mathcal{O}$ in $\varpi^{-1}\mathcal{O}/\mathcal{O}$. For any $u \in \mathcal{O}^\times$, we have

$$\chi_W(S(ua_0)) = \omega_\pi(u)\chi_W(S(a_0)) = \chi_W(S(a_0)).$$

Hence we obtain

$$(q-1)\chi_W(S(a_0)) + \chi_W(S(0)) = 0.$$

Recall that from the proof of Lemma 5.5, we have $W_c \cap Y_{a_0} = \mathbb{K}_c x_{\varpi^{-1}} \mathbb{K}_c$ and $W_c \cap Y_0 = \mathbb{K}_c x_1 \mathbb{K}_c$. Hence $S(a_0) = T(1)$ and $S(0) = T(0)$. This proves the claim of the lemma. □

This finishes the proof of Theorem 5.1. □

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