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Reflection Positivity of $\mathcal{N} = 1$ Wess-Zumino model on the lattice with exact U(1)$_R$ symmetry

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Abstract: By using overlap Majorana fermions, the $\mathcal{N} = 1$ chiral multiple can be formulated so that the supersymmetry is manifest and the vacuum energy is cancelled in the free limit, thanks to the bilinear nature of the free action. It is pointed out, however, that in this formulation the reflection positivity is violated in the bosonic part of the action, although it is satisfied in the fermionic part. It is found that the positivity of the spectral density of the bosonic two-point correlation function is ensured only for the spacial momenta $a|p_k| \lesssim 1.72$ $(k = 1, 2, 3)$. It is then argued that in formulating $\mathcal{N} = 1$ Wess-Zumino model with the overlap Majorana fermion, one may adopt a simpler nearest-neighbor bosonic action, discarding the free limit manifest supersymmetry. The model still preserves the would-be U$(1)_R$ symmetry and satisfies the reflection positivity.

Keywords: overlap fermion, reflection positivity, admissibility condition.

Contents

1. Introduction

The chiral multiplet of $\mathcal{N}=1$ supersymmetry[1] can be formulated on the lattice so that the supersymmetry is preserved and the vacuum energy is cancelled in the free limit, thanks to the bilinear nature of the free action. By using overlap (Majorana) fermion [2, 3, 4] for the fermionic component, species doublers[5, 6, 7] are successfully removed and $\mathrm{U}(1)_R$ symmetry can be maintained at the same time[8, 9, 10]. With this chiral multiplet, one may formulate lattice $\mathcal{N}=1$ Wess-Zumino model with exact $\mathrm{U}(1)_R$ symmetry[11, 12, 13, 14, 15, 16]. A numerical study of this lattice $\mathcal{N}=1$ Wess-Zumino model has recently been reported in [17].

The purpose of this short article is, however, to show that in this formulation of the chiral multiplet, the reflection positivity[18, 19, 20, 21, 22] is violated in the bosonic part of the action, although it is satisfied in the fermionic part, as shown recently in [23]. We will also examine the spectral density of the bosonic two-point correlation function (cf. [24]). It is found that the positivity of the spectral density is ensured only for the momenta $a|p_k| \lesssim 1.72$ $(k=1,2,3)$, and the mode with a negative density appears at the energy as low as $aE \simeq 0.69$ for the momenta $a\boldsymbol{p}=(\pi,0,0),(0,\pi,0),(0,0,\pi)$.

We will then argue that in formulating the lattice $\mathcal{N}=1$ Wess-Zumino model with the overlap (Majorana) fermion, one may adopt the simpler nearest-neighbor bosonic action, discarding the free limit manifest supersymmetry. The model so constructed still preserves the $\mathrm{U}(1)_R$ symmetry and satisfies the reflection positivity.

This paper is organized as follows. In section 2, we review briefly the $\mathcal{N}=1$ chiral multiple on the lattice formulated with overlap Majorana fermion. In section 3, we show that the reflection positivity is not fulfilled in the bosonic part of the action. The spectral density of the bosonic two-point correlation function is also examined. In section 4, we show that it is possible to formulate lattice Wess-Zumino model which possesses both the reflection positivity and the exact $\mathrm{U}(1)_R$ symmetry, by adopting the simpler nearest-neighbor bosonic action. Section 5 is devoted to discussion.

2. $\mathcal{N}=1$ chiral multiple with overlap Majorana fermion

The action of the free $\mathcal{N}=1$ chiral multiplet is given by

$$S_0 = a^4 \sum_x \Big\{ \frac{1}{2}\chi^T C D_1 \chi + \phi^* D_1^2 \phi + F^* F + \frac{1}{2}\chi^T C D_2 \chi + F D_2 \phi + F^* D_2 \phi^* \Big\}. \tag{2.1}$$

In this expression, we have used a decomposition of the overlap Dirac operator [2][3], $D = D_1 + D_2$, where

$$D_1 = \frac{1}{2}\gamma_\mu(\partial_\mu^* + \partial_\mu)(A^\dagger A)^{-1/2}, \qquad D_2 = \frac{1}{a}\Big\{1 - (1 + \frac{1}{2}a^2 \partial_\mu^* \partial_\mu)(A^\dagger A)^{-1/2}\Big\}, \tag{2.2}$$

and

$$A = 1 - aD_{\mathrm{w}}, \qquad D_{\mathrm{w}} = \frac{1}{2}\Big\{\gamma_\mu(\partial_\mu^* + \partial_\mu) - a\partial_\mu^* \partial_\mu\Big\}. \tag{2.3}$$

Note that D_1 and D_2 have different spin structures with respect to spinor space. In particular, we have $\{\gamma_5, D_1\} = 0$ and $[\gamma_5, D_2] = 0$. In terms of this decomposition, the Ginsparg-Wilson relation $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$ [8] is expressed as

$$2D_2 = a(-D_1^2 + D_2^2), \tag{2.4}$$

and as a consequence, we have relations

$$\gamma_5(1 - \frac{1}{2}aD)\gamma_5(1 - \frac{1}{2}aD) = 1 - \frac{1}{2}aD_2, \qquad \gamma_5(1 - \frac{1}{2}aD)\gamma_5 D = D_1. \tag{2.5}$$

It is also understood that the 4×4 identity matrix in operators D_1^2 and D_2 is omitted when these operators are acting on bosonic fields.

It is straightforward to see that the above free action S_0 is invariant under "lattice $\mathcal{N}=1$ supersymmetry":

$$\begin{aligned}
&\delta_\epsilon \chi = -\sqrt{2}P_+(D_1\phi + F)\epsilon - \sqrt{2}P_-(D_1\phi^* + F^*)\epsilon, \\
&\delta_\epsilon \phi = \sqrt{2}\epsilon^T C P_+ \chi, \qquad \delta_\epsilon \phi^* = \sqrt{2}\epsilon^T C P_- \chi, \\
&\delta_\epsilon F = \sqrt{2}\epsilon^T C D_1 P_+ \chi, \qquad \delta_\epsilon F^* = \sqrt{2}\epsilon^T C D_1 P_- \chi,
\end{aligned} \tag{2.6}$$

where ϵ is a 4 component Grassmann parameter. We also note that the free action S_0 possesses three types of U(1) symmetry [10]. The first is a rather trivial one acting only on bosonic fields and is defined by the transformation:

$$\begin{aligned}
&\delta_\alpha \chi = 0, \\
&\delta_\alpha \phi = i\alpha\phi, \\
&\delta_\alpha F = -i\alpha F,
\end{aligned} \tag{2.7}$$

where α is an infinitesimal real parameter. The second one is nothing but the chiral symmetry introduced by Lüscher,

$$\delta_\alpha \chi = i\alpha\gamma_5(1 - \frac{1}{2}aD)\chi, \tag{2.8}$$

Thirdly, somewhat surprisingly, the *bosonic* sector of S_0 possesses a U(1) symmetry analogous to eq. (2.8):

$$\begin{aligned}
\delta_\alpha \phi &= +i\alpha\{(1 - \frac{1}{2}aD_2)\phi - \frac{1}{2}aF^*\}, \\
\delta_\alpha F &= +i\alpha\{(1 - \frac{1}{2}aD_2)F - \frac{1}{2}aD_1^2\phi^*\}
\end{aligned} \tag{2.9}$$

due to the Ginsparg-Wilson relation. The lattice action S_0 is not invariant under a uniform rotation of the complex phase of bosonic fields, ϕ, F, due to the presence of terms $FD_2\phi$ and $F^*D_2\phi^*$. The above provides a lattice counterpart of this uniform phase rotation of bosonic fields under which the free action S_0 is invariant. Using a linear combination of the above three U(1) symmetries, it is possible to define the $\mathrm{U}(1)_R$ symmetry [10] in the interacting system.

$$\begin{aligned}
\delta_\alpha \chi &= +i\alpha\gamma_5(1 - \frac{1}{2}aD)\chi, \\
\delta_\alpha \phi &= -3i\alpha\phi + i\alpha\{(1 - \frac{1}{2}aD_2)\phi - \frac{1}{2}aF^*\}, \\
\delta_\alpha F &= +3i\alpha F + i\alpha\{(1 - \frac{1}{2}aD_2)F - \frac{1}{2}aD_1^2\phi^*\}.
\end{aligned} \tag{2.10}$$

3. Violation of the reflection positivity in the bosonic part

3.1 Reflection positivity condition

In this subsection, we will formulate the reflection positivity condition. It has been rigorously shown that the lattice theory satisfying the reflection positivity condition corresponds to the quantum theory with unitary time evolution [18, 19, 20]. Here we consider the generic case in which there are both a bosonic field ϕ and a fermionic field ψ. Let us assume that $S(\phi, \psi, \bar{\psi})$ is the action of a lattice model[1] and its partition function Z is given by the path integration

$$Z = \int [\mathcal{D}\phi\mathcal{D}\phi^*][\mathcal{D}\psi\mathcal{D}\bar{\psi}]\, \mathrm{e}^{-S(\phi,\psi,\bar{\psi})}. \tag{3.1}$$

[1] In the following, we write the bosonic field argument of a function like $S(\phi)$ instead of $S(\phi, \phi^*)$ for the notational simplicity. This notation never means that S is an analytic function of ϕ.

We set the lattice spacing a to be unity, and assume the finite volume hypercubic lattice $\Lambda = \{-L+1, -L+2, \ldots, L-1, L\}^d \subset \mathbb{Z}^d$. We impose the *anti*-periodic boundary condition in the time direction for the fermionic field ψ, while the periodic boundary condition for the bosonic field ϕ. In the spacial directions, periodic boundary conditions are imposed for both fields.

To formulate the reflection positivity condition, we first introduce the time reflection operator θ as follows. For each site $x = (t, \boldsymbol{x}) \in \Lambda$, we denote $\theta x = (-t+1, \boldsymbol{x})$. This is the time reflection with respect to the $t = 1/2$ plane. We define the operation of θ for bosonic fields as

$$(\theta\phi)(x) = \phi(\theta x) \tag{3.2}$$

and for functions of bosonic fields $\mathcal{F}(\phi)$ as

$$(\theta\mathcal{F})(\phi) = \mathcal{F}^*(\theta\phi), \tag{3.3}$$

where * means complex conjugation. For fermionic fields, the θ reflection is defined as

$$(\theta\bar{\psi})(x) = \gamma_0\psi(\theta x), \tag{3.4}$$

$$(\theta\psi)(x) = \bar{\psi}(\theta x)\gamma_0. \tag{3.5}$$

We extend this θ operation to the whole field algebra $\mathcal{A}$. We define the field algebra $\mathcal{A}$, the algebra of observables, as the Grassmann algebra generated by the fermionic fields with the coefficients of the continuous functions of bosonic fields which are integrable with respect to the bosonic Gaussian functional measure. For $\mathcal{F}, \mathcal{G} \in \mathcal{A}$, the θ operation is defined by the relations

$$\theta(\mathcal{F}\mathcal{G}) = \theta(\mathcal{G})\theta(\mathcal{F}), \tag{3.6}$$

$$\theta(\alpha\mathcal{F} + \beta\mathcal{G}) = \alpha^*\theta(\mathcal{F}) + \beta^*\theta(\mathcal{G}). \tag{3.7}$$

For instance, if $\mathcal{F}$ has the form of

$$\mathcal{F}(\phi, \psi, \bar{\psi}) = f(\phi)\bar{\psi}_{a_1}(x_1)\ldots\bar{\psi}_{a_n}(x_n)\psi_{b_1}(y_1)\ldots\psi_{b_m}(y_m), \tag{3.8}$$

its θ reflection should be

$$\theta(\mathcal{F})(\phi, \psi, \bar{\psi}) = f^*(\theta\phi)(\bar{\psi}\gamma_0)_{b_m}(\theta y_m)\ldots(\bar{\psi}\gamma_0)_{b_1}(\theta y_1)(\gamma_0\psi)_{a_n}(\theta x_n)\ldots(\gamma_0\psi)_{a_1}(\theta x_1). \tag{3.9}$$

Let Λ_+ (resp. Λ_-) be the set of lattice sites with positive (resp. non-positive) time components, and $\mathcal{A}_\pm$ be the subalgebras of $\mathcal{A}$, which depends only upon fields on $\Lambda_\pm$. In this notation, θ is a map from $\Lambda_\pm$ into $\Lambda_\mp$ and from $\mathcal{A}_\pm$ into $\mathcal{A}_\mp$.

Reflection positivity condition is defined through this θ map. For a lattice theory with the expectation functional $\langle\cdot\rangle$ defined as

$$\langle\mathcal{F}\rangle = \frac{1}{Z}\int[\mathcal{D}\phi\mathcal{D}\phi^*][\mathcal{D}\psi\mathcal{D}\bar{\psi}]\,\mathrm{e}^{-S(\phi,\psi,\bar{\psi})}\mathcal{F}(\phi, \psi, \bar{\psi}), \qquad \mathcal{F} \in \mathcal{A}, \tag{3.10}$$

we say the theory is reflection positive with respect to θ if any function $\mathcal{F}_+ \in \mathcal{A}_+$ fulfills the inequality

$$\langle\theta(\mathcal{F}_+)\mathcal{F}_+\rangle \geq 0. \tag{3.11}$$

3.2 Reflection positivity of the free overlap boson

In this subsection, we investigate the reflection positivity of the bosonic sector of the free chiral multiplet (2.1). It will be shown in the following that the bosonic sector does *not* satisfy the reflection positivity condition. After integrating out the auxiliary field F, we have the overlap boson system which is defined through the lattice action on Λ

$$S_b(\phi) = \sum_{x\in\Lambda} \phi^*(x)\Box_\Lambda \phi(x), \tag{3.12}$$

where we have defined

$$\Box_\Lambda(x,y) = \sum_{n\in\mathbb{Z}^4} \Box(x+2nL,y), \tag{3.13}$$

$$D^\dagger D = \Box\cdot\hat{1}, \tag{3.14}$$

with $\hat{1}$ being the unit spinor matrix. This $\Box$ given above is the bosonic overlap operator on $\mathbb{Z}^d$ and $\Box_\Lambda$ is that on Λ with periodic boundary conditions. The field algebra $\mathcal{A}$ of this overlap boson system is defined as the set of all continuous functions of bosonic field configulations $\phi = \{\phi(x)\}_{x\in\Lambda}$, which are integrable with respect to the bosonic Gaussian measure

$$[\mathcal{D}\phi][\mathcal{D}\phi^*]\,\mathrm{e}^{-S_b(\phi)}. \tag{3.15}$$

The expectation of this theory is defined by the bosonic path integration

$$\langle F\rangle = \frac{1}{Z}\int [\mathcal{D}\phi][\mathcal{D}\phi^*]\,\mathrm{e}^{-S_b(\phi)}\mathcal{F}(\phi), \qquad \mathcal{F}\in\mathcal{A}. \tag{3.16}$$

The standard way of investigating the reflection positivity of lattice field theory is to prove that the action can be written in the form of

$$-S_b(\phi) = B(\phi) + \theta(B)(\phi) + \sum_s \theta(C_s)(\phi)C_s(\phi), \qquad B, C_s\in\mathcal{A}_+, \tag{3.17}$$

where in the third term C_s are elements of $\mathcal{A}_+$ parametrized by some discrete parameter s [20]. To see that the equation (3.17) indeed implies the reflection positivity (3.11), we first note that for an arbitrary $\mathcal{F}_+\in\mathcal{A}_+$,

$$\begin{aligned}\langle\theta(\mathcal{F}_+)\mathcal{F}_+\rangle_0 &:= \int [\mathcal{D}\phi][\mathcal{D}\phi^*]\theta(\mathcal{F}_+)\mathcal{F}_+ \\ &= \int \prod_{x\in\Lambda_+} d\phi(x)d\phi^*(x)\,\mathcal{F}_+(\phi)\cdot\int\prod_{x\in\Lambda_-} d\phi(x)d\phi^*(x)\,\theta(\mathcal{F}_+)(\phi) \\ &= \left|\int\prod_{x\in\Lambda_+} d\phi(x)d\phi^*(x)\,\mathcal{F}_+(\phi)\right|^2 \geq 0.\end{aligned} \tag{3.18}$$

If the action is given in the form of (3.17), we obtain for all $\mathcal{F} \in \mathcal{A}_+$,

$$
\begin{aligned}
&\left\langle \mathrm{e}^{-S_b}\theta(\mathcal{F}_+)\mathcal{F}_+\right\rangle_0 \\
&= \left\langle \mathrm{e}^{B+\theta(B)+\sum_s \theta(C_s)C_s}\theta(\mathcal{F}_+)\mathcal{F}_+\right\rangle_0 \\
&= \left\langle \theta(\mathrm{e}^B)\,\mathrm{e}^B \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\sum_s \theta(C_s)C_s\Big)^n \theta(\mathcal{F}_+)\mathcal{F}_+\right\rangle_0 \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{s_1}\cdots\sum_{s_n}\left\langle \theta(\mathrm{e}^B)\,\mathrm{e}^B\,\theta(C_{s_1})C_{s_1}\ldots\theta(C_{s_n})C_{s_n}\,\theta(\mathcal{F}_+)\mathcal{F}_+\right\rangle_0 \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{s_1}\cdots\sum_{s_n}\left\langle \theta\Big(\mathrm{e}^B C_{s_1}\ldots C_{s_n}\mathcal{F}_+\Big)\,\mathrm{e}^B C_{s_1}\ldots C_{s_n}\mathcal{F}_+\Big)\right\rangle_0 .
\end{aligned}
\tag{3.19}
$$

This last expression is clearly positive from (3.18). This immediately implies the reflection positivity because

$$
\langle\theta(\mathcal{F}_+)\mathcal{F}_+\rangle = \frac{\left\langle \mathrm{e}^{-S_b}\theta(\mathcal{F}_+)\mathcal{F}_+\right\rangle_0}{\langle \mathrm{e}^{-S_b}\rangle_0} \geq 0, \qquad \mathcal{F}_+ \in \mathcal{A}_+. \tag{3.20}
$$

We note that the third term in (3.17) may be given by an integration over a continuous parameter s as

$$
\int ds\,\theta(C_s)(\phi)C_s(\phi), \qquad C_s \in \mathcal{A}_+. \tag{3.21}
$$

This type of the action appears in the case of overlap fermions. See Ref.[23] for detail.

Therefore, to prove the reflection positivity of the 'overlap boson' system reduces to find the decomposition of the action (3.12) into (3.17). We first note that S_b can be written as

$$
\begin{aligned}
S_b = &\sum_{x,y\in\Lambda_+}\phi^*(x)\Box(x,y)\phi(y) + \sum_{x,y\in\Lambda_-}\phi^*(x)\Box(x,y)\phi(y) \\
&+ 2\sum_{x\in\Lambda_+,y\in\Lambda_-}\phi^*(x)\Box(x,y)\phi(y),
\end{aligned}
\tag{3.22}
$$

where $\Box(x,y)$ is the kernel of the operator $\Box$ on Λ. To establish the decomposition (3.17), we should find that (i) the second term is the θ reflection of the first term, and that (ii) the last term is written in the form of

$$
-\int \theta(C_s)C_s\,ds \tag{3.23}
$$

for some $C_s \in \mathcal{A}_+$ parametrized by some parameter s. Note that this second condition is equivalent to say that

$$
-\sum_{x\in\Lambda_+,y\in\Lambda_-}\phi^*(x)\Box(x,y)\phi(y) = \int f(s)\,\theta(C_s)(\phi)\,C_s(\phi)\,ds \tag{3.24}
$$

for some *non-negative* function $f(s)$. In this bosonic system, while (i) holds true, the property (ii) breaks down, as will be shown below.

To show this, we will derive the spectral representation of the kernel $\Box(x,y)$. First, the Fourier transformation $\Box(p)$ is given by:

$$\begin{aligned}\Box(p) &= 1 - \frac{1-\sum_\mu(1-\cos p_\mu)}{\sum_\mu \sin^2 p_\mu + \left[1-\sum_\mu(1-\cos p_\mu)\right]^2} \\ &= 1 + \frac{b(\boldsymbol{p}) - \cos p_0}{\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos p_0}},\end{aligned} \tag{3.25}$$

where

$$a(\boldsymbol{p}) = 1 + \sum_j \sin^2 p_j + b(\boldsymbol{p})^2, \tag{3.26}$$

$$b(\boldsymbol{p}) = \sum_j (1-\cos p_j). \tag{3.27}$$

From this formula we obtain the following representation of the kernel $\Box(x,y)$,

$$\Box(x,y)\Big|_{x_0\neq y_0} = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} I(x_0-y_0), \tag{3.28}$$

where we have defined

$$I(x_0-y_0) = \int \frac{dp_0}{2\pi}\, \mathrm{e}^{ip_0(x_0-y_0)} \frac{b(\boldsymbol{p})-\cos p_0}{\sqrt{a(\boldsymbol{p})-2b(\boldsymbol{p})\cos p_0}}. \tag{3.29}$$

Furthermore, by Cauchy's integration theorem, $I(x)$ defined above can be estimated as

$$\begin{aligned} I(x_0)\Big|_{x_0\neq 0} &= \int \frac{dp_0}{2\pi}\, \mathrm{e}^{ip_0x_0} \frac{b(\boldsymbol{p})-\cos p_0}{\sqrt{a(\boldsymbol{p})-2b(\boldsymbol{p})\cos p_0}} \\ &= \int_{E_1}^{\infty} \frac{dE}{\pi}\, \mathrm{e}^{-|E|x_0} \frac{b(\boldsymbol{p})-\cosh E}{\sqrt{2b(\boldsymbol{p})\cosh E - a(\boldsymbol{p})}},\end{aligned} \tag{3.30}$$

where E_1 is the positive solution of

$$2b(\boldsymbol{p})\cosh E_1 - a(\boldsymbol{p}) = 0. \tag{3.31}$$

Since the kernel of the operator on the finite lattice $\Box_\Lambda(x,y)$ is defined as (3.13), it is straightforward to derive the following spectral representation of $\Box_\Lambda$,

$$\begin{aligned}\Box_\Lambda(x,y)\Big|_{x_0>y_0} &= \frac{1}{(2L)^3}\sum_{\boldsymbol{p}} \mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \sum_{n_0\in\mathbb{Z}} I(x_0+2n_0L-y_0) \\ &= \frac{1}{(2L)^3}\sum_{\boldsymbol{p}} \mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \int_{E_1}^{\infty} \frac{dE}{\pi} \frac{1}{1-\mathrm{e}^{-2EL}} \times \\ &\quad \times \left[\mathrm{e}^{-E(x_0-y_0)} + \mathrm{e}^{-E(2L-x_0+y_0)}\right] \frac{b(\boldsymbol{p})-\cos p_0}{\sqrt{a(\boldsymbol{p})-2b(\boldsymbol{p})\cos p_0}}.\end{aligned} \tag{3.32}$$

The second term represents a finite volume effect.

Now we can see the property (3.24) does not hold. From eq.(3.32), we obtain

$$
\begin{aligned}
&-\sum_{x\in\Lambda_+,y\in\Lambda_-}\phi^*(x)\Box_\Lambda(x,y)\phi(y)\\
&=\frac{1}{(2L)^3}\sum_{\boldsymbol{p}}\int_{E_1}^{\infty}\frac{dE}{\pi}\frac{1}{1-\mathrm{e}^{-2EL}}\frac{\cosh E-b(\boldsymbol{p})}{\sqrt{2b(\boldsymbol{p})\cosh E-a(\boldsymbol{p})}}\times\\
&\quad\times\sum_{x\in\Lambda_+,y\in\Lambda_-}\Big\{\mathrm{e}^{-E(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\phi^*(x)\phi(y)+\mathrm{e}^{-2EL}\mathrm{e}^{+E(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\phi^*(x)\phi(y)\Big\}\\
&=\frac{1}{(2L)^3}\sum_{\boldsymbol{p}}\int_{E_1}^{\infty}\frac{dE}{\pi}\frac{1}{1-\mathrm{e}^{-2EL}}\frac{\cosh E-b(\boldsymbol{p})}{\sqrt{2b(\boldsymbol{p})\cosh E-a(\boldsymbol{p})}}\times\\
&\quad\times\Big\{\theta\big(C_{E,\boldsymbol{p}}\big)(\phi)\big(C_{E,\boldsymbol{p}}\big)(\phi)+\theta\big(e^{-EL}C_{-E,\boldsymbol{p}}\big)(\phi)\big(e^{-EL}C_{-E,\boldsymbol{p}}\big)(\phi)\Big\},
\end{aligned}
\tag{3.33}
$$

where we define

$$
C_{E,\boldsymbol{p}}(\phi)=\sum_{x\in\Lambda_+}\mathrm{e}^{-Ex_0}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}\phi^*(x)\in\mathcal{A}_+. \tag{3.34}
$$

In this case, $(E,\boldsymbol{p})$ plays a role of the parameter s in (3.24). For the condition (3.24) to be satisfied, the coefficient factor $\cosh E-b(\boldsymbol{p})$ should be non-negative for any $(E,\boldsymbol{p})$ satisfying $E_1\le E$, but this is *not* the case. In fact, $\cosh E-b(\boldsymbol{p})$ can become *both* positive and negative in general, which prevents us from proving the reflection positivity.

3.3 Källén-Lehmann representation of the free overlap boson propagator

Another way to see the violation of unitarity of the overlap boson system is to study the Källén-Lehmann representation of the propagator, $(\Box)^{-1}$, which should carry all the information of Hilbert space of state vectors and the spectrum of Hamiltonian [24]. It is sufficient to consider the infinite volume propagator,

$$
\begin{aligned}
\Delta_+(x,y)&=\frac{1}{\Box}(x,y)\Big|_{x_0>y_0}\\
&=\int\frac{d^4p}{(2\pi)^4}\mathrm{e}^{ip(x-y)}\frac{1}{\Box(p)}.
\end{aligned}
\tag{3.35}
$$

It is possible to evaluate this propagator explicitly to obtain the spectral representation

$$
\Delta_+(x,y)=\int_0^\infty\frac{dE}{\pi}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{-E(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}\rho(E,\boldsymbol{p}). \tag{3.36}
$$

The spectral density $\rho(E,\boldsymbol{p})$ is given by

$$
\rho(E,\boldsymbol{p})=(\text{isolated poles})+\frac{(\cosh E-b(\boldsymbol{p}))\sqrt{2b(p)\cosh E-a(\boldsymbol{p})}}{\cosh^2 E-a(\boldsymbol{p})+b(\boldsymbol{p})^2}\theta(E-E_1). \tag{3.37}
$$

In the second term, there appears the factor $\cosh E-b(\boldsymbol{p})$ which is not positive definite and this violates the positivity of $\rho(E,\boldsymbol{p})$.

3.4 Estimation of the violation in the momentum space

From the explicit form of the spectral density (3.37), we can find where in the Brillouin zone the reflection positivity is violated. One notes that there is the region in the spacial Brillouin zone where the spectral density $\rho(E, \boldsymbol{p})$ can not become negative. Let us call this region $\mathcal{S}$. The region $\mathcal{S}$ is characterized by the condition that the negative value of $\cosh E - b(\boldsymbol{p})$ be avoided. The necessary and sufficient condition on spacial momenta $\boldsymbol{p}$ to avoid negative $\cosh E - b(\boldsymbol{p})$ is that

$$\cosh E - b(\boldsymbol{p}) \geq 0, \qquad \forall E \geq E_1, \tag{3.38}$$

which is equivalent to the condition

$$\cosh E_1 \geq b(\boldsymbol{p}) \qquad \text{i.e.} \ \frac{1 + \sum_k \sin^2 p_k - (\sum_k (1 - \cos p_k))^2}{2 \sum_k (1 - \cos p_k)} \geq 0. \tag{3.39}$$

By using the identity $\sin^2 p_k + \cos^2 p_k = 1$ and completing the square with respect to $\cos p_k$, the 'safe' region $\mathcal{S}$ is given by

$$\mathcal{S} = \left\{ \boldsymbol{p} \in [-\pi, \pi]^{d-1} \ : \ \sum_{k=1}^{d-1} (\cos p_k - \frac{1}{2})^2 \leq \frac{d+1}{4} \right\}, \tag{3.40}$$

where d is the spacetime dimension.

Now let us estimate the size of $\mathcal{S}$ to investigate whether we can ignore the violation of the reflection positivity or not. In the case of $d = 4$, there are three spacial momentum components. First, we consider the case in which $p_1 = p_2 = p_3 =: p$. In this direction, the safe momentum region has the extent

$$-1.72 \leq p \leq 1.72. \tag{3.41}$$

Second, we consider another direction $p_1 = p$, $p_2 = p_3 = 0$. In this case, in the safe region $\mathcal{S}$, p is restricted by [2]

$$-1.95 \leq p \leq 1.95. \tag{3.43}$$

These regions are a little bit lager than or the same as $[-\pi/2, \pi/2]^{d-1}$.

When the spacial momenta $\boldsymbol{p}$ does not belong to $\mathcal{S}$, the spectral density $\rho(E, \boldsymbol{p})$ has to become negative on the energy interval $E_1 \leq E < E_c$, where E_1 and E_c are determined by

$$\cosh E_1 = \frac{a(\boldsymbol{p})}{2b(\boldsymbol{p})}, \qquad \cosh E_c = \boldsymbol{b}(\boldsymbol{p}), \tag{3.44}$$

[2] This restriction value of spacial momenta $\boldsymbol{p}$ is exactly the same as for the two dimensional case. In the case of $d = 2$, the spacial momenta has only one component and $\mathcal{S}$ is given as

$$\mathcal{S} = \{-1.95 \leq p \leq 1.95\}. \tag{3.42}$$

since $\rho(E,\boldsymbol{p}) < 0$ is equivalent to $a(\boldsymbol{p})/2b(\boldsymbol{p}) \leq \cosh E < b(\boldsymbol{p})$ when $\boldsymbol{p} \notin \mathcal{S}$. We will numerically estimate E_1 and E_c, the lower and upper bound of the energy interval on which the spectral density become negative. For instance, if $d = 4$, these energy values are computed as shown in the following table:

$\boldsymbol{p}$	$b(\boldsymbol{p})$	$a(\boldsymbol{p})$	$a(\boldsymbol{p})/2b(\boldsymbol{p})$	$E_1 = \cosh^{-1} a(\boldsymbol{p})/2b(\boldsymbol{p})$	$E_c = \cosh^{-1} b(\boldsymbol{p})$
(π,π,π)	6	37	37/12	1.79...	2.48...
$(\pi,\pi,0)$	4	17	17/8	1.39...	2.06...
$(\pi,0,0)$	2	5	5/4	0.69...	1.32...

Whether these values are large enough or not should depend on the physics one wants to see through the overlap boson.

4. Refletion positivity of lattice Wess-Zumino model

To remedy the violation of the reflection positivity, one may adopt the simpler nearest-neighbor action for the boson fields, ϕ and F as follows[3]:

$$S_0' = a^4 \sum_x \Big\{ -\frac{1}{2}\chi^T C D \chi + \phi^*(-\partial_\mu^*\partial_\mu)\phi + F^* F \Big\}. \tag{4.1}$$

This action still possesses three types of U(1) symmetry, Eq. (2.7), (2.8) and

$$\begin{aligned} \delta_\alpha \phi &= +i\alpha\phi, \\ \delta_\alpha F &= +i\alpha F, \end{aligned} \tag{4.2}$$

instead of Eq. (2.9).

In this formulation of the chiral multiplet, the action of the lattice $\mathcal{N} = 1$ Wess-Zumino model may be given as follows:

$$\begin{aligned} S = a^4 \sum_x \Big\{ & -\frac{1}{2}\chi^T C D \chi + \phi^*(-\partial_\mu^*\partial_\mu)\phi + F^* F + \frac{1}{a} X^T C X \\ & - g\tilde{\chi}^T C \phi P_+ \tilde{\chi} - g^* \tilde{\chi}^T C \phi^* P_- \tilde{\chi} + g F \phi^2 + g^* F^* \phi^{*2} \Big\}, \end{aligned} \tag{4.3}$$

where $X(x)$ is an auxiliary Majorana fermion field and $\tilde{\chi}(x) = \chi(x) + X(x)$. Then one may define the $\mathrm{U}(1)_R$ symmetry as follows:

$$\begin{aligned} \delta_\alpha \chi &= +i\alpha\gamma_5(1 - \frac{1}{2}aD)\chi, \\ \delta_\alpha \phi &= -2i\alpha\phi, \\ \delta_\alpha F &= +4i\alpha F \end{aligned} \tag{4.4}$$

[3]Here, we have changed the sign convention of the fermionic action by introducing new Majorana field $\chi' = i\chi$. Of course this does not change any physical results. It is simply because this convention has been used in the proof of the reflection positivity for the overlap fermions in our previous work [23].

The reflection positivity is now satisfied in this formulation of the Wess-Zumino model. The θ-reflection is defined for the bosonic fields χ, F in the same way as in the generic case (3.3),

$$\theta\phi(x) = \phi(\theta x) \tag{4.5}$$

$$\theta F(x) = F(\theta x) \tag{4.6}$$

and for the fermionic fields χ, X as in (3.4),

$$(\theta\bar{\chi})(x) = \gamma_0\chi(\theta x), \qquad (\theta\chi)(x) = \bar{\chi}(\theta x)\gamma_0, \tag{4.7}$$

$$(\theta\bar{X})(x) = \gamma_0 X(\theta x), \qquad (\theta X)(x) = \bar{X}(\theta x)\gamma_0. \tag{4.8}$$

Note that this definition of θ reflection does not contradict to the Majorana conditions $\bar{\chi} = \chi^T C$ and $\bar{X} = X^T C$. Our field algebra $\mathcal{A}$ here is that of the polynomial algebra of fermionic fields whose coefficients are the well-behaved functions of the bosonic fields. We extend θ operation to whole algebra $\mathcal{A}$, by the relations (3.6) and (3.7).

To prove the reflection positivity of the Wess-Zumino model, it is sufficient to show that the action (4.3) can be rewritten in the form of (3.17)

$$-S = B + \theta(B) + \sum_s \theta(C_s)C_s, \qquad B, C_s \in \mathcal{A}_+, \tag{4.9}$$

[23]. Let us first consider the free part of (4.3). The first term in (4.3), the overlap Majorana fermion, can be written in the form of (3.17) as is shown in ref [23]. On the other hand, the second term can be written in the form of (4.9), as is well-known. Furthermore, the third and fourth terms in (4.3) is θ reflection of themselves and do not contain any 'time hopping terms'. Therefore, these terms can be written in the form of

$$\begin{aligned} & a^4 \sum_{x\in\Lambda} \Big\{ F^* F + \frac{1}{a} X^T C X \Big\} \\ & = a^4 \sum_{x\in\Lambda_+} \Big\{ F^* F + \frac{1}{a} X^T C X \Big\} + \theta\Big[a^4 \sum_{x\in\Lambda_+} \Big\{ F^* F + \frac{1}{a} X^T C X \Big\} \Big]. \end{aligned} \tag{4.10}$$

The rest of the terms in (4.3) are interaction terms,

$$S_{\text{int}} := a^4 \sum_x \Big\{ -g\tilde{\chi}^T C\phi P_+ \tilde{\chi} - g^* \tilde{\chi}^T C\phi^* P_- \tilde{\chi} + gF\phi^2 + g^* F^* \phi^{*2} \Big\}, \tag{4.11}$$

which are all strictly local. They are equal to theta-reflection of themselves again, and do not contain any nonlocal 'time hopping' terms either. This means that S_{int} can also be written as

$$S_{\text{int}} = B + \theta(B) \tag{4.12}$$

with

$$B = \sum_{x\in\Lambda_+} \Big\{ -g\tilde{\chi}^T C\phi P_+ \tilde{\chi} - g^* \tilde{\chi}^T C\phi^* P_- \tilde{\chi} + gF\phi^2 + g^* F^* \phi^{*2}, \Big\} \tag{4.13}$$

which obviously belongs to $\mathcal{A}_+$. Therefore, one concludes that this lattice Wess-Zumino model satisfies the reflection positivity condition.

5. Discussion

Preserving R symmetry exactly is a useful way in formulating supersymmetric field theories on the lattice. This point has been emphasized by Elliot, Giedt and Moore [25] in their formulation of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. The discrete R symmetry in the two-dimensional $\mathcal{N} = 2$ Wess-Zumino model [26] has played an important role in the numerical study of the correspondence to $\mathcal{N} = 2$ conformal field theories[27].

In formulating the exact R symmetry on the lattice, however, there is a freedom in the choice of the bosonic part of the action. When one can preserve some part of the extended supersymmetries in the theories with $\mathcal{N} \geq 2$ [26, 28], it seems useful to adopt the bosonic actions to preserve the supersymmetries, although one should take into care a possible effect of the violation of the reflection positivity. But, for the theories of $\mathcal{N} = 1$, it seems difficult to preserve the supersymmetry in general[29]. The free limit supersymmetry may still help in the convergence to the supersymmetric limit in the interacting models. Otherwise, the reflection positivity condition may give a possible guideline to choose a bosonic action.

It would be interesting to examine further the inter-relation among the reflection positivity, the vacuum energy cancellation(the exact supersymmetry) and the exact $\mathrm{U}(1)_R$ symmetry of free chiral multiplet on the lattice. If one adopts the Majorana Wilson fermion for the fermionic component of the chiral multiplet, one can show that the bosonic part of the supersymmetric action now fulfills the reflection positivity condition. In this case, the $\mathrm{U}(1)_R$ symmetry is not manifest. But, through the block spin transformation, it is recovered in the fixed point action [8]. In this course of the renormalization group transformatons, it seems possible to maintain the vacuum energy cancellation by adjusting the parameters in the block-spin kernels and the normalization factors. Then, if the reflection positivity could also be maintained through the block-spin transformation, all the three conditions could be fulfills in the fixed point approach[30, 31, 32, 33, 34].

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A. Spectral density ρ

In this appendix, we derive eq.(3.37). Define a function $f(z)$ as

$$f(z) = \frac{1}{\Box}(p_0 = z, \boldsymbol{p}). \tag{A.1}$$

It is straightforward to find the explicit form

$$f(z) = \frac{\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos z}\big(\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos z} - (b(\boldsymbol{p}) - \cos z)\big)}{a(\boldsymbol{p}) - b(\boldsymbol{p})^2 - \cos^2 z}. \tag{A.2}$$

f has a branch cut where inside the square root becomes negative,

$$a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos z < 0 \Longleftrightarrow \cos z > \frac{a(\boldsymbol{p})}{2b(\boldsymbol{p})} = \frac{1 + \sum_j \sin^2 p_j + b(\boldsymbol{p})^2}{2b(\boldsymbol{p})}. \tag{A.3}$$

Since

$$\frac{1 + \sum_j \sin^2 p_j + b(\boldsymbol{p})^2}{2b(\boldsymbol{p})} \geq \sqrt{1 + \sum_j \sin^2 p_j} \geq 1, \tag{A.4}$$

Equation $a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos z = 0$ has two pure imaginary solutions $\pm iE_1$ $(E_1 > 0)$:

$$E_1 = \cosh^{-1}\left(\frac{a(\boldsymbol{p})}{2b(\boldsymbol{p})}\right), \qquad E_1 > 0. \tag{A.5}$$

f has isolated poles where the denominator vanishes,

$$\cos^2 z = a(\boldsymbol{p}) - b(\boldsymbol{p})^2 = 1 + \sum_j \sin^2 p_j \tag{A.6}$$

To solve this equation, put $z = x + iy$, $x, y \in \mathbb{R}$ and it becomes

$$\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y - 2i \cos x \cosh y \sin x \sinh y = a(\boldsymbol{p}) - b(\boldsymbol{p})^2 \geq 1. \tag{A.7}$$

Noting that the imaginary part must vanish, one finds that (A.7) is equivalent to

$$\begin{cases} \cos x = 0 \\ \sinh^2 y = a(\boldsymbol{p}) - b(\boldsymbol{p})^2 \end{cases} \quad \text{or} \quad \begin{cases} \sin x = 0 \\ \cosh^2 y = a(\boldsymbol{p}) - b(\boldsymbol{p})^2. \end{cases} \tag{A.8}$$

The solution of these equations are :

$$z = x + iy = k\pi \pm iE_0 \quad \text{or} \quad z = x + iy = (\frac{1}{2} + l)\pi \pm iE_2 \quad (k, l \in \mathbb{Z}), \tag{A.9}$$

where E_0 and E_2 are defined as the unique positive solution of $\cosh E_0 = a(\boldsymbol{p}) - b(\boldsymbol{p})^2$ and $\sinh E_2 = a(\boldsymbol{p}) - b(\boldsymbol{p})^2$ respectively.

From these observations, the analytic structure of the function f can be drawn in the complex z-plane as in the figure below. The application of Cauchy's integration theorem to the contour drawn in the figure tells us that

$$\int_{-\pi}^{\pi} \frac{dp_0}{2\pi}\, \mathrm{e}^{ip_0(x_0 - y_0)} f(p_0) = 2\pi i \times (\text{residue from poles}) + \int_{E_1}^{\infty} \frac{idE}{2\pi}\, (f_+(iE) - f_-(iE)), \tag{A.10}$$

where in f_+ one chooses the branch in which $\sqrt{-1} = i$, in f_- the branch in which $\sqrt{-1} = -i$.

It is not difficult to see that only the pole $z = \pm iE_1$ has a non-zero residue and therefore contributes the first term of eq (A.10). The other poles are removable singularities. The contributing residue coming from $z = iE_1$ can be computed to give the result

$$\mathrm{Res}\,(f, z = iE_1) = \frac{\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cosh E_1}(\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cosh E_1} - (b(\boldsymbol{p}) - \cosh E_1))}{4\pi i \cosh E_1 \sinh E_1}\chi_{\mathcal{S}}(\boldsymbol{p}) \tag{A.11}$$

where $\chi_{\mathcal{S}}$ is a characteristic function of $\mathcal{S}$ defined as

$$\chi_{\mathcal{S}}(\boldsymbol{p}) = \begin{cases} 1 & (\boldsymbol{p} \in \mathcal{S}) \\ 0 & (\boldsymbol{p} \notin \mathcal{S}). \end{cases} \tag{A.12}$$

The second term in (A.10) is also easily calculated as

$$\int_{E_1}^{\infty} \frac{idE}{2\pi}\,(f_+(iE) - f_-(iE)) = \int_0^{\infty} \frac{dE}{\pi}\,\frac{(\cosh E - b(\boldsymbol{p}))\sqrt{2b(p)\cosh E - a(\boldsymbol{p})}}{\cosh^2 E - a(\boldsymbol{p}) + b(\boldsymbol{p})^2}\theta(E - E_1) \tag{A.13}$$

Substituting (A.11) and (A.13) into (A.10), we get the spectral density ρ

$$\begin{aligned}\rho(E, \boldsymbol{p}) = &\frac{\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cosh E}(\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cosh E} - (b(\boldsymbol{p}) - \cosh E))}{2\cosh E \sinh E}\pi\chi_{\mathcal{S}}(\boldsymbol{p})\delta(E - E_1) \\ &+ \frac{(\cosh E - b(\boldsymbol{p}))\sqrt{2b(p)\cosh E - a(\boldsymbol{p})}}{\cosh^2 E - a(\boldsymbol{p}) + b(\boldsymbol{p})^2}\theta(E - E_1).\end{aligned} \tag{A.14}$$

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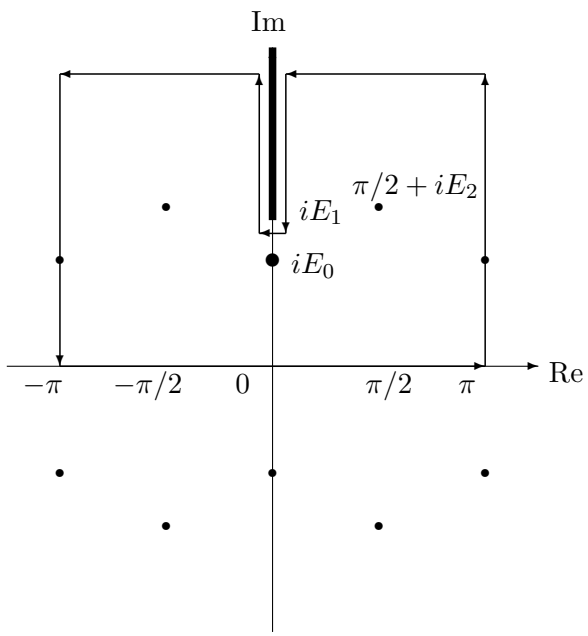

Figure 1: Integration contour of f. Only the $z = iE_0$ pole has non-vanishing residue.

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