ZETA ELEMENTS IN THE K-THEORY OF DRINFELD MODULAR VARIETIES

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Abstract. Beilinson [Be] constructs special elements in the second K-group of an elliptic modular curve, and shows that the image under the regulator map is related to the special values of the L-functions of elliptic modular forms. In this paper, we give an analogue of this result in the context of Drinfeld modular varieties.

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1. Introduction

The theme of this article is a function field analogue of Beilinson's result on elliptic modular curves concerning his conjectures on motives over $\mathbb{Q}$ (see [Be, Section 5]). The conjectures (among other things) relate the image of the regulator map and the special values of L-functions. Our main theorem (Theorem 1.1 below) is the Drinfeld modular analogue of Beilinson's theorem ([Be, Theorem 5.1.2]) or more precisely its refinement due to Kato ([Ka, p.127, Theorem 2.6]) which computes explicitly the regulator map in terms of the special value of L-function.

Let us give some notation to state our results. We refer to later sections for the precise statement. Let F be the function field of a projective smooth geometrically irreducible curve C over a finite field. We fix a closed point ∞ of C and let $A = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ denote the coordinate ring of the affine curve $C \setminus \{\infty\}$. Let q_∞ denote the cardinality of the residue field at ∞.

Let d be a positive integer. Let $J \subset I \subset A$ be nonzero ideals. We let $N_{I,J} = (A/I)^{\oplus d-1} \oplus (A/J)$. It is an A-module of finite length. We consider the moduli space $\mathcal{M}^d_{N_{I,J}}$ of rank d Drinfeld modules with level $N_{I,J}$ structures (see Section 2.4.7).

Let $\mathbb{A}^\infty = \prod'_{\wp \neq \infty} F_\wp$ denote the ring of finite adeles. Given an element $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$, we construct, in Section 2.4.7, an element

$$\kappa_{I,J,\gamma} \in K_d(\mathcal{M}^d_{N_{I,J}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

using theta functions and Siegel units. Let Z denote the center of GL_d, let $\mathcal{A}_1$ denote the space of $Z(F_\infty)$-invariant automorphic forms on $\mathrm{GL}_{d,F}$ (see Section 4.1.1), and let $\mathcal{A}_1^\circ \subset \mathcal{A}_1$ denote the subspace of $Z(F_\infty)$-invariant cusp forms. The main result of this paper is a formula which describes the image of the element $\kappa_{I,J,\gamma}$ under the

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regulator map

$$\mathrm{reg}_{L_1,L_2} : K_d(\mathcal{M}^d_{N_{I,J}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to (\mathcal{A}_1)^{\mathbb{K}^{\infty}_{I,J}}$$

defined in Section 7.5. Here the symbols L_1, L_2 in the subscript denote certain $\widehat{A}$-lattices of $\mathbb{A}^{\infty\oplus d}$ corresponding to I and J defined in Section 2.4.7, and $(\mathcal{A}_1)^{\mathbb{K}^{\infty}_{I,J}} \subset \mathcal{A}_1$ denotes the subspace of $Z(F_\infty) \times \mathbb{K}^{\infty}_{I,J}$-invariant automorphic forms where $\mathbb{K}^{\infty}_{I,J} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ is the open compact subgroup corresponding to I and J defined in Section 2.4.7.

In Section 5.4.7, we construct a $Z(F_\infty) \times \mathbb{K}^{\infty}_{I,J}$-invariant automorphic form

$$\eta_{\mathbb{K}^{\infty}_{I,J},\gamma} : \mathrm{GL}_d(F)\backslash \mathrm{GL}_d(\mathbb{A})/(Z(F_\infty) \times \mathbb{K}^{\infty}_{I,J}) \to \mathbb{C}$$

which is an analogue of a modular symbol.

We are ready to state our main result.

Theorem 1.1. *(See Theorem 8.2) Let $f \in (\mathcal{A}_1^{\mathrm{o}})^{\mathbb{K}^{\infty}_{I,J}}$ be a $Z(F_\infty) \times \mathbb{K}^{\infty}_{I,J}$-invariant cusp form satisfying Condition (2) of Section 4.2.2 (this is roughly the condition that the cusp form is a Hecke eigenform). Then we have*

$$\langle f, \mathrm{reg}_{L_1,L_2}(\kappa_{I,J,\gamma})\rangle = \frac{1}{\log\, q_\infty} \lim_{s\to 0} \frac{\partial}{\partial s} L^{I,J}(f, s - \frac{d-1}{2})\langle f, \eta_{\mathbb{K}^{\infty}_{I,J},\gamma}\rangle.$$

Here $L^{I,J}(f,s)$ is the L-function of f (with local factors at the primes dividing I removed) which is defined in Section 4.2.2, and $\langle\ ,\ \rangle : \mathcal{A}_1^{\mathrm{o}} \times \mathcal{A}_1 \to \mathbb{C}$ is the Petersson inner product defined in Section 4.1.2.

As an application of Theorem 1.1, we obtain the following result (Theorem 1.2) which accounts for the part of the Beilinson conjecture on the surjectivity of the regulator map (sometimes called the weak Beilinson conjecture).

We write $P^{\mathrm{o}} : \mathcal{A}_1 \to \mathcal{A}_1^{\mathrm{o}}$ for the projection map to the space of cusp forms (Section 4.2.2). Let $\mathcal{A}^{\mathrm{o}}_{\mathrm{St}} \subset \mathcal{A}_1^{\mathrm{o}}$ denote the subspace which is characterized by the condition that the corresponding representation at the prime ∞ is the Steinberg representation (see Section 9.1 for the precise statement).

Theorem 1.2. *The image of the homomorphism*

$$K_d(\mathcal{M}^d_{N_{I,J}}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\mathrm{reg}_{L_1,L_2}} \mathcal{A}_1 \xrightarrow{P^{\mathrm{o}}} \mathcal{A}_1^{\mathrm{o}}$$

equals $(\mathcal{A}^{\mathrm{o}}_{\mathrm{St}})^{\mathbb{K}^{\infty}_{L_1,L_2}}$.

The outline of the proof of Theorem 1.1 is as follows. We construct in Section 2.4 a subspace of the K-group using the units on the modular varieties. In the case of elliptic modular curves, these elements are called Beilinson elements (or Beilinson-Kato elements).

We then compute (Theorem 6.3) an integral, which we call zeta integral. This is the pairing between a Hecke eigen cusp form and a certain automorphic form. We see that it equals the product of the automorphic L-function associated to the Hecke eigen cusp form and a certain factor without the complex parameter. The computation is done using the norm property of Euler systems (see Proposition 6.2). This formula is given purely in terms of automorphic forms. A similar statement was proved using the Rankin-Selberg method in the elliptic modular case.

The limit of the zeta integral, as the complex parameter tends to zero, is related to the pairing between a Hecke eigen cusp form and the image by the regulator map of the elements in K-groups constructed above. This is the zeta value formula

(Theorem 1.1). This can be seen using an analogue of the Kronecker limit formula (Proposition 3.4). Theorem 1.2 above is proved in Section 9 using this formula and some computation of the Borel-Moore homology groups (see Corollary 5.16).

In comparison with the Beilinson conjectures, there are two open problems which we have not solved. Let us give some remarks concerning these problems. The Beilinson conjectures are stated for the integral part of the motivic cohomology groups, which are certain parts of the algebraic K-groups, of projective smooth schemes over $\mathbb{Q}$. One problem is that the Drinfeld modular varieties $\mathcal{M}^d_{N_{I,J}}$ are affine of pure dimension $d-1$ over F. Hence we have not constructed elements in the K-groups of projective smooth schemes over F for $d \geq 2$. In the case of elliptic modular curves, Beilinson has resolved this problem by considering the compactified elliptic modular curves and using the Drinfeld-Manin theorem. The same method can be applied in the case of Drinfeld modular curves, that is, for $d = 2$, and the details are written in our other paper [Ko-Ya3]. However, for $d \geq 3$, we do not know how to construct a good compactification of $\mathcal{M}^d_{N_{I,J}}$. Even if we assume the existence of such a good compactification, we still do not how to overcome the problem. The other problem is that we do not know if the subspace of the K-groups of the Drinfeld modular varieties we constructed is contained in the integral part. For Drinfeld modular curves, we have affirmative results in our paper [Ko-Ya3]. We do not know how to resolve these problems in higher dimensions.

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2. Construction of zeta elements

We give the construction of some special elements in the K-theory of Drinfeld modular varieties. In Section 2.1, we recall some facts on Drinfeld modular varieties. The function field analogue of Siegel units and theta functions are defined in Sections 2.2 and 2.3. In the case of elliptic modular curves, the algebraic construction of theta functions is due to Kato ([Ka]), while similar functions appear in the earlier work of Coates and Wiles [Coa-Wi], as rational functions on CM elliptic curves. The construction of special elements follows the idea of Beilinson.

2.1. **Drinfeld modular varieties.**

2.1.1. *Notations.* Let C be a smooth projective geometrically irreducible curve over the finite field $\mathbb{F}_q$ of q elements. Let F denote the function field of C. Fix a closed point ∞ of C. Let q_∞, F_∞, $|\ |_\infty : F_\infty \to q_\infty^{\mathbb{Z}} \cup \{0\}$ denote the cardinality of the residue field of C at ∞, the completion of F at ∞, and the absolute value at ∞, respectively. Let $A = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ be the coordinate ring of the affine $\mathbb{F}_q$-scheme $C \setminus \{\infty\}$.

2.1.2. We recall the definition of a Drinfeld module ([Dr]).

We fix an integer $d \geq 1$. Let S be an A-scheme. A *Drinfeld module* of rank d over S is a scheme E in A-modules over S satisfying the following conditions:

(1) Zariski locally on S, the scheme E is isomorphic to $\mathbb{G}_a$ as a commutative group scheme.
(2) If we denote the A-action on E by $\varphi : A \to \mathrm{End}_{S\text{-group}}(E)$, then, for every $a \in A \setminus \{0\}$, the morphism $\varphi(a) : E \to E$ on E is finite, locally free of constant degree $|a|_\infty^d$.
(3) The A-action on $\mathrm{Lie}\, E$ induced by φ coincides with the A-action on $\mathrm{Lie}\, E$ which comes from the structure homomorphism $A \to \Gamma(S, \mathcal{O}_S)$.

2.1.3. Let N be a torsion A-module. Let $U_N = \mathrm{Spec}\, A \setminus \mathrm{Supp}\, N$ be the spectrum of the localization of A by the elements in A which are invertible on $\mathrm{Spec}\, A \setminus \mathrm{Supp}\, N$. Let S be a U_N-scheme, and (E, φ) be a Drinfeld module of rank d over S. A *level N-structure* on (E, φ) is a monomorphism $\psi : N_S \hookrightarrow E$ from the constant group scheme N_S to E in the category of schemes in A-modules over S.

2.1.4. Let us consider the sheaf $\mathcal{M}_N^d$ of groupoids on the big étale site of U_N-schemes which associates, to a U_N-scheme S, the groupoid of triples (E, φ, ψ) where (E, φ) is a Drinfeld module over S and ψ is a level N-structure. If $N \neq 0$ (resp. if N is of finite length), the functor $\mathcal{M}_N^d$ is representable by an affine U_N-scheme (resp. by a smooth Deligne-Mumford U_N-stack). The representability is stated and proved in [Dr, Proposition 5.3] (see also [Lau, Theorem 1.4.1]), in the case $N = (I^{-1}/A)^{\oplus d}$ with a non-zero ideal $I \subsetneqq A$. The method of the proof in [Dr] may be applied to our case.

Let $N \hookrightarrow N'$ be an injection of torsion A-modules. We let $r_{N',N} : \mathcal{M}_{N'}^d \to \mathcal{M}_N^d \times_{U_N} U_{N'}$ denote the morphism $(E, \varphi, \psi) \mapsto (E, \varphi, \psi|_N)$ where $\psi|_N$ is the restriction of ψ to the submodule N.

Let $N \twoheadrightarrow N''$ be a surjection of torsion A-modules such that the kernel is of finite length. We let $m_{N,N''} : \mathcal{M}_N^d \to \mathcal{M}_{N''}^d \times_{U_{N''}} U_N$ denote the morphism $(E, \varphi, \psi) \mapsto (E'', \varphi'', \psi'')$ where $E'' = E/\psi(\mathrm{Ker}(N \to N''))$ and φ'', ψ'' are those induced by the quotient map ([Lau, Lemma 1.4.1]).

2.2. **Theta functions.**

In this section, we construct an element

$$\theta_{E/S} \in \Gamma(E \setminus S, \mathcal{O}_{E/S}^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$$

which we call the theta function associated to a Drinfeld module E over S.

2.2.1. Let (E, φ) be a Drinfeld module of rank d over a reduced A-scheme S. Let $\pi : E \to S$ denote the structure morphism. We regard S as a closed subscheme of E via the zero section $S \hookrightarrow E$.

Lemma 2.1. *Let the notation be as above. There exists an element $f \in \Gamma(E \setminus S, \mathcal{O}_{E \setminus S}^\times)$ satisfying the following properties:*

(1) *For $a \in A \setminus \{0\}$, let $N_a : \Gamma(E \setminus \mathrm{Ker}\, \varphi(a), \mathcal{O}_E^\times) \to \Gamma(E \setminus S, \mathcal{O}_E^\times)$ denote the norm map with respect to the finite flat morphism $\varphi(a) : E \setminus \mathrm{Ker}\, \varphi(a) \to E \setminus S$. Then $N_a(f) = f$ for any $a \in A \setminus \{0\}$.*
(2) *The order $\mathrm{ord}_S(f)$ of zero of f at the closed subscheme S is equal to $q_\infty^d - 1$.*

Proof. Let us consider the sequence

$$0 \to \mathcal{O}_S^\times \to \pi_* \mathcal{O}_{E\setminus S}^\times \xrightarrow{\mathrm{ord}_S} \mathbb{Z} \to 0$$

of Zariski sheaves of abelian groups on S. It is exact since we assumed S to be reduced. The multiplicative monoid $A \setminus \{0\}$ acts on $\mathcal{O}_{E\setminus S}$ by the norm map N_a for $a \in A \setminus \{0\}$. We let the monoid $A \setminus \{0\}$ act on the sheaf $\mathcal{O}_S^\times$ in such a way that the action on $a \in A$ on $\mathcal{O}_S^\times$ is given by the $|a|_\infty^d$-power map. We let the monoid $A \setminus \{0\}$ act trivially on the sheaf $\mathbb{Z}$. Then the above exact sequence is an exact sequence of $A \setminus \{0\}$-modules, and defines an element of the extension module $\mathrm{Ext}^1_{\mathbb{Z}[A\setminus\{0\}]_S}(\mathbb{Z}, \mathcal{O}_S^\times)$ in the abelian category of Zariski sheaves of $A \setminus \{0\}$-modules on S. Since $A \setminus \{0\}$ acts trivially on $\mathbb{Z}$ and via the character $|\ |_\infty^d : A \setminus \{0\} \to q_\infty^{d\mathbb{Z}_{\geq 0}}$ on $\mathcal{O}_S^\times$, we have $(|a|_\infty^d - 1)\mathrm{Ext}^1_{\mathbb{Z}[A\setminus\{0\}]_S}(\mathbb{Z}, \mathcal{O}_S^\times) = 0$ for any $a \in A \setminus \{0\}$. Since the greatest common divisor of $|a|_\infty^d - 1$ as a runs through $A \setminus \{0\}$ is $q_\infty^d - 1$, the extension group $\mathrm{Ext}^1_{\mathbb{Z}[A\setminus\{0\}]_S}(\mathbb{Z}, \mathcal{O}_S^\times)$ is annihilated by $q_\infty^d - 1$. In particular, the above exact sequence splits after pulling back by $q_\infty^d - 1 : \mathbb{Z} \to \mathbb{Z}$. Now let f be the image of $1 \in \mathbb{Z}$ by the section which gives the splitting. □

2.2.2. Since the choice of the element $f \in \Gamma(E \setminus S, \mathcal{O}_{E\setminus S}^\times)$ in Lemma 2.1 is unique up to $\mathrm{Hom}_{\mathbb{Z}[A\setminus\{0\}]_S}(\mathbb{Z}, \mathcal{O}_S^\times) \cong \mu_{q_\infty^d - 1}(S)$, the element $f^{q_\infty^d - 1} \in \Gamma(E \setminus S, \mathcal{O}_{E\setminus S}^\times)$ does not depend on the choice of f. Hence the element

$$f \otimes (1/(q_\infty^d - 1)) = f^{q_\infty^d - 1} \otimes (1/(q_\infty^d - 1)^2) \tag{2.1}$$

in $\Gamma(E \setminus S, \mathcal{O}_{E\setminus S}^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined without ambiguity. We call it the *theta function* of (E, φ) and denote it by $\theta_{E/S}$. The normalizing factor $1/(q_\infty^d - 1)$ in (2.1) is chosen so that the formula in Theorem 8.2 is simplest. See also the remark by Kato ([Ka, p.122, Remark 1.5]).

Zariski locally on S, there is an isomorphism $E \cong \mathbb{G}_{a,S} = \mathrm{Spec}\ \mathbb{Z}[T] \times_{\mathrm{Spec}\ \mathbb{Z}} S$ of commutative group schemes. Fix such an isomorphism and let $f' \in \Gamma(E, \mathcal{O}_E)$ denote the element corresponding to the coordinate function T on $\mathbb{G}_{a,S}$. It follows from the definition that the function $\theta_{E/S}$ is of the form $\theta_{E/S} = cf'$ for some constant $c \in \Gamma(S, \mathcal{O}_S^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$. The constant c is explicitly calculated in the following way. Take an element $a \in A \setminus \{0\}$ and write $N_a(f') = c'f'$. Then the relation $N_a(cf') = cf'$ implies that $c = c'^{-1} \otimes 1/(|a|_\infty^d - 1)$. Therefore

$$\theta_{E/S} = \frac{f'^{|a|_\infty^d}}{N_a(f')} \otimes \frac{1}{|a|_\infty^d - 1} \in \Gamma(E \setminus S, \mathcal{O}_{E\setminus S}^\times) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{2.2}$$

The following properties are easily checked:

Proposition 2.2. (1) *Let $g : S' \to S$ be a morphism from another reduced scheme S' to S, and let $g_E : E \times_S S' \to E$ denote the base change morphism induced by g. Then we have $g_E^* \theta_{E/S} = \theta_{E\times_S S'/S'}$.*

(2) *Let $h : E \to E'$ be an isogeny (that is, a morphism of schemes in A-modules which is finite flat as a morphism of schemes) from another Drinfeld module E' of rank d over S to E. Then $N_h \theta_{E/S} = \theta_{E'/S}$ where N_h is the norm map associated with h.*

□

2.3. **Siegel units.**

Let N be a nonzero torsion A-module. We let $E_N^d \to \mathcal{M}_N^d$ denote the universal Drinfeld module, and $\psi : N_{\mathcal{M}_N^d} \hookrightarrow E_N^d$ the universal level structure. If N is of finite length, then $\mathcal{M}_N^d$ is smooth over $U_N = \operatorname{Spec} A \setminus \operatorname{Supp} N$, so in particular it is reduced. Hence we have a theta function $\theta_{E_N^d/\mathcal{M}_N^d} \in \Gamma(E_N^d \setminus \mathcal{M}_N^d, \mathcal{O}^\times_{E_N^d \setminus \mathcal{M}_N^d}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathcal{M}_N^d$ is regarded as a subscheme of E_N^d by the zero section. For $b \in N \setminus \{0\}$, we let $\psi_b : \mathcal{M}_N^d \to E_N^d$ denote the restriction of ψ to the subscheme $\mathcal{M}_N^d = \{b\} \times \mathcal{M}_N^d \subset N_{\mathcal{M}_N^d}$ and put $g_{N,b} = \psi_b^* \theta_{E_N^d/\mathcal{M}_N^d} \in \mathcal{O}(\mathcal{M}_N^d)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. We call the element $g_{N,b}$ a Siegel unit.

Let N' be an A-module of finite length generated by at most d elements (so in particular that $\mathcal{M}_N^d$ is nonempty), and N be a sub A-module of N'. By Proposition 2.2(1), we have $r^*_{N',N} g_{N,b} = g_{N',b}$ for any $b \in N \setminus \{0\} \subset N' \setminus \{0\}$. Let $\alpha : N \to N''$ be a surjection of A-modules of finite length. It follows from Proposition 2.2(1)(2) that $m^*_{N,N''} g_{N'',b''} = \prod_{b \in N, \alpha(b) = b''} g_{N,b}$ for any $b'' \in N'' \setminus \{0\}$.

2.4. **Elements in K-theory.**

2.4.1. *Notation.* We use the notation C, F, ∞, q_∞, A introduced in Section 2.1.1. We also let $\mathcal{O}_\infty$ denote the ring of integers of the local field F_∞ and let $\widehat{A} = \varprojlim_I A/I$ where the limit is taken over all nonzero ideals I of A. We let $\mathbb{A}^\infty = \widehat{A} \otimes_A F$ and $\mathbb{A} = F_\infty \times \mathbb{A}^\infty$ denote the rings of finite adeles and adeles, respectively.

Let us consider the d-dimensional vector space $V = F^{\oplus d}$ over F. We regard it as the set of row vectors. We write $V_\infty = V \otimes_F F_\infty$, $\mathcal{O}_{V_\infty} = \mathcal{O}_\infty^{\oplus d} \subset V_\infty$, $V^\infty = V \otimes_F \mathbb{A}^\infty$, and $\mathcal{O}_{V^\infty} = \widehat{A}^{\oplus d} \subset V^\infty$. For a ring R, we let $\operatorname{Mat}_d(R)$ denote the ring of $d \times d$-matrices with entries in R.

2.4.2. We define schemes $\mathcal{M}^d$ and $\mathcal{M}_1^d$ as the limit with respect to the level structures of Drinfeld modular varieties of rank d in this paragraph. They are both equipped with the action of $\mathrm{GL}_d(\mathbb{A}^\infty)$ and are canonically isomorphic as $\mathrm{GL}_d(\mathbb{A}^\infty)$-schemes.

For a torsion A-module N, let us use the shorthand $\mathcal{M}_{N,F}^d = \mathcal{M}_N^d \times_{U_N} \operatorname{Spec} F$. We use the same notation $m_{*,*}$ and $r_{*,*}$ as in Section 2.1.4 for the corresponding morphisms of schemes over $\operatorname{Spec} F$.

An $\widehat{A}$-lattice L in V^∞ is a free $\widehat{A}$-module of rank d contained in V^∞ such that the canonical map $L \otimes_{\widehat{A}} \mathbb{A}^\infty \to V^\infty$ is an isomorphism. Let us consider the set of pairs of $\widehat{A}$-lattices (L_1, L_2) in V^∞ such that $L_1 \subset L_2$. We consider it as an ordered set by setting $(L_1, L_2) < (L_1', L_2')$ if and only if $L_1' \subset L_1 \subset L_2 \subset L_2'$. Let $\mathcal{M}^d$ denote the inverse limit $\varprojlim_{(L_1,L_2)} \mathcal{M}^d_{L_2/L_1,F}$ where the transition map for $(L_1, L_2) < (L_1', L_2')$ is given by the composite $r_{L_2'/L_1, L_2/L_1} m_{L_2'/L_1', L_2'/L_1}$. The group $\mathrm{GL}_d(\mathbb{A}^\infty)$ acts from the left on this inverse limit by the multiplication from the right on V^∞ (hence acts on the set of sublattices of V^∞).

We also consider the inverse limit $\mathcal{M}_1^d = \varprojlim_{L \subset \widehat{A}^{\oplus d} \subset V^\infty} \mathcal{M}^d_{\widehat{A}^{\oplus d}/L,F}$ where the limit is taken over $\widehat{A}$-lattices L contained in $\widehat{A}^{\oplus d}$ with transition maps given by $m_{*,*}$.

We can prove that the canonical map $\mathcal{M}^d \to \mathcal{M}_1^d$ induced from the inclusion of the index sets is an isomorphism. Let us give a brief sketch of the proof. The scheme $\mathcal{M}_1^d$ is equipped with the action of $\mathrm{GL}_d(\mathbb{A}^\infty)$. This can be shown using an argument similar to the one given by Drinfeld ([Dr, p.577, D)], see also [Lau, p.15, (1.7)]). It is known that $a \in F^\times \subset \mathbb{A}^{\infty\times} \cong Z(\mathrm{GL}_d(\mathbb{A}^\infty))$, where Z denotes the center, acts trivially. Hence the canonical morphism $\mathcal{M}^d \to \mathcal{M}^d_{L_2/L_1,F}$ and the composition $\mathcal{M}^d \to \mathcal{M}^d_{aL_2/aL_1,F} \cong \mathcal{M}^d_{L_2/L_1,F}$, where the first map is the canonical morphism and the second map is the morphism induced by the multiplication-by-a map $L_2/L_1 \cong aL_2/aL_1$ where $a \in F^\times$, coincide. Now consider the set of pairs of lattices $\{(a\widehat{A}^{\oplus d}, aL)\}$ with $\widehat{A}^{\oplus d} \subset L$ and $a \in F^\times$. Then this set is cofinal with the system for $\mathcal{M}^d$. Hence $\mathcal{M}^d \to \mathcal{M}^d_1$ is an isomorphism and is $\mathrm{GL}_d(\mathbb{A}^\infty)$-equivariant.

2.4.3. Let us give a remark concerning the scheme $\mathcal{M}^d_2 = \mathcal{M}^d_{V^\infty/\widehat{A}^{\oplus d}}$, which, by definition, is equal to the projective limit $\varprojlim_{\widehat{A}^{\oplus d} \subset L \subset V^\infty} \mathcal{M}^d_{L/\widehat{A}^{\oplus d},F}$ with transition maps $r_{*,*}$. This remark will not be used in the sequel so the reader may skip this paragraph. The scheme $\mathcal{M}^d_2$ is the scheme which appears in the paper by Drinfeld [Dr] (and other sources). It is isomorphic to $\mathcal{M}^d$ in a canonical way (a reasoning similar to the one given above applies).

One reason we consider $\mathcal{M}^d_1$ instead of $\mathcal{M}^d_2$ is that it is easier to construct elements in the K-groups of $\mathcal{M}^d_1$ than in that of $\mathcal{M}^d_2$ and to prove properties such as the Euler system relations. We refer to [Ko-Ya1] for the details.

2.4.4. Let $\mathcal{S}(V^\infty)$ denote the space of $\mathbb{Z}$-valued Schwartz-Bruhat functions on V^∞, that is, the space of $\mathbb{Z}$-valued functions on V^∞ that are locally constant and compactly supported. We let $\mathcal{S}'(V^\infty) \subset \mathcal{S}(V^\infty)$ denote the subspace of those functions f such that $f(0) = 0$. We have a $\mathrm{GL}_d(\mathbb{A}^\infty)$-equivariant isomorphism $\mathcal{S}(V^\infty)^{\otimes d} \cong \mathcal{S}(\mathrm{Mat}_d(\mathbb{A}^\infty))$ by setting

$$(f_1 \otimes \cdots \otimes f_d)((x_{ij})) = f_1(x_{1j}) f_2(x_{2j}) \cdots f_d(x_{dj})$$

where $f_i \in \mathcal{S}(V^\infty)$ for each i and $(x_{ij}) \in \mathrm{Mat}_d(\mathbb{A}^\infty)$. By multiplication of the inverse from the right on V^∞ and on $\mathrm{Mat}_d(\mathbb{A}^\infty)$, we have a left action of $\mathrm{GL}_d(\mathbb{A}^\infty)$ on $\mathcal{S}(V^\infty)$ and on $\mathcal{S}(\mathrm{Mat}_d(\mathbb{A}^\infty))$. We see that $\mathcal{S}'(V^\infty)^{\otimes d} \subset \mathcal{S}(V^\infty)^{\otimes d}$ is a $\mathrm{GL}_d(\mathbb{A}^\infty)$-submodule.

2.4.5. Let us define a $\mathrm{GL}_d(\mathbb{A}^\infty)$-equivariant homomorphism

$$\kappa : \mathcal{S}'(V^\infty)^{\otimes d} \to K_d(\mathcal{M}^d) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We first define a $\mathrm{GL}_d(\mathbb{A}^\infty)$-equivariant homomorphism

$$\mathcal{S}'(V^\infty) \to K_1(\mathcal{M}^d) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \varinjlim_{(L_1,L_2)} (K_1(\mathcal{M}^d_{L_2/L_1,F}) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Note that any element of $\mathcal{S}'(V^\infty)$ can be expressed as a $\mathbb{Z}$-linear combination of characteristic functions of the sets of the form $b + L$ where $L \subset V^\infty$ is an $\widehat{A}$-lattice and $b \in V^\infty \setminus L$. We take an $\widehat{A}$-lattice $L' \subset V^\infty$ large enough so that $b \in L'$ and $L \subset L'$. We have a Siegel unit $g_{L'/L,b} \in \mathcal{O}(\mathcal{M}^d_{L'/L,F})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ where we wrote b again for the image in L'/L. The image of $g_{L'/L,b}$ in the limit $\varinjlim_{(L_1,L_2)} \mathcal{O}(\mathcal{M}_{L_2/L_1,F})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ is independent of the choice of L' by Proposition 2.2.

Let $L \supset L''$ and let $b \in V^\infty \setminus L$. Then we have an equality

$$\mathrm{ch}_{b+L} = \sum_{b'' \in V^\infty/L'',\, b'' \bmod L = b} \mathrm{ch}_{b''+L''}$$

in $\mathcal{S}(V^\infty)$, where ch means the characteristic function of its subscript. We have a similar equality $g_{L'/L,b} = \prod_{b'' \in L'/L'', b'' \bmod L = b} g_{L'/L'',b''}$ in $\varinjlim_{(L_1,L_2)} \mathcal{O}(\mathcal{M}_{L_2/L_1,F})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ which follows from a remark at the end of Section 2.3. Hence the map

$$\mathcal{S}'(V^\infty) \to \varinjlim_{(L_1,L_2)} \mathcal{O}(\mathcal{M}^d_{L_2/L_1})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$$

which sends ch_{b+L} to $g_{L'/L,b}$ is well-defined. It is easy to see that the map is $\mathrm{GL}_d(\mathbb{A}^\infty)$-equivariant.

Now composing this with the symbol map $\mathcal{O}(-)^\times \to K_1(-)$ and the product structure $K_1(-)^{\otimes d} \to K_d(-)$ of K-theory, we obtain the desired homomorphism κ.

2.4.6. Let $I, J \subsetneq A$ be nonzero ideals with $J \subset I$ and let $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$. We put $N_{I,J} = (A/I)^{\oplus(d-1)} \oplus (A/J)$. We define an element $\kappa_{I,J,\gamma} \in K_d(\mathcal{M}^d_{N_{I,J},F}) \otimes_{\mathbb{Z}} \mathbb{Q}$ where $\mathcal{M}^d_{N_{I,J},F} = \mathcal{M}^d_{N_{I,J}} \otimes_{U_{N_{I,J}}} \operatorname{Spec} F$. We let

$$\begin{aligned} Y_{I,J} = \ & \{(g_{ij}) \in \mathrm{Mat}_d(\widehat{A}) \mid (g_{ij})_{1 \le j \le d} \equiv (\delta_{ij})_{1 \le j \le d} \bmod I \text{ for } 1 \le i \le d-1 \\ & \text{and } \bmod J \text{ for } i = d\}, \end{aligned}$$

(where δ_{ij} is the Kronecker delta) and regard it as an open compact subset of $\mathrm{Mat}_d(\mathbb{A}^\infty)$. Let $\mathrm{ch}_{\gamma \cdot Y_{I,J}} \in \mathcal{S}(\mathrm{Mat}_d(\mathbb{A}^\infty))$ denote the characteristic function of the set $\gamma \cdot Y_{I,J}$. Then $\mathrm{ch}_{\gamma \cdot Y_{I,J}}$ belongs to $\mathcal{S}'(V^\infty)^{\otimes d}$. We let $\kappa_{I,J,\gamma} = \kappa(\mathrm{ch}_{\gamma \cdot Y_{I,J}})$.

2.4.7. Let $L_1 = (I\widehat{A})^{\oplus(d-1)} \oplus (J\widehat{A})$ and $L_2 = \widehat{A}^{\oplus d}$ be $\widehat{A}$-lattices in V^∞. We let $N_{I,J} = L_2/L_1 = (A/I)^{\oplus(d-1)} \oplus (A/J)$. Let $\mathbb{K}^\infty_{I,J} = Y_{I,J} \cap \mathrm{GL}_d(\widehat{A}) \subset \mathrm{GL}_d(\widehat{A})$ be an open compact subgroup. Then $(K_d(\mathcal{M}^d) \otimes_{\mathbb{Z}} \mathbb{Q})^{\mathbb{K}^\infty_{I,J}} \cong (K_d(\mathcal{M}^d_1) \otimes_{\mathbb{Z}} \mathbb{Q})^{\mathbb{K}^\infty_{I,J}} \cong K_d(\mathcal{M}^d_{N_{I,J},F}) \otimes_{\mathbb{Z}} \mathbb{Q}$ since rational K-theory satisfies étale descent. (The proof of the fact that $r_{*,*}$ is étale is found in [Lau, p.8, LEMMA (1.4.2)]. The fact that $m_{*,*}$ is étale can be proved by using the fact that any morphism between étale schemes over a base scheme is étale.) As $\mathrm{ch}_{\gamma Y_{I,J}}$ is $\mathbb{K}^\infty_{I,J}$-invariant, so is $\kappa_{I,J,\gamma}$, and hence we obtain an element in $K_d(\mathcal{M}^d_{N_{I,J},F}) \otimes_{\mathbb{Z}} \mathbb{Q}$. This element is also denoted $\kappa_{I,J,\gamma}$.

3. Kronecker limit formula

We prove a function field analogue of the Kronecker limit formula. The case $d = 1$ is due to Gross and Rosen [Gro-Ro]. The first author follows the same line to prove the general case [Ko]. Here we give a simpler, more conceptual proof. First, we recall the analytic study at infinity of Drinfeld modular varieties. The reader is referred to [De-Hu] for more details. We then give an analytic description of the theta functions and Siegel units which were defined in Sections 2.2 and 2.3. In Section 3.4, Eisenstein series with a complex parameter s are defined. The limit as s tends to 0 is expressed in terms of those analytic functions (Proposition 3.4).

3.1. **Simplicial complexes.**

3.1.1. Let us recall the notion of (abstract) simplicial complex. A simplicial complex is a pair (Y_0, Δ) of a set Y_0 and a set Δ of finite subsets of Y_0 which satisfies the following conditions:

- If $S \in \Delta$ and $T \subset S$, then $T \in \Delta$.
- If $v \in Y_0$, then $\{v\} \in \Delta$.

In this paper we call a simplicial complex in the sense above a strict simplicial complex, and use the terminology “simplicial complex” in a little broader sense, since we will treat as simplicial complexes some arithmetic quotients of Bruhat-Tits building, in which two different simplices may have the same set of vertices. Bruhat-Tits building itself is a strict simplicial complex. Our primary example of a (nonstrict) simplicial complex is $X_{\mathbb{K}^\infty,\bullet}$ to be introduced in Section 5.3.1.

We adopt the following definition of a simplicial complex: a simplicial complex is a collection $Y_\bullet = (Y_i)_{i\ge 0}$ of the sets indexed by non-negative integers, equipped with the following additional data

- a subset $V(\sigma) \subset Y_0$ with cardinality $i+1$, for each $i \ge 0$ and for each $\sigma \in Y_i$ (we call $V(\sigma)$ the set of vertices of σ), and
- an element in Y_j, for each $i \ge j \ge 0$, for each $\sigma \in Y_i$, and for each subset $V' \subset V(\sigma)$ with cardinality $j+1$ (we denote this element in Y_j by the symbol $\sigma \times_{V(\sigma)} V'$ and call it a face of σ)

which satisfy the following conditions:

- For each $\sigma \in Y_0$, the equality $V(\sigma) = \{\sigma\}$ holds,
- For each $i \ge 0$, for each $\sigma \in Y_i$, and for each non-empty subset $V' \subset V(\sigma)$, the equality $V(\sigma \times_{V(\sigma)} V') = V'$ holds.
- For each $i \ge 0$ and for each $\sigma \in Y_i$, the equality $\sigma \times_{V(\sigma)} V(\sigma) = \sigma$ holds, and
- For each $i \ge 0$, for each $\sigma \in Y_i$, and for each non-empty subsets $V', V'' \subset V(\sigma)$ with $V'' \subset V'$, the equality $(\sigma \times_{V(\sigma)} V') \times_{V'} V'' = \sigma \times_{V(\sigma)} V''$ holds.

Let us call the elements of the form $\sigma \times_V (\sigma)$ for j and V' as above, a j-dimensional face of σ. We remark here that the symbol $\times_{V(\sigma)}$ does not mean a fiber product in any way.

Any strict simplicial complex gives a simplicial complex in the sense above in the following way. Let (Y_0, Δ) be a strict simplicial complex. We identify Y_0 with the set of subsets of Y_0 with cardinality 1. For $i \ge 1$ let Y_i denote the set of the elements in Δ which has cardinality $i+1$ as a subset of Y_0. For $i \ge 1$ and for $\sigma \in Y_i$, we put $V(\sigma) = \sigma$ regarded as a subset of Y_0. For a non-empty subset $V \subset V(\sigma)$, of cardinality $i'+1$, we put $\sigma \times_{V(\sigma)} V = V$ regarded as an element of $Y_{i'}$. Then it is easily checked that the collection $Y_\bullet = (Y_i)_{i\ge 0}$ together with the assignments $\sigma \mapsto V(\sigma)$ and $(\sigma, V) \mapsto \sigma \times_{V(\sigma)} V$ forms a simplicial complex.

3.1.2. There is an alternative, less complicated, equivalent definition of a simplicial complex in the sense above, which we will describe in this paragraph. As it will not be used in this article, the reader may skip this paragraph. For a set S, let $\mathcal{P}^{\mathrm{fin}}(S)$ denote the category whose object are the non-empty finite subsets of S and whose morphisms are the inclusions. Then giving a simplicial complex in our sense is equivalent to giving a pair (Y_0, F) of a set Y_0 and a presheaf F of sets on $\mathcal{P}^{\mathrm{fin}}(Y_0)$ such that $F(\{\sigma\}) = \{\sigma\}$ holds for every $\sigma \in Y_0$. This equivalence is explicitly described as follows: given a simplicial complex $Y_\bullet$, the corresponding F is the presheaf which associates, to a non-empty finite subset $V \subset Y_0$ with cardinality $i+1$, the set of elements $\sigma \in Y_i$ satisfying $V(\sigma) = V$.

This alternative definition of a simplicial complex is smarter, nevertheless we have adopted the former definition since it is nearer to the definition of a simplicial complex in the usual sense.

We call an element in Y_i an i-simplex in $Y_\bullet$. For an i-simplex $\sigma \in Y_i$, we call an element of the form $\sigma \times_{V(\sigma)} V'$ for some non-empty subset $V' \subset V(\sigma)$ a face of σ.

Let $Y_\bullet$ and $Z_\bullet$ be simplicial complexes. A map from $Y_\bullet$ to $Z_\bullet$ is a collection $f = (f_i)_{i \geq 0}$ of maps $f_i : Y_i \to Z_i$ of sets which satisfies the following conditions:

- for any $i \geq 0$ and for any $\sigma \in Y_i$, the restriction of f_0 to $V(\sigma)$ is injective and the image of $f|_{V(\sigma)}$ is equal to the set $V(f_i(\sigma))$, and
- for any $i \geq j \geq 0$, for any $\sigma \in Y_i$, and for any non-empty subset $V' \subset V(\sigma)$ with cardinality $j+1$ we have $f_j(\sigma \times_{V(\sigma)} V') = f_i(\sigma) \times_{V(f_i(\sigma))} f_0(V')$.

3.1.3. Let $Y_\bullet$ be a simplicial complex. We associate a CW complex $|Y_\bullet|$ which we call the geometric realization of $Y_\bullet$. Let $I(Y_\bullet)$ denote the disjoint union $\coprod_{i \geq 0} Y_i$. We define a partial order on the set $I(Y_\bullet)$ by saying that $\tau \leq \sigma$ if and only if τ is a face of σ. For $\sigma \in I(Y_\bullet)$, we let Δ_σ denote the set of maps $f : V(\sigma) \to \mathbb{R}_{\geq 0}$ satisfying $\sum_{v \in V(\sigma)} f(v) = 1$. We regard Δ_σ as a topological space whose topology is induced from that of the real vector space $\mathrm{Map}(V(\sigma), \mathbb{R})$. If τ is a face of σ, we regard the space Δ_τ as the closed subspace of Δ_σ which consists of the maps $V(\sigma) \to \mathbb{R}_{\geq 0}$ whose support is contained in the subset $V(\tau) \subset V(\sigma)$. We let $|Y_\bullet|$ denote the colimit $\varinjlim_{\sigma \in I(Y_\bullet)} \Delta_\sigma$ in the category of topological spaces and call it the geometric realization of $Y_\bullet$. It follows from the definition that the geometric realization $|Y_\bullet|$ has a canonical structure of CW-complex.

3.2. **Analytic theory of Drinfeld modular varieties.**

3.2.1. *Notations.* We use the notation introduced in Sections 2.1.1 and 2.4.1. We let V^* denote the dual of V; the elements are regarded as column vectors in F. We write $V_\infty^* = V^* \otimes_F F_\infty$ and $\mathcal{O}_{V_\infty^*} = \mathcal{O}_\infty^\oplus \subset V_\infty^*$. Let $\varpi_\infty \in \mathcal{O}_\infty$ be a uniformizer.

For a scheme Y of finite type over $\operatorname{Spec} F_\infty$, we denote by Y^{an} the rigid analytic space over F_∞ associated to Y. We will often identify the underlying set of Y^{an} with the set of closed points of Y.

3.2.2. *Drinfeld symmetric space (*[Dr], [Ge]*).* Let $\widetilde{V}_\infty$ denote the locally free sheaf of rank d on $\operatorname{Spec} F_\infty$ associated to the F_∞ vector space V_∞. Let $\mathbb{V}_\infty^*$ (resp. $\mathbb{P}(V_\infty^*)$) denote the vector bundle (resp. the projective space bundle) over $\operatorname{Spec} F_\infty$ associated to the locally free sheaf $\widetilde{V}_\infty$ over $\operatorname{Spec} F_\infty$. Let $\mathbb{V}_\infty^{*,\mathrm{an}}$, $\mathbb{P}(V_\infty^*)^{\mathrm{an}}$ denote the rigid analytic space over F_∞ associated to the schemes $\mathbb{V}_\infty^*$, $\mathbb{P}(V_\infty^*)$ over $\operatorname{Spec} F_\infty$, respectively. The canonical right action of the group $\mathrm{GL}_d(F_\infty)$ on V_∞ induces a canonical left action of $\mathrm{GL}_d(F_\infty)$ on $\mathbb{V}_\infty^{*,\mathrm{an}}$ and $\mathbb{P}(V_\infty^*)^{\mathrm{an}}$. It follows from the definition of $\mathbb{V}_\infty^*$ that for any F_∞-algebra R, the F_∞-vector space of R-valued points $\mathbb{V}_\infty^*(R)$ of $\mathbb{V}_\infty^*$ is canonically isomorphic to $V_\infty^* \otimes_{F_\infty} R$.

Let $\mathcal{H}_0$ denote the set of subbundles of $\mathbb{V}_\infty^*$ of rank $d-1$. Let $\widetilde{\mathfrak{X}}$ denote the subset

$$\widetilde{\mathfrak{X}} = \mathbb{V}_\infty^{*,\mathrm{an}} \setminus (\cup_{H_0 \in \mathcal{H}_0} H_0^{\mathrm{an}}) \subset \mathbb{V}_\infty^{*,\mathrm{an}}$$

of the underlying set of $\mathbb{V}_\infty^{*,\mathrm{an}}$. We let $\mathfrak{X} = \mathbb{P}(V_\infty^*)^{\mathrm{an}} \setminus (\cup_{H \in \mathcal{H}} H^{\mathrm{an}})$ where $\mathcal{H}$ denotes the set of F_∞-rational hyperplanes in $\mathbb{P}(V_\infty^*)$. Let $0 = \operatorname{Spec} F_\infty \subset \mathbb{V}_\infty^*$ denote image of the zero section. The analytification $(\mathbb{V}_\infty^* \setminus 0)^{\mathrm{an}} \to \mathbb{P}(V_\infty^*)^{\mathrm{an}}$ of the canonical morphism $\mathbb{V}_\infty^* \setminus 0 \to \mathbb{P}(V_\infty^*)$ gives a canonical map $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ of sets.

In [Dr, Proposition 6.1], Drinfeld shows that the set $\mathfrak{X}$ is an admissible open subset of $\mathbb{P}(V_\infty^*)^{\mathrm{an}}$, and hence $\mathfrak{X}$ has a canonical structure of a rigid analytic space over F_∞. It follows from the same argument as in [Sc-St, p.51, (C)] that the set $\widetilde{\mathfrak{X}}$

is an admissible open subset of $\mathbb{V}_\infty^{*,\mathrm{an}}$. Hence $\widetilde{\mathfrak{X}}$ has a canonical structure of a rigid analytic space over F_∞. The canonical map $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ in the last paragraph gives a morphism of rigid analytic spaces over F_∞. An alternative construction of the rigid analytic space $\widetilde{\mathfrak{X}}$ is given by defining $\widetilde{\mathfrak{X}}$ to be the fiber product $\mathfrak{X} \times_{\mathbb{P}(V_\infty^*)^{\mathrm{an}}} (\mathbb{V}_\infty^* \setminus \{0\})^{\mathrm{an}}$ in the category of rigid analytic spaces over F_∞.

3.2.3. *Bruhat-Tits building (*[Br-Ti]*).* In this section, we recall the definition of the Bruhat-Tits building of PGL_d over F_∞, which is a simplicial complex.

An $\mathcal{O}_\infty$-lattice in V_∞ is a free $\mathcal{O}_\infty$-submodule of V_∞ of rank d. We denote by $\mathrm{Lat}_{\mathcal{O}_\infty}(V_\infty)$ the set of $\mathcal{O}_\infty$-lattices in V_∞. We regard the set $\mathrm{Lat}_{\mathcal{O}_\infty}(V_\infty)$ as a partially ordered set whose elements are ordered by the inclusions of $\mathcal{O}_\infty$-lattices. Two $\mathcal{O}_\infty$-lattices L, L' of V_∞ are called homothetic if $L = \varpi_\infty^j L'$ for some $j \in \mathbb{Z}$. Let $\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$ denote the set of homothety classes of $\mathcal{O}_\infty$-lattices V_∞. We denote by cl the canonical surjection $\mathrm{cl} : \mathrm{Lat}_{\mathcal{O}_\infty}(V_\infty) \to \overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$. We say that a subset S of $\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$ is totally ordered if $\mathrm{cl}^{-1}(S)$ is a totally ordered subset of $\mathrm{Lat}_{\mathcal{O}_\infty}(V_\infty)$. The pair $(\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty), \Delta)$ of the set $\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$ and the set Δ of totally ordered finite subsets of $\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$ forms a strict simplicial complex. The Bruhat-Tits building of PGL_d over F_∞ is a simplicial complex $\mathcal{BT}_\bullet$ which is isomorphic to the simplicial complex associated to this strict simplicial complex. In the next paragraphs we explicitly describe the simplicial complex $\mathcal{BT}_\bullet$.

For an integer $i \geq 0$, let $\widetilde{\mathcal{BT}}_i$ be the set of sequences $(L_j)_{j\in\mathbb{Z}}$ of $\mathcal{O}_\infty$-lattices in V_∞ indexed by $j \in \mathbb{Z}$ such that $L_j \supsetneqq L_{j+1}$ and $\varpi_\infty L_j = L_{j+i+1}$ hold for all $j \in \mathbb{Z}$. In particular, if $(L_j)_{j\in\mathbb{Z}}$ is an element in $\widetilde{\mathcal{BT}}_0$, then $L_j = \varpi_\infty^j L_0$ for all $j \in \mathbb{Z}$. We identify the set $\widetilde{\mathcal{BT}}_0$ with the set $\mathrm{Lat}_{\mathcal{O}_\infty}(V_\infty)$ by associating the $\mathcal{O}_\infty$-lattice L_0 to an element $(L_j)_{j\in\mathbb{Z}}$ in $\mathcal{BT}_0$. We say that two elements $(L_j)_{j\in\mathbb{Z}}$ and $(L'_j)_{j\in\mathbb{Z}}$ in $\widetilde{\mathcal{BT}}_i$ are equivalent if there exists an integer ℓ satisfying $L'_j = L_{j+\ell}$ for all $j \in \mathbb{Z}$. We denote by $\mathcal{BT}_i$ the set of the equivalence classes in $\widetilde{\mathcal{BT}}_i$. For $i = 0$, the identification $\widetilde{\mathcal{BT}}_0 \cong \mathrm{Lat}_{\mathcal{O}_\infty}(V_\infty)$ gives an identification $\mathcal{BT}_0 \cong \overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$.

Let $\sigma \in \mathcal{BT}_i$ and take a representative $(L_j)_{j\in\mathbb{Z}}$ of σ. For $j \in \mathbb{Z}$, let us consider the class $\mathrm{cl}(L_j)$ in $\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$. Since $\varpi_\infty L_j = L_{j+i+1}$, we have $\mathrm{cl}(L_j) = \mathrm{cl}(L_{j+i+1})$. Since $L_j \supsetneqq L_k \supsetneqq \varpi_\infty L_j$ for $0 \leq j < k \leq i$, the elements $\mathrm{cl}(L_0), \ldots, \mathrm{cl}(L_i) \in \overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$ are distinct. Hence the subset $V(\sigma) = \{\mathrm{cl}(L_j) \mid j \in \mathbb{Z}\} \subset \mathcal{BT}_0$ has cardinality $i+1$ and does not depend on the choice of $(L_j)_{j\in\mathbb{Z}}$. It is easy to check that the map from $\mathcal{BT}_i$ to the set of finite subsets of $\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$ which sends $\sigma \in \mathcal{BT}_i$ to $V(\sigma)$ is injective and that the set $\{V(\sigma) \mid \sigma \in \mathcal{BT}_i\}$ is equal to the set of totally ordered subsets of $\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty)$ with cardinality $i+1$. In particular, for any $j \in \{0, \ldots, i\}$ and for any subset $V' \subset V(\sigma)$ of cardinality $j+1$, there exists a unique element in $\mathcal{BT}_j$, which we denote by $\sigma \times_{V(\sigma)} V'$, such that $V(\sigma \times_{V(\sigma)} V')$ is equal to V'. Thus the collection $\mathcal{BT}_\bullet = \coprod_{i\geq 0} \mathcal{BT}_i$ together with the data $V(\sigma)$ and $\sigma \times_{V(\sigma)} V'$ forms a simplicial complex which is canonically isomorphic to the simplicial complex associated to the strict simplicial complex $(\overline{\mathrm{Lat}}_{\mathcal{O}_\infty}(V_\infty), \Delta)$ which we introduced in the first paragraph of Section 3.2.3. We call the simplicial complex $\mathcal{BT}_\bullet$ the Bruhat-Tits building of PGL_d over F_∞.

The simplicial complex is $\mathcal{BT}_\bullet$ is of dimension at most $d-1$, by which we mean that $\mathcal{BT}_i$ is an empty set for $i > d-1$. It follows from the fact that $\widetilde{\mathcal{BT}}_i$ is an empty set for $i > d-1$, which we can check as follows. Let $i > d-1$ and assume that there exists an element $(L_j)_{j\in\mathbb{Z}}$ in $\widetilde{\mathcal{BT}}_i$. Then for $j = 0, \ldots, i+1$, the quotient

L_j/L_{i+1} is a subspace of the d-dimensional $(\mathcal{O}_\infty/\varpi_\infty\mathcal{O}_\infty)$-vector space $L_0/L_{i+1} = L_0/\varpi_\infty L_0$. These subspaces must satisfy $L_0/L_{i+1} \supsetneqq L_1/L_{i+1} \supsetneqq \cdots \supsetneqq L_{i+1}/L_{i+1}$. It is impossible since $i+1>d$.

We let $\mathcal{BT}_{j,*}$ denote the quotient $\widetilde{\mathcal{BT}}_j/F_\infty^\times$. This set is identified with the set of pairs (σ, v) with $\sigma \in \mathcal{BT}_j$ and $v \in \mathcal{BT}_0$ a vertex of σ, which we call a pointed j-simplex. Here the element $(L_i)_{i\in\mathbb{Z}} \bmod F_\infty^\times$ of $\widetilde{\mathcal{BT}}_j/F_\infty^\times$ corresponds to the pair $((L_i)_{i\in\mathbb{Z}}, L_0)$ via this identification.

We identify the set $\widetilde{\mathcal{BT}}_0$ with the coset $\mathrm{GL}_d(F_\infty)/\mathrm{GL}_d(\mathcal{O}_\infty)$ by associating to an element $g \in \mathrm{GL}_d(F_\infty)/\mathrm{GL}_d(\mathcal{O}_\infty)$ the lattice $\mathcal{O}_{V_\infty} g^{-1}$. Let $\mathcal{I} = \{(a_{ij}) \in \mathrm{GL}_d(\mathcal{O}_\infty) \,|\, a_{ij} \bmod \varpi_\infty = 0 \text{ if } i > j\}$ be the Iwahori subgroup. Similarly, we identify the set $\widetilde{\mathcal{BT}}_{d-1}$ with the coset $\mathrm{GL}_d(F_\infty)/\mathcal{I}$ by associating to an element $g \in \mathrm{GL}_d(F_\infty)/\mathcal{I}$ the chain of lattices $(L_i)_{i\in\mathbb{Z}}$ characterized by $L_i = \mathcal{O}_{V_\infty}\Pi_i g^{-1}$ for $i = 0, \ldots, d$. Here, for $i = 0, \ldots, d$, we let Π_i denote the diagonal $d \times d$ matrix $\Pi_i = \mathrm{diag}(\varpi_\infty, \ldots, \varpi_\infty, 1, \ldots, 1)$ with ϖ_∞ appearing i times and 1 appearing $d-i$ times.

The reader is referred to Section 7.2.1 where we recall the relationship between the building $\mathcal{BT}_\bullet$ and the Drinfeld symmetric space $\mathfrak{X}$.

3.2.4. We give an explicit description of a rigid analytic subspace $\widetilde{\mathfrak{U}}_L \subset \widetilde{\mathfrak{X}}$ associated to a lattice $L \in \widetilde{\mathcal{BT}}_0$ in this section. A similar description for rigid analytic subspaces $\mathfrak{U}_\sigma \subset \mathfrak{X}$ associated to a simplex $\sigma \in \mathcal{BT}_\bullet$ is known and the details are recalled in Section 7.2.1.

The underlying set of $\mathbb{V}_\infty^{*,\mathrm{an}}$ is canonically isomorphic to the set of closed points of $\mathbb{V}_\infty^*$. For $x \in \mathbb{V}_\infty^*$, let $F_\infty(x)$ denote the residue field at x. Since $F_\infty(x)$ is a finite extension of F_∞, the absolute value $|\ |_\infty$ on F_∞ is uniquely extended to that on $F_\infty(x)$, which we denote by the same symbol $|\ |_\infty$. We let $\mathcal{O}_{F_\infty(x)}$ denote the valuation ring $\mathcal{O}_{F_\infty(x)} = \{a \in F_\infty(x) \mid |a|_\infty \le 1\}$ of $F_\infty(x)$ with respect to the absolute value $|\ |_\infty$. We denote by κ_∞ the residue field of $\mathcal{O}_\infty$ and let $\kappa_\infty(x)$ denote the residue field of $\mathcal{O}_{F_\infty(x)}$. Since the closed point in $\mathbb{V}_\infty^*$ corresponding to x is a $F_\infty(x)$-rational point, it gives an element $\boldsymbol{\tau}_x$ in $V_\infty^* \otimes_{F_\infty} F_\infty(x)$ via the isomorphism mentioned in the first paragraph of Section 3.2.2.

We let $\widetilde{\mathfrak{U}}_{\mathcal{O}_{V_\infty}}$ denote the set of elements $x \in \mathbb{V}_\infty^{*,\mathrm{an}}$ such that $\boldsymbol{\tau}_x \in \mathcal{O}_{V_\infty^*} \otimes_{\mathcal{O}_\infty} \mathcal{O}_{F_\infty(x)}$ and that $\boldsymbol{\tau}_x \otimes 1 \in (\mathcal{O}_{V_\infty^*} \otimes_{\mathcal{O}_\infty} \mathcal{O}_{F_\infty(x)}) \otimes_{\mathcal{O}_{F_\infty(x)}} \kappa_\infty(x) = (\mathcal{O}_{V_\infty^*} \otimes_{\mathcal{O}_\infty} \kappa_\infty) \otimes_{\kappa_\infty} \kappa_\infty(x)$ does not belong to $\overline{H} \otimes_{\kappa_\infty} \kappa_\infty(x)$ for any proper κ_∞-vector subspace $\overline{H} \subsetneqq \mathcal{O}_{V_\infty^*} \otimes_{\mathcal{O}_\infty} \kappa_\infty$. It is easy to check that the subset $\widetilde{\mathfrak{U}}_{\mathcal{O}_{V_\infty}}$ of $\mathbb{V}_\infty^{*,\mathrm{an}}$ is contained in the subset $\widetilde{\mathfrak{X}} \subset \mathbb{V}_\infty^{*,\mathrm{an}}$. Let $L \in \widetilde{\mathcal{BT}}_0$ be an $\mathcal{O}_\infty$-lattice in V_∞. Take $g \in \mathrm{GL}_d(F_\infty)$ such that $L = \mathcal{O}_{V_\infty} g^{-1}$. We set $\widetilde{\mathfrak{U}}_L = g\widetilde{\mathfrak{U}}_{\mathcal{O}_{V_\infty}}$. In other words, $\widetilde{\mathfrak{U}}_L = \{x \in \mathbb{V}_\infty^{*,\mathrm{an}} \,|\, \mathbf{v}\boldsymbol{\tau}_x \in \mathcal{O}_{F_\infty(x)}^\times \text{ for all } \mathbf{v} \in L \setminus \varpi_\infty L\}$.

It is easy to check that the subset $\widetilde{\mathfrak{U}}_L$ is an open affinoid subset of $\mathbb{V}_\infty^{*,\mathrm{an}}$ which is contained in $\widetilde{\mathfrak{X}}$. The coordinate $\mathcal{O}_\infty$-algebra $\widetilde{B}_L^o$ of the formal model of the affinoid $\widetilde{\mathfrak{U}}_L$ has the following explicit description. Let $\mathrm{Sym}^\bullet L = \bigoplus_{n\ge 0} \mathrm{Sym}^n_{\mathcal{O}_\infty} L$ denote the symmetric algebra on L over $\mathcal{O}_\infty$. Let $S_L \subset L \setminus \varpi_\infty L$ be a complete set of representatives of the set $(L/\varpi_\infty L) \setminus \{0\}$ with respect to the surjection

$$L \setminus \varpi_\infty L \twoheadrightarrow (L/\varpi_\infty L) \setminus \{0\}.$$

Then $\widetilde{B}_L^o$ is isomorphic to the ϖ_∞-adic completion of the $\mathcal{O}_\infty$-subalgebra $\mathrm{Sym}^\bullet L[S_L^{-1}]$ of the field of fractions $\mathrm{Frac}\,\mathrm{Sym}^\bullet L$ of $\mathrm{Sym}^\bullet L$ generated by $\mathrm{Sym}^\bullet L$ and the set

$\{b^{-1} \mid b \in S_L\}$. The $\mathcal{O}_\infty$-algebra $\mathrm{Sym}^\bullet L[S_L^{-1}]$ is regular and excellent ([EGAIV, 7.8, p.214]) since it is a localization of the polynomial algebra $\mathrm{Sym}^\bullet L$ over $\mathcal{O}_\infty$. It follows that the $\mathcal{O}_\infty$-algebra $\widetilde{B}_L^o$ is regular. Since ϖ_∞ is a prime element in $\mathrm{Sym}^\bullet L[S_L^{-1}]$, the ideal $\varpi_\infty \widetilde{B}_L^o$ of $\widetilde{B}_L^o$ generated by ϖ_∞ is a prime ideal of height one. Since $\widetilde{B}_L^o$ is regular, its localization at $\varpi_\infty \widetilde{B}_L^o$ is a discrete valuation ring. Hence the prime ideal $\varpi_\infty \widetilde{B}_L^o$ defines via the homomorphism η a valuation v_L on the field of fractions of $\widetilde{B}_L^o$.

3.2.5. *The order of rigid analytic functions on $\widetilde{\mathfrak{X}}$ at lattices.* We define a homomorphism

$$\mathrm{ord}_L : \Gamma(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widetilde{\mathfrak{X}}}^\times) \to \mathbb{Z}$$

of abelian groups for $L \in \widetilde{\mathcal{BT}}_0$, to be the composite

$$\Gamma(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widetilde{\mathfrak{X}}}^\times) \to \Gamma(U_L, \mathcal{O}_{U_L}^\times) \to \mathbb{Z}$$

where the first map is the restriction and the second map is the map given by the valuation v_L. Let $h \in \Gamma(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widetilde{\mathfrak{X}}}^\times)$ be an invertible rigid analytic function. Then by the definition of the homomorphism ord_L we have $\mathrm{ord}_L h = \inf_{x \in U_L} \log_{q_\infty} |h(x)|_\infty$.

Given a lattice $L \in \widetilde{\mathcal{BT}}_0$ and a row vector $\mathbf{a} \in V_\infty$, we let $\mathrm{ord}_L(\mathbf{a}) = \sup\{n \in \mathbb{Z} \mid \mathbf{a} \in \varpi_\infty^n L\}$, and $|\mathbf{a}|_L = q_\infty^{-\mathrm{ord}_L(\mathbf{a})}$. Note that $|\mathbf{a}|_L = 1$ if and only if $\mathbf{a} \in L \setminus \varpi_\infty L$. The abuse of notation is justified by the following proposition.

Proposition 3.1. *Given a lattice $L \in \widetilde{\mathcal{BT}}_0$ and a row vector $\mathbf{a} \in V_\infty$, let $f_{\mathbf{a}}$ be the rigid analytic function on $\widetilde{\mathfrak{X}}$ characterized by $f_{\mathbf{a}}(x) = \mathbf{a}\boldsymbol{\tau}_x$ for every $x \in \widetilde{\mathfrak{X}}$. Then we have* $\mathrm{ord}_L f_{\mathbf{a}} = \mathrm{ord}_L(\mathbf{a})$.

Proof. We may without loss of generality that $L = \mathcal{O}_{V_\infty}$. Hence, if $\mathbf{a} = (a_1, \ldots, a_d)$ with $a_i \in F_\infty (1 \le i \le d)$, then $\mathrm{ord}_L f_{\mathbf{a}} = \inf_{1 \le i \le d} (-\log_{q_\infty} |a_i|_\infty)$. The claim follows. $\square$

3.2.6. *Convention.* Given an element $g \in \mathrm{GL}_d(\mathbb{A})$, we always denote by g_∞ the component at infinity, and g^∞ the finite part. Given a function f on $\mathrm{GL}_d(\mathbb{A})$, we write $f(g) = f(g_\infty, g^\infty)$ for $g = (g_\infty, g^\infty) \in \mathrm{GL}_d(\mathbb{A})$.

An A-lattice in V is a projective A-submodule in V of rank d. Recall that we defined an $\widehat{A}$-lattice L in V^∞ to be a free $\widehat{A}$-module of rank d contained in V^∞ such that the canonical map $L \otimes_{\widehat{A}} \mathbb{A}^\infty \to V^\infty$ is an isomorphism. Let $\mathrm{Lat}_{\widehat{A}}(V^\infty)$ (resp. $\mathrm{Lat}_A(V)$) denote the set of $\widehat{A}$-lattices in V^∞ (resp. A-lattices in V). There are canonical isomorphisms $\mathrm{GL}_d(\mathbb{A}^\infty)/\mathrm{GL}_d(\widehat{A}) \cong \mathrm{Lat}_{\widehat{A}}(V^\infty) \cong \mathrm{Lat}_A(V)$, where $g^\infty \in \mathrm{GL}_d(\mathbb{A}^\infty)/\mathrm{GL}_d(\widehat{A})$ is sent to $\mathcal{O}_{V^\infty} g^{\infty -1}$ in $\mathrm{Lat}_{\widehat{A}}(V^\infty)$, and to $\mathcal{O}_{V^\infty} g^{\infty -1} \cap V$ in $\mathrm{Lat}_A(V)$.

Given $\Lambda \in \mathrm{Lat}_A(V)$, let $\widehat{\Lambda} = \Lambda \otimes_A \widehat{A} \in \mathrm{Lat}_{\widehat{A}}(V^\infty)$ be the corresponding $\widehat{A}$-lattice in V^∞. If $g^\infty \in \mathrm{GL}_d(\mathbb{A}^\infty)$ is given, we denote by $\Lambda g^{\infty -1} \in \mathrm{Lat}_A(V)$ the A-lattice corresponding to the $\widehat{A}$-lattice $\widehat{\Lambda} g^{\infty -1}$ via the isomorphism above. Suppose we are given an element $\mathbf{b} \in V/\Lambda$. We denote by $\mathbf{b} g^{\infty -1} \in V/(\Lambda g^{\infty -1})$ the image of $\mathbf{b} \in V/\Lambda$ under the sequence of isomorphisms

$$V/\Lambda \cong V^\infty/\widehat{\Lambda} \cong V^\infty/\widehat{\Lambda} g^{\infty -1} \cong V/\Lambda g^{\infty -1}$$

where the first and third maps are those induced by the canonical inclusion $V \subset V^\infty$ and the second is the map induced by the multiplication-by-$g^{\infty -1}$ on V^∞.

3.2.7. *Uniformization* ([Dr]). Let $L_1 \subset L_2 \subset V^\infty$ be $\widehat{A}$-lattices. Let $\mathbb{K}^\infty_{L_1,L_2} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ be the compact open subgroup consisting of those elements g^∞ such that $L_i g^{\infty-1} = L_i\,(i = 1, 2)$ and the induced action on L_2/L_1 is the identity. It is an open compact subgroup of $\mathrm{GL}_d(\widehat{A})$. We have a canonical isomorphism ([Dr], see [Bl-St] for a different construction)

$$(\mathcal{M}^d_{L_2/L_1} \times_{U_{L_2/L_1}} \operatorname{Spec} F_\infty)^{\mathrm{an}} \cong \mathrm{GL}_d(F)\backslash(\mathfrak{X} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}) \tag{3.1}$$

of rigid analytic spaces over $\operatorname{Spec} F_\infty$. Here in the right hand side, $\mathfrak{X}\times\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}$ denotes the disjoint union of copies of $\mathfrak{X}$ indexed by the set $\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}$, and $\mathrm{GL}_d(F)\backslash(\mathfrak{X} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2})$ denotes its quotient under the diagonal action of $\mathrm{GL}_d(F)$.

We define a map

$$\mathrm{ord}_{L_1,L_2} : \mathcal{O}(\mathcal{M}^d_{L_2/L_1})^\times \to \mathrm{Map}(\mathrm{GL}_d(F)\backslash(\widetilde{\mathcal{BT}}_0) \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}, \mathbb{C})$$

as the composite

$$\begin{aligned}&\mathcal{O}(\mathcal{M}^d_{L_2/L_1})^\times \xrightarrow{(1)} \mathcal{O}((\mathcal{M}^d_{L_2/L_1} \times_{U_{L_2/L_1}} \operatorname{Spec} F_\infty)^{\mathrm{an}})^\times \\ &= \mathcal{O}(\mathrm{GL}_d(F)\backslash(\mathfrak{X} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}))^\times \\ &\xrightarrow{(2)} \mathrm{Map}(\mathrm{GL}_d(F)\backslash(\widetilde{\mathcal{BT}}_0) \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}, \mathbb{C})\end{aligned}$$

where the map (1) is the analytification, and the map (2) is the map induced by the functions ord_L for $L \in \widetilde{\mathcal{BT}}_0$, as defined in Section 3.2.5, composed with the pullback by the canonical quotient map $\widetilde{\mathfrak{X}} \to \mathfrak{X}$.

Let $I, J \subsetneqq A$ be two nonzero ideals with $J \subset I$. We remark that if we let $L_2 = \widehat{A}^{\oplus d}$ and $L_1 = (I\widehat{A})^{\oplus(d-1)} \oplus (J\widehat{A})$, we recover $\mathbb{K}^\infty_{I,J} = \mathbb{K}^\infty_{L_1,L_2}$ of Section 2.4.7.

3.3. **Analytic descriptions of theta functions and Siegel units.**

3.3.1. Let $x \in \widetilde{\mathfrak{X}}$ and let $\boldsymbol{\tau}_x \in V^*_\infty \otimes_{F_\infty} F_\infty(x)$ be as in Section 3.2.4. Let $\Lambda \subset V$ be an A-lattice. We let $\Lambda_x = \{\boldsymbol{\lambda}\boldsymbol{\tau}_x \mid \boldsymbol{\lambda} \in \Lambda\} \subset F_\infty(x)$. We define $\sigma_{\Lambda_x}(z) \in \Gamma(\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)}, \mathcal{O}_{\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)}})$ to be the rigid analytic function

$$\sigma_{\Lambda_x}(z) = z \prod_{\lambda\in\Lambda_x\setminus\{0\}} \left(1 - \frac{z}{\lambda}\right)$$

where z is the coordinate function of $\mathbb{G}_{a,F_\infty(x)}$. The function σ_{Λ_x} induces an isomorphism of rigid analytic groups $\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)}/\Lambda_x \xrightarrow{\cong} \mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)}$. (We refer to [De-Hu, p.46] for a similar statement over C_∞ instead of $F_\infty(x)$. Their argument readily applies to our case.) On the additive group $\mathbb{G}_{a,F_\infty(x)}$ over $F_\infty(x)$, there exists a unique structure $\varphi_{\Lambda_x} : A \to \mathrm{End}(\mathbb{G}_{a,F_\infty(x)})$ of Drinfeld module of rank d over $F_\infty(x)$ such that the diagram

$$\begin{array}{ccc}\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)} & \xrightarrow{a} & \mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)} \\ {\scriptstyle\sigma_{\Lambda_x}}\downarrow & & {\scriptstyle\sigma_{\Lambda_x}}\downarrow \\ \mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)} & \xrightarrow{\varphi_{\Lambda_x}(a)^{\mathrm{an}}} & \mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)}\end{array}$$

commutes ([De-Hu, (2.1) Theorem, p.46]).

Let $\theta^{(q^d_\infty-1)^2,\mathrm{an}}_{(\mathbb{G}_{a,F_\infty(x)},\varphi_{\Lambda_x})/F_\infty(x)} \in \Gamma(\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)} \setminus \{0\}, \mathcal{O}^\times_{\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)}\setminus\{0\}})$ be the analytification of the function $f^{q^d_\infty-1}$ defined in Section 2.2.2 for the Drinfeld module

$(\mathbb{G}_{a,F_\infty(x)}, \varphi_{\Lambda_x})$ of rank d over $F_\infty(x)$. Let $\theta_x^{(q_\infty^d-1)^2,\mathrm{an}} \in \Gamma(\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)} \backslash \Lambda_x, \mathcal{O}^\times_{\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)}\backslash\Lambda_x})$ denote the composite of the function $\theta^{(q_\infty^d-1)^2,\mathrm{an}}_{(\mathbb{G}_{a,F_\infty(x)},\varphi_{\Lambda_x})/F_\infty(x)}$ and the restriction of the function σ_{Λ_x} to $\mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)} \setminus \Lambda_x$. It follows from the formula (2.2) that the function $\theta_x^{(q_\infty^d-1)^2,\mathrm{an}}$ has the following description. For any $a \in A \setminus \{0\}$, we have

$$\begin{aligned}\left(\theta_x^{(q_\infty^d-1)^2,\mathrm{an}}(z)\right)^{(|a|_\infty^d-1)/(q_\infty^d-1)} &= \left(\frac{\sigma_{\Lambda_x}(z)^{|a|_\infty^d-1}}{(\mathrm{N}_a(\sigma_{\Lambda_x}))(z)}\right)^{q_\infty^d-1} \\ &= \left(\frac{\sigma_{\Lambda_x}(z)^{|a|_\infty^d-1}}{\prod_{\mathbf{a}\in\Lambda/a}\sigma_{\Lambda_x}\left(\frac{z}{a}+\frac{\mathbf{a}\boldsymbol{\tau}_x}{a}\right)}\right)^{q_\infty^d-1}.\end{aligned} \tag{3.2}$$

Let $\mathbf{b} \in (V/\Lambda)\setminus\{0\}$. Let $L_1 = \Lambda \otimes_A \widehat{A} \subset V^\infty$ and L_2 be the $\widehat{A}$-lattice generated by L_1 and $\widetilde{\mathbf{b}} \otimes 1 \in V \otimes_A \widehat{A} \cong V^\infty$ where $\widetilde{\mathbf{b}} \in V$ is a lift of $\mathbf{b} \in V/\Lambda$. We introduce an element $g^{\mathrm{an}}_{\Lambda,\mathbf{b}} \in \mathcal{O}(\mathfrak{X})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and its adelic version $g^{\mathbb{A}}_{\Lambda,\mathbf{b}} \in \mathcal{O}(\mathrm{GL}_d(F)\backslash(\mathfrak{X} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}))^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. We let $g^{(q_\infty^d-1)^2}_{\Lambda,\mathbf{b}}(x) = \theta_x^{(q_\infty^d-1)^2,\mathrm{an}}(\mathbf{b}\boldsymbol{\tau}_x) \in F_\infty(x)$ for $x \in \widetilde{\mathfrak{X}}$. It follows from the formula (3.2) that for any $a \in A\backslash\{0\}$, there exists an invertible rigid analytic function $g^{\mathrm{an}}_{\Lambda,\mathbf{b},a}$ on $\mathfrak{X}$ satisfying $g^{\mathrm{an}}_{\Lambda,\mathbf{b},a}(\overline{x}) = (g^{(q_\infty^d-1)^2}_{\Lambda,\mathbf{b}}(x))^{(|a|_\infty^d-1)/(q_\infty^d-1)}$ for every $x \in \widetilde{\mathfrak{X}}$, where $\overline{x}$ denotes the image of x under the canonical map $\widetilde{\mathfrak{X}} \to \mathfrak{X}$. Since the rigid analytic space $\mathfrak{X}$ is reduced, such a function $g^{\mathrm{an}}_{\Lambda,\mathbf{b},a}$ is unique. We put $g^{\mathrm{an}}_{\Lambda,\mathbf{b}} = g^{\mathrm{an}}_{\Lambda,\mathbf{b},a} \otimes 1/((|a|_\infty^d - 1)(q_\infty^d - 1)) \in \mathcal{O}(\mathfrak{X})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows from the uniqueness of $g^{\mathrm{an}}_{\Lambda,\mathbf{b},a}$ that the element $g^{\mathrm{an}}_{\Lambda,\mathbf{b}}$ does not depend on the choice of a.

For $a \in A \setminus \{0\}$, we define $g^{\mathbb{A}}_{\Lambda,\mathbf{b},a} \in \mathcal{O}(\mathfrak{X} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2})^\times$ by setting $g^{\mathbb{A}}_{\Lambda,\mathbf{b},a}(x, g^\infty) = g^{\mathrm{an}}_{\Lambda g^{\infty-1},\mathbf{b}g^{\infty-1},a}(x)$. It follows from the formula (3.2) that $g^{\mathbb{A}}_{\Lambda,\mathbf{b},a}(\gamma x, \gamma g^\infty) = g^{\mathbb{A}}_{\Lambda,\mathbf{b},a}(x, g^\infty)$ for any $\gamma \in \mathrm{GL}_d(F)$. Hence $g^{\mathbb{A}}_{\Lambda,\mathbf{b},a}$ is an element in $(\mathcal{O}(\mathfrak{X}\times\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2})^\times)^{\mathrm{GL}_d(F)} = \mathcal{O}(\mathrm{GL}_d(F)\backslash\mathfrak{X}\times\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2})^\times$. We put $g^{\mathbb{A}}_{\Lambda,\mathbf{b}} = g^{\mathbb{A}}_{\Lambda,\mathbf{b},a}\otimes 1/((|a|_\infty^d-1)(q_\infty^d-1)) \in \mathcal{O}(\mathrm{GL}_d(F)\backslash\mathfrak{X} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows from the uniqueness of $g^{\mathrm{an}}_{\Lambda,\mathbf{b},a}$ that the element $g^{\mathbb{A}}_{\Lambda,\mathbf{b}}$ does not depend on the choice of a.

3.3.2. For $x \in \widetilde{\mathfrak{X}}$ and for $g^\infty \in \mathrm{GL}_d(\mathbb{A}^\infty)$, let $(E_{x,g^\infty}, \varphi_{x,g^\infty})$ be the Drinfeld module of rank d over $F_\infty(x)$ with the level (L_2/L_1)-structure $\psi_{x,g^\infty} : (L_2/L_1)_{\mathrm{Spec}\, F_\infty(x)} \to E_{x,g^\infty}$ corresponding to the image of the point $(x, g\mathbb{K}^\infty_{L_1,L_2})$ under the composite

$$\begin{aligned}\widetilde{\mathfrak{X}} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2} &\to \mathrm{GL}_d(F)\backslash(\mathfrak{X} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}) \\ &\xrightarrow{\cong} (\mathcal{M}^d_{L_2/L_1} \times_{U_{L_2/L_1}} \mathrm{Spec}\, F_\infty)^{\mathrm{an}}\end{aligned}$$

where the last map is the canonical isomorphism (3.1). It follows from the construction of the isomorphism (3.1) given in [Dr, Proposition 6.6, p.583] that there is an isomorphism $\alpha : (E_{x,g^\infty}, \varphi_{x,g^\infty}) \cong (\mathbb{G}_{a,F_\infty(x)}, \varphi_{(\Lambda g^{\infty-1})_x})$ of Drinfeld modules over $F_\infty(x)$ such that the analytification of the composite $(L_2/L_1)_{\mathrm{Spec}\, F_\infty(x)} \xrightarrow{\psi_{x,g^\infty}} E_{x,g^\infty} \xrightarrow{\alpha} \mathbb{G}_{a,F_\infty(x)}$ is equal to the composite

$$L_2/L_1 \xrightarrow{(1)} L_2 g^{\infty-1}/L_1 g^{\infty-1} \xrightarrow{(2)} (A\widetilde{\mathbf{b}} + \Lambda)g^{\infty-1}/\Lambda g^{\infty-1} \xrightarrow{(3)} \mathbb{G}^{\mathrm{an}}_{a,F_\infty(x)}$$

where the map (1) is the isomorphism supplied by the left multiplication by $g^{\infty-1}$, the map (2) is the inverse of the isomorphism $(A\widetilde{\mathbf{b}}+\Lambda)g^{\infty-1}/\Lambda g^{\infty-1} \xrightarrow{\cong} L_2 g^{\infty-1}/L_1 g^{\infty-1}$ given by the inclusions $(A\widetilde{\mathbf{b}} + \Lambda)g^{\infty-1} \subset L_2 g^{\infty-1}$ and $\Lambda g^{\infty-1} \subset L_1 g^{\infty-1}$, and the map (3) is the morphism which sends the class of $\boldsymbol{\lambda} \in (A\widetilde{\mathbf{b}} + \Lambda)g^{\infty-1}$ to $\sigma_{(\Lambda g^{\infty-1})_x}(\boldsymbol{\lambda}\boldsymbol{\tau}_x)$.

Recall that we defined in Section 2.4.5 a Siegel unit $g_{L_2/L_1,b} \in \mathcal{O}(\mathcal{M}^d_{L_2/L_1})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ where b is the class of $\widetilde{\mathbf{b}} \otimes 1$ in L_2/L_1. It follows from the isomorphism in the previous paragraph and the construction of $g^{\mathbb{A}}_{\Lambda,\mathbf{b}}$ that the element in $\mathcal{O}((\mathcal{M}^d_{L_2/L_1} \times_{U_{L_2/L_1}} \operatorname{Spec} F_\infty)^{\mathrm{an}})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to $g^{\mathbb{A}}_{\Lambda,\mathbf{b}}$ via the canonical isomorphism (3.1), coincides with the analytification of $g_{L_2/L_1,b}$.

3.4. **Eisenstein series.**

We define Eisenstein series $E_{\Lambda,\mathbf{b}}$ in this section. We also define its adelic version $\mathbb{E}_{\Lambda,\mathbf{b}}$.

3.4.1. We define $\mathbb{C}((q_\infty^{-s}))$-valued functions $E_{\Lambda,\mathbf{b}}$ on the set $\widetilde{\mathcal{BT}}_0$ of $\mathcal{O}_\infty$-lattices in V_∞. (Here, q_∞^{-s} is regarded as an indeterminate.) Given an A-lattice $\Lambda \subset V$ and $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, we let

$$E_{\Lambda,\mathbf{b}}(L) = \sum_{\mathbf{x} \in V,\, \mathbf{x} \bmod \Lambda = \mathbf{b}} |\mathbf{x}|_L^{-s}.$$

The sum is convergent in the (q_∞^{-s})-adic topology.

The following lemma is checked easily.

Lemma 3.2. *Let $\Lambda \supset \Lambda'$ be two A-lattices in V, and $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$. Then*

(1) $E_{\Lambda,\mathbf{b}} = \sum_{\mathbf{b}' \in V/\Lambda', \mathbf{b}' \bmod \Lambda = \mathbf{b}} E_{\Lambda',\mathbf{b}'}$,
(2) *If $a \in A \setminus \{0\}$, then $E_{a\Lambda,a\mathbf{b}} = E_{\Lambda,\mathbf{b}} |a|_\infty^{-s}$.*

□

3.4.2. Given an A-lattice $\Lambda \subset V$ and $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, we let

$$\mathbb{E}_{\Lambda,\mathbf{b}}(g_\infty, g^\infty) = E_{\Lambda g^{\infty-1}, \mathbf{b} g^{\infty-1}}(\mathcal{O}_{V_\infty} g_\infty^{-1}),$$

for $(g_\infty, g^\infty) \in \mathrm{GL}_d(\mathbb{A})$.

We note that $\mathbb{E}_{\Lambda,\mathbf{b}}$ is a $\mathbb{C}((q_\infty^{-s}))$-valued function

$$\mathbb{E}_{\Lambda,\mathbf{b}} : \mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / (\mathrm{GL}_d(\mathcal{O}_\infty) \times \mathbb{K}^\infty_{L_1,L_2}) \to \mathbb{C}((q_\infty^{-s}))$$

on the double coset space $\mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / (\mathrm{GL}_d(\mathcal{O}_\infty) \times \mathbb{K}^\infty_{L_1,L_2})$ where the $\widehat{A}$-lattices L_1 and L_2 are as in Section 3.3.2 and the open compact subgroup $\mathbb{K}^\infty_{L_1,L_2}$ is as in Section 3.2.7.

3.4.3. We write $V_{\mathbb{A}} = V \otimes_F \mathbb{A} = V_\infty \times V^\infty$. Let $\Lambda \subset V$ be an A-lattice and $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$. We put $\widehat{\Lambda} = \Lambda \otimes_A \widehat{A} \subset V^\infty$. Let us define a $\mathbb{C}((q_\infty^{-s}))$-valued function $\phi_{\Lambda,\mathbf{b}}$ on $V_{\mathbb{A}}$. For $\mathbf{x} = (\mathbf{x}_\infty, \mathbf{x}^\infty) \in V_{\mathbb{A}}$, where $\mathbf{x}_\infty$ (resp. $\mathbf{x}^\infty$) denotes the component at ∞ (resp. the finite part) of $\mathbf{x}$, we put

$$\phi_{\Lambda,\mathbf{b}}(\mathbf{x}) = \phi_\infty(\mathbf{x}_\infty) \phi^\infty_{\Lambda,\mathbf{b}}(\mathbf{x}^\infty)$$

where $\phi^\infty_{\Lambda,\mathbf{b}}$ is defined to be the characteristic function on $\widetilde{\mathbf{b}} \otimes 1 + \widehat{\Lambda} \subset V^\infty$ (see Section 3.3.2 for notation), and $\phi_\infty(\mathbf{x}_\infty) = |\mathbf{x}_\infty|^{-s}_{\mathcal{O}_{V_\infty}}$.

Lemma 3.3. *If $g \in \mathrm{GL}_d(\mathbb{A})$, then $\mathbb{E}_{\Lambda,\mathbf{b}}(g) = \sum_{\mathbf{x} \in V} \phi_{\Lambda,\mathbf{b}}(\mathbf{x}g)$.*

Proof. This is immediate from the definition of $\mathbb{E}_{\Lambda,\mathbf{b}}$ and of $\phi_{\Lambda,\mathbf{b}}$. □

3.5. **Limit formula.**

We give a short proof of the function field analogue of the Kronecker limit formula. This was already proved in [Gro-Ro] for the rank one case and in [Ko] for an arbitrary rank, but here we give another proof, mainly to fix some notation concerning adeles.

Proposition 3.4. *Let $\Lambda \subset V$ be an A-lattice, $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, and $(g_\infty, g^\infty) \in \mathrm{GL}_d(\mathbb{A})$. We let $L_1 = \Lambda \otimes_A \widehat{A} \subset V^\infty$ and L_2 be the $\widehat{A}$-lattice in V^∞ generated by L_1 and $\widetilde{\mathbf{b}} \otimes 1 \in (V/\Lambda) \otimes_A \widehat{A}$ where $\widetilde{\mathbf{b}}$ is a lift of $\mathbf{b}$. Then $(1 - q_\infty^{d-s})\mathbb{E}_{\Lambda,\mathbf{b}}(g_\infty, g^\infty)$ is a Laurent polynomial in q_∞^{-s} which is divisible by $1 - q_\infty^{-s}$. Moreover we have*

$$\mathrm{ord}_{L_1,L_2} g^{\mathbb{A}}_{\Lambda,\mathbf{b}} = \frac{1}{1-q_\infty^{-s}} \mathbb{E}_{\Lambda,\mathbf{b}}\Big|_{s=0}.$$

Proof. We prove the non-adelic version. Let L be an $\mathcal{O}_\infty$-lattice in V_∞. Using Proposition 3.1 and the analytic description of theta functions given in Section 3.3.1, we have

$$\begin{aligned}
&\mathrm{ord}_L g^{\mathrm{an}}_{\Lambda,\mathbf{b}} \\
&= \frac{1}{|a|_\infty^d - 1} \times \left[|a|_\infty^d \left\{ \mathrm{ord}_L(\mathbf{b}) + \sum_{\boldsymbol{\lambda} \in \Lambda \setminus \{0\}} (\mathrm{ord}_L(\boldsymbol{\lambda} - \mathbf{b}) - \mathrm{ord}_L(\boldsymbol{\lambda})) \right\} \right. \\
&\quad \left. - \sum_{\mathbf{a} \in \Lambda/a} \left\{ \mathrm{ord}_L\left(\frac{\mathbf{a}+\mathbf{b}}{a}\right) + \sum_{\boldsymbol{\lambda} \in \Lambda \setminus \{0\}} \left(\mathrm{ord}_L\left(\boldsymbol{\lambda} - \frac{\mathbf{a}+\mathbf{b}}{a}\right) - \mathrm{ord}_L(\boldsymbol{\lambda}) \right) \right\} \right]
\end{aligned}$$

for any $a \in A \setminus \{0\}$. We note that the summands $\mathrm{ord}_L(\boldsymbol{\lambda} - \mathbf{b}) - \mathrm{ord}_L(\boldsymbol{\lambda})$ and $\mathrm{ord}_L(\boldsymbol{\lambda} - \frac{\mathbf{a}+\mathbf{b}}{a}) - \mathrm{ord}_L(\boldsymbol{\lambda})$ are zero for almost all $\boldsymbol{\lambda} \in \Lambda \setminus \{0\}$.

Let $E^*_\Lambda(L) = \sum_{\lambda \in \Lambda \setminus \{0\}} |\lambda|_L^{-s}$. The expression above equals

$$\begin{aligned}
&\frac{1}{|a|_\infty^d - 1} \frac{1}{\log q_\infty} \frac{\partial}{\partial s} \left\{ |a|_\infty^d (E_{\Lambda,\mathbf{b}}(L) - E^*_\Lambda(L)) \right. \\
&\quad \left. - \sum_{\mathbf{a} \in \Lambda/a} (E_{\Lambda,(\mathbf{a}+\mathbf{b})/a}(L) - E^*_\Lambda(L)) \right\} \Bigg|_{s=0}.
\end{aligned}$$

The terms in E^*_Λ cancel and we obtain

$$\frac{1}{|a|_\infty^d - 1} \frac{1}{\log q_\infty} \frac{\partial}{\partial s} \left\{ |a|_\infty^d E_{\Lambda,\mathbf{b}}(L) - \sum_{\mathbf{a} \in \Lambda/a} E_{\Lambda,(\mathbf{a}+\mathbf{b})/a}(L) \right\} \Bigg|_{s=0}.$$

From Lemma 3.2, we have

$$\sum_{\mathbf{a} \in \Lambda/a} E_{\Lambda,(\mathbf{a}+\mathbf{b})/a}(L) = E_{a^{-1}\Lambda,\mathbf{b}/a}(L) = E_{\Lambda,\mathbf{b}}(L)|a|_\infty^s.$$

Since $E_{\Lambda,\mathbf{b}}(L) - E^*_\Lambda(L)$ and $E_{\Lambda,(\mathbf{a}+\mathbf{b})/a}(L) - E^*_\Lambda(L)$ are finite sums, we see that $(1 - q_\infty^{d-s})E_{\Lambda,\mathbf{b}}(L)$ is a Laurent polynomial in q_∞^{-s} divisible by $1 - q_\infty^{-s}$ and the expression above is equal to

$$\frac{1}{1-q_\infty^d} \left[\frac{q_\infty^d - q_\infty^s}{1 - q_\infty^s} E_{\Lambda,\mathbf{b}}(L) \right]_{s=0}.$$

The proposition now follows from the definition of adelic Eisenstein series $\mathbb{E}_{\Lambda,\mathbf{b}}$ and the (adelic) Siegel unit $g^{\mathbb{A}}_{\Lambda,\mathbf{b}}$. $\square$

4. Automorphic forms

We recall the definition of automorphic forms and Hecke operators. The definition of the local L-factor is given in terms of Hecke eigenvalues. For a new vector (see Condition (3) in Section 4.2.2), the definition agrees with the (usual) definition by Godement and Jacquet (Lemma 4.1).

4.1. **Automorphic forms.**

4.1.1. *Automorphic forms.* Let R be a commutative ring. By an R-valued automorphic form for the general linear group $\mathrm{GL}_{d,F}$ over F, we mean an R-valued function on $\mathrm{GL}_d(F)\backslash\mathrm{GL}_d(\mathbb{A})$ which is invariant under right translation by an open compact subgroup of $\mathrm{GL}_d(\mathbb{A})$. The set of R-valued automorphic forms, denoted $\mathcal{A}_R$, is an R-module on which the group $\mathrm{GL}_d(\mathbb{A})$ acts by right translation, that is, for an automorphic form f, an element $x \in \mathrm{GL}_d(F)\backslash\mathrm{GL}_d(\mathbb{A})$, and an element $g \in \mathrm{GL}_d(\mathbb{A})$, we put $(gf)(x) = f(xg)$. For an R-valued character χ_∞ of the ∞-component $Z(F_\infty)$ of $Z(\mathbb{A})$, let $\mathcal{A}_R(\chi_\infty)$ denote the R-module of R-valued automorphic forms on which $Z(F_\infty)$ acts via χ_∞. For two nonzero ideals I, J of A with $J \subset I$, let $\mathcal{A}_R(I, J, \chi_\infty)$ denote the $\mathbb{K}^\infty_{I,J}$-invariant part of $\mathcal{A}_R(\chi_\infty)$ where $\mathbb{K}^\infty_{I,J}$ is as in Section 2.4.7.

In this paper we will deal with several subspaces of $\mathcal{A}_\mathbb{C}$ whose inclusive relations are expressed in the following diagram

$$\begin{array}{ccccc} \mathcal{A}_\mathbb{C} & \supset & \mathcal{A}_1 & \supset & \mathcal{A}_{\mathrm{St}} \\ \cup & & \cup & & \cup \\ \mathcal{A}^{\mathrm{o}}_\mathbb{C} & \supset & \mathcal{A}^{\mathrm{o}}_1 & \supset & \mathcal{A}^{\mathrm{o}}_{\mathrm{St}}. \end{array}$$

Among the five subspaces (other than $\mathcal{A}_\mathbb{C}$ itself) in the diagram above, the three subspaces $\mathcal{A}^{\mathrm{o}}_\mathbb{C}$, $\mathcal{A}_1$ and $\mathcal{A}^{\mathrm{o}}_1$ are stable under the action of $\mathrm{GL}_d(\mathbb{A})$ and will be introduced in the next paragraph. The remaining two subspaces $\mathcal{A}_{\mathrm{St}}$ and $\mathcal{A}^{\mathrm{o}}_{\mathrm{St}}$ are stable under the action of the subgroup $\mathrm{GL}_d(\mathbb{A}^\infty) \subset \mathrm{GL}_d(\mathbb{A})$ and will be introduced in Section 9.1.

We let $\mathcal{A}^{\mathrm{o}}_\mathbb{C} \subset \mathcal{A}_\mathbb{C}$ denote the space of cusp forms. We put $\mathcal{A}_1 = \mathcal{A}_\mathbb{C}(1)$, where 1 denotes the trivial character, and $\mathcal{A}^{\mathrm{o}}_1 = \mathcal{A}_1 \cap \mathcal{A}^{\mathrm{o}}_\mathbb{C}$. We let $\mathcal{A}^{\mathrm{o}}_\mathbb{C}(\chi_\infty) = \mathcal{A}^{\mathrm{o}}_\mathbb{C} \cap \mathcal{A}_\mathbb{C}(\chi_\infty)$ and $\mathcal{A}^{\mathrm{o}}_\mathbb{C}(I, J, \chi_\infty) = \mathcal{A}^{\mathrm{o}}_\mathbb{C} \cap \mathcal{A}_\mathbb{C}(I, J, \chi_\infty)$ where χ_∞, I, J are as above.

4.1.2. For each place $\wp$ of F, we let $F_\wp$ denote the completion of F at $\wp$ and $\mathcal{O}_\wp$ denote the ring of integers of $F_\wp$. We fix a Haar measure $dg_\wp$ of $\mathrm{GL}_d(F_\wp)$ such that $\prod_\wp dg_\wp$ defines a Haar measure of $\mathrm{GL}_d(\mathbb{A})$ with $\mathrm{vol}(\mathrm{GL}_d(\prod_\wp \mathcal{O}_\wp)) = 1$.

Let $R = \mathbb{C}((q_\infty^{-s}))$. We define a $\mathbb{C}$-bilinear map

$$\langle\ ,\ \rangle : \mathcal{A}^{\mathrm{o}}_1 \times \mathcal{A}_R(|\ |_\infty^{-sd}) \to R,$$

where $|\ |_\infty^{-sd}$ is regarded as a character $F_\infty^\times \to R^\times$, by setting

$$\langle f_1, f_2\rangle = \int_{Z(F_\infty)\mathrm{GL}_d(F)\backslash\mathrm{GL}_d(\mathbb{A})} f_1(g) f_2(g) |\det g|^s dg.$$

Here the bracket $|\ |$ denotes the idelic norm, that is, for $(a_\wp) \in \mathbb{A}^\times$, we let $|(a_\wp)| = \prod_\wp |a_\wp|_\wp$ with $|\ |_\wp$ the absolute value at $\wp$. We note that the above integral is convergent since the support of f_1 is compact modulo center by Harder's result ([Ha]).

4.2. L-functions.

4.2.1. *Hecke operators.* Let $J \subset I$ be nonzero ideals of A, and let $\wp$ be a prime ideal. We put

$$e_\wp = \begin{cases} 0 & \text{if } \wp \mid I, \\ d-1 & \text{if } \wp \nmid I, \quad \wp \mid J, \\ d & \text{if } \wp \nmid J. \end{cases}$$

We write $\mathrm{diag}(a_1, \dots, a_d)$ for the diagonal $(d \times d)$-matrix whose diagonal entries are $a_1, \dots, a_d$. Let $\varpi_\wp$ denote the element in $\mathbb{A}^\times$ whose component at $\wp$ is a (fixed) uniformizer and whose components at other places are 1.

We define the Hecke operators $T_{\wp,r}$ and the dual Hecke operators $T^*_{\wp,r}$ for the open compact subgroup $\mathbb{K}^\infty_{I,J} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ where $r = 0, \dots, e_\wp$. We define $T_{\wp,r}$ (resp. $T^*_{\wp,r}$) to be the operator given by the double coset

$$\begin{array}{l} \mathbb{K}^\infty_{I,J}\mathrm{diag}(\varpi_\wp, \dots, \varpi_\wp, 1, \dots, 1)\mathbb{K}^\infty_{I,J} \\ (\text{resp. } \mathbb{K}^\infty_{I,J}\mathrm{diag}(\varpi_\wp^{-1}, \dots, \varpi_\wp^{-1}, 1, \dots, 1)\mathbb{K}^\infty_{I,J}) \end{array}$$

where $\varpi_\wp$ (resp. $\varpi_\wp^{-1}$) appears r times. In particular, the operators $T_{\wp,0}$ and $T^*_{\wp,0}$ are the identity.

4.2.2. Let $f \in \mathcal{A}_\mathbb{C}$ be a $\mathbb{C}$-valued automorphic form. Suppose that f satisfies the following conditions for some nonzero ideals $J \subset I \subsetneq A$ of A.

(1) The open compact subgroup $\mathbb{K}^\infty_{I,J}$ of $\mathrm{GL}_d(\mathbb{A}^\infty)$ acts trivially on f.

(2) Let $\wp$ be a nonzero prime ideal of A, and define the integer $e_\wp$ as in Section 4.2.1. Then f is an eigenform with respect to the operator $T_{\wp,r}$ for all $r \le e_\wp$.

Let $a_{\wp,r}$ denote the eigenvalue of the operator $T_{\wp,r}$ on f. We define the L-function $L^{I,J}(f,s)$ of f to be the infinite product

$$L^{I,J}(f,s) = \prod_\wp \left[\sum_{r=0}^{e_\wp} a_{\wp,r} q_\wp^{\frac{r(r-1)}{2} - r(s + \frac{d-1}{2})} \right]^{-1}$$

in $\mathbb{C}((q^{-s}))$ where $\wp$ runs through the nonzero prime ideals of A. The infinite product $L^{I,J}(f,s)$ is convergent for the (q^{-s})-adic topology. We also consider the following condition.

(3) There exists a cuspidal automorphic representation $\pi = \otimes'_v \pi_v \subset \mathcal{A}_\mathbb{C}$ such that f is of the form $(\otimes_{v \nmid I\infty} f_v) \otimes f_{I\infty} \in \pi$, where $f_v \in \pi_v$ is a new vector ("vecteur essentiel" in [Ja-Pi-Sh, p.211,(4.4)]) for each $v \nmid I\infty$, and $f_{I\infty} \in \pi_\infty \otimes \bigotimes_{v|I} \pi_v$.

We note that (3) implies (2). We will need the following lemma.

Lemma 4.1. *Let the notation be as above. Suppose Condition (3) is satisfied. Let $\wp$ be a prime ideal such that $e_\wp \ge d-1$. Then*

$$L(\pi_\wp, s) = \left[\sum_{r=0}^{e_\wp} a_{\wp,r} q_\wp^{\frac{r(r-1)}{2} - r(s + \frac{d-1}{2})} \right]^{-1}$$

where the left hand side is the local L-factor of Godement and Jacquet (see [Go-Ja]*).*

Proof. This is well known in the case $e_\wp = d$ (see, for example, [Cog, Lecture 7]). The case $e_\wp \ge d-1$ is [Ko-Ya4, Theorem 4.2]. $\square$

4.3. **Projection to cusp forms.**

The pairing introduced below appears in Theorem 6.3. The map P^{o} given in this section is used in Theorem 9.1.

4.3.1. Let $\mathcal{A}_1 = \mathcal{A}_{\mathbb{C}}(1) \subset \mathrm{Map}(\mathrm{GL}_d(F)F_\infty^\times \backslash \mathrm{GL}_d(\mathbb{A}), \mathbb{C})$ denote the space of $F_\infty^\times$-invariant $\mathbb{C}$-valued automorphic forms, as defined in Section 4.1.1. Let $\mathcal{A}_1^{\mathrm{o}} \subset \mathcal{A}_1$ denote the subspace of $F_\infty^\times$-invariant cusp forms. We consider the map $\langle\ ,\ \rangle : \mathcal{A}_1^{\mathrm{o}} \times \mathcal{A}_1 \to \mathbb{C}$ defined by the integral

$$\langle f_1, f_2 \rangle = \int_{Z(F_\infty)\mathrm{GL}_d(F)\backslash \mathrm{GL}_d(\mathbb{A})} f_1(g)\overline{f_2(g)}dg$$

where the bar denotes the complex conjugate.

4.3.2. For any open compact subgroup $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A})$, the space $(\mathcal{A}_1^{\mathrm{o}})^{\mathbb{K}}$ is finite dimensional. (This follows from Harder's result [Ha, COROLLARY 1.2.3, p.256].) Since $\langle f, f \rangle > 0$ for any nonzero $f \in \mathcal{A}_1^{\mathrm{o}}$, the restriction of $\langle , \rangle$ to $(\mathcal{A}_1^{\mathrm{o}})^{\mathbb{K}} \times (\mathcal{A}_1^{\mathrm{o}})^{\mathbb{K}}$ is non-degenerate. Hence for any $f \in \mathcal{A}_1^{\mathbb{K}}$, there exists a unique $f^{\mathrm{o}} \in (\mathcal{A}_1^{\mathrm{o}})^{\mathbb{K}}$ satisfying $\langle f', f \rangle = \langle f', f^{\mathrm{o}} \rangle$ for all $f' \in (\mathcal{A}_1^{\mathrm{o}})^{\mathbb{K}}$. We claim that the equality $\langle f'', f \rangle = \langle f'', f^{\mathrm{o}} \rangle$ holds for any $f'' \in \mathcal{A}_1^{\mathrm{o}}$. By setting $f'''(g) = \frac{1}{\mathrm{vol}(\mathbb{K})} \int_{\mathbb{K}} f''(gk)dk$, we obtain

$$\langle f'', f \rangle = \langle f''', f \rangle = \langle f''', f^{\mathrm{o}} \rangle = \langle f'', f^{\mathrm{o}} \rangle.$$

Passing to the inductive limit, the map $f \mapsto f^{\mathrm{o}}$ gives a surjective $\mathrm{GL}_d(\mathbb{A})$-equivariant homomorphism $P^{\mathrm{o}} : \mathcal{A}_1 \to \mathcal{A}_1^{\mathrm{o}}$, which is a left inverse to the inclusion $\mathcal{A}_1^{\mathrm{o}} \hookrightarrow \mathcal{A}_1$.

5. Borel-Moore homology of the Bruhat-Tits building

In Section 8, we will compute the image of the elements $\kappa_{I,J,\gamma}$ under the regulator map. Borel-Moore homology is the dual of cohomology with compact support, and the Borel-Moore homology groups of top degree of the Bruhat-Tits building is a synonym of the group of harmonic cochains ([Ga, Definition 3.15]) of top degree. We describe the Borel-Moore homology of (the top degree of the quotient of) the Bruhat-Tits building in this section.

5.1. **Borel-Moore homology.**

Recall from the introduction that there appeared a subspace $\mathcal{A}_{\mathrm{St}}^{\mathrm{o}}$ of the space $\mathcal{A}_{\mathbb{C}}$ of automorphic forms on $\mathrm{GL}_d(\mathbb{A})$ in the statement of Theorem 1.2. In Section 5.1, the setup which leads to the definition of $\mathcal{A}_{\mathrm{St}}^{\mathrm{o}}$ is given.

Let $Y_\bullet$ be a simplicial complex. As we have remarked at the end of Section 3.1.3, the geometric realization $|Y_\bullet|$ has a canonical structure of CW-complex. The notions of homology and cohomology for $|Y_\bullet|$ are well known. If $Y_\bullet$ is locally finite (see Section 5.1.2), we have two more similar notions: cohomology with compact support and Borel-Moore homology. Borel-Moore homology is the dual of cohomology with compact support, as cohomology is the dual of homology.

Let us recall these four notions of (co)homologies. Usually the homology groups of $Y_\bullet$ are defined to be the homology groups of a complex $C_\bullet$ whose component in degree i is the free abelian group generated by the i-simplices of $Y_\bullet$. For a precise definition of the boundary homomorphism of the complex $C_\bullet$, we need to choose an orientation of each simplex. In this paper we adopt an alternative, equivalent definition of homology groups which does not require any choice of orientations.

The latter definition seems a little complicated at first glance, however it will soon turn out to be a better way for describing the (co)homology of the arithmetic quotients Bruhat-Tits building, which seems to have no canonical, good choice of orientations.

5.1.1. We recall in Sections 5.1.1 and 5.1.2 the precise definitions of the (co)homology, the cohomology with compact support and the Borel-Moore homology of a simplicial complex. When computing (co)homology, one usually fixes an orientation of each simplex once and for all, but we do not. This results in an apparently different definition, but they indeed agree with the usual definition. This will be useful when defining the map ι in Section 5.2.1, since the Bruhat-Tits building is not naturally oriented.

We introduce the notion of orientation of a simplex. Let $Y_\bullet$ be a simplicial complex and let $i \geq 0$ be a non-negative integer. For an i-simplex $\sigma \in Y_i$, we let $T(\sigma)$ denote the set of all bijections from the finite set $\{1, \dots, i+1\}$ of cardinality $i+1$ to the set $V(\sigma)$ of vertices of σ. The symmetric group S_{i+1} acts on the set $\{1, \dots, i+1\}$ from the left and hence on the set $T(\sigma)$ from the right. Through this action the set $T(\sigma)$ is a right S_{i+1}-torsor.

We define the set $O(\sigma)$ of orientations of σ to be the $\{\pm 1\}$-torsor $O(\sigma) = T(\sigma) \times_{S_{i+1}, \mathrm{sgn}} \{\pm 1\}$ which is the push-forward of the S_{i+1}-torsor $T(\sigma)$ with respect to the signature character $\mathrm{sgn} : S_{i+1} \to \{\pm 1\}$. When $i \geq 1$, the $\{\pm 1\}$-torsor $O(\sigma)$ is isomorphic, as a set, to the quotient $T(\sigma)/A_{i+1}$ of $T(\sigma)$ by the action of the alternating group $A_{i+1} = \operatorname{Ker} \mathrm{sgn} \subset S_{i+1}$. When $i = 0$, the $\{\pm 1\}$-torsor $O(\sigma)$ is isomorphic to the product $O(\sigma) = T(\sigma) \times \{\pm 1\}$, on which the group $\{\pm 1\}$ acts via its natural action on the second factor.

Let $i \geq 1$ and let $\sigma \in Y_i$. For $v \in V(\sigma)$ let σ_v denote the $(i-1)$-simplex $\sigma_v = \sigma \times_{V(\sigma)} (V(\sigma) \setminus \{v\})$. There is a canonical map $s_v : O(\sigma) \to O(\sigma_v)$ of $\{\pm 1\}$-torsors defined as follows. Let $\nu \in O(\sigma)$ and take a lift $\widetilde{\nu} : \{1, \dots, i+1\} \xrightarrow{\cong} V(\sigma)$ of ν in $T(\sigma)$. Let $\widetilde{\iota}_v : \{1, \dots, i\} \hookrightarrow \{1, \dots, i+1\}$ denote the unique order-preserving injection whose image is equal to $\{1, \dots, i+1\} \setminus \{\widetilde{\nu}^{-1}(v)\}$. It follows from the definition of $\widetilde{\iota}_v$ that the composite $\widetilde{\nu} \circ \widetilde{\iota}_v : \{1, \dots, i\} \to V(\sigma)$ induces a bijection $\widetilde{\nu}_v : \{1, \dots, i\} \xrightarrow{\cong} V(\sigma) \setminus \{v\} = V(\sigma_v)$. We regard $\widetilde{\nu}_v$ as an element in $T(\sigma_v)$. We define $s_v : O(\sigma) \to O(\sigma_v)$ to be the map which sends $\nu \in O(\sigma)$ to $(-1)^{\widetilde{\nu}^{-1}(v)}$ times the class of $\widetilde{\nu}_v$. It is easy to check that the map s_v is well-defined.

Let $i \geq 2$ and $\sigma \in Y_i$. Let $v, v' \in V(\sigma)$ with $v \neq v'$. We have $(\sigma_v)_{v'} = (\sigma_{v'})_v$. Let us consider the two composites $s_{v'} \circ s_v : O(\sigma) \to O((\sigma_v)_{v'})$ and $s_v \circ s_{v'} : O(\sigma) \to O((\sigma_{v'})_v)$. It is easy to check that the equality

$$s_{v'} \circ s_v(\nu) = (-1) \cdot s_v \circ s_{v'}(\nu) \tag{5.1}$$

holds for every $\nu \in O(\sigma)$.

5.1.2. We say that a simplicial complex $Y_\bullet$ is locally finite if for any $i \geq 0$ and for any $\tau \in Y_i$, there exist only finitely many $\sigma \in Y_{i+1}$ such that τ is a face of σ. We recall the four notions of homology or cohomology for a locally finite simplicial complex. Let $Y_\bullet$ be a simplicial complex (resp. a locally finite simplicial complex). For an integer $i \geq 0$, we let $Y_i' = \coprod_{\sigma \in Y_i} O(\sigma)$ and regard it as a $\{\pm 1\}$-set. Given an abelian group M, we regard the abelian groups $\bigoplus_{\nu \in Y_i'} M$ and $\prod_{\nu \in Y_i'} M$ as $\{\pm 1\}$-modules in such a way that the component at $\epsilon \cdot \nu$ of $\epsilon \cdot (m_\nu)$ is equal to ϵm_ν for $\epsilon \in \{\pm 1\}$ and for $\nu \in Y_i'$.

For $i \geq 1$, we let $\widetilde{\partial}_{i,\oplus} : \bigoplus_{\nu \in Y'_i} M \to \bigoplus_{\nu \in Y'_{i-1}} M$ (resp. $\widetilde{\partial}_{i,\prod} : \prod_{\nu \in Y'_i} M \to \prod_{\nu \in Y'_{i-1}} M$) denote the homomorphism of abelian groups which sends $m = (m_\nu)_{\nu \in Y'_i}$ to the element $\widetilde{\partial}_i(m)$ whose coordinate at $\nu' \in O(\sigma') \subset Y'_{i-1}$ is equal to

$$\widetilde{\partial}_i(m)_{\nu'} = \sum_{(v,\sigma,\nu)} m_\nu \tag{5.2}$$

where in the sum in the right hand side (v, σ, ν) runs over the triples of $v \in Y_0 \backslash V(\sigma')$, an element $\sigma \in Y_i$, and $\nu \in O(\sigma)$ which satisfy $V(\sigma) = V(\sigma') \amalg \{v\}$ and $s_v(\nu) = \nu'$. Note that the sum on the right hand side is a finite sum for $\widetilde{\partial}_{i,\oplus}$ by definition. One can see also that the sum is a finite sum in the case of $\widetilde{\partial}_{i,\prod}$ using the locally finiteness of $Y_\bullet$. Each of $\widetilde{\partial}_{i,\oplus}$ and $\widetilde{\partial}_{i,\prod}$ is a homomorphism of $\{\pm 1\}$-modules. Hence it induces a homomorphism $\partial_{i,\oplus} : (\bigoplus_{\nu \in Y'_i} M)_{\{\pm 1\}} \to (\bigoplus_{\nu \in Y'_{i-1}} M)_{\{\pm 1\}}$ (resp. $\partial_{i,\prod} : (\prod_{\nu \in Y'_i} M)_{\{\pm 1\}} \to (\prod_{\nu \in Y'_{i-1}} M)_{\{\pm 1\}}$) of abelian groups, where the subscript $\{\pm 1\}$ means the coinvariants. It follows from the formula (5.1) and the definition of $\partial_{i,\oplus}$ and $\partial_{i,\prod}$ that the family of abelian groups $((\bigoplus_{\nu \in Y'_i} M)_{\{\pm 1\}})_{i \geq 0}$ (resp. $((\prod_{\nu \in Y'_i} M)_{\{\pm 1\}})_{i \geq 0}$) indexed by the integer $i \geq 0$, together with the homomorphisms $\partial_{i,\oplus}$ (resp. $\partial_{i,\prod}$) for $i \geq 1$, forms a complex of abelian groups. The homology groups of this complex are the homology groups $H_*(Y_\bullet, M)$ (resp. the Borel-Moore homology groups $H_*^{\mathrm{BM}}(Y_\bullet, M)$) of $Y_\bullet$ with coefficients in M.

The family of abelian groups $(\mathrm{Map}_{\{\pm 1\}}(Y'_i, M))_{i \geq 0}$ (resp. $(\mathrm{Map}^{\mathrm{fin}}_{\{\pm 1\}}(Y'_i, M))_{i \geq 0}$ where the superscript fin means finite support) of the $\{\pm 1\}$-equivariant maps from Y'_i to M forms a complex of abelian groups in a similar manner. (One uses the locally finiteness of $Y_\bullet$ for the latter.) The cohomology groups of this complex are the cohomology groups $H^*(Y_\bullet, M)$ (resp. the cohomology groups with compact support $H_c^*(Y_\bullet, M)$) of $Y_\bullet$ with coefficients in M.

5.1.3. It follows from the definition that the following universal coefficients theorem holds. That is, for a simplicial complex $Y_\bullet$, there exist canonical short exact sequences

$$0 \to H_i(Y_\bullet, \mathbb{Z}) \otimes M \to H_i(Y_\bullet, M) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{i-1}(Y_\bullet, \mathbb{Z}), M) \to 0$$

and

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(Y_\bullet, \mathbb{Z}), M) \to H^i(Y_\bullet, M) \to \mathrm{Hom}_{\mathbb{Z}}(H_i(Y_\bullet, \mathbb{Z}), M) \to 0.$$

for any abelian group M.

Similarly, for a locally finite simplicial complex $Y_\bullet$, we have short exact sequences

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_c^{i+1}(Y_\bullet, \mathbb{Z}), M) \to H_i^{\mathrm{BM}}(Y_\bullet, M) \to \mathrm{Hom}_{\mathbb{Z}}(H_c^i(Y_\bullet, \mathbb{Z}), M) \to 0$$

and

$$0 \to H_c^i(Y_\bullet, \mathbb{Z}) \otimes M \to H_c^i(Y_\bullet, M) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_c^{i+1}(Y_\bullet, \mathbb{Z}), M) \to 0$$

for any abelian group M. The canonical inclusions

$$\Big(\bigoplus\nolimits_{\nu \in Y'_i} M\Big)_{\{\pm 1\}} \hookrightarrow \Big(\prod\nolimits_{\nu \in Y'_i} M\Big)_{\{\pm 1\}} \text{ and}$$
$$\mathrm{Map}^{\mathrm{fin}}_{\{\pm 1\}}(Y'_i, M) \hookrightarrow \mathrm{Map}_{\{\pm 1\}}(Y'_i, M)$$

for $i \geq 0$ induce homomorphisms $H_i(Y_\bullet, M) \to H_i^{\mathrm{BM}}(Y_\bullet, M)$ and $H_c^i(Y_\bullet, M) \to H^i(Y_\bullet, M)$ of abelian groups, respectively.

5.1.4. Let $f = (f_i)_{i\geq 0} : Y_\bullet \to Z_\bullet$ be a map of simplicial complexes. For each integer $i \geq 0$ and for each abelian group M, the map f canonically induces homomorphisms $f_* : H_i(Y_\bullet, M) \to H_i(Z_\bullet, M)$ and $f^* : H^i(Z_\bullet, M) \to H^i(Y_\bullet, M)$. We say that the map f is finite if the subset $f_i^{-1}(\sigma)$ of Y_i is finite for any $i \geq 0$ and for any $\sigma \in Z_i$. If $Y_\bullet$ and $Z_\bullet$ are locally finite, and if f is finite, then f canonically induces homomorphisms $f_* : H_i^{\mathrm{BM}}(Y_\bullet, M) \to H_i^{\mathrm{BM}}(Z_\bullet, M)$ and $f^* : H_c^i(Z_\bullet, M) \to H_c^i(Y_\bullet, M)$.

5.2. **Borel-Moore homology of Bruhat-Tits building.**

In Section 5.2.1, we construct a homomorphism (5.3) from the Borel-Moore homology of the building to the space of function on $\mathrm{GL}_d(F_\infty)$. We prove in Lemma 5.1 that the homomorphism (5.3) is injective and we determine its image in Corollary 5.5.

Let $d \geq 1$ be an integer. Let $\mathcal{BT}_\bullet$ be the Bruhat-Tits building of PGL_d over F_∞, which is introduced in Section 3.2.3. For an integer $i \geq 0$, let $\widetilde{\mathcal{BT}}_i$ and $\mathcal{BT}_{i,*}$ be as in Section 3.2.3.

5.2.1. We define a canonical, $\mathrm{GL}_d(F_\infty)$-equivariant homomorphism

$$H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet, \mathbb{C}) \to \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C}) \tag{5.3}$$

of complex vector spaces. Here the group $\mathrm{GL}_d(F_\infty)$ acts on the space $\mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})$ by the right translation. Let us define $\widetilde{\iota} : \mathcal{BT}_{d-1,*} \to \coprod_{\nu \in \mathcal{BT}_{d-1}} T(\sigma)$ as follows. Take $(\sigma = (L_i)_{i\in\mathbb{Z}}, L_0)$ in $\mathcal{BT}_{d-1,*}$. We define $\widetilde{\iota}(\sigma, L_0)$ to be the bijection in $T(\sigma)$ from $\{1, \ldots, d\}$ to $V(\sigma)$ which sends i to the class of L_{i-1} in $\mathcal{BT}_0$. We denote by ι the composite

$$\iota : \mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times \cong \mathcal{BT}_{d-1,*} \xrightarrow{\widetilde{\iota}} \coprod_{\sigma \in \mathcal{BT}_{d-1}} T(\sigma) \to \coprod_{\sigma \in \mathcal{BT}_{d-1}} O(\sigma) = (\mathcal{BT}_{d-1})'.$$

Here $\mathcal{I}$ is the Iwahori subgroup introduced in Section 3.2.3, and $\mathcal{I}F_\infty^\times$ denotes the subgroup of $\mathrm{GL}_d(F_\infty)$ generated by the center $F_\infty^\times$ of $\mathrm{GL}_d(F_\infty)$ and and $\mathcal{I}$. The first map is the isomorphism induced by the isomorphism $\widetilde{\mathcal{BT}}_{d-1} \cong \mathrm{GL}_d(F_\infty)/\mathcal{I}$ given in Section 3.2.3. Let us consider the composite

$$\left(\prod_{\nu \in (\mathcal{BT}_{d-1})'} \mathbb{C} \right)_{\{\pm 1\}} \xrightarrow{\mu} \mathrm{Map}_{\{\pm 1\}}((\mathcal{BT}_{d-1})', \mathbb{C}) \to \mathrm{Map}(\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times, \mathbb{C}). \tag{5.4}$$

Here the first map μ is the isomorphism which sends the class of an element $a = (a_\nu) \in \prod_{\nu \in (\mathcal{BT}_{d-1})'} \mathbb{C}$ to the map $(\mathcal{BT}_{d-1})' \to \mathbb{C}$, which sends $\nu \in (\mathcal{BT}_{d-1})'$ to $a_\nu - a_{(-1)\cdot\nu}$. The second map is the homomorphism induced by the map ι. Since $H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet, \mathbb{C})$ is a subspace of the source of the map (5.4) and since the target of the map (5.4) is a subspace of $\mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})$, the map (5.4) induces the desired homomorphism (5.3). It follows from the construction that this homomorphism is $\mathrm{GL}_d(F_\infty)$-equivariant.

Lemma 5.1. *The homomorphism (5.3) is injective.*

Proof. We have defined the homomorphism (5.3) to be the composite of several homomorphisms which are obviously injective except for the second homomorphism in (5.4) which we denote by ι^*. We prove the injectivity of ι^*. Let $S \subset (\mathcal{BT}_{d-1})'$

denote the image of the map ι in Section 5.2.1. The composite $\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times$ of ι with the canonical surjection $(\mathcal{BT}_{d-1})' \to \mathcal{BT}_{d-1}$ is surjective, since it is equal to the canonical map $\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times \cong \mathcal{BT}_{d-1,*} \to \mathcal{BT}_{d-1}$. Hence we have $S \cup (-1) \cdot S = (\mathcal{BT}_{d-1})'$. It follows that the restriction homomorphism $\mathrm{Map}_{\{\pm 1\}}((\mathcal{BT}_{d-1})', \mathbb{C}) \to \mathrm{Map}(S, \mathbb{C})$ is injective. The homomorphism ι^* is the composite of this (injective) restriction homomorphism with the pullback homomorphism $\mathrm{Map}(S, \mathbb{C}) \to \mathrm{Map}(\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times, \mathbb{C})$ with respect to the surjection $\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times \twoheadrightarrow S$ induced by the map ι. Hence the homomorphism ι^* is injective. □

5.2.2. We say that an element $f \in \mathrm{Map}(\mathcal{BT}_{d-1,*}, \mathbb{C})$ is a ($\mathbb{C}$-valued) harmonic cochain if the following two conditions are satisfied:

(1) Let $\sigma_n = ((L_i)_{i\in\mathbb{Z}}, L_n) \in \mathcal{BT}_{d-1,*}$ be a pointed $(d-1)$-simplex for $n \in \mathbb{Z}$. Then $f(\sigma_n) = (-1)^{n-n'} f(\sigma_{n'})$ holds for $n, n' \in \mathbb{Z}$.
(2) Let $\tau_+ \in \mathcal{BT}_{d-2,*}$ be a pointed $(d-2)$-simplex. Let $\{\tau_+^i\}_{0\le i\le q}$ be the set of pointed $(d-1)$-simplices, each of which contains τ_+ as its (pointed) face. Then $\sum_{i=0}^q f(\tau_+^i) = 0$.

Lemma 5.2. *The space of harmonic cochains coincides with the image of the map* (5.3).

Proof. Let f be a harmonic cochain. Let us show that it lies in the image of the map (5.3). As f satisfies the condition (1) above, we can find an element $(f_\nu)_{\nu\in(\mathcal{BT}_{d-1})'}$ of $\prod_{\nu\in(\mathcal{BT}_{d-1})'} \mathbb{C}$ which maps to f via the map (5.4).

By the definition of $H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet, \mathbb{C})$ given in Section 5.1.2, an element of $\prod_{\nu\in(\mathcal{BT}_{d-1})'} \mathbb{C}$ lies in the image of $H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet, \mathbb{C})$ if and only if (5.2) is zero. Using Condition (2) of the definition of harmonic cochain, one can verify that it holds true. □

5.2.3. We give an alternative description of the homomorphism (5.3). Since $\mathcal{BT}_d$ is an empty set, it follows from Section 5.1.3 that we have a canonical isomorphism $H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet, M) \cong \mathrm{Hom}_{\mathbb{C}}(H_c^{d-1}(\mathcal{BT}_\bullet, \mathbb{Z}), M)$ for any abelian group M. From this it follows that

$$(5.5) \qquad \begin{aligned} H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet, \mathbb{C}) &\cong \mathrm{Hom}_{\mathbb{C}}(H_c^{d-1}(\mathcal{BT}_\bullet, \mathbb{C}), \mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(H_c^{d-1}(\mathcal{BT}_\bullet, \mathbb{C}), \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})) \end{aligned}$$

where the second isomorphism follows from the Frobenius reciprocity.

Let $\sigma_0 \in \mathcal{BT}_{d-1,*} \cong \mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times$ be the pointed $(d-1)$-simplex corresponding to the coset $\mathcal{I}F_\infty^\times$. Let $[\iota(\sigma_0)] : (\mathcal{BT}_{d-1})' \to \mathbb{C}$ denote the $\{\pm 1\}$-equivariant map with finite support which sends $\epsilon \cdot \iota(\sigma_0) \in (\mathcal{BT}_{d-1})'$ to ϵ for $\epsilon \in \{\pm 1\}$ and which sends the other elements in $(\mathcal{BT}_{d-1})'$ to zero. Since $\mathcal{BT}_d$ is an empty set, the element $[\iota(\sigma_0)] \in \mathrm{Map}_{\{\pm 1\}}^{\mathrm{fin}}((\mathcal{BT}_{d-1})', \mathbb{C})$ defines a class in $H_c^{d-1}(\mathcal{BT}_\bullet, \mathbb{C})$. We also denote this class by $[\iota(\sigma_0)]$. Let us consider the composite

$$(5.6) \qquad \begin{aligned} H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet, \mathbb{C}) &\xrightarrow{(1)} \mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(H_c^{d-1}(\mathcal{BT}_\bullet, \mathbb{C}), \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})) \\ &\xrightarrow{(2)} \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C}) \end{aligned}$$

where (1) is the isomorphism (5.5) and (2) is given by the evaluation at $[\iota(\sigma_0)]$.

Lemma 5.3. *The homomorphism (5.6) coincides with the homomorphism (5.3).*

Proof. We put $M = \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})$. It follows from the definition that the homomorphism (5.6) is equal to the restriction to $H^{\mathrm{BM}}_{d-1}(\mathcal{BT}_\bullet, \mathbb{C}) \subset (\prod_{\nu \in (\mathcal{BT}_{d-1})'} \mathbb{C})_{\{\pm 1\}}$ of the composite

$$
(5.7)\qquad \begin{aligned}
\Big(\prod\nolimits_{\nu \in (\mathcal{BT}_{d-1})'} \mathbb{C}\Big)_{\{\pm 1\}} &\cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{Map}^{\mathrm{fin}}_{\{\pm 1\}}((\mathcal{BT}_{d-1})', \mathbb{C}), \mathbb{C}) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{Map}^{\mathrm{fin}}_{\{\pm 1\}}((\mathcal{BT}_{d-1})', \mathbb{C}), M) \\
&\to M
\end{aligned}
$$

where the second isomorphism follows from the Frobenius reciprocity and the last map is the homomorphism given by the evaluation at $[\iota(\sigma_0)]$. It is straightforward to check that the homomorphism (5.7) sends the class of the element $(a_\nu) \in \prod_{\nu \in (\mathcal{BT}_{d-1})'} \mathbb{C}$ to the map $\mathrm{GL}_d(F_\infty) \to \mathbb{C}$ which sends $g \in \mathrm{GL}_d(F_\infty)$ to $a_{\iota(g\mathcal{I}F_\infty^\times)} - a_{(-1)\cdot\iota(g\mathcal{I}F_\infty^\times)}$. Hence the homomorphism (5.7) is equal to the composite of (5.4) with the canonical inclusion $\mathrm{Map}(\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty, \mathbb{C}) \subset M$. Hence it follows from the definition that the homomorphism (5.3) is equal to the restriction to $H^{\mathrm{BM}}_{d-1}(\mathcal{BT}_\bullet, \mathbb{C}) \subset (\prod_{\nu \in (\mathcal{BT}_{d-1})'} \mathbb{C})_{\{\pm 1\}}$ of the homomorphism (5.7), which proves the claim. $\square$

5.2.4. We give a consequence of Lemma 5.3. It is shown by Borel ([Bo, 6.2, 6.4]) that there is an isomorphism $\mathrm{St}_d \cong H^{d-1}_c(\mathcal{BT}_\bullet, \mathbb{C})$ of $\mathbb{C}[\mathrm{GL}_d(F_\infty)]$-modules, where St_d is the Steinberg representation (see [Lau, p.193] for definition). By Schur's lemma this isomorphism is unique up to scalar; we fix one. It is known that the subspace $\mathrm{St}_d^{\mathcal{I}} \subset \mathrm{St}_d$ of $\mathcal{I}$-invariant vectors is one-dimensional. Let us fix a basis $e_0 \in \mathrm{St}_d^{\mathcal{I}}$. Since $[\iota(\sigma_0)] \in H^{d-1}_c(\mathcal{BT}_\bullet, \mathbb{C})$ is $\mathcal{I}$-invariant, it corresponds to a scalar multiple ce_0 of e_0 under the fixed isomorphism $H^{d-1}_c(\mathcal{BT}_\bullet, \mathbb{C}) \cong \mathrm{St}_d$.

Lemma 5.4. *The scalar c is non-zero.*

Proof. Since the group $\mathrm{GL}_d(F_\infty)$ acts transitively on the set $\mathcal{BT}_{d-1}$ which is a quotient of the set $\widetilde{\mathcal{BT}}_{d-1} \cong \mathrm{GL}_d(F_\infty)/\mathcal{I}$, the space $\mathrm{Map}_{\{\pm 1\}}((\mathcal{BT}_{d-1})', \mathbb{C})$ of $(d-1)$-cochains computing the group $H^{d-1}_c(\mathcal{BT}_\bullet, \mathbb{C})$ is generated, as a $\mathbb{C}[\mathrm{GL}_d(F_\infty)]$-module, by the map $[\iota(\sigma_0)] : (\mathcal{BT}_{d-1})' \to \mathbb{C}$. It follows that the $\mathbb{C}[\mathrm{GL}_d(F_\infty)]$-module $H^{d-1}_c(\mathcal{BT}_\bullet, \mathbb{C})$ is generated by the element $[\iota(\sigma_0)] \in H^{d-1}_c(\mathcal{BT}_\bullet, \mathbb{C})$. Hence the $\mathbb{C}[\mathrm{GL}_d(F_\infty)]$-module St_d is generated by the element ce_0. Hence we have $c \neq 0$. $\square$

It follows from Lemma 5.3 that the homomorphism (5.3) is the composite of the isomorphism (5.5) with the composite

$$
\begin{aligned}
& \mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(H^{d-1}_c(\mathcal{BT}_\bullet, \mathbb{C}), \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})) \\
\cong\ & \mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d, \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})) \\
\to\ & \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})
\end{aligned}
$$

where the last map is given by the evaluation at ce_0. Hence, by Lemma 5.4, we have:

Corollary 5.5. *The image of the homomorphism (5.3) is equal to the image of the map $\mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d, \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})) \to \mathrm{Map}(\mathrm{GL}_d(F_\infty), \mathbb{C})$ given by the evaluation at e_0.* $\square$

5.2.5. We need the following fact in Section 5.3.1 and in Section 5.4.4.

Lemma 5.6. *Let $i \geq 0$ be an integer, let $\sigma \in \mathcal{BT}_i$ and let $v, v' \in V(\sigma)$ be two vertices with $v \neq v'$. Suppose that an element $g \in \mathrm{GL}_d(F_\infty)$ satisfies $|\det g|_\infty = 1$. Then we have $gv \neq v'$.*

Proof. Let $\widetilde{\sigma}$ be an element $(L_j)_{j\in\mathbb{Z}}$ in $\widetilde{\mathcal{BT}}_i$ such that the class of $\widetilde{\sigma}$ in $\mathcal{BT}_i$ is equal to σ. There exist two integers $j, j' \in \mathbb{Z}$ such that v, v' is the class of L_j, $L_{j'}$, respectively. Assume that $gv = v'$. Then there exists an integer $k \in \mathbb{Z}$ such that $L_j g^{-1} = \varpi_\infty^k L_{j'} = L_{j'+(i+1)k}$. Let us fix a Haar measure $d\mu$ of the F_∞-vector space V_∞. As is well-known, the push-forward of $d\mu$ with respect to the automorphism $V_\infty \to V_\infty$ given by the right multiplication by γ is equal to $|\det \gamma|_\infty^{-1} d\mu$ for every $\gamma \in \mathrm{GL}_d(F_\infty)$. Since $|\det g|_\infty = 1$, it follows from the equality $L_j g^{-1} = L_{j'+(i+1)k}$ that the two $\mathcal{O}_\infty$-lattices L_j and $L_{j'+(i+1)k}$ have a same volume with respect to $d\mu$. Hence we have $j = j' + (i+1)k$, which implies $L_j = \varpi_\infty^k L_{j'}$. It follows that the class of L_j in $\mathcal{BT}_0$ is equal to the class of $L_{j'}$, which contradicts the assumption $v \neq v'$. ☐

5.3. **Borel-Moore homology of some arithmetic quotients of Bruhat-Tits building.**

We define a certain simplicial complex in Section 5.3.1, whose Borel-Moore homology groups play a major role in this article. The homomorphism constructed in Section 5.2.1 induces a homomorphism from the Borel-Moore homology of the simplicial complex to the space of automorphic forms. Using an isomorphism of Borel (which is recalled in Section 5.2.3), we see in Corollary 5.7 that the image of this homomorphism inside the space of cusp forms is the $\mathcal{A}_{\mathrm{St}}^{\mathrm{o}}$ (see Section 9.1 for the precise definition). We note that these functions lying in the image of the homomorphism constructed are usually called “harmonic cochains”.

5.3.1. For an open compact subgroup $\mathbb{K}^\infty \subset \mathrm{GL}_d(\mathbb{A}^\infty)$, we define the simplicial complex $\widetilde{X}_{\mathbb{K}^\infty,\bullet}$ as the disjoint union $\widetilde{X}_{\mathbb{K}^\infty,\bullet} = \mathcal{BT}_\bullet \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty$ of copies of $\mathcal{BT}_\bullet$ indexed by $\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty$. The group $\mathrm{GL}_d(\mathbb{A})$ canonically acts on $\widetilde{X}_{\mathbb{K}^\infty,\bullet}$ from the left. For $i \geq 0$, we let $X_{\mathbb{K}^\infty,i}$ denote the quotient $X_{\mathbb{K}^\infty,i} = \mathrm{GL}_d(F)\backslash \widetilde{X}_{\mathbb{K}^\infty,i}$ under the action of $\mathrm{GL}_d(F) \subset \mathrm{GL}_d(\mathbb{A})$. Let us introduce the structure of simplicial complex on the collection $X_{\mathbb{K}^\infty,\bullet} = (X_{\mathbb{K}^\infty,i})_{i\geq 0}$.

Let $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$. For each $i \geq 0$, consider the inclusion $\mathcal{BT}_i \hookrightarrow \widetilde{X}_{\mathbb{K}^\infty,i}$ which sends $\sigma \in \mathcal{BT}_i$ to $(\sigma, \gamma\mathbb{K}^\infty)$. This induces an injection

$$\mathrm{GL}_d(F) \cap \gamma\mathbb{K}^\infty\gamma^{-1}\backslash\mathcal{BT}_i \hookrightarrow X_{\mathbb{K}^\infty,i}.$$

We give an explanation for the notation $\mathrm{GL}_d(F)\cap\gamma\mathbb{K}^\infty\gamma^{-1}\backslash\mathcal{BT}_i$ in the source of this injection. It should be read as follows. We regard the group $\mathrm{GL}_d(F)$ as a subgroup of $\mathrm{GL}_d(\mathbb{A}^\infty)$ via the diagonal embedding $\mathrm{GL}_d(F) \to \mathrm{GL}_d(\mathbb{A}^\infty)$ when we take the intersection $\mathrm{GL}_d(F) \cap \gamma\mathbb{K}^\infty\gamma^{-1}$. We then regard $\mathrm{GL}_d(F) \cap \gamma\mathbb{K}^\infty\gamma^{-1}$ as a subgroup of $\mathrm{GL}_d(F_\infty)$ via the homomorphism $\mathrm{GL}_d(F) \cap \gamma^{-1}\mathbb{K}^\infty\gamma \hookrightarrow \mathrm{GL}_d(F) \hookrightarrow \mathrm{GL}_d(F_\infty)$ when we let it act on $\mathcal{BT}_i$.

If g is an element in $\mathrm{GL}_d(F) \cap \gamma\mathbb{K}^\infty\gamma^{-1}$, the product formula $\prod_v |\det g|_v = 1$ (where v runs over all places of F) implies that $|\det g|_\infty = 1$. Hence for $(\sigma, \gamma\mathbb{K}^\infty) \in \widetilde{X}_{\mathbb{K}^\infty,i}$, it follows from Lemma 5.6, that the image of the set of vertices $V((\sigma, \gamma\mathbb{K}^\infty))$ under the surjection $\widetilde{X}_{\mathbb{K}^\infty,0} \twoheadrightarrow X_{\mathbb{K}^\infty,0}$ is a subset of $X_{\mathbb{K}^\infty,0}$ with cardinality $i+1$.

We denote this subset by $V(\mathrm{cl}(\sigma,\gamma\mathbb{K}^\infty))$, since it is easily checked that the set $V(\mathrm{cl}(\sigma,\gamma\mathbb{K}^\infty))$ depends only on the class $\mathrm{cl}(\sigma,\gamma\mathbb{K}^\infty)$ of $(\sigma,\gamma\mathbb{K}^\infty)$ in $X_{\mathbb{K}^\infty,i}$. Thus the collection $X_{\mathbb{K}^\infty,\bullet}=(X_{\mathbb{K}^\infty,i})_{i\ge 0}$ has a structure of a simplicial complex.

5.3.2. The structure of the simplicial complex is uniquely characterized by the property that the collection of the canonical surjection $\widetilde{X}_{\mathbb{K}^\infty,i}\twoheadrightarrow X_{\mathbb{K}^\infty,i}$ is a map of simplicial complexes $\widetilde{X}_{\mathbb{K}^\infty,\bullet}\twoheadrightarrow X_{\mathbb{K}^\infty,\bullet}$.

This means in particular that if $\sigma\in X_{\mathbb{K}^\infty,i}$ is an i-simplex and if $\widetilde{\sigma}\in\widetilde{X}_{\mathbb{K}^\infty,i}$ is an i-simplex which maps to σ, then $V(\sigma)$ is the image of $V(\widetilde{\sigma})$ under the map $\widetilde{X}_{\mathbb{K}^\infty,0}\to X_{\mathbb{K}^\infty,0}$.

For a non-empty subset $V\subset V(\sigma)$ of cardinality $i'+1$, let $\widetilde{V}\subset V(\widetilde{\sigma})$ be the unique subset which maps bijectively onto V under the map $\widetilde{X}_{\mathbb{K}^\infty,0}\to X_{\mathbb{K}^\infty,0}$. Then the i'-simplex $\sigma\times_{V(\sigma)}V$ is equal to the image of $\widetilde{\sigma}\times_{V(\widetilde{\sigma})}\widetilde{V}$ under the surjection $\widetilde{X}_{\mathbb{K}^\infty,i'}\to X_{\mathbb{K}^\infty,i'}$. It is straightforward to check that the simplicial complex $X_{\mathbb{K}^\infty,\bullet}$ is locally finite.

5.3.3. The homomorphism (5.3) in Section 5.2.1 induces the homomorphism

$$\begin{aligned} H^{\mathrm{BM}}_{d-1}(\widetilde{X}_{\mathbb{K}^\infty,\bullet},\mathbb{C}) &\cong \prod_{g\mathbb{K}^\infty\in\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty} H^{\mathrm{BM}}_{d-1}(\mathcal{BT}_\bullet,\mathbb{C}) \\ &\to \prod_{g\mathbb{K}^\infty\in\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty}\mathrm{Map}(\mathrm{GL}_d(F_\infty),\mathbb{C}) \\ &\cong \mathrm{Map}(\mathrm{GL}_d(\mathbb{A})/\mathbb{K}^\infty,\mathbb{C}) \end{aligned}\tag{5.8}$$

which is $\mathrm{GL}_d(\mathbb{A})$-equivariant. It follows from the definition of Borel-Moore homology and the fact that the simplicial complex $\widetilde{X}_{\mathbb{K}^\infty,\bullet}$ has no i-simplex for $i\ge d$ that the $\mathrm{GL}_d(F)$-invariant subspace of the source of (5.8) is isomorphic to $H^{\mathrm{BM}}_{d-1}(X_{\mathbb{K}^\infty,\bullet},\mathbb{C})$. Hence we have a homomorphism

$$H^{\mathrm{BM}}_{d-1}(X_{\mathbb{K}^\infty,\bullet},\mathbb{C})\to(\mathrm{Map}(\mathrm{GL}_d(\mathbb{A})/\mathbb{K}^\infty,\mathbb{C}))^{\mathrm{GL}_d(F)}=\mathcal{A}_{\mathbb{C}}^{\mathbb{K}^\infty}.\tag{5.9}$$

The following is a consequence of Lemma 5.1 and Corollary 5.5.

Corollary 5.7. *The homomorphism (5.9) is injective and its image is equal to the image of the homomorphism*

$$\mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d,\mathcal{A}_{\mathbb{C}}^{\mathbb{K}^\infty})\to\mathcal{A}_{\mathbb{C}}^{\mathbb{K}^\infty}$$

given by the evaluation at $e_0\in\mathrm{St}_d^{\mathcal{I}}$. □

5.4. **Apartments.**

Here we recall the definition of the apartments which are simplicial subcomplexes of the Bruhat-Tits building. We then associate to each apartment a class in the Borel-Moore homology of a quotient of the Bruhat-Tits building. This class is an analogue of a modular symbol, and for its construction we require a lemma (Lemma 5.11) from our other paper [Ko-Ya2] where this analogue of a modular symbol is the main subject for study. The classes $\eta_{\mathbb{K}^\infty,\gamma}\in\mathcal{A}_{\mathbb{C}}^{\mathbb{K}^\infty}$ defined in Section 5.4.7 will be of use in Section 6.2, especially in the proof of Proposition 6.5.

We recall one more lemma (Lemma 5.12) from [Ko-Ya2]. This lemma states that a class in the Borel-Moore homology which comes from the homology is expressed as a linear combination of the classes of apartments. Corollary 5.16 is the form we will use, and will appear in the proof of Lemma 9.4.

For the general theory of Bruhat-Tits building and apartments, the reader is referred to the book [Ab-Br].

5.4.1. We give an explicit description of the simplicial complex $A_\bullet$ below without making use of the theory of root systems. For the viewpoint in the general theory of root systems, we refer the reader to [Ab-Br, p. 523, 10.1.7 Example].

Put $A_0 = \mathbb{Z}^{\oplus d}/\mathbb{Z}(1,\dots,1)$. For two elements $x = (x_j), y = (y_j) \in \mathbb{Z}^{\oplus d}$, we write $x \le y$ when $x_j \le y_j$ for all $1 \le j \le d$. We say that a subset $\widetilde{\sigma} \subset \mathbb{Z}^{\oplus d}$ is small if for any two elements $x, y \in \widetilde{\sigma}$ we have either $x \le y \lneq x + (1,\dots,1)$ or $y \le x \lneq y + (1,\dots,1)$. Explicitly, this means that $\widetilde{\sigma}$ is a finite set and is of the form $\widetilde{\sigma} = \{x_0,\dots,x_i\}$ for some elements $x_0,\dots,x_i$ satisfying $x_0 \lneq \cdots \lneq x_i \lneq x_{i+1} = x_0 + (1,\dots,1)$. We say that a finite subset $\sigma \subset A_0$ has a small lift to $\mathbb{Z}^{\oplus d}$ if there exists a small subset $\widetilde{\sigma} \subset \mathbb{Z}^{\oplus d}$ which maps bijectively onto σ under the canonical surjection $\mathbb{Z}^{\oplus d} \twoheadrightarrow A_0$. For $i \ge 0$, we let A_i denote the set of the subsets $\sigma \subset A_0$ with cardinality $i+1$ which has a small lift to $\mathbb{Z}^{\oplus d}$. It is clear that the pair $(A_0, \coprod_{i\ge 0} A_i)$ forms a strict simplicial complex and the collection $A_\bullet = (A_i)_{i\ge 0}$ is the simplicial complex associated to the strict simplicial complex $(A_0, \coprod_{i\ge 0} A_i)$. We note that A_i is an empty set for $i \ge d$, since by definition there is no small subset of $\mathbb{Z}^{\oplus d}$ with cardinality larger than d.

5.4.2. Let $v_1,\dots,v_d$ be a basis of $V_\infty = F_\infty^{\oplus d}$. We define a map $\iota_{v_1,\dots,v_d} : A_\bullet \to \mathcal{BT}_\bullet$ of simplicial complexes.

Let $\widetilde{\iota}_{v_1,\dots,v_d} : \mathbb{Z}^{\oplus d} \to \widetilde{\mathcal{BT}}_0$ denote the map which sends the element $(n_1,\dots,n_d) \in \mathbb{Z}^d$ to the $\mathcal{O}_\infty$-lattice $\mathcal{O}_\infty \varpi_\infty^{n_1} v_1 \oplus \mathcal{O}_\infty \varpi_\infty^{n_2} v_2 \oplus \cdots \oplus \mathcal{O}_\infty \varpi_\infty^{n_d} v_d$. Let $i \ge 0$ be an integer and let $\sigma \in A_i$. Take a small subset $\widetilde{\sigma} \subset \mathbb{Z}^{\oplus d}$ with cardinality $i+1$ which maps bijectively onto σ under the surjection $\mathbb{Z}^{\oplus d} \to \mathbb{Z}^{\oplus d}/\mathbb{Z}(1,\dots,1) = A_0$. By definition the set $\widetilde{\sigma}$ is of the form $\widetilde{\sigma} = \{x_0,\dots,x_i\}$ where $x_0,\dots,x_i \in \mathbb{Z}^{\oplus d}$ satisfy $x_0 \lneq \cdots \lneq x_i \lneq x_{i+1}$ where we have put $x_{i+1} = x_0 + (1,\dots,1)$. For each integer $j \in \mathbb{Z}$ we write j in the form $j = m(i+1) + r$ with $m \in \mathbb{Z}$ and $r \in \{0,\dots,i\}$, and put $x_j = x_r + m(1,\dots,1)$ and $L_j = \widetilde{\iota}_{v_1,\dots,v_d}(x_j)$. The sequence $(L_j)_{j\in\mathbb{Z}}$ of $\mathcal{O}_\infty$-lattices gives an element $\widetilde{\iota}_{v_1,\dots,v_d,i}(\widetilde{\sigma})$ in $\widetilde{\mathcal{BT}}_i$. We denote by $\iota_{v_1,\dots,v_d,i}(\sigma)$ the class of $\widetilde{\iota}_{v_1,\dots,v_d,i}(\widetilde{\sigma})$ in $\mathcal{BT}_i$.

Lemma 5.8. *The class $\iota_{v_1,\dots,v_d,i}(\sigma)$ does not depend on the choice of a small lift $\widetilde{\sigma}$.*

Proof. The inverse image of σ under the canonical surjection $\mathbb{Z}^{\oplus d} \to \mathbb{Z}^{\oplus d}/\mathbb{Z}(1,\dots,1)$ is equal to $\{x_j \mid j \in \mathbb{Z}\}$. Since $x_j \lneq x_{j'}$ for $j \lneq j'$ and $x_{j+i+1} = x_j + (1,\dots,1)$, any small subset $\widetilde{\sigma}'$ of $\mathbb{Z}^{\oplus d}$ with cardinality $i+1$ which maps bijectively onto σ is of the form $\widetilde{\sigma}' = \{x_l, x_{l+1},\dots,x_{l+i}\}$ for some $l \in \mathbb{Z}$. The element $\widetilde{\iota}_{v_1,\dots,v_d,i}(\widetilde{\sigma}')$ is the sequence $(L'_j)_{j\in\mathbb{Z}}$, where $L'_j = L_{j+l}$. Hence the two elements $\widetilde{\iota}_{v_1,\dots,v_d,i}(\widetilde{\sigma})$ and $\widetilde{\iota}_{v_1,\dots,v_d,i}(\widetilde{\sigma}')$ gives the same element in $\mathcal{BT}_i$. $\square$

It is easily checked that the map $\iota_{v_1,\dots,v_d,i} : A_i \to \mathcal{BT}_i$ is injective for every $i \ge 0$ and that the collection of the maps $\iota_{v_1,\dots,v_d,i}$ forms a map $\iota_{v_1,\dots,v_d} : A_\bullet \to \mathcal{BT}_\bullet$ of simplicial complexes. We define a simplicial subcomplex $A_{v_1,\dots,v_d,\bullet}$ of $\mathcal{BT}_\bullet$ to be the image of the map $\iota_{v_1,\dots,v_d}$ so that $A_{v_1,\dots,v_d,i}$ is the image of the map $\iota_{v_1,\dots,v_d,i}$ for each $i \ge 0$. We call the subcomplex $A_{v_1,\dots,v_d,\bullet}$ of $\mathcal{BT}_\bullet$ the apartment in $\mathcal{BT}_\bullet$ corresponding to the basis $v_1,\dots,v_d$. Since the map $\iota_{v_1,\dots,v_d,i}$ is injective for every $i \ge 0$, the map $\iota_{v_1,\dots,v_d}$ induces an isomorphism $A_\bullet \xrightarrow{\cong} A_{v_1,\dots,v_d,\bullet}$ of simplicial complexes.

5.4.3. We introduce a special element β in the group $H^{\mathrm{BM}}_{d-1}(A_\bullet, \mathbb{Z})$, which is an analogue of the fundamental class. Let $\sigma \in A_{d-1}$ and take a small lift $\widetilde{\sigma} \subset \mathbb{Z}^{\oplus d}$ to

$\mathbb{Z}^{\oplus d}$. By definition the set $\widetilde{\sigma}$ is of the form $\widetilde{\sigma} = \{x_1, \ldots, x_d\}$ with $x_0 \lneq x_1 \lneq \cdots \lneq x_d$ where we have put $x_0 = x_d - (1, \ldots, 1)$. It follows from this property that for each integer i with $1 \le i \le d$ there exists a unique integer $w(i)$ with $1 \le w(i) \le d$ such that the $w(i)$-th coordinate of $x_i - x_{i-1}$ is equal to 1 and the other coordinates of $x_i - x_{i-1}$ are equal to zero. Since we have $\sum_{i=1}^{d}(x_i - x_{i-1}) = x_d - x_0 = (1, \ldots, 1)$, the map $w : \{1, \ldots, d\} \to \{1, \ldots, d\}$ is injective. Hence it defines an element w in the symmetric group S_d. The maps $\{1, \ldots, d\} \to A_0 = \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \ldots, 1)$ which sends i to the class of $x_{w^{-1}(i)}$ in A_0 gives an element $[\widetilde{\sigma}]$ in $T(\sigma)$.

Lemma 5.9. *The element $[\widetilde{\sigma}] \in T(\sigma)$ does not depend on the choice of a lift $\widetilde{\sigma}$.*

Proof. For each integer $j \in \mathbb{Z}$ we write j of the form $j = md + r$ with $m \in \mathbb{Z}$ and $r \in \{0, \ldots, d-1\}$ and put $x_j = x_r + m(1, \ldots, 1)$. As we have mentioned in the proof of Lemma 5.8, The inverse image of σ under the canonical surjection $\mathbb{Z}^{\oplus d} \to \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \ldots, 1)$ is equal to $\{x_j \mid j \in \mathbb{Z}\}$ and any small lift $\widetilde{\sigma}'$ of σ to $\mathbb{Z}^{\oplus d}$ is of the form $\widetilde{\sigma}' = \{x_l, x_{l+1}, \ldots, x_{l+d-1}\}$ for some $l \in \mathbb{Z}$. For each $i \in \{1, \ldots, d\}$, the unique integer $j \in \{l, l+1, \ldots, l+d-1\}$ such that the i-th coordinate of $x_j - x_{j-1}$ is equal to 1 and the other coordinates of $x_j - x_{j-1}$ are equal to zero is congruent to $w^{-1}(i)$ modulo d. Hence the class of x_j in A_0 does not depend on the choice of a small lift $\widetilde{\sigma}'$. This proves the claim. $\square$

We denote by $[\sigma]$ the class of $[\widetilde{\sigma}]$ in $O(\sigma)$. We let $\widetilde{\beta}$ denote the element $\widetilde{\beta} = (\beta_\nu)_{\nu \in A'_{d-1}}$ in $\prod_{\nu \in A'_{d-1}} \mathbb{Z}$ where $\beta_\nu = 1$ if $\nu = [\sigma]$ for some $\sigma \in A_{d-1}$ and $\beta_{\sigma'} = 0$ otherwise. We denote by β the class of $\widetilde{\beta}$ in $(\prod_{\nu \in A'_{d-1}} \mathbb{Z})_{\{\pm 1\}}$.

Proposition 5.10. *The element $\beta \in (\prod_{\nu \in A'_{d-1}} \mathbb{Z})_{\{\pm 1\}}$ is a $(d-1)$-cycle in the chain complex which computes the Borel-Moore homology of $A_\bullet$.*

Proof. The assertion is clear for $d = 1$ since the $(d-2)$-nd component of the complex is zero. Suppose that $d \ge 2$. Let τ be an element in A_{d-2}. Take a small lift $\widetilde{\tau} \subset \mathbb{Z}^{\oplus d}$ of τ to $\mathbb{Z}^{\oplus d}$. By definition the set $\widetilde{\tau}$ is of the form $\widetilde{\tau} = \{x_1, \ldots, x_d\}$ with $x_0 \lneq x_1 \lneq \cdots \lneq x_{d-1}$ where we have put $x_0 = x_d - (1, \ldots, 1)$. There is a unique $i \in \{1, \ldots, d-1\}$ such that $x_i - x_{i-1}$ has two non-zero coordinates. There are exactly two elements in $\mathbb{Z}^{\oplus d}$ which is larger than x_{i-1} and which is smaller than x_i. We denote these two elements by y_1 and y_2. We put $\widetilde{\sigma}_j = \widetilde{\tau} \cup \{y_j\}$ for $j = 1, 2$. The sets $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ are small subsets of $\mathbb{Z}^{\oplus d}$ of cardinality d and their images σ_1, σ_2 under the surjection $\mathbb{Z}^{\oplus d} \twoheadrightarrow \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \ldots, 1)$ are elements in A_{d-1}. For $j = 1, 2$, let w_j denote the element w in the symmetric group S_d which appeared in the first paragraph of Section 5.4.3 for $\sigma = \sigma_j$. It follows from the definition of σ_j that we have $w_1 = w_2(i, i+1)$, where $(i, i+1)$ denotes the transposition of i and $i+1$. It is easily checked that the set of the elements in A_{d-1} which has τ as a face is equal to $\{\sigma_1, \sigma_2\}$. Since we have $\mathrm{sgn}(w_1) = -\mathrm{sgn}(w_2)$, it follows that the component in $(\prod_{\nu \in O(\tau)} \mathbb{Z})_{\{\pm 1\}}$ of the image of β under the boundary map $(\prod_{\nu \in A'_{d-1}} \mathbb{Z})_{\{\pm 1\}} \to (\prod_{\nu' \in A'_{d-2}} \mathbb{Z})_{\{\pm 1\}}$ is equal to zero. This proves the claim. $\square$

5.4.4. Let us define simplicial complexes $\Gamma\backslash\mathcal{BT}_\bullet$ in this paragraph.

Let $\mathbb{K}^\infty \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ be an open compact subgroup. Let $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$ and put $\Gamma = \mathrm{GL}_d(F) \cap \gamma\mathbb{K}^\infty\gamma^{-1}$. As we have explained in Section 5.3.1, any element $g \in \Gamma$ satisfies $|\det g|_\infty = 1$. Hence it follows from Lemma 5.6 that for each $i \ge 0$ and for each $\sigma \in \mathcal{BT}_i$, the image of $V(\sigma)$ under the surjection $\mathcal{BT}_0 \twoheadrightarrow \Gamma\backslash\mathcal{BT}_0$

is a subset of $\Gamma\backslash\mathcal{BT}_0$ with cardinality $i+1$. We denote this subset by $V(\mathrm{cl}(\sigma))$, since it is easily checked that it depends only on the class $\mathrm{cl}(\sigma)$ of σ in $\Gamma\backslash\mathcal{BT}_i$. Thus the collection $\Gamma\backslash\mathcal{BT}_\bullet = (\Gamma\backslash\mathcal{BT}_i)_{i\geq 0}$ has a canonical structure of a simplicial complex such that the collection of the canonical surjection $\mathcal{BT}_i \twoheadrightarrow \Gamma\backslash\mathcal{BT}_i$ is a map of simplicial complexes $\mathcal{BT}_\bullet \twoheadrightarrow \Gamma\backslash\mathcal{BT}_\bullet$.

5.4.5. We need two statements (Lemma 5.11, Lemma 5.12) whose proof will be given in our forthcoming paper [Ko-Ya2]. Let Γ be as in Section 5.4.4. For an F-basis $v_1,\ldots,v_d$ (that is, a basis of $F^{\oplus d}$ regarded as a basis of $F_\infty^{\oplus d}$), we consider the composite

$$A_\bullet \xrightarrow{\iota_{v_1,\ldots,v_d}} \mathcal{BT}_\bullet \to \Gamma\backslash\mathcal{BT}_\bullet. \tag{5.10}$$

Lemma 5.11. *The map (5.10) is a finite map of simplicial complexes in the sense of Section 5.1.4.*

Proof. See [Ko-Ya2]. □

It follows from Lemma 5.11 that the map (5.10) induces a homomorphism

$$H_{d-1}^{\mathrm{BM}}(A_\bullet,\mathbb{Z}) \to H_{d-1}^{\mathrm{BM}}(\Gamma\backslash\mathcal{BT}_\bullet,\mathbb{Z}).$$

We let $\beta_{v_1,\ldots,v_d} \in H_{d-1}^{\mathrm{BM}}(\Gamma\backslash\mathcal{BT}_\bullet,\mathbb{Z})$ denote the image under this homomorphism of the element $\beta \in H_{d-1}^{\mathrm{BM}}(A_\bullet,\mathbb{Z})$ introduced in Section 5.4.3. We call this the class of the apartment $A_{v_1,\ldots,v_d,\bullet}$.

Lemma 5.12. *The image of the canonical map*

$$H_{d-1}(\Gamma\backslash\mathcal{BT}_\bullet,\mathbb{C}) \to H_{d-1}^{\mathrm{BM}}(\Gamma\backslash\mathcal{BT}_\bullet,\mathbb{C})$$

is contained in the sub $\mathbb{C}$-vector space generated by the classes of apartments associated to an F-basis.

Proof. See [Ko-Ya2]. □

5.4.6. Let the notation be as in Section 5.4.4. Since $\mathcal{BT}_d$ is an empty set, the abelian group $H_{d-1}^{\mathrm{BM}}(\Gamma\backslash\mathcal{BT}_\bullet,\mathbb{Z})$ is canonically isomorphic to the Γ-invariant part of $H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet,\mathbb{Z})$. We describe the image of $\beta_{v_1,\ldots,v_d}$ under the composite

$$H_{d-1}^{\mathrm{BM}}(\Gamma\backslash\mathcal{BT}_\bullet,\mathbb{Z}) \hookrightarrow H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet,\mathbb{Z}) \to H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet,\mathbb{C}) \to \mathrm{Map}(\mathrm{GL}_d(F_\infty),\mathbb{C})$$

where the last map is the homomorphism (5.3).

Let $T \subset \mathrm{GL}_d$ be the maximal torus of diagonal matrices and let $N(T(F_\infty))$ denote be the normalizer of $T(F_\infty)$ in $\mathrm{GL}_d(F_\infty)$. As a set, $N(T(F_\infty))$ is the disjoint union $N(T(F_\infty)) = \coprod_{w\in S_d} \dot{w}T(F_\infty)$ where w runs over the symmetric group S_d and $\dot{w} = (\delta_{i,w(j)})$ denotes the permutation matrix associated with w. Let

$$\overline{\phi} : N(T(F_\infty))/T(F_\infty) \to \{\pm 1\} \tag{5.11}$$

denote the map which sends the coset $\dot{w}T(F_\infty)$ to $\mathrm{sgn}(w)$.

Lemma 5.13. *We have $N(T(F_\infty)) \cap \mathcal{I}F_\infty^\times = T(\mathcal{O}_\infty)F_\infty^\times$.*

Proof. If $g \in \mathcal{I}F_\infty^\times$, then the diagonal entries of g are non-zero and have the same ∞-adic valuation. This implies that $\dot{w}T(F_\infty)\cap\mathcal{I}F_\infty^\times$ is empty except for $w=1$. Since $T(F_\infty)\cap\mathcal{I}F_\infty^\times = T(\mathcal{O}_\infty)F_\infty^\times$, the claim follows. □

Lemma 5.13 shows that the canonical map

$$N(T(F_\infty))/T(\mathcal{O}_\infty)F_\infty^\times \to \mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times \tag{5.12}$$

is injective. Let $g \in \mathrm{GL}_d(F)$ denote the matrix whose i-th row is equal to v_i for $1 \le i \le g$. Let $\mu_g : N(T(F_\infty))/T(\mathcal{O}_\infty)F_\infty^\times \to \mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times$ denote the injection which is the composite of (5.12) with the automorphism of $\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times$ given by the left multiplication by g^{-1}. Let $\phi_{v_1,\dots,v_d} : \mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times \to \{-1,0,1\}$ denote the map characterized by the following properties:

- For $h \in N(T(F_\infty))$, the map $\phi_{v_1,\dots,v_d}$ sends the image of $hT(\mathcal{O}_\infty)F_\infty^\times \in N(T(F_\infty))/T(\mathcal{O}_\infty)F_\infty^\times$ under the injection μ_g to $\overline{\phi}(hT(F_\infty)) \in \{\pm 1\}$. Here $\overline{\phi}$ is the map in (5.11).
- If an element $g'\mathcal{I}F_\infty^\times$ does not belong to the image of μ_g, then $\phi_{v_1,\dots,v_d}(g'\mathcal{I}F_\infty^\times) = 0$.

Lemma 5.14. *The composite* $\mathrm{GL}_d(F_\infty) \to \mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times \xrightarrow{\phi_{v_1,\dots,v_d}} \{-1,0,1\} \subset \mathbb{C}$ *is equal to the image of the class of the apartment* $A_{v_1,\dots,v_d,\bullet}$ *under the homomorphism (5.3).*

Proof. Let $(L_j)_{j\in\mathbb{Z}}$ be an element in $\widetilde{\mathcal{BT}}_{d-1}$ whose class in $\mathcal{BT}_{d-1}$ belongs to the subset $A_{v_1,\dots,v_d,d-1} \subset \mathcal{BT}_{d-1}$. Let σ be the unique element in A_{d-1} such that the class of (L_j) in $\mathcal{BT}_{d-1}$ is equal to the image of σ under the map $\iota_{v_1,\dots,v_d,d-1} : A_{d-1} \to \mathcal{BT}_{d-1}$. It follows from the definition of the map $\iota_{v_1,\dots,v_d,d-1}$ that there exists a small lift $\widetilde{\sigma} = \{x_0,\dots,x_{d-1}\}$, with $x_0 \lneq \cdots \lneq x_{d-1} \lneq x_d = x_0 + (1,\dots,1)$, of σ to $\mathbb{Z}^{\oplus d}$ such that for $i = 0,\dots,d$, the image of x_i under the map $\widetilde{\iota}_{v_1,\dots,v_d}$ in Section 5.4.2 is equal to $L_i \in \widetilde{\mathcal{BT}}_0$. We write $x_0 = (m_1,\dots,m_d)$, where $m_1,\dots,m_d \in \mathbb{Z}$. Let w be the element in the symmetric group S_d which was constructed from $\widetilde{\sigma}$ in Section 5.4.3. Then for $i = 0,\dots,d-1$, the element $x_i \in \mathbb{Z}^{\oplus d}$ is characterized by the following property: the $w(j)$-th coordinate of x_i is equal to $m_j + 1$ if $0 \le j \le i$, and is equal to m_j if $i < j \le d-1$. Hence it follows from the definition of the map $\widetilde{\iota}_{v_1,\dots,v_d}$, that for $i = 0,\dots,d$, the $\mathcal{O}_\infty$-lattice L_i is of the form

$$L_i = \bigoplus_{1\le j\le i} \varpi_\infty^{m_j+1}\mathcal{O}_\infty v_{w(j)} \oplus \bigoplus_{i<j\le d} \varpi_\infty^{m_j}\mathcal{O}_\infty v_{w(j)}.$$

It is straightforward from the definition of the isomorphism $\widetilde{\mathcal{BT}}_{d-1} \cong \mathrm{GL}_d(F_\infty)/\mathcal{I}$ in Section 3.2.3 to check that the element $(L_j)_{j\in\mathbb{Z}}$ in $\widetilde{\mathcal{BT}}_{d-1}$ corresponds via this isomorphism to the coset $g^{-1}\dot{w}\mathrm{diag}(\varpi_\infty^{-m_1},\dots,\varpi_\infty^{-m_d})\mathcal{I}$. The class $[\sigma] \in O(\sigma)$ is $\mathrm{sgn}(w)$ times the class of the element in $T(\sigma)$ given by the map $\{1,\dots,d\} \to A_0$ which sends i to the class of x_i. Consider the composite map $c : H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet,\mathbb{C}) \to \mathrm{Map}(\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times,\mathbb{C})$ of the homomorphism (5.4) with the canonical inclusion $H_{d-1}^{\mathrm{BM}}(\mathcal{BT}_\bullet,\mathbb{C}) \hookrightarrow (\prod_{\nu\in(\mathcal{BT}_{d-1})'}\mathbb{C})_{\{\pm1\}}$. Then the image of the class of the apartment $A_{v_1,\dots,v_d}$ under this map c is equal to the map $\phi_{v_1,\dots,v_d}$. Thus the claim follows from the definition of the homomorphism (5.3). $\square$

Let $p_{d-1} : \mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times \to \Gamma\backslash\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times$ denote the canonical surjection. It follows from Lemma 5.11 that for every $a \in \Gamma\backslash\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times$ the sum

$$\overline{\phi}_{v_1,\dots,v_d}(a) = \sum_{b\in p_{d-1}^{-1}(a)} \phi_{v_1,\dots,v_d}(b)$$

is a finite sum and hence gives an element $\overline{\phi}_{v_1,\dots,v_d}(a) \in \mathbb{Z}$. The following statement is a consequence of Lemma 5.14.

Corollary 5.15. *The map* $\overline{\phi}_{v_1,\dots,v_d} : \Gamma\backslash \mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times \to \mathbb{Z} \subset \mathbb{C}$ *is equal to the image of* $\beta_{v_1,\dots,v_d} \in H_{d-1}^{\mathrm{BM}}(\Gamma\backslash\mathcal{BT}_\bullet, \mathbb{Z})$ *under the composite*

$$\begin{aligned} &H_{d-1}^{\mathrm{BM}}(\Gamma\backslash\mathcal{BT}_\bullet, \mathbb{Z}) \to H_{d-1}^{\mathrm{BM}}(\Gamma\backslash\mathcal{BT}_\bullet, \mathbb{C}) \\ &\hookrightarrow \Big(\prod_{\nu \in (\Gamma\backslash\mathcal{BT}_{d-1})'} \mathbb{C}\Big)_{\{\pm 1\}} \xrightarrow{(1)} \mathrm{Map}(\Gamma\backslash\mathrm{GL}_d(F_\infty)/\mathcal{I}F_\infty^\times, \mathbb{C}) \end{aligned}$$

where the map (1) is the homomorphism induced by the homomorphism (5.4). $\square$

5.4.7. Let $\mathbb{K}^\infty \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ be an open compact subgroup. For $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$, we construct a function

$$\eta_{\mathbb{K}^\infty,\gamma} : \mathrm{GL}_d(F)\backslash(\mathcal{BT}_{d-1,*} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty) \to \mathbb{C}$$

in $\mathcal{A}_{\mathbb{C}}^{\mathbb{K}^\infty}$ as follows. Take a complete set of representatives $\{g_j\}_{j\in J} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ of $\mathrm{GL}_d(F)\backslash\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty$. We then have $\coprod_{j\in J}\Gamma_j\backslash\mathcal{BT}_{d-1,*} \cong \mathrm{GL}_d(F)\backslash(\mathcal{BT}_{d-1,*} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty)$ where $\Gamma_j = \mathrm{GL}_d(F) \cap g_j\mathbb{K}^\infty g_j^{-1}$ for each $j \in J$. Suppose $\gamma = gg_jk$ with $g \in \mathrm{GL}_d(F), k \in \mathbb{K}^\infty$ and $j \in J$. Let v_i denote the i-th row of the matrix g. Then we put $\eta_{\mathbb{K}^\infty,\gamma} = \phi_{v_1,\dots,v_d}$ on $\Gamma_j\backslash\mathcal{BT}_{d-1,*}$ and zero on $\Gamma_{j'}\backslash\mathcal{BT}_{d-1,*}$ for $j' \neq j$. This is independent of the choice of the g_j's, g, and k.

It follows from Corollary 5.15 that $\eta_{\mathbb{K}^\infty,\gamma} \in \mathcal{A}_{\mathbb{C}}^{\mathbb{K}^\infty}$ is equal to the image under the homomorphism (5.9) of the element $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}^\infty,\bullet}, \mathbb{C}) = \bigoplus_{j'\in J} H_{d-1}^{\mathrm{BM}}(\Gamma_{j'}\backslash\mathcal{BT}_\bullet, \mathbb{C})$ which is equal to the class of $A_{v_1,\dots,v_j,\bullet}$ on $\Gamma_j\backslash\mathcal{BT}_\bullet$ and which is zero on $\Gamma_{j'}\backslash\mathcal{BT}_\bullet$ for $j' \neq j$. Let us also denote by $\eta_{\mathbb{K}^\infty,\gamma}$ the element in $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}^\infty,\bullet}, \mathbb{C})$ described above.

Corollary 5.16. *The image of the canonical map*

$$H_{d-1}(X_{\mathbb{K}^\infty,\bullet}, \mathbb{C}) \to H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}^\infty,\bullet}, \mathbb{C})$$

is contained in the subspace generated by the elements of the form $\eta_{\mathbb{K}^\infty,\gamma}$ *with* $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$.

Proof. Let $\{g_j\}_{j\in J}$ be as above. Then $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}^\infty,\bullet}, \mathbb{C}) \cong \bigoplus_{j\in J} H_{d-1}^{\mathrm{BM}}(\Gamma_j\backslash\mathcal{BT}_\bullet, \mathbb{C})$. Fix $j_0 \in J$. From Lemma 5.12, it follows that the subspace generated by the elements $\eta_{\mathbb{K}^\infty,\gamma}$, where γ runs over the set $\{gg_{j_0} \mid g \in \mathrm{GL}_d(F)\}$, contains the subspace of elements $e = (e_j)_{j\in J} \in H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}^\infty,\bullet}, \mathbb{C})$ with $e_j = 0$ if $j \neq j_0$ and with e_{j_0} coming from $H_{d-1}(X_{\mathbb{K}^\infty,\bullet}, \mathbb{C})$. The claim follows. $\square$

6. Zeta Integral

The aim of this section is to prove Theorem 6.3. It states that the integral of a Hecke eigen cusp form against a certain automorphic form with a complex parameter is expressed as the product of the L-function of the cusp form and a certain integral not involving the complex parameter. The case $d = 2$ is the analogue of the case of $\mathrm{GL}_{2,\mathbb{Q}}$, and may be proved using the Rankin-Selberg method. Here we use instead the fact, whose proof is given in our other paper [Ko-Ya1], that the automorphic forms constructed using the homomorphism $\mathcal{E}$ satisfy certain trace relations.

In Section 6.1 (especially in Proposition 6.2), we express the fact that this trace relation holds as that a certain family of automorphic forms is an "Euler system". This is an abuse of terminology, since an Euler system in the usual sense is a family of elements in certain Galois cohomology groups. The reader familiar with the theory of Euler systems will easily find that the notion of Euler system in this section is an automorphic counterpart of that in usual sense. The reader who is not familiar with the theory of Euler systems should regard the properties stated in Proposition 6.2 as the definition of Euler system in our sense.

6.1. **Euler systems.**

We recall below the statement (Proposition 6.2) that certain automorphic forms constructed from distributions satisfies certain trace relation. The proof is given in our other paper [Ko-Ya1].

6.1.1. Let $R = \mathbb{C}((q_\infty^{-s}))$. We define a homomorphism

$$\mathcal{E} : \mathcal{S}'(V^\infty)^{\otimes d} \to \mathcal{A}_R(|\ |_\infty^{-sd})$$

as follows. (See Section 2.4.4 for the definition of $\mathcal{S}'(V^\infty)$.) Let $\Phi_\infty : \mathrm{Mat}_d(F_\infty) \to \mathbb{C}((q_\infty^{-s}))$ denote the function which sends $M_\infty \in \mathrm{Mat}_d(F_\infty)$ to $\det(\phi_\infty(M_{\infty,i}\Pi_{j-1}^{-1}))$, where ϕ_∞ is as in Section 3.4.3, Π_{j-1} is as in Section 3.2.3, and $M_{\infty,i}$ is the i-th row of M_∞. For $\Phi^\infty \in \mathcal{S}'(V^\infty)^{\otimes d}$ and $g = (g^\infty, g_\infty) \in \mathrm{GL}_d(\mathbb{A})$, we regard Φ^∞ as an element in $\mathcal{S}(\mathrm{Mat}_d(\mathbb{A}^\infty))$ via the isomorphism $S(V^\infty)^{\otimes d} \cong \mathcal{S}(\mathrm{Mat}_d(\mathbb{A}^\infty))$ in Section 2.4.4 and we put

$$\mathcal{E}(\Phi^\infty)(g) = \sum_{M \in \mathrm{Mat}_d(F)} \Phi^\infty(Mg^\infty)\Phi_\infty(Mg_\infty). \tag{6.1}$$

Let us show that the sum on the right hand side is convergent. Note that any $\Phi^\infty \in S'(V^\infty)^{\otimes d}$ is a linear combination of functions of the form $\phi_{\Lambda_1,\mathbf{b}_1} \otimes \cdots \otimes \phi_{\Lambda_d,\mathbf{b}_d}$ where each $\Lambda_i \subset V$ is an A-lattice and $\mathbf{b}_i \in (V/\Lambda_i) \setminus \{0\}$ (We refer to Section 3.4.3 for the definition of $\phi_{*,*}^\infty$).

Lemma 6.1. *Let the notation be as above. For $\Phi^\infty = \phi_{\Lambda_1,\mathbf{b}_1}^\infty \otimes \cdots \otimes \phi_{\Lambda_d,\mathbf{b}_d}^\infty$ and for $g \in \mathrm{GL}_d(\mathbb{A})$, we have*

$$\mathcal{E}(\Phi^\infty)(g) = \det((\mathbb{E}_{\Lambda_i,\mathbf{b}_i}(g\Pi_{j-1}^{-1}))_{1 \le i,j \le d}),$$

where $\mathbb{E}_{\Lambda_i,\mathbf{b}_i}$ is the Eisenstein series defined in Section 3.4.2.

Proof. This follows from the definitions using Lemma 3.3. □

This lemma implies that the sum on the right hand side of (6.1) above is convergent.

6.1.2. Let $J \subset I \subsetneqq A$ be nonzero ideals. Given $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$, we put $\mathcal{E}_{I,J,\gamma} = \mathcal{E}(\mathrm{ch}_{\gamma \cdot Y_{I,J}})$.

Proposition 6.2. *Let $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$. The system of automorphic forms $(\mathcal{E}_{I,J,\gamma})_{I,J}$ indexed by two nonzero ideals $J \subset I \subsetneqq A$ is an Euler system. That is, the following statement holds (see the comments in the second paragraph of Section 6 on our usage of this terminology).*

Let $I', J' \subset A$ be nonzero ideals satisfying $J' \subset I' \subset I$ and $J' \subset J$. We let $\wp$ be a prime ideal dividing I', and assume that $\mathrm{Supp}\,(I/I') \subset \{\wp\}$ *and that* $\mathrm{Supp}\,(J/J') \subset \{\wp\}$. *Let $e_\wp$ be as in Section 4.2.1. Then*

$$\mathrm{Tr}_{I,J}^{I',J'} \mathcal{E}_{I',J',\gamma} = \sum_{r=0}^{e_\wp} (-1)^r q_\wp^{r(r-1)/2} T_{\wp,r}^* \mathcal{E}_{I,J,\gamma}.$$

Here $T^*_{\wp,r}$ *is the dual Hecke operator defined in Section 4.2.1, and* $\mathrm{Tr}_{I,J}^{I',J'} : \mathcal{A}_R(I',J',|\ |_\infty^{-ds}) \to \mathcal{A}_R(I,J,|\ |_\infty^{-ds})$ *is the trace map.*

Proof. See [Ko-Ya1]. See also the thesis by Grigorov [Gr, p.25, Theorem 1.4.6] where a relevant portion of the proof is presented. □

6.2. **Zeta Integral.**

Let $J \subset I$ be two nonzero ideals of A. For an element $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$, we write $\eta_{I,J,\gamma} = \eta_{\mathbb{K}^\infty_{I,J},\gamma} \in \mathcal{A}_\mathbb{C}(I,J,1)$, where $\eta_{\mathbb{K}^\infty_{I,J},\gamma} \in \mathcal{A}_\mathbb{C}(I,J,1)$ is as in Section 5.4.7.

Theorem 6.3. *Let f be an element in $\mathcal{A}^\circ_\mathbb{C}(I,J,1)$. Suppose Conditions (1)(2) of Section 4.2.2 are satisfied. We have*

$$\langle f, \mathcal{E}_{I,J,\gamma}\rangle = (1-q_\infty^{-s})^{d-1} L^{I,J}(f, s-\frac{d-1}{2})\langle f, \eta_{I,J,\gamma}\rangle.$$

Corollary 6.4. *If $L^{I,J}(f,-\frac{d-1}{2}) \neq 0$, then $\langle f, \eta_{I,J,\gamma}\rangle = 0$.*

Proof of Corollary 6.4. From Proposition 3.4 it follows that the left hand side of the equation in Theorem 6.3 has a zero of order d at $s = 0$. Then counting the number of zeros on the right hand side gives the claim. □

We refer to Remark 8.3 for an implication of this corollary.

6.2.1. **Proof of Theorem 6.3: Step 1. Application of Euler systems**. For any nonzero ideal I' of A with $I' \subset J$, we consider the element

$$\mathcal{E}_{I,J,I',\gamma} = \mathrm{Tr}_{I,J}^{I',I'}(\mathcal{E}_{I',I',\gamma})$$

in $\mathcal{A}_R(I,J,|\ |_\infty^{-sd})$. By Proposition 6.2, we have

$$\mathcal{E}_{I,J,I',\gamma} = \prod_{\wp | I',\, \wp\nmid I}\left(\sum_{r=0}^{e_\wp}(-1)^r q_\wp^{r(r-1)/2} T^*_{\wp,r}\right)\mathcal{E}_{I,J,\gamma},$$

where $e_\wp$ is as in Section 4.2.1. Thus

$$\langle f, \mathcal{E}_{I,J,I',\gamma}\rangle = \langle \prod_{\wp | I',\, \wp\nmid I}(\sum_{r=0}^{e_\wp}(-1)^r q_\wp^{\frac{r(r-1)}{2}-rs} T_{\wp,r}) f, \mathcal{E}_{I,J,\gamma}\rangle$$

and hence

$$\langle f, \mathcal{E}_{I,J,\gamma}\rangle = \prod_{\wp | I',\, \wp\nmid I}\left[\sum_{r=0}^{e_\wp}(-1)^r a_{\wp,r} q_\wp^{\frac{r(r-1)}{2}-rs}\right]^{-1}\langle f, \mathcal{E}_{I,J,I',\gamma}\rangle.$$

Next we consider the limit of $\mathcal{E}_{I,J,I',\gamma}$ as I' gets smaller. We note that for all $I' \subset J$, the function $\mathcal{E}_{I,J,I',\gamma}$ is invariant under the action of $\mathcal{I} \times \mathbb{K}^\infty_{I,J}$ where $\mathcal{I}$ is the Iwahori subgroup, since $\mathcal{I} = \bigcap_{j=1}^d \Pi_{j-1}^{-1}\mathrm{GL}_d(\mathcal{O}_\infty)\Pi_{j-1}$.

We put

$$Y_{I,J,I'} = \{g \in Y_{I,J} \mid g \bmod I' \in \mathrm{GL}_d(A/I')\}.$$

where $Y_{I,J}$ is as in Section 2.4.6. It is easy to check that the set $Y_{I,J,I'}$ is written as the disjoint sum

$$Y_{I,J,I'} = \coprod_{h \in \mathbb{K}^\infty_{I',I'}\backslash\mathbb{K}^\infty_{I,J}} Y_{I',I'}\cdot h.$$

It follows that $\mathrm{Tr}_{I,J}^{I',I'}(\mathrm{ch}_{\gamma\cdot Y_{I',I'}}) = \mathrm{ch}_{\gamma\cdot Y_{I,J,I'}}$. Hence we have $\mathcal{E}_{I,J,I',\gamma} = \mathcal{E}(\mathrm{ch}_{\gamma\cdot Y_{I,J,I'}})$. Since $\gamma \cdot Y_{I,J,I'}$ is a subset of $\gamma \cdot Y_{I,J}$ and since $\gamma \cdot Y_{I,J,I''} \subset \gamma \cdot Y_{I,J,I''}$ for $I'' \subset I'$, the limit of $\mathcal{E}_{I,J,I',\gamma}(g) \in R$ as I' gets smaller exists for any $g = (g^\infty, g_\infty) \in \mathrm{GL}_d(\mathbb{A})$ and is equal to

$$\mathcal{E}_{I,J,I',\gamma}(g) = \sum_{\substack{M\in \mathrm{Mat}_d(F)\\ Mg^\infty \in \bigcap_{I'} \gamma\cdot Y_{I,J,I'}}} \Phi_\infty(Mg_\infty).$$

Since $\bigcap_{I'} \gamma \cdot Y_{I,J,I'} = \gamma\mathbb{K}^\infty_{I,J}$, we have

$$(6.2)\qquad \begin{aligned} \mathcal{E}_{I,J,\lim,\gamma}(g) &= \textstyle\sum_{X\in \mathrm{GL}_d(F), Xg^\infty\in\gamma\mathbb{K}^\infty_{I,J}} \Phi_\infty(Xg_\infty) \\ &= \textstyle\sum_{X\in \mathrm{GL}_d(F), Xg^\infty\in\gamma\mathbb{K}^\infty_{I,J}} \det((|\mathbf{1}_i X g_\infty \Pi_{j-1}^{-1}|^{-s}_{\mathcal{O}_{V_\infty}})_{1\le i,j\le d}), \end{aligned}$$

where $\mathbf{1}_i = (0,\dots,0,1,0\dots,0)$ with 1 placed in the i-th place, with respect to the (q_∞^{-s})-adic topology.

Recall that since f is a cusp form, the support of the function $f(g)$ is contained in $Z(\mathbb{A})\mathbb{K}$ for some compact open subset $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A})$. We note also that $Z(F_\infty)$ acts trivially on $f(g)\mathcal{E}_{I,J,I',\gamma}(g)|\det g|^{-s}$. It follows that the inner product $\langle f, \mathcal{E}_{I,J,I',\gamma}\rangle$ is a sum over the finite set $\mathrm{GL}_d(F)Z(F_\infty)\backslash \mathrm{Supp}\,(f)/(\mathbb{K}\cap(\mathcal{I}\times\mathbb{K}^\infty_{I,J}))$ which does not depend on I'. Hence the limit of $\langle f, \mathcal{E}_{I,J,I',\gamma}\rangle$ with respect to I' commutes with the bracket $\langle , \rangle$. Thus we obtain

$$\langle f, \mathcal{E}_{I,J,\gamma}\rangle = L^{I,J}(f, s-\frac{d-1}{2})\langle f, \mathcal{E}_{I,J,\lim,\gamma}\rangle.$$

6.2.2. **Proof of Theorem 6.3: Step 2. Unfolding the integral.** Now to prove the theorem, it suffices to prove the following proposition.

Proposition 6.5. *Let the notation be as above. We have*

$$\langle f, \mathcal{E}_{I,J,\lim,\gamma}\rangle = (1-q_\infty^{-s})^{d-1}\langle f, \eta_{I,J,\gamma}\rangle.$$

Proof. Given two nonzero ideals I, J of A with $J \subset I \subsetneq A$, we define a function $\widetilde{\phi}_{I,J,\gamma}$ on $\mathrm{GL}_d(\mathbb{A})$ as follows. For $g = (g_\infty, g^\infty) \in \mathrm{GL}_d(\mathbb{A})$, we let

$$\widetilde{\phi}_{I,J,\gamma}(g) = \widetilde{\phi}^\infty_{I,J,\gamma}(g^\infty)\widetilde{\phi}_\infty(g_\infty),$$

where $\widetilde{\phi}_\infty(g_\infty) = \det(|\mathbf{1}_i g_\infty \Pi_{j-1}^{-1}|^{-s}_{\mathcal{O}_{V_\infty}})$, and $\widetilde{\phi}^\infty_{I,J,\gamma}$ is the characteristic function of $\gamma\mathbb{K}^\infty_{I,J}$. We have

$$\mathcal{E}_{I,J,\lim,\gamma} = \sum_{\gamma'\in \mathrm{GL}_d(F)} \widetilde{\phi}_{I,J,\gamma}(\gamma' g).$$

Hence $\langle f, \mathcal{E}_{I,J,\lim,\gamma}\rangle$ is equal to

$$\begin{aligned} &\int_{Z(F_\infty)\mathrm{GL}_d(F)\backslash \mathrm{GL}_d(\mathbb{A})} f(g) \sum_{\gamma'\in\mathrm{GL}_d(F)} \widetilde{\phi}_{I,J,\gamma}(\gamma' g)|\det g|^s dg \\ = &\int_{Z(F_\infty)\backslash \mathrm{GL}_d(\mathbb{A})} f(g)\widetilde{\phi}_{I,J,\gamma}(g)|\det g|^s dg \\ = &\ \mathrm{vol}(\mathbb{K}^\infty_{I,J})\int_{Z(F_\infty)\backslash \mathrm{GL}_d(F_\infty)} f(g_\infty,\gamma)\widetilde{\phi}_\infty(g_\infty)|\det g_\infty|^s_\infty dg_\infty. \end{aligned}$$

Now let us fix $g_\infty \in \mathrm{GL}_d(F_\infty)$ and consider the value $\widetilde{\phi}_\infty(g_\infty)$. Let us write $H_{i,j}(g_\infty) = |\mathbf{1}_i g_\infty \Pi_{j-1}|^{-s}_{\mathcal{O}_{V\infty}}$ and let $H(g_\infty) = (H_{i,j}(g_\infty))_{1\le i,j\le d}$ so that $\widetilde{\phi}_\infty(g_\infty) = \det H(g_\infty)$. For each $i = 1,\dots,d$, there exists a unique $n_i = n_i(g_\infty) \in \{1,\dots,d\}$ such that $H_{i,j}(g_\infty) = H_{i,1}(g_\infty)$ for $1 \le j \le n_i$ and $H_{i,j}(g_\infty) = q_\infty^{-s}H_{i,1}(g_\infty)$ for

$n_i + 1 \leq j \leq d$. If $n_{i_1} = n_{i_2}$ for some $i_i \neq i_2$, the i_1-st row and i_2-nd row of $H(g_\infty)$ are linearly dependent and hence $\det H(g_\infty) = 0$. Suppose that $n_1, \dots, n_d$ are distinct, and let w denote the element in the symmetric group S_d which sends $i \in \{1, \dots, d\}$ to n_i. This occurs exactly when $g_\infty \in \dot{w} T(F_\infty)\mathcal{I}$ where $T \subset \mathrm{GL}_d$ denotes the subgroup of diagonal matrices and $\dot{w} \in \mathrm{GL}_d(F_\infty) \subset \mathrm{GL}_d(\mathbb{A})$ is the permutation matrix associated with w. Then we have $\prod_{i=1}^d H_{i,1}(g_\infty) = |\det(g_\infty)|_\infty^{-s}$ and hence

$$\det H(g_\infty) = \mathrm{sgn}(w) \det H(\dot{w}^{-1} g_\infty) = \mathrm{sgn}(w)\, |\det(g_\infty)|_\infty^{-s} \det D(s),$$

where $D(s)$ is the $d \times d$ matrix

$$D(s) = \begin{pmatrix} 1 & q_\infty^{-s} & \cdots\cdots\cdots & & q_\infty^{-s} \\ 1 & 1 & q_\infty^{-s} & \cdots & q_\infty^{-s} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & q_\infty^{-s} \\ 1 & \cdots\cdots\cdots\cdots & & & 1 \end{pmatrix}.$$

Simple calculation shows that $\det D(s) = (1 - q_\infty^{-s})^{d-1}$.

This shows that the left hand side of the equation in Theorem 6.3 equals

$$(1 - q_\infty^{-s})^{d-1} \mathrm{vol}(\mathbb{K}_{I,J}^\infty) \sum_{w \in S_d} \int_{Z(F_\infty)\backslash \dot{w} T(F_\infty)\mathcal{I}} f(g_\infty, \gamma) \mathrm{sgn}(w) dg_\infty.$$

One can then verify that this equals the right hand side by checking the definition of $\eta_{I,J,\gamma}$ and the definition of $\beta_{*,\dots,*}$ in Section 5.4.4. □

This completes the proof of Theorem 6.3. □

7. Regulator

In this section, we construct a homomorphism which we call regulator map from the K-group of Drinfeld modular varieties to the space of automorphic forms.

7.1. **K-theory for rigid analytic spaces.**

For a rigid analytic space $\mathfrak{Y}$, we may define the K-theory ring $K_*(\mathfrak{Y})$ using the exact category of locally free coherent $\mathcal{O}_\mathfrak{Y}$-modules. Let us collect the basic properties in this section.

7.1.1. Let $\mathfrak{Y}$ be a rigid analytic space (of finite type) over F_∞. We let $K_m(\mathfrak{Y})$ denote the m-th K-group constructed from the exact category of locally free coherent $\mathcal{O}_\mathfrak{Y}$-modules. The graded abelian abelian group $K_*(\mathfrak{Y})$ becomes a graded ring by giving the product structure following [Wa, §9]. For a morphism $\mathfrak{Y}_1 \to \mathfrak{Y}_2$ of rigid analytic spaces over F_∞, we have a pullback morphism $K_*(\mathfrak{Y}_1) \to K_*(\mathfrak{Y}_2)$ of graded rings.

7.1.2. Let Y be a scheme of finite type over F_∞. Let Y^{an} denote the rigid analytic space associated with Y. There is an exact functor $- \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y^{\mathrm{an}}}$ from the exact category of locally free coherent $\mathcal{O}_Y$-modules to that of $\mathcal{O}_{Y^{\mathrm{an}}}$-modules. This functor induces a morphism of graded rings $K_*(Y) \to K_*(Y^{\mathrm{an}})$. Here the ring structure of $K_*(Y)$ is as defined in [Gi], which uses the recipe of [Wa, §9].

7.1.3. Let B be an affinoid algebra over F_∞. It is known ([Bos-Gu-Re, p.378, Theorem 3], [Bos-Gu-Re, p.374, Proposition 2] and [Fr-vdPu, p.98, Proposition 4.7.2]) that the abelian category of finitely generated projective B-modules and the category of locally free coherent $\mathcal{O}_{\mathrm{Spm}\, B}$-modules are equivalent. This induces an isomorphism of graded rings $K_*(\mathrm{Spec}\, B) \cong K_*(\mathrm{Spm}\, B)$.

7.1.4. Let $\mathfrak{Y}$ be a rigid analytic space over F_∞. Let us construct the symbol map $\mathcal{O}(\mathfrak{Y})^\times \to K_1(\mathfrak{Y})$. Let $f \in \mathcal{O}(\mathfrak{Y})^\times$. Then f gives a morphism $f : \mathfrak{Y} \to \mathbb{G}_{m,F_\infty}^{\mathrm{an}}$ of rigid analytic spaces. We define the image of f in $K_1(\mathfrak{Y})$ under the symbol map to be the image of the coordinate function t in $\mathcal{O}(\mathrm{Spec}\, F_\infty[t, t^{-1}]) = \mathcal{O}(\mathbb{G}_{m,F_\infty})$ via the map

$$\mathcal{O}(\mathbb{G}_{m,F_\infty})^\times = K_1(\mathbb{G}_{m,F_\infty}) \xrightarrow{(1)} K_1(\mathbb{G}_{m,F_\infty}^{\mathrm{an}}) \xrightarrow{(2)} K_1(\mathfrak{Y})$$

where the map (1) is as in Section 7.1.2 and is the map (2) is the pullback by f.

To see that it is a group homomorphism, let us first recall the following fact. Let $m, \mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ denote the multiplication, the first projection, and the second projection respectively. Let $t \in \mathcal{O}(\mathbb{G}_m)^\times \cong K_1(\mathbb{G}_m)$ denote the coordinate function. Then we have an equality $m^*t = \mathrm{pr}_1^*t + \mathrm{pr}_2^*t$ in $K_1(\mathbb{G}_m \times \mathbb{G}_m)$.

Now let $f_1, f_2, f_3 = f_1 f_2 \in \mathcal{O}(\mathfrak{Y})^\times$ and regard each of them as a morphism $\mathfrak{Y} \to \mathbb{G}_{m,F_\infty}^{\mathrm{an}}$. Since $m \circ (f_1, f_2) = f_3$, we obtain $f_3^*t = (f_1, f_2)^* m^* t = (f_1, f_2)^*(\mathrm{pr}_1^*t + \mathrm{pr}_2^*t) = f_1^*t + f_2^*t$ using the fact above. Here we used the same notation $m, \mathrm{pr}_1, \mathrm{pr}_2, t$ for its analytification by abuse. This shows that the symbol map is a group homomorphism.

We remark that the symbol map for schemes may be defined in an analogous manner.

7.1.5. Let $f : \mathfrak{Y}_1 \to \mathfrak{Y}_2$ be a morphism of rigid analytic spaces over F_∞. The symbol maps $\mathcal{O}(\mathfrak{Y}_i)^\times \to K_1(\mathfrak{Y}_i)$ for $i = 1, 2$ defined in the previous section are compatible in the sense that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{O}(\mathfrak{Y}_2)^\times \arrow[r] \arrow[d, "f^*"'] & K_1(\mathfrak{Y}_2) \arrow[d, "f^*"'] \\ \mathcal{O}(\mathfrak{Y}_1)^\times \arrow[r] & K_1(\mathfrak{Y}_1) \end{tikzcd}$$

where the horizontal arrows are the symbol maps. The commutativity follows immediately from the definitions.

7.1.6. Let Y be a scheme of finite type over F_∞ and let Y^{an} denote the associated rigid analytic space. Then the diagram

$$\begin{tikzcd} \mathcal{O}(Y)^\times \arrow[r, "(1)"] \arrow[d, "(2)"'] & K_1(Y) \arrow[d, "(3)"] \\ \mathcal{O}(Y^{\mathrm{an}})^\times \arrow[r, "(4)"] & K_1(Y^{\mathrm{an}}) \end{tikzcd}$$

where the map (1) is the symbol map (for a scheme), the map (2) is the analytification map, the map (3) is the map in Section 7.1.2, and the map (4) is the symbol map (Section 7.1.4, is commutative. The commutativity follows from the fact that the symbol map $\mathcal{O}(Y)^\times \to K_1(Y)$ for a scheme can be constructed in a similar way as we did in Section 7.1.4 for rigid analytic spaces.

7.1.7. Let B be an affinoid algebra over F_∞. Then the diagram

$$\begin{array}{ccc} B^\times & \longrightarrow & K_1(\operatorname{Spm} B) \\ \Big\| & & \Big\downarrow \cong \\ B^\times & \longrightarrow & K_1(\operatorname{Spec} B) \end{array}$$

where the horizontal arrows are the symbol maps and the right vertical arrow is the map defined in Section 7.1.3, is commutative. The commutativity can be checked in the same manner as in Section 7.1.6.

7.2. **Affinoid covering of Drinfeld symmetric spaces.**

The Drinfeld symmetric space $\mathfrak{X}$ has a canonical covering of affinoids $\mathcal{U}_\sigma$ where σ is a simplex in the Bruhat-Tits buildings $\mathcal{BT}_\bullet$. The aim of this section is to give an explicit description of this covering and of the coordinate rings of the affinoid and the corresponding formal model.

All the results in this subsection is well-known and some of them are found in the literature. In particular, the explicit local description of the formal model of $\mathfrak{X}$ is found in [Ge, p.75-76, III.1.3]. However we reproduce the results here because we need to point out some ring theoretic properties and to describe explicitly the relation with the continuous map $\lambda : \mathfrak{X} \to |\mathcal{BT}_\bullet|$ in Section 7.2.1.

7.2.1. Let $|\mathcal{BT}_\bullet|$ be the geometric realization of the Bruhat-Tits building $\mathcal{BT}_\bullet$. In [Dr, p.579, §6, 3)], Drinfeld constructs a canonical continuous map $\lambda : \mathfrak{X} \to |\mathcal{BT}_\bullet|$ from the underlying topological space of $\mathfrak{X}$ to $|\mathcal{BT}_\bullet|$. Let $\sigma \in \coprod_{0\le i\le d-1} \mathcal{BT}_i$ be a simplex of $\mathcal{BT}_\bullet$. Let $|\sigma| \subset \mathcal{BT}$ denote the geometric realization of the simplicial subcomplex of $\mathcal{BT}_\bullet$ which consists of the faces of σ. It follows from [Dr, Proposition 6.1, p.579] that the subset $\lambda^{-1}(|\sigma|) \subset \mathfrak{X}$ is an admissible open subset of $\mathfrak{X}$ such that the restriction $\mathfrak{U}_\sigma$ of $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ to $\lambda^{-1}(|\sigma|)$ is an affinoid. The map $\lambda : \mathfrak{X} \to |\mathcal{BT}_\bullet|$ is $\mathrm{GL}_d(F_\infty)$-equivariant. The action of $g \in \mathrm{GL}_d(F_\infty)$ induces for each $\sigma \in \coprod_{0\le i\le d-1} \mathcal{BT}_i$ a canonical isomorphism $\mathfrak{U}_\sigma \xrightarrow{\cong} \mathfrak{U}_{g\sigma}$.

Let $\mathcal{U}_\sigma$ denote the formal model of $\mathfrak{U}_\sigma$ over Spf $\mathcal{O}_\infty$, and let B_σ^o denote the coordinate $\mathcal{O}_\infty$-algebra of $\mathcal{U}_\sigma$. In this section, we give a list of properties of the ring B_σ^o which will be used later.

7.2.2. Let i be an integer with $0 \le i \le d-1$ and let $\sigma \in \mathcal{BT}_i$ be an i-simplex of $\mathcal{BT}_\bullet$. Let us take a representative $(L_j)_{j\in\mathbb{Z}} \in \widetilde{\mathcal{BT}}_i$ of σ. We use the notation in Section 3.2.2 and in Section 3.2.4. For $x \in \widetilde{\mathfrak{X}} = \mathbb{V}_\infty^{*,\mathrm{an}} \setminus (\cup_{H_0\in\mathcal{H}_0} H_0^{\mathrm{an}})$, we let $[x]$ denote the image of x under the canonical morphism $\widetilde{\mathfrak{X}} \to \mathfrak{X}$. It follows from the construction of the map λ that the subset $\lambda^{-1}(|\sigma|) \subset \mathfrak{X}$ is equal to the set of classes $[x] \in \mathfrak{X}$ of $x \in \widetilde{\mathfrak{X}}$ satisfying the following condition.

- For $j = 0, \ldots, i$ and for any $v, v' \in L_j$ with $v \notin L_{j+1}$ and $v' \notin \varpi_\infty L_0$, we have $|v\boldsymbol{\tau}_x|_\infty \ge |v'\boldsymbol{\tau}_x|_\infty \ge |\varpi_\infty v\boldsymbol{\tau}_x|_\infty$.

From this we have the following explicit description of the $\mathcal{O}_\infty$-algebra B_σ^o. For $j = 0, \ldots, i+1$, choose a finite subset $S_j \subset L_j \setminus L_{j+1}$ such that the composite

$$S_j \hookrightarrow L_j\backslash L_{j+1} \twoheadrightarrow (L_j/\varpi_\infty L_j)\backslash(L_{j+1}/\varpi_\infty L_j) \twoheadrightarrow ((L_j/\varpi_\infty L_j)\backslash(L_{j+1}/\varpi_\infty L_j))/\kappa_\infty^\times$$

is surjective. Then the $\mathcal{O}_\infty$-algebra B_σ^o is isomorphic to the ϖ_∞-adic completion of the $\mathcal{O}_\infty$-subalgebra of $\operatorname{Frac}\operatorname{Sym}^\bullet L_0$ generated by the set

$$\bigcup_{0\le j\le j'\le i+1} \{\ell/\ell' \mid \ell \in S_j,\ \ell' \in S_{j'}\}.$$

7.2.3. For a more explicit description of the $\mathcal{O}_\infty$-algebra B_σ^o, we choose the sets $S_0,\dots,S_i$ in the following way. Take a complete set $S \subset \mathcal{O}_\infty$ of representatives of $\kappa_\infty = \mathcal{O}_\infty/\varpi_\infty\mathcal{O}_\infty$ in $\mathcal{O}_\infty$. For $j=0,\dots,i+1$, let d_j denote the the dimension of L_0/L_j over κ_∞. Take an $\mathcal{O}_\infty$-basis $e_1,\dots,e_d$ of L_0 such that for $j=0,\dots,i$ the set $\{\varpi_\infty e_1,\dots,\varpi_\infty e_{d_j}, e_{d_j+1},\dots,e_d\}$ forms an $\mathcal{O}_\infty$-basis of L_j. For $j=1,\dots,d-1$ we put $e_{d+j}=\varpi_\infty e_j$. For $j=0,\dots,i$, we put

$$S_j = \bigcup_{d_j<j'\le d_{j+1}} \{e_{j'} + a_1 e_{j'+1} + a_2 e_{j'+2}\cdots a_{d-1}e_{j'+d-1} \mid a_1,\dots,a_{d-1}\in S\}.$$

This particular choice of the sets $S_0,\dots,S_i$ gives the following description of the $\mathcal{O}_\infty$-algebra B_σ^o. For $j\in\mathbb{Z}/d\mathbb{Z}$, define an element $T_j \in \operatorname{Frac}\operatorname{Sym}^\bullet L_0$ as follows. For $j=1,\dots,d-1$ we put $T_j = e_{j+1}/e_j$. For $j=0$ we put $T_0=\varpi_\infty e_0/e_d$. Let y be the product

$$y = \prod_{j\in\mathbb{Z}/d\mathbb{Z}}\ \prod_{a_1,\dots,a_{d-1}\in S} (1 + a_1T_j + a_2T_jT_{j+1} + \cdots + a_{d-1}T_j\cdots T_{j+d-2})$$

and let z be the product

$$z = \prod_{j\in\{0,\dots,d-1\}\setminus\{d_0,\dots,d_{d-1}\}} T_j.$$

Then $T_0,\dots,T_{d-1}$ are algebraically independent over $\mathcal{O}_\infty$ and the $\mathcal{O}_\infty$-algebra B_σ^o is isomorphic to the ϖ_∞-adic completion of the $\mathcal{O}_\infty$-algebra

$$R_\sigma = \mathcal{O}_\infty[T_0,T_1,\dots,T_{d-1},y^{-1},z^{-1}]/(T_0T_1\cdots T_{d-1}-\varpi_\infty).$$

Let I be a subset of $\{0,1,\dots,i\}$ and let σ' be a face of σ corresponding to the subset $\{\mathrm{cl}(L_i)\ |i\in I\}$ of the set $V(\sigma)=\{\mathrm{cl}(L_0),\dots,\mathrm{cl}(L_i)\}\subset \mathcal{BT}_0$ of vertices of σ. Then the $\mathcal{O}_\infty$-algebra B_σ^o is isomorphic to the ϖ_∞-adic completion of the $\mathcal{O}_\infty$-algebra $R_\sigma[T_I^{-1}]$ where $T_I=\prod_{j\in\{0,\dots,i\}\setminus I} T_{d_j}$. The homomorphism $B_\sigma^o \to B_{\sigma'}^o$ obtained by taking ϖ_∞-adic completion of the canonical homomorphism $R_\sigma \to R_\sigma[T_I^{-1}]$ is equal to the homomorphism $B_\sigma^o \to B_{\sigma'}^o$ induced from the inclusion $\lambda^{-1}(|\sigma'|)\subset\lambda^{-1}(|\sigma|)$ of admissible open subsets of $\mathfrak{X}$.

7.2.4. Let the notation be as in Section 7.2.3. If follows easily from the definition that the $\mathcal{O}_\infty$-algebra R_σ has the following properties:

(1) The ring R_σ is a regular integral domain of dimension d,
(2) The ring R_σ is flat and finitely generated over $\mathcal{O}_\infty$,
(3) The $\mathcal{O}_\infty$-algebra R_σ has semistable reduction, that is, the special fiber $\operatorname{Spec} R_\sigma/\varpi_\infty R_\sigma$ is a simple normal crossing divisor of $\operatorname{Spec} R_\sigma$.

The property (2) implies that the ring R_σ is excellent ([EGAIV, 7.8, p.214]). Hence its ϖ_∞-adic completion B_σ^o is also a regular noetherian integral domain.

7.2.5. For each simplex $\sigma \in \coprod_{0\le i\le d-1} \mathcal{BT}_i$, let X_σ denote the intersection of all irreducible components in $\mathrm{Spec}\,\overline{B}_\sigma$, where $\overline{B}_\sigma = B_\sigma^o/\varpi_\infty B_\sigma^o$. Since $\overline{B}_\sigma$ is isomorphic to $R_\sigma/\varpi_\infty R_\sigma$, it follows from the definition of R_σ that the schemes $\mathrm{Spec}\,\overline{B}_\sigma$ and X_σ have the following properties.

(1) The set of irreducible components of $\mathrm{Spec}\,\overline{B}_\sigma$ is canonically isomorphic to the set of vertices of σ.
(2) Each irreducible component of $\mathrm{Spec}\,\overline{B}_\sigma$ is smooth of dimension $d-1$ over $\mathrm{Spec}\,\kappa_\infty$.
(3) The scheme X_σ is non-empty, irreducible and smooth of dimension $d-1-i$ over $\mathrm{Spec}\,\kappa_\infty$.
(4) For each vertex $v \in V(\sigma)$, let $\sigma' = \sigma \times_{V(\sigma)} (V(\sigma)\setminus\{v\})$ be the face of σ corresponding to the subset $V(\sigma)\setminus\{v\} \subset V(\sigma)$, and let $X_{\sigma,v}$ denote the intersection of all irreducible components of $\mathrm{Spec}\,\overline{B}_\sigma$ except for the component corresponding to v. Then $X_{\sigma,v}$ is non-empty, irreducible and smooth of dimension $d-i$ over $\mathrm{Spec}\,\kappa_\infty$. The scheme X_σ is a closed subscheme of $X_{\sigma,v}$ and the canonical morphism $\mathrm{Spec}\,\overline{B}_{\sigma'} \to \mathrm{Spec}\,\overline{B}_\sigma$ induces an isomorphism from $X_{\sigma'}$ to the open complement $X_{\sigma,v}\setminus X_\sigma$ of X_σ in $X_{\sigma,v}$.
(5) If $i = d-1$, then X_σ is isomorphic to $\mathrm{Spec}\,\kappa_\infty$.
(6) If $i = d-2$, then X_σ is isomorphic to the projective line over $\mathrm{Spec}\,\kappa_\infty$ minus all the κ_∞-rational points.

7.2.6. Let $v \in \mathcal{BT}_v$ be a 0-simplex of $\mathcal{BT}_\bullet$. It follows from the properties of B_v^o described above that the ideal $\varpi_\infty B_v^o$ of B_v^o generated by ϖ_∞ is a prime ideal and that the localization of the ring B_v^o at $(\varpi_\infty) \subset B_v^o$ is a discrete valuation ring. Hence the prime ideal $\varpi_\infty B_v^o$ defines a valuation on (the field of fractions of) B_v, which will be denoted by $v : B_v \to \mathbb{Z}$ by abuse of notation. For a unit $f \in \mathcal{O}(\mathfrak{X})^\times$ on $\mathfrak{X}$, we write $v(f)$ for the image of f under the map $\mathcal{O}(\mathfrak{X})^\times \to B_v^\times \to \mathbb{Z}$ where the first map is the pullback by the canonical open immersion and the second map is the valuation v.

7.3. **Regulator for Drinfeld symmetric spaces.**

7.3.1. For an integer $i \ge 0$, let $\sigma \in \mathcal{BT}_i$ be an i-simplex. We have a canonical homomorphism

$$K_*(\mathfrak{X}) \xrightarrow{(1)} K_*(\mathrm{Spm}\,B_\sigma) \xrightarrow{(2)} K_*(\mathrm{Spec}\,B_\sigma) \tag{7.1}$$

where the map (1) is the pullback map with respect to the open immersion $\mathfrak{U}_\sigma \hookrightarrow \mathfrak{X}$. The map (2) is the map constructed in Section 7.1.3

One can then apply the localization sequence to the triple $\mathrm{Spec}\,\overline{B}_\sigma \subset \mathrm{Spec}\,B_\sigma^o \supset \mathrm{Spec}\,B_\sigma$, and obtain a boundary homomorphism $K_{m+1}(\mathrm{Spec}\,B_\sigma) \to K_m(\mathrm{Spec}\,\overline{B}_\sigma)$ for each integer $m \ge 0$. In this way we obtain a canonical homomorphism

$$b_\sigma : K_*(\mathfrak{X}) \to K_{*-1}(\mathrm{Spec}\,\overline{B}_\sigma)$$

as the composite of the boundary map with the map (7.1)

7.3.2. Let $\sigma \in \mathcal{BT}_i$ be an i-simplex and let τ be a face of σ. Then there is a canonical open immersion $\mathfrak{U}_\tau \to \mathfrak{U}_\sigma$, which in turn induces a flat map of rings

$B_\tau^o \to B_\sigma^o$. This implies that the diagram

$$\begin{array}{ccc} K_*(\operatorname{Spec} B_\tau) & \longrightarrow & K_{*-1}(\operatorname{Spec} \overline{B}_\tau) \\ \downarrow & & \downarrow \\ K_*(\operatorname{Spec} B_\sigma) & \longrightarrow & K_{*-1}(\operatorname{Spec} \overline{B}_\sigma) \end{array}$$

where the vertical arrows are the boundary maps and the vertical arrows are the pullback maps, is commutative.

7.3.3. For a pointed simplex $\sigma_+ \in \mathcal{BT}_{\bullet,*}$, we let $b_{\sigma_+} : K_*(\mathfrak{X}) \to K_{*-1}(\operatorname{Spec} \overline{B}_\sigma)$ denote the map b_σ where $\sigma \in \mathcal{BT}_\bullet$ is the (not pointed) underlying simplex of σ_+. Using the remarks in Section 7.3.2, we see that the maps b_{σ_+} assemble together to give a $\mathrm{GL}_d(F_\infty)$-equivariant homomorphism

$$K_*(\mathfrak{X}) \to \varprojlim_{\sigma_+ \in \mathcal{BT}_{\bullet,*}} K_{*-1}(\operatorname{Spec} \overline{B}_\sigma). \tag{7.2}$$

7.3.4. Let $\sigma_+ = ((L_i)_{i\in\mathbb{Z}}, L_0) \in \mathcal{BT}_{m-1,*}$ be a pointed $(m-1)$-simplex. We let $\sigma_j = (\cdots \supsetneqq L_0 \supsetneqq \cdots \supsetneqq L_j \supsetneqq L_m \supsetneqq \cdots) \in \mathcal{BT}_j$ for $0 \le j \le m-1$ so that σ_j is a face of σ_{j+1} and σ_{m-1} is the underlying simplex of σ_+. We obtain a homomorphism

$$K_{m-1}(\operatorname{Spec} \overline{B_\sigma}) \to K_{m-1}(X_{\sigma_0}) \to \cdots \to K_0(X_{\sigma_{m-1}}) \cong \mathbb{Z}.$$

where the first map is the pullback map and each of the rest is the boundary map (See Section 7.2.5 for the notation X_σ). Thus we obtain a $\mathrm{GL}_d(F_\infty)$-equivariant homomorphism

$$\mathrm{reg}_{\mathfrak{X},m} : K_m(\mathfrak{X}) \to \mathrm{Map}(\mathcal{BT}_{m-1,*}, \mathbb{Z}),$$

as the composition of the map above with the map (7.2), which we call the (m-th) regulator for $\mathfrak{X}$.

7.3.5. In this section, we show that the image of the regulator map is contained in the space of harmonic cochains.

Let $m \ge 0$ be an integer. Let $\sigma \in \mathcal{BT}_i$ with $i \le m-1$. For $j = 0, \dots, i$, let us take a j-simplex $\sigma_j \in \mathcal{BT}_j$ such that $\sigma_i = \sigma$ and σ_j is a face of σ_{j+1} for $j = 0, \dots, i-1$. We obtain a sequence of homomorphisms

$$K_{m-1}(X_{\sigma_0}) \to K_{m-2}(X_{\sigma_1}) \to \cdots \to K_{m-1-i}(X_\sigma)$$

where each arrow is the boundary map in a localization sequence. We consider the map $K_{m-1}(\operatorname{Spec} \overline{B}_\sigma) \to K_{m-1-i}(X_\sigma)$ obtained as the composite of the map above with the pullback map $K_{m-1}(\operatorname{Spec} \overline{B}_\sigma) \to K_{m-1}(X_{\sigma_0})$.

Lemma 7.1. *The homomorphism $K_{m-1}(\operatorname{Spec} \overline{B}_\sigma) \to K_{m-1-i}(X_\sigma)$ defined above is independent, up to sign, of the choice of σ_j's.*

Proof. Let σ_j, σ'_j $(0 \le j \le i)$ be two choices. We claim that the two homomorphisms $G_{m-1}(\overline{B}_\sigma) \to G_{m-1-i}(X_\sigma)$ for σ_j's and for σ'_j's coincide. We may assume that there exists j_0 with $0 \le j_0 \le i-1$ such that $\sigma_{j_0} \ne \sigma'_{j_0}$ and $\sigma_j = \sigma'_j$ for $j \ne j_0$. We put $Y = X_{\sigma_{j_0-1}}$ if $j_0 \ge 1$ and put $Y = \operatorname{Spec} B_\sigma$ if $j_0 = 0$. Let X denote the locally closed reduced subscheme of $\operatorname{Spec} \overline{B}_\sigma$ whose underlying set is $X_{\sigma_{j_0}} \cup X_{\sigma'_{j_0}} \cup X_{\sigma_{j_0+1}}$. Then the connecting homomorphism $K_{m-j_0}(Y) \to K_{m-j_0-1}(X_{\sigma_{j_0}})$ is equal to the composition of the connecting homomorphism $K_{m-j_0}(Y) \to K_{m-j_0-1}(X)$

and the restriction $K_{m-j_0-1}(X) \to K_{m-j_0-1}(X_{\sigma_{j_0}})$ with respect to the open immersion $X_{\sigma_{j_0}} \subset X$. Similar fact also holds for the connecting homomorphism $K_{m-j_0}(Y) \to K_{m-j_0-1}(X_{\sigma'_{j_0}})$. Since the two homomorphisms

$$K_{m-j_0-1}(X) \to K_{m-j_0-1}(X_{\sigma_{j_0}}) \to K_{m-j_0-2}(X_{\sigma_{j_0+1}})$$

and

$$K_{m-j_0-1}(X) \to K_{m-j_0-1}(X_{\sigma'_{j_0}}) \to K_{m-j_0-2}(X_{\sigma_{j_0+1}})$$

differ by the multiplication by -1, the claim follows. $\square$

Lemma 7.2. *The image of* $\mathrm{reg}_{\mathfrak{X},d-1}$ *is contained in the space of harmonic cochains.*

Proof. Let $\kappa \in K_d(\mathfrak{X})$ and consider $f = \mathrm{reg}_{\mathfrak{X},d-1}(\kappa)$. We need to verify Conditions (1)(2) of Section 5.2.2. Condition (1) follows from Lemma 7.1. To verify Condition (2), let us use the notation τ_+ and τ_+^i there. Let $\sigma_j \in \mathcal{BT}_j$ be a j-simplex for $0 \le j \le d-2$ such that $\sigma_{d-2} = \tau$ and σ_j is a face of σ_{j+1}. Then using Lemma 7.1, we see that the value $(\mathrm{reg}_{\mathfrak{X}}\kappa)(\tau_+^i)$ is, up to sign (but the same sign for all i), the image of κ under the map

$$K_{d+1}(\mathfrak{X}) \to K_d(X_{\sigma_0}) \to K_{d-1}(X_{\sigma_1}) \to \cdots \to K_1(X_{\sigma_{d-1}}) \xrightarrow{\partial_i} K_0(X_{\tau_i})$$

where the first map is the composition of the restriction map and the boundary map, and the rest are boundary maps; we labeled the last map for later use. We note that $X_{\sigma_{d-1}} = X_\sigma$ is the projective line minus all the κ_∞-rational points, and the set $\{X_{\tau_j}\}_{0\le j\le q}$ is the set of all the κ_∞-rational points of X_σ. By explicit computation, we know that $K_1(X_\sigma) \cong \mathcal{O}(X_\sigma)^\times$ and $K_0(X_{\tau_j}) \cong \mathbb{Z}$ for all j. Furthermore, the boundary map ∂_j sends a unit $f \in \mathcal{O}(X_\sigma)^\times$ to its order at X_{τ_j}. Thus the sum is zero as claimed. $\square$

7.4. Regulator on symbols.

7.4.1. We define a map $\mathrm{reg}_u : \mathcal{O}(\mathfrak{X})^{\times\otimes d} \to \mathrm{Hom}(\mathcal{BT}_{d-1,*}, \mathbb{Z})$ as follows. Let $f_1 \otimes \cdots \otimes f_d \in \mathcal{O}(\mathfrak{X})^{\times\otimes d}$ and $((L_i)_{i\in\mathbb{Z}}, L_0) \in \mathcal{BT}_{d-1,*}$ be a pointed $(d-1)$-simplex. Then we put

$$(\mathrm{reg}_u(f_1 \otimes \cdots \otimes f_d))((L_i)_{i\in\mathbb{Z}}, L_0) = \det \begin{pmatrix} \mathrm{ord}_{L_0} f_1 & \dots & \mathrm{ord}_{L_{d-1}} f_1 \\ \vdots & \ddots & \vdots \\ \mathrm{ord}_{L_0} f_d & \dots & \mathrm{ord}_{L_{d-1}} f_d \end{pmatrix},$$

where the symbol ord is as in Section 3.2.5.

We have a symbol map $\mathrm{sym} : \mathcal{O}(-)^{\times\otimes d} \to K_d(-)$ which is obtained from the symbol map $\mathcal{O}(-)^\times \to K_1(-)$ (Section 7.1.4) and the product structure $K_1(-)^{\otimes d} \to K_d(-)$ (Section 7.1.1).

Proposition 7.3. *Let the notation be as above. We have* $\mathrm{reg}_u = \mathrm{reg}_{\mathfrak{X}} \circ \mathrm{sym}$.

Let $\sigma_+ = ((L_i)_{i\in\mathbb{Z}}, L_0) \in \mathcal{BT}_{d-1,*}$. We let $\sigma_j = (\cdots \supsetneq L_0 \supsetneq \cdots \supsetneq L_j \supsetneq L_d \supsetneq \dots) \in \mathcal{BT}_j$ (as in Section 7.3.4). Then by definition, $\mathrm{reg} \circ \mathrm{sym}(F)$ evaluated at σ_+ is the image of the element F under the composition

$$\begin{aligned} K_d(\mathfrak{X}) \xrightarrow{(1)} K_d(\mathrm{Spec}\, B_\sigma) \xrightarrow{\partial_d} K_{d-1}(\mathrm{Spec}\, \overline{B}_\sigma) \xrightarrow{(2)} K_{d-1}(X_{\sigma_0}) \\ \xrightarrow{\partial_{d-1}} \dots \xrightarrow{\partial_1} K_0(X_{\sigma_{d-1}}) \cong \mathbb{Z} \end{aligned} \tag{7.3}$$

where the map (1) is the map (7.1), the map (2) is the pullback map, and each ∂_j for $1 \le j \le d$ is the boundary map in a localization sequence.

7.4.2. Let $\sigma \in \mathcal{BT}_{d-1}$ be the class of σ_+. We use the notation in Section 7.2.2.

Let $\widehat{R}_\sigma$ denote the completion of R_σ at the maximal ideal $(T_0, \ldots, T_{d-1})$. Let $\mathfrak{p}_j = (T_j, \ldots, T_{d-1}) \subset \widehat{R}_\sigma$ for $j = 0, \ldots, d-1$ and $\mathfrak{p}_d = (0)$. Each $\mathfrak{p}_j$ is a prime ideal; we let $\kappa(\mathfrak{p}_j)$ denote the field of fractions of $\widehat{R}_\sigma/\mathfrak{p}_j\widehat{R}_\sigma$. For $j = 0, \ldots, d-1$, note that $\widehat{R}_{\sigma,\mathfrak{p}_j}/\mathfrak{p}_{j+1}\widehat{R}_{\sigma,\mathfrak{p}_j}$ is a discrete valuation ring. Its field of fractions is $\kappa(\mathfrak{p}_{j+1})$ and the residue field is $\kappa(\mathfrak{p}_j)$. We obtain a homomorphism

$$K_d(\operatorname{Spec}\kappa(\mathfrak{p}_d)) \to K_{d-1}(\operatorname{Spec}\kappa(\mathfrak{p}_{d-1})) \to \cdots \to K_0(\operatorname{Spec}\kappa(\mathfrak{p}_0)) = \mathbb{Z} \tag{7.4}$$

where each map is the boundary map in the localization sequence associated to the triple $\operatorname{Spec}\kappa(\mathfrak{p}_{j+1}) \subset \operatorname{Spec}\widehat{R_\sigma}/\mathfrak{p}_j\widehat{R_\sigma} \supset \operatorname{Spec}\kappa(\mathfrak{p}_j)$.

For $j = 0, \ldots, d-1$, the element $T_{d-j-1} \in \widehat{R}_\sigma$ is a prime element of the regular noetherian ring R. Hence the localization $\widehat{R}_{\sigma,(T_{d-j-1})}$ of $\widehat{R}_\sigma$ at the prime ideal (T_{d-j-1}) is a discrete valuation ring. Let v'_j denote the valuation $(\operatorname{Frac}\widehat{R}_\sigma)^\times \to \mathbb{Z}$ given by the discrete valuation ring $\widehat{R}_{\sigma,(T_{d-j-1})}$.

Lemma 7.4. *Let $h_1, \ldots, h_d \in \widehat{R}_\sigma[1/\varpi_\infty]^\times$. The image of $h_1 \otimes \cdots \otimes h_d \in (\widehat{R}_\sigma[1/\varpi_\infty]^\times)^{\otimes d}$ under the map $(\widehat{R}_\sigma[1/\varpi_\infty]^\times)^{\otimes d} \to (\kappa(\mathfrak{p}_d)^\times)^{\otimes d} \to K_d(\operatorname{Spec}\kappa(\mathfrak{p}_d)) \xrightarrow{(1)} \mathbb{Z}$, where the map (1) is* (7.4)*, equals* $\det(v'_{j-1}(h_i))_{1\le i,j\le d}$.

Proof. Note that $\widehat{R}_\sigma[1/\varpi_\infty]^\times$ is generated by $\widehat{R}_\sigma^\times$ and $T_0, \ldots, T_{d-1}$. Hence the claim above follows by using the computation of boundary maps of K-theory in localization sequences of Gillet described in [Gi, Theorem 7.21, p.274]. Note also that since the target group $\mathbb{Z}$ is torsion free, the 2-torsion appearing in the formula of Gillet may be ignored. $\square$

7.4.3. Let v_j denote the class of L_j in $\mathcal{BT}_0$. Then the composite $B^o_\sigma \to \widehat{R} \to \widehat{R}_{(T_{d-j})}$ factors through the canonical homomorphism $B^o_\sigma \to B^o_{v_j}$. Let $\xi_j : B^o_{v_j} \to R_{(T_{d-j})}$ denote the induced homomorphism.

Lemma 7.5. *the diagram*

$$\begin{array}{ccccc}
\mathcal{O}(\mathfrak{X})^\times & \longrightarrow & B^o_{v_j}[1/\varpi_\infty]^\times & \xrightarrow{\xi_j} & \widehat{R}_{(T_{d-j})}[1/\varpi_\infty]^\times \\
{\scriptstyle \operatorname{ord}_{L_j}}\big\downarrow & & {\scriptstyle v_j}\big\downarrow & & {\scriptstyle v'_j}\big\downarrow \\
\mathbb{Z} & = & \mathbb{Z} & = & \mathbb{Z}
\end{array}$$

is commutative.

Proof. Since $\widetilde{B}^o_{L_j,(\varpi_\infty)} \to B^o_{v_j,(\varpi_\infty)} \to \widehat{R}_{(T_{d-j})}$, where $\widetilde{B}^o_{L_j}$ is as in Section 3.2.5, are the homomorphism of discrete valuation rings with uniformizer ϖ_∞, the commutativity follows from Lemma 7.6 below. $\square$

Lemma 7.6. *Let L be an $\mathcal{O}_\infty$-lattice of V_∞ and let $v \in \mathcal{BT}_0$ denote its class. Then for $f \in \mathcal{O}(\mathfrak{X})^\times$ we have $\operatorname{ord}_L(f) = v(f)$, where in the left hand side we regard f as an element in $\mathcal{O}(\widetilde{\mathfrak{X}})^\times$ via the pullback by the canonical quotient map $\widetilde{\mathfrak{X}} \to \mathfrak{X}$.*

Proof. This follows from the definitions given in Sections 3.2.5, 7.2.6. $\square$

7.4.4. *Proof of Proposition 7.3.* One can check that the homomorphism (7.3) equals the composition

$$K_d(\mathfrak{X}) \to K_d(\operatorname{Spec} B_\sigma) \xrightarrow{(1)} K_d(\operatorname{Spec} \kappa(\mathfrak{p}_0)) \xrightarrow{(2)} K_0(\operatorname{Spec} \kappa(\mathfrak{p}_d)) \cong \mathbb{Z}$$

where the map (1) is the pullback map and the map (2) is the map (7.4). Then the claim follows from Lemma 7.4 and Lemma 7.5. □

7.5. **Regulator for Drinfeld modular varieties.**

Let $L_1 \subset L_2 \subset V^\infty$ be $\widehat{A}$-lattices. Using the uniformization of Drinfeld modular varieties (see Section 3.2.7), we obtain a homomorphism, which is denoted reg_{L_1,L_2}, as the composite

$$\begin{aligned} &K_d(\mathcal{M}^d_{L_2/L_1,F}) \xrightarrow{(1)} K_d(\mathcal{M}^d_{L_2/L_1,F} \times_{\operatorname{Spec} F} \operatorname{Spec} F_\infty) \\ &\xrightarrow{(2)} \mathrm{Map}_{\mathrm{GL}_d(F)}(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}, K_d(\mathfrak{X})) \\ &\xrightarrow{(3)} \mathrm{Map}_{\mathrm{GL}_d(F)}(\mathcal{BT}_{d-1,*} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}, \mathbb{Z}) \xrightarrow{(4)} (\mathcal{A}_1)^{\mathbb{K}^\infty_{L_1,L_2}}, \end{aligned}$$

where the map (1) is the pullback map, the map (2) is the analytification, the map (3) is the map induced by $\mathrm{reg}_{\mathfrak{X}}$, and the map (4) is the inclusion. We write $\mathrm{reg} : K_d(\mathcal{M}^d) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathcal{A}_1$ for the map induced by the limit $\varinjlim_{(L_1,L_2)} \mathrm{reg}_{L_1,L_2}$.

Again using the uniformization, we define reg_{u,L_1,L_2} as the composite

$$\begin{aligned} &\mathcal{O}(\mathcal{M}^d_{L_2/L_1,F})^{\times \otimes d} \xrightarrow{(1)} \mathrm{Map}_{\mathrm{GL}_d(F)}(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}, \mathcal{O}(\mathfrak{X})^{\times \otimes d}) \\ &\xrightarrow{(2)} \mathrm{Map}_{\mathrm{GL}_d(F)}(\mathcal{BT}_{d-1,*} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{L_1,L_2}, \mathbb{Z}) \xrightarrow{(3)} (\mathcal{A}_1)^{\mathbb{K}^\infty_{L_1,L_2}} \end{aligned}$$

where the map (1) is the analytification using the uniformization, the map (2) is the map obtained using reg_u, and the map (3) is the inclusion.

The following is a corollary to Proposition 7.3. We omit the proof.

Corollary 7.7. *Let the notation be as above. We have an equality of two maps:*

$$\mathrm{reg}_{u,L_1,L_2} = \mathrm{reg}_{L_1,L_2} \circ \mathrm{sym} : \mathcal{O}(\mathcal{M}^d_{L_2/L_1,F})^{\times \otimes d} \to (\mathcal{A}_1)^{\mathbb{K}^\infty_{L_1,L_2}}.$$

□

8. Zeta value formula

The aim of this section is to prove Theorem 8.2. This states that the image of the regulator map is expressed in terms of the L-function. This may be regarded as the function field analogue of Kato's refinement ([Ka, p.127, Theorem 2.6]) of Beilinson's theorem [Be, Theorem 5.1.2].

Recall that we defined a homomorphism κ in Section 2.4.5 and a homomorphism $\mathcal{E}$ in Section 6.1.1. We defined a homomorphism reg in Section 7.5.

Lemma 8.1. *Let $\Phi^\infty \in \mathcal{S}'(V^\infty)^{\otimes d}$. Then*

$$\mathrm{reg}(\kappa(\Phi^\infty)) = \lim_{s \to 0} \frac{1}{(1 - q_\infty^{-s})^d} \mathcal{E}(\Phi^\infty),$$

where

$$\mathrm{reg} : K_d(\mathcal{M}^d) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathcal{A}_1$$

is the regulator map defined in Section 7.5.

Proof. This follows from Lemma 6.1 and Proposition 3.4 using Corollary 7.7. □

Theorem 8.2. *Let $J \subset I$ be nonzero ideals of A. Let $f \in \mathcal{A}^{\mathrm{o}}_{\mathbb{C}}(I, J, 1)$ be a cusp form satisfying Conditions (1)(2) of Section 4.2.2. Let L_1 and L_2 be $\widehat{A}$-lattices as defined in Section 2.4.7. Let $\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)$. Then*

$$\langle f, \mathrm{reg}_{L_1,L_2}(\kappa_{I,J,\gamma})\rangle = \frac{1}{\log\, q_\infty} \lim_{s\to 0} \frac{\partial}{\partial s} L^{I,J}(f, s - \frac{d-1}{2})\langle f, \eta_{\mathbb{K}^\infty_{I,J},\gamma}\rangle.$$

Proof. Divide both sides of the equation in Theorem 6.3 by $(1 - q_\infty^{-s})^d$. Then compute the limit as s tends to 0 using Lemma 8.1. The claim follows. $\square$

Remark 8.3. *Theorem 8.2 describes the value at $s = -\frac{d-1}{2}$ of the first derivative of the L-function $L(f, s)$. We note that when $L(f, -\frac{d-1}{2}) \neq 0$, then the right hand side of the formula of Theorem 8.2 is zero by Corollary 6.4.*

9. Nontriviality

The aim of this section is to prove Theorem 9.1. This states that the cusp form part of the image of the regulator map of (the limit of) Drinfeld modular varieties contains the space $\mathcal{A}^{\mathrm{o}}_{\mathrm{St}}$ (defined below). This may be regarded as the function field analogue of the surjectivity of the regulator map of the Beilinson conjectures.

The proof uses the zeta value formula (Theorem 8.2), Corollary 5.16, and some standard results from the theory of automorphic forms.

9.1. **The image of the regulator map.**

We defined an element $[\iota(\sigma_0)] \in H^{d-1}_c(\mathcal{BT}_\bullet, \mathbb{C})$ in Section 5.2.3. We also write $[\iota(\sigma_0)]$ for the corresponding element in St_d under the isomorphism (see Section 5.2.3) of Borel. Note that it is an Iwahori spherical vector, i.e., a nonzero element which is invariant under the action of the Iwahori subgroup $\mathcal{I}$. We let $\mathcal{A}_{\mathrm{St}}$ denote the image of the map $\mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d, \mathcal{A}_1) \to \mathcal{A}_1$ given by the evaluation at $[\iota(\sigma_0)]$.

We put $\mathcal{A}^{\mathrm{o}}_{\mathrm{St}} = \mathcal{A}_{\mathrm{St}} \cap \mathcal{A}^{\mathrm{o}}_1$. Let us consider the composite of the projection $P^{\mathrm{o}} : \mathcal{A} \to \mathcal{A}^{\mathrm{o}}$ defined in Section 4.3 and the homomorphism in the statement of Corollary 5.7. We claim that the group $\mathcal{A}^{\mathrm{o}}_{\mathrm{St}}$ is equal to the image of this composite map. Note that the center $F_\infty^\times$ of $\mathrm{GL}_d(F_\infty)$ acts trivially on St_d. It follows that the image of the latter homomorphism is contained in $\mathcal{A}_1$. Hence by Corollary 5.7, it is also equal to the image of the composite with the projection P^{o} of the homomorphism (5.9).

Theorem 9.1. *Let the notation be as above. Then the image of the composite map*

$$K_d(\mathcal{M}^d) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\mathrm{reg}} \mathcal{A}_1 \xrightarrow{P^{\mathrm{o}}} \mathcal{A}^{\mathrm{o}}_1,$$

equals $\mathcal{A}^{\mathrm{o}}_{\mathrm{St}}$.

Remark 9.2. Although we omit the proof, it is not difficult to show that the homomorphisms reg and P^{o} are defined over $\mathbb{Q}$ in the sense that we have a sequence of homomorphisms

$$K_d(\mathcal{M}^d) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\mathrm{reg}_{\mathbb{Q}}} \mathcal{A}_1 \cap \mathcal{A}_{\mathbb{Q}} \xrightarrow{P^{\mathrm{o}}_{\mathbb{Q}}} \mathcal{A}^{\mathrm{o}}_1 \cap \mathcal{A}_{\mathbb{Q}},$$

which gives the sequence in Theorem 9.1 after tensoring by $\mathbb{C}$. We have a corollary that the image of the sequence above equals $\mathcal{A}^{\mathrm{o}}_{\mathrm{St}} \cap \mathcal{A}_{\mathbb{Q}}$. We also remark that the variant of Corollary 1.2 holds.

9.2. **Proof of Theorem 9.1.**

Using Lemma 7.2 and Corollary 5.5, we see that the image of the regulator map reg is contained in $\mathcal{A}_{\mathrm{St}}$. It remains to show that $P^{\mathrm{o}} \circ \mathrm{reg}$ surjects onto $\mathcal{A}^{\mathrm{o}}_{\mathrm{St}}$.

Let $\pi = \otimes'_v \pi_v \subset \mathcal{A}^{\mathrm{o}}_1$ be an irreducible cuspidal automorphic representation such that π_∞ is isomorphic to the Steinberg representation St_d. By [Sh, p.190, COROLLARY], the representation π_v is generic for each v. Thus we can take a nonzero $f = \otimes_v f_v \in \pi$ such that f_v is a new vector for $v \neq \infty$ and f_∞ is an Iwahori spherical vector. Let I_π be the prime-to-∞-part of the conductor of π. Then it is known that $f \in \pi^{\mathbb{K}^\infty_{A,I_\pi}}$. We note that f satisfies Condition (3) of Section 4.2.2.

We consider the map

$$K_d(\mathcal{M}^d) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\mathrm{reg}} \mathcal{A}_1 \xrightarrow{\langle f, - \rangle} \mathbb{C}. \tag{9.1}$$

Lemma 9.3. *If the homomorphism (9.1) is nonzero for every irreducible cuspidal automorphic representation $\pi = \otimes'_v \pi_v \subset \mathcal{A}^{\mathrm{o}}_1$ such that $\pi_\infty \cong \mathrm{St}_d$, then Theorem 9.1 holds.*

Proof. As is well known, there is a direct sum decomposition $\mathcal{A}^{\mathrm{o}}_1 = \oplus_{\pi'} \pi'$ where $\pi' \subset \mathcal{A}_1$ runs over the irreducible automorphic representations contained in $\mathcal{A}^{\mathrm{o}}_1$. The multiplicity one theorem says that no two direct summands are isomorphic.

Let $M = \mathrm{Image}[P^{\mathrm{o}} \circ \mathrm{reg}] \cap \mathcal{A}^{\mathrm{o}}_{\mathrm{St}}$. One can check that the maps P^{o} and reg are $\mathrm{GL}_d(\mathbb{A}^\infty)$-equivariant. So $M \subset \mathcal{A}^{\mathrm{o}}_{\mathrm{St}}$ is a $\mathrm{GL}_d(\mathbb{A}^\infty)$-submodule. From the assumption of the lemma, we have, for any irreducible automorphic representation $\pi = \otimes'_v \pi_v \subset \mathcal{A}^{\mathrm{o}}_1$ such that $\pi_\infty \cong \mathrm{St}_d$, the anti-linear map

$$M \subset \mathcal{A}^{\mathrm{o}}_1 \xrightarrow{\langle f, - \rangle} \mathbb{C}$$

where f is constructed from π as above, is nonzero. Let $h = (h_{\pi'})_{\pi'} \in M \subset \mathcal{A}^{\mathrm{o}}_{\mathrm{St}}$ be an element which is not in the kernel of the map above. Take a sufficiently small compact open subgroup $\mathbb{K}^\infty \subset \mathrm{GL}_d(\widehat{A})$ such that all $h_{\pi'}$ are fixed under $\mathbb{K}^\infty$.

From the strong multiplicity one theorem, it follows that one can find a Hecke operator T (for $\mathbb{K}^\infty$) such that $T = \mathrm{id}$ on $\pi^{\mathbb{K}^\infty}$ and $T = 0$ on ${\pi'}^{\mathbb{K}^\infty}$ for π' such that $\pi' \neq \pi$ and $h_{\pi'} \neq 0$. One applies this T to h as an element in $M^{\mathbb{K}^\infty}$, and sees that $h_\pi \in M$. The claim follows. $\square$

Lemma 9.4. *The map (9.1) is nonzero.*

Proof. We choose an auxiliary prime v_0 as follows. If $d = 1$ and π is the trivial representation, we fix an arbitrary finite prime v_0. Otherwise, we fix a finite prime v_0 satisfying the following condition: The local L-factor $L(\pi_{v_0}, s)$ does not have a pole at $s = -(d-1)/2$. The existence of such v_0 is obvious when $d = 1$, and follows from [Ja-Sh, p. 515, (2.5)] when $d \geq 2$.

We obtain the following isomorphism

$$\mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d, \mathcal{A}_{\mathbb{C}}(v_0, I_\pi v_0, 1)) \cong H^{\mathrm{BM}}_{d-1}(X_{\mathbb{K}^\infty_{v_0, I_\pi v_0}, \bullet}, \mathbb{C})$$

using Corollary 5.6. There is an element in the left hand side which sends $\iota(\sigma_0) \in \mathrm{St}_d$ to f since f_∞ is an Iwahori spherical vector. As the support of f is compact modulo center, the corresponding element on the right hand side lies in the image of the canonical map $H_{d-1}(X_{\mathbb{K}^\infty_{v_0, I_\pi v_0}, \bullet}, \mathbb{C}) \to H^{\mathrm{BM}}_{d-1}(X_{\mathbb{K}^\infty_{v_0, I_\pi v_0}, \bullet}, \mathbb{C})$. Using

Corollary 5.16, we can express f as a sum

$$f = \sum_{\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)} a_\gamma \eta_{v_0, I_\pi v_0, \gamma} \in \mathcal{A}_{\mathbb{C}}(v_0, I_\pi v_0, 1)$$

with $a_\gamma \in \mathbb{C}$. We let

$$\kappa_f = \sum_{\gamma \in \mathrm{GL}_d(\mathbb{A}^\infty)} a_\gamma \kappa_{v_0, I_\pi v_0, \gamma} \in K_d(\mathcal{M}^d_{v_0, I_\pi v_0, F}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Using Lemma 4.1, it follows from Theorem 8.2 that

$$\langle f, \operatorname{reg} \kappa_f \rangle = \frac{1}{\log q_\infty} \lim_{s \to 0} \frac{\partial}{\partial s} L^{\{\infty, v_0\}}(\pi, s - \frac{d-1}{2}) \langle f, f \rangle,$$

where $L^{\{\infty, v_0\}}(\pi, s)$ is the global L-function of the representation π without the local factors at ∞ and at v_0. If $d = 1$ and π is the trivial representation, the global L-function $L(\pi, s) = L^{\{\infty, v_0\}}(\pi, s) L(\pi_{v_0}, s) L(\pi_\infty, s)$ has a simple pole at $s = 0$. Otherwise, by [Ja-Sh, p. 557, (5.4)], $L(\pi, s)$ has neither a pole nor a zero at $s = -(d-1)/2$. Since $L(\pi_\infty, s) = L(\mathrm{St}_d, s) = (1 - q_\infty^{-s + \frac{d-1}{2}})^{-1}$, it follows with our choice of v_0 that $\lim_{s \to 0} \frac{\partial}{\partial s} L^{\{\infty, v_0\}}(\pi, s - \frac{d-1}{2})$ is nonzero. □

This completes the proof of Theorem 9.1. □

Proof of Theorem 1.2. Recall that we defined the action of $\mathrm{GL}_d(\mathbb{A}^\infty)$ on the moduli space $\mathcal{M}^d$ in Section 2.4.2. Take the $\mathbb{K}^\infty_{L, \widehat{A}^{\oplus d}}$-invariant part of the statement of Theorem 9.1 as we have done in Section 2.4.7. Then use the étale descent of rational K-theory to conclude. □

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