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COXETER ELEMENTS FOR VANISHING CYCLES OF TYPES $\mathrm{A}_{\frac{1}{2}\infty}$ AND $\mathrm{D}_{\frac{1}{2}\infty}$

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Abstract. We introduce two real entire functions $f_{\mathrm{A}_{\frac{1}{2}\infty}}$ and $f_{\mathrm{D}_{\frac{1}{2}\infty}}$ in two variables, having only two critical values 0 and 1. Associated maps $\mathbf{C}^2 \to \mathbf{C}$ define topologically locally trivial fibrations over $\mathbf{C}\setminus\{0,1\}$. The critical points over 0 and 1 are infinitely many ordinary double points, whose associated vanishing cycles in the generic fiber span its middle homology group and their intersection diagram forms bi-partitely decomposed quivers of type $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$, respectively. Coxeter element of type $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$ are introduced as the product of the monodromies of the fibrations around 0 and 1. We describe the spectra of the intersection form (normalized in the iterval $[0,4]$) and the Coxeter elements (normalized in the interval $(-\frac{1}{2},\frac{1}{2})$).

Contents

[1]Present paper is planned as the first part of a paper "Primitive forms of types $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$" in preparation. We publish the present part (the spectra of Coxeter elements) separately, because of its own interests.

1. Functions of types $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$

We introduce functions of type $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$ and associated fibrations.

1.1. Definition of $f_{\mathrm{A}_{\frac{1}{2}\infty}}$ and $f_{\mathrm{D}_{\frac{1}{2}\infty}}$.

Definition. The function f_P of type $P \in \{\mathrm{A}_{\frac{1}{2}\infty}, \mathrm{D}_{\frac{1}{2}\infty}\}$ [1] is a *real entire function*[2] in two variables x and y given by

$$(1.1.1) \qquad f_{\mathrm{A}_{\frac{1}{2}\infty}}(x,y) \ := \ xs^2(x) - y^2 \ = \ 1 - c^2(x) - y^2$$

$$(1.1.2) \qquad f_{\mathrm{D}_{\frac{1}{2}\infty}}(x,y) \ := \ xs^2(x) - xy^2 \ = \ 1 - c^2(x) - xy^2.$$

Here $s(x)$ and $c(x)$ are real entire functions [3] in a variable x given by

$$(1.1.3) \qquad s(x) \ := \ \frac{\sin\sqrt{x}}{\sqrt{x}} \ = \ \prod_{n=1}^{\infty}\left(1 - \frac{x}{n^2\pi^2}\right)$$

$$(1.1.4) \qquad c(x) \ := \ \cos\sqrt{x} \ = \ \prod_{n=1}^{\infty}\left(1 - \frac{4x}{(2n-1)^2\pi^2}\right).$$

1.2. Real level sets $X_{\mathrm{A}_{\frac{1}{2}\infty},0,\mathbf{R}}$ and $X_{\mathrm{D}_{\frac{1}{2}\infty},0,\mathbf{R}}$.

We introduce the real level-0 set of the function f_P of type P by

$$X_{P,0,\mathbf{R}} \ := \ \mathbf{R}^2 \cap f_P^{-1}(0) \ .$$

Conceptual figures of them are drawn in the following.

Figure 1

$X_{A_{\frac{1}{2}\infty},0,\mathbb{R}}$

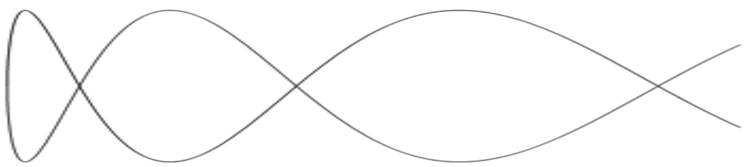
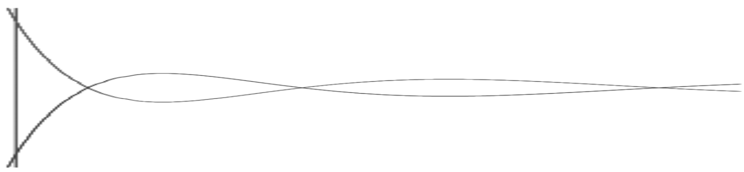


Figure 2

$X_{D_{\frac{1}{2}\infty},0,\mathbb{R}}$



[1] In the present paper, the expression "*of type P*" automatically implies $P \in \{\mathrm{A}_{\frac{1}{2}\infty}, \mathrm{D}_{\frac{1}{2}\infty}\}$. Meaning for this name is given in §2.2 Quiver and its Remark.

[2] We mean by a *real entire function of n-variables* a holomorphic function on $\mathbf{C}^n$ which is real valued on the real form $\mathbf{R}^n$ of $\mathbf{C}^n$.

[3] In the sequel of the present paper, we shall freely use the following equalities: $c(0)=s(0)=1$, $xs^2(x)+c^2(x)=1$, $s'(x)=\frac{1}{2x}(c(x)-s(x))$ and $c'(x)=-\frac{1}{2}s(x)$ without referring to them explicitly (here $f'(x)$=the differentiation of $f(x)$).

Terminology 1. By a *bounded connected component* (*bcc* for short) of type P, we mean a bounded connected component of $\mathbf{R}^2 \setminus X_{P,0,\mathbf{R}}$.

2. By a *node* of type P, we mean a point on the real curve $X_{P,0,\mathbf{R}}$ where two local smooth irreducible components are crossing normally.

3. We say that a node of type P is *adjacent* to a bcc of type P if the node belongs to the closure of the bcc.

We state some immediate observations on the level set $X_{P,0,\mathbf{R}}$, which can be easily verified by a use of absolutely convergent infinite products (1.1.3) and (1.1.4).

Observation 1. *For $n=0,1,2,\cdots$, there exists exactly one bounded connected component of type P, containing the interval $(n^2\pi^2,(n+1)^2\pi^2)$ on the x-axis and contained in the domain $(n^2\pi^2,(n+1)^2\pi^2)\times y$-axis.*

2. *For $n=1,2,3,\cdots$, the point $c_{P,0}^{(n)} := (n^2\pi^2, 0)$ on the x-axis is a node of type P, which is adjacent to two bcc containing the interval $((n-1)^2\pi^2, n^2\pi^2)$ and the interval $(n^2\pi^2,(n+1)^2\pi^2)$.*

1.3. **Fibrations over $\mathbf{C}\setminus\{0,1\}$.** .

For each type $P \in \{\mathrm{A}_{\frac{1}{2}\infty}, \mathrm{D}_{\frac{1}{2}\infty}\}$, let us consider a holomorphic map

$$f_P \;:\; \mathbf{X}_P \;\longrightarrow\; \mathbf{C}, \tag{1.3.5}$$

where the domain $\mathbf{X}_P := \mathbf{C}^2$ of f_P is regarded as a contractible Stein manifold equipped with the real form $\mathbf{R}^2$. The fiber $X_{P,t} := f_P^{-1}(t)$ over $t \in \mathbf{C}$ is an *open* Riemann surface, closely embedded in $\mathbf{C}^2$.

Remark. As we shall see in sequel, the fiber $X_{P,t}$ ($t \in \mathbf{C}$) has infinite genus. It is "wild" in the sense that the closure $\bar{X}_{P,t}$ in $\mathbf{P}^2_{\mathbf{C}}$ is equal to $X_{P,t} \cup \mathbf{P}^1_{\mathbf{C}}$ (i.e. the "ends" of $X_{P,t}$ is the $\mathbf{P}^1_{\mathbf{C}}$, this fact can be easily shown by the value distribution theory of one variable). By putting

$$\bar{\mathbf{X}}_P \;:=\; \mathbf{X}_P \cup (\mathbf{P}^1_{\mathbf{C}} \times \mathbf{C}) \;:= \cup_{t\in\mathbf{C}} (\bar{X}_{P,t}, t) \;\subset \mathbf{P}^2_{\mathbf{C}} \times \mathbf{C}, \tag{1.3.6}$$

we obtain a proper map, i.e. a "compactification" of (1.3.5):

$$\bar{f}_P \;:\; \bar{\mathbf{X}}_P \;\longrightarrow\; \mathbf{C}. \tag{1.3.7}$$

However, the spaces $\bar{X}_{P,t}$ and $\bar{\mathbf{X}}_P$ are not manifolds with boundary (note that their "boundaries" $\mathbf{P}^1_{\mathbf{C}}$ and $\mathbf{P}^1_{\mathbf{C}} \times \mathbf{C}$, respectively, have the same dimension as the "interior" $X_{P,t}$ and $\mathbf{X}_P$).

By a lack of tools to handle such objects at present, we shall not use this compactification in the present paper. Nevertheless, in the following Theorem 3, we show that f_P induces a locally topologically trivial fibration over $\mathbf{C}\setminus\{0,1\}$. The proof is an elementary handwork, however it is not standard due to the transcendental nature of f_P mentioned. Therefore, we write the proof down to the earth.

Theorem. *For each type $P\in\{\mathrm{A}_{\frac{1}{2}\infty},\mathrm{D}_{\frac{1}{2}\infty}\}$, we have the followings.*

1. *The function f_P has only two critical values 0 and 1. That is, the set of critical points C_P of f_P is contained in two fibers $X_{P,0}$ and $X_{P,1}$.*

2. i) *The critical set C_P lies in the real form $\mathbf{R}^2$ of $\mathbf{X}_P$.*

ii) *The Hessian form of $f_P|_{\mathbf{R}^2}$ at a critical point is non-degenerate. More precisely, the Hessian form is indefinite at a point in $C_{P,0}:=C_P\cap X_{P,0}$ and is negative definite at a point in $C_{P,1}:=C_P\cap X_{P,1}$.*

iii) *We have the natural bijections:*

$$C_{P,0} \simeq \{\textit{nodes of type } P\} \quad (\textit{identity map}), \tag{1.3.8}$$

$$C_{P,1} \simeq \{\textit{bcc's of type } P\} \quad (c\mapsto B_c:= \substack{\textit{the bcc}\\ \textit{containing}}\ c) \tag{1.3.9}$$

3. *The restriction of the map f_P to the smooth fibers:*

$$f_P|_{\mathbf{X}_P\setminus(X_{P,0}\cup X_{P,1})}:\mathbf{X}_P\setminus(X_{P,0}\cup X_{P,1})\to\mathbf{C}\setminus\{0,1\} \tag{1.3.10}$$

is a topologically locally trivial fibration.

Proof. **1.** We proceed direct calculations separately for each type.

$\mathrm{A}_{\frac{1}{2}\infty}$**:** The defining equations for $C_{\mathrm{A}_{\frac{1}{2}\infty}}$ are $\partial_x f_{\mathrm{A}_{\frac{1}{2}\infty}}=cs=0, \partial_y f_{\mathrm{A}_{\frac{1}{2}\infty}}=-2y=0$. Hence, $C_{\mathrm{A}_{\frac{1}{2}\infty}}=\{(x,0)\mid s(x)=0 \text{ or } c(x)=0\}$, where we have

$$f_{\mathrm{A}_{\frac{1}{2}\infty}}(x,0)=\begin{cases}0 & \text{if } s(x)=0,\\ 1 & \text{if } c(x)=0.\end{cases}$$

$\mathrm{D}_{\frac{1}{2}\infty}$**:** The defining equations for $C_{\mathrm{D}_{\frac{1}{2}\infty}}$ are $\partial_x f_{\mathrm{D}_{\frac{1}{2}\infty}}=cs-y^2=0, \partial_y f_{\mathrm{D}_{\frac{1}{2}\infty}}=-2xy=0$. Hence, $C_{\mathrm{D}_{\frac{1}{2}\infty}}=\{(0,\pm1)\}\cup\{(x,0)\mid s(x)=0 \text{ or } c(x)=0\}$, where we have

$$f_{\mathrm{D}_{\frac{1}{2}\infty}}(0,\pm1)=0 \quad\text{and}\quad f_{\mathrm{D}_{\frac{1}{2}\infty}}(x,0)=\begin{cases}0 & \text{if } s(x)=0,\\ 1 & \text{if } c(x)=0.\end{cases}$$

2. i) Due to the descriptions of C_P in **1.**, we have only to show that the zero loci of $s(x)=0$ and $c(x)=0$ are real numbers. This follows from the fact that the infinite product expressions (1.1.3) and (1.1.4) are absolutely convergent and the zero loci of $s(x)=0$ and $c(x)=0$ are given by the union of zero locus of factors of the expressions, respectively.

ii) Let us calculate the Hessian at a critical point.

The statement for the two critical points $(0,\pm1)$ on $X_{\mathrm{D}_{\frac{1}{2}\infty},0}$ can be verified directly. The other critical points are on the x-axis, i.e. one always has $y=0$. Since $\partial_x\partial_y f_P\,|_{y=0}=0$ for each type $P\in\{\mathrm{A}_{\frac{1}{2}\infty}.\mathrm{D}_{\frac{1}{2}\infty}\}$, the Hessian is a diagonal matrix of the form

$$[\partial_x(c(x)s(x)),-2]_{diag} \quad \text{for type } P=\mathrm{A}_{\frac{1}{2}\infty},$$

$$[\partial_x(c(x)s(x)), -2x]_{diag} \quad \text{for type } P = \mathrm{D}_{\frac{1}{2}\infty},$$

where the second diagonal component is always negative. We calculate the sign of the first diagonal component by
$\partial_x(c(x)s(x)) \mid_{c=0} = -\frac{1}{2}s^2 = -\frac{1}{2x} < 0$ and $\partial_x(c(x)s(x)) \mid_{s=0} = \frac{1}{2x} > 0$, implying the statement **ii)**.

iii) Combining the explicit descriptions of the set $C_{P,0}$, $C_{P,1}$ in Proof of **1.** with Observations 2. and 3. in §1.2, the correspondences are defined and are injective (see Figure 1 and 2.). So, we need only to show their surjectivity. But, this is again trivial since i) any node of a curve is a critical point of the defining equation of the curve, where Hessian is indefinite, and ii) inside of any bounded connected component of a complement of a real curve in $\mathbf{R}^2$, there exists at least a point where f_P takes local maximum, then the Hessian at the point should be negative definite since we saw in **2. ii)** that it is already non-degenerate.

3. Let us show that the fibration (1.3.10) is locally topologically trivial. Since our map is neither proper nor extendable to a suitably stratified proper map (recall 1.3 Remark.), we cannot use standard technique such as Thom-Ehrshman theorems. Instead, we use an elementary fact that $X_{P,t}$ is a ramified covering space: namely, in view of the equations (1.3.8) and (1.3.9), the projection map $(x, y) \in \mathbf{C}^2 \mapsto x \in \mathbf{C}$ to the x-plane induces a proper and ramified double covering maps $\pi_{P,t}$:

$$(1.3.11) \qquad X_{\mathrm{A}_{\frac{1}{2}\infty},t} \to \mathbf{C}\ (t\in\mathbf{C}) \ \text{ and } \ X_{\mathrm{D}_{\frac{1}{2}\infty},t} \to \mathbf{C}\setminus\{0\}\ (t\in\mathbf{C}\setminus\{0\}),$$

(for $X_{\mathrm{D}_{\frac{1}{2}\infty},0}$, see [4]). Let us denote by $\mathbf{C}_P$ the base space of this covering, i.e. $\mathbf{C}_P := \mathbf{C}$ if $P = \mathrm{A}_{\frac{1}{2}\infty}$ and $:= \mathbf{C} \setminus \{0\}$ if $P = \mathrm{D}_{\frac{1}{2}\infty}$. In view of the defining equation of $X_{P,t}$, the covering is ramifying at $X_{P,t} \cap \{y=0\}$, i.e. at solutions $x \in \mathbf{C}_P$ of the equation

$$(1.3.12) \qquad xs^2(x) - t \ = \ 0,$$

which, apparently, has infinitely many solutions, depending on $t \in \mathbf{C}$.

We, now, state an elementary but a crucial fact on the function xs^2.

Fact. *The correspondence* $\pi : \mathbf{C}_P \to \mathbf{C}$, $x \mapsto t := xs^2(x) = \sin^2(\sqrt{x})$ *is ramifying exactly and only at the inverse images of the points* 0 *and* 1, *and induces a (topological) covering map over* $\mathbf{C} \setminus \{0, 1\}$.

Proof of **Fact.** The critical points of the map $t = xs^2(x)$ are given by the equation $s(x)c(x) = 0$, and are exactly the points where $t = 0$ or 1) (recall Proof of **1.**). Thus, the restricted map $\pi' := \pi|_{\pi^{-1}(\mathbf{C}\setminus\{0,1\})}$

[4] Since the fiber $X_{\mathrm{D}_{\frac{1}{2}\infty},0}$ contains an irreducible component $L := \{x = 0\}$, the map on $X_{\mathrm{D}_{\frac{1}{2}\infty},0}$ is not a covering, but its restriction to $X_{\mathrm{D}_{\frac{1}{2}\infty},0} \setminus L$ is a covering.

over $\mathbf{C}\setminus\{0,1\}$ is a locally homeomorphism. To see that π' is a covering (i.e. a proper map on each component of an inverse image of a simply connected open subset of $\mathbf{C}\setminus\{0,1\}$), we need to show that the inverse map of $xs^2(x)=t$ as a multivalued function in t is analytically continuable everywhere on the set $\mathbf{C}\setminus\{0,1\}$. Since the equation is equivalent to $\sqrt{x}=\pm\sin^{-1}(\sqrt{t})$, this fact follows from the fact that the multivalued function $\sin^{-1}(u)$ has singular points (i.e. points where the function cannot be analytically continued) only at $u=\pm 1$, easily seen from the integral expression $\sin^{-1}(u)=\int_0^u \frac{du}{\sqrt{1-u^2}}$. □

Owing to **Fact**, we find a disc neighbourhood $\mathfrak{U}$ for any $t_0\in\mathbf{C}\setminus\{0,1\}$ so that $\pi^{-1}(\mathfrak{U})$ decomposes into components homeomorphic to U. For each $x_i\in\pi^{-1}(t_0)$ ($i\in I$ index set), let $s_i(t)$ be the function on $t\in\mathfrak{U}$, defining a section of π such that $s_i(t_0)=x_i$ (actually, $s_i(t)=\big(\sqrt{x_i}+\int_{\sqrt{t_0}}^{\sqrt{t}}\frac{du}{\sqrt{1-u^2}}\big)^2$ for choices of $\sqrt{t_0}$ and $\sqrt{x_i}$ such that $\sqrt{t_0}=\sin(\sqrt{x_i})$ and path of integral in the connected component of $\pm\sqrt{\mathfrak{U}}$ containing $\sqrt{t_0}$).

We can find a differentiable map $\varphi:\mathfrak{U}\times\mathbf{C}_P\to\mathbf{C}_P$ such that i) $\varphi(t_0,x)=x$, ii) for each $t\in U$, the $\varphi_t:=\varphi(t,\cdot)$ is a diffeomorphism of $\mathbf{C}_P$, and iii) for each $i\in I$, $\varphi(t,s_i(t))$ is constant (equal to $s_i(t_0)=x_i$). The diffeomorphism φ_t can be uniquely lifted to a diffeomorphism $\hat{\varphi}_t: X_{P,t}\simeq X_{P,t_0}$ of the double covers such that $\varphi_t\circ\pi_{P,t}=\pi_{P,t_0}\circ\hat{\varphi}_t$. The $\hat{\varphi}_t$ gives the local trivialization of (1.3.10). □

This completes a proof of Theorem 1., 2. and 3. □

2. Vanishing cycles

We show that the middle homology group of a generic fiber of the map (1.3.5) has basis consisting of vanishing cycles. The intersection form among them forms the *principal quiver*[5] of type $\mathrm{A}_{\frac{1}{2}\infty}$ or $\mathrm{D}_{\frac{1}{2}\infty}$.

2.1. **Middle homology groups.** In the present paragraph, we describe the middle homology group of the general fibers of (1.3.10) in terms of vanishing cycles of the function f_P of type $P\in\{\mathrm{A}_{\frac{1}{2}\infty},\mathrm{D}_{\frac{1}{2}\infty}\}$.

Vanishing cycles: For a critical point $c\in C_P=C_{P,0}\sqcup C_{P,1}$, we define an oriented 1-cycle $\gamma_{P,c}$ in $X_{P,t}$ for $t\in(0,1)$ as follows.

Due to Theorem 2, we can choose holomorphic local coordinates (u,v) in a neighborhood $\mathfrak{U}$ of c in $\mathbf{X}_P$ such that i) u and v are real valued on $\mathfrak{U}_{\mathbf{R}}:=\mathfrak{U}\cap\mathbf{R}^2$, ii) $\frac{\partial(u,v)}{\partial(x,y)}|_{\mathfrak{U}_R}>0$ and iii) $f_P|_{\mathfrak{U}}=u^2-v^2$ if

[5] We mean by a *quiver* an oriented graph. It is called *principal*, if the set of vertices's has a bipartite decomposition $\Gamma_0\sqcup\Gamma_1$ such that the head (resp. tail) of any edge belongs to Γ_0 (resp. Γ_1) (e.g. Figure 3 and 4). See [Sa2,3].

$c \in C_{P,0}$ and $f_P|_{\mathfrak{U}}=1-u^2-v^2$ if $c \in C_{P,1}$. Then, define cycles:

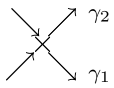
$$(2.1.13)\qquad \gamma_{P,c} := \begin{cases} (\sqrt{t}\cos(\theta), \sqrt{-1}\sqrt{t}\sin(\theta))\ (0 \le \theta \le 2\pi), & \text{if } c\in C_{P,0} \\ (\sqrt{1-t}\cos(\theta), \sqrt{1-t}\sin(\theta))\ (0\le\theta\le 2\pi), & \text{if } c\in C_{P,1}. \end{cases}$$

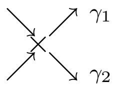
Fact. *The oriented cycle $\gamma_{P,c}$ in the surface $X_{P,t}$ is, up to free homotopy, unique and independent of a choice of coordinates (u, v).*

Definition. We shall denote the homology class in $\mathrm{H}_1(X_{P,t}, \mathbf{Z})$ of the cycle $\gamma_{P,c}$ by the same $\gamma_{P,c}$, and call it the *vanishing cycle* of the function f_P at the critical point $c \in C_P$ (vanishing along the path $t\downarrow 0$ or $t\uparrow 1$).

Sign convention of intersection numbers of 1-cycles on $X_{P,t}$.

i) Let I be the skew symmetric intersection form between two oriented 1-cycles on a oriented surface. Then we define the convention of the sign of intersection number locally as follows:

Fig.3 $\quad I(\gamma_1, \gamma_2) = 1$ if 

, $\quad I(\gamma_1, \gamma_2) = -1$ if 

ii) The orientation of the surface $X_{P,t}$ is $\sqrt{-1}dz\wedge d\bar{z}=2dx\wedge dy$ for a local holomorphic coordinate $z=x+iy$ on $X_{P,t}$. *Eg.* Cycles γ_x and γ_y locally homotopic to x-axis and y-axis intersects as $I(\gamma_x, \gamma_y)=1$ at z=0.

Theorem. 4. *The middle homology group of $X_{P,t}$, $t\in(0,1)$ is given by*

$$(2.1.14)\qquad \mathrm{H}_1(X_{P,t}, \mathbf{Z}) \ \simeq\ \mathrm{H}_P := \ \mathrm{H}_{P,0} \ \oplus\ \mathrm{H}_{P,1},$$

where

$$(2.1.15)\qquad \mathrm{H}_{P,0} \ :=\ \oplus_{c\in C_{P,0}} \mathbf{Z}\gamma_{P,c}$$

$$(2.1.16)\qquad \mathrm{H}_{P,1} \ :=\ \oplus_{c\in C_{P,1}} \mathbf{Z}\gamma_{P,c}$$

are formally defined free abelian group spanned by vanishing cycles.

5. *Let $I_P : \mathrm{H}_1(X_{P,t}, \mathbf{Z}) \times \mathrm{H}_1(X_{P,t}, \mathbf{Z}) \to \mathbf{Z}$ be the intersection form on the middle homology group. Then we have*

$$(2.1.17)\qquad I_P \ =\ J_P -\ {}^tJ_P$$

where J_P and ${}^tJ_P$ are integral bilinear forms on H_P given by

$$(2.1.18)\quad J_P(\gamma_{P,c}, \gamma_{P,c'}) := \begin{cases} 1 & \text{if } c = c', \\ -1 & \text{if } c \in C_{P,0},\ c' \in C_{P,1} \text{ and } c \in \overline{B}_{c'}, \\ 0 & \text{else,} \end{cases}$$

and

$$(2.1.19)\quad {}^tJ_P(\gamma_{P,c}, \gamma_{P,c'}) := \begin{cases} 1 & \text{if } c = c', \\ -1 & \text{if } c \in C_{P,1},\ c' \in C_{P,0} \text{ and } c' \in \overline{B}_{c}, \\ 0 & \text{else.} \end{cases}$$

Remark. The meaning to use the form J_P shall be clarified in §2.3.

Proof. We first calculate intersection numbers between vanishing cycles $\gamma_{P,c}$ and $\gamma_{P,c'}$ as given in **5.**

Suppose both critical points c, c' belong to $C_{P,0}$ (resp. $C_{P,1}$). If $c \neq c'$ then we, for t close enough to 0 (resp. 1), the supports of the vanishing cycles are close to c and c' so that they are disjoint, i.e. $\gamma_{P,c} \cap \gamma_{P.c'} = \emptyset$ and we get $I_P(\gamma_{P,c}, \gamma_{P,c'}) = 0$. Then, this equality holds for any $t \in (0,1)$. If $c = c'$, then $I_P(\gamma_{P,c}, \gamma_{P,c}) = 0$ due to skew-symmetry of I_P.

Next, we consider a cycle $\gamma_{P,c}$ for $c \in C_{P,0}$ and a cycle $\gamma_{P,c'}$ for $c' \in C_{P,1}$. From their expressions in (2.1.13), we observe the following two facts:

i) The cycle $\gamma_{P,c}$ intersects only with each of connected component of $\mathbf{R}^2 \setminus X_{P,0,\mathbf{R}}$ adjacent to c at one point $(u,v) = (\varepsilon\sqrt{t}, 0)$ for $\varepsilon \in \{\pm 1\}$.

ii) The underlying set $|\gamma_{P,c'}|$ is presented by a circle of radius $1-t$ in the bcc $B_{c'}$ containing c', i.e. it is equal to $\{(u',v') \in B_{c'} \mid f_P(u',v') = t\}$.

These means that cycles $\gamma_{P,c}$ and $\gamma_{P,c'}$ for the same $t \in (0,1)$ intersect if and only if the critical point c is adjacent to the bounded component $B_{c'}$, and, then, they intersect transversely at one point, say p. Let (u', v') be the coordinates for the cycle $\gamma_{P,c'}$ in (2.1.13). Then, by an orientation preserving orthogonal linear transformation of the coordinates, the intersection point p may be given by $(u', v') = (\sqrt{1-t}, 0)$

We determine the sign of the intersection as follows: in a neighbourhood of p, we have an equality $f_P = u^2 - v^2 = 1 - u'^2 - v'^2$. Then the differentiation at p of the equation gives $df|_p = \varepsilon\sqrt{t}du|_p = -\sqrt{1-t}du'|_p$. Since $du \wedge dv|_p = c du' \wedge dv'|_p$ for some positive $c \in \mathbf{R}_{>0}$, we get

a) $$\frac{\partial v}{\partial v'}|_p \;=\; \varepsilon c \frac{\sqrt{t}}{\sqrt{1-t}}.$$

On the other hand, since du and du' are co-normal vectors to $X_{P,t}$ at p (i.e. $df|_p \mathbin{//} du|_p \mathbin{//} du'|_p$), we use dv and dv' as for complex coordinates of the 1-dimensional complex tangent space $T(X_{P,t})_p$ at p, which are compatible with the sign convention ii) of the surface $X_{P,t}$.

Using these coordinates, the infinitesimal direction $\frac{\partial}{\partial \theta}|_p$ of $\gamma_{P,c}$ at p is evaluated by

b) $$\frac{\partial v}{\partial \theta}|_p \;=\; \varepsilon\sqrt{-1}\sqrt{t}$$

and the infinitesimal direction $\frac{\partial}{\partial \theta'}|_p$ of $\gamma_{c',1}$ at p is evaluate by

c) $$\frac{\partial v'}{\partial \theta'}|_p \;=\; \sqrt{1-t}.$$

Combining a), b) and c), we obtain that the angle from the cycle $\gamma_{P,c'}$ to the cycle $\gamma_{P,c}$ at their intersection point p is given by the angle of the complex number

d) $$\left(\frac{\partial v}{\partial \theta}|_p / \frac{\partial v'}{\partial \theta'}|_p\right) \;/\; \frac{\partial v}{\partial v'}|_p \;=\; \frac{\sqrt{-1}}{c},$$

i.e. the angle is $\frac{\pi}{2}$. Then due to our sign convention, we obtain

$$I_P(\gamma_{P,c}, \gamma_{P,c'}) = -1 \quad \text{and} \quad I_P(\gamma_{P,c'}, \gamma_{P,c}) = 1,$$

which is independent of the sign $\varepsilon \in \{\pm 1\}$. Thus, (2.1.17) is shown.

Finally in the following i)-v), we prove **4.**

We formally put (2.1.15) and (2.1.16).

i) Let us first show a natural isomorphism.

$$\mathrm{H}_1(X_{P,0},\mathbf{Z}) \;\simeq\; \mathrm{H}_{P,1}. \tag{2.1.20}$$

Proof of (2.1.20). We first show that $X_{P,0,\mathbf{R}}$ is a deformation retract of $X_{P,0}$. For the proof of it, recall the double cover expression of $X_{P,0}$ over $\mathbf{C}_P$, used in the proof of **Theorem 3**. In case of type $P=\mathrm{A}_{\frac{1}{2}\infty}$, the deformation retract of the plane $\mathbf{C}_P$ to the half real axis $\mathbf{R}_{\geq 0}$ induces the retract of the covering space $X_{P,0}$ to its real form $X_{P,0,\mathbf{R}}$. In case of type $P=\mathrm{D}_{\frac{1}{2}\infty}$, we do the retraction irreducible-componentwisely to the real axis $\mathbf{R}$ (details are left to the reader). Thus, in view of Figure 1 and 2, we have a natural isomorphism:

$$\mathrm{H}_1(X_{P,0},\mathbf{Z}) \;\simeq\; \mathrm{H}_1(X_{P,0,\mathbf{R}},\mathbf{Z}) \;\simeq\; \mathrm{H}_{P,1}. \qquad \Box\,)$$

ii) Using the double cover expressions of fibers $X_{P,t}$ in the proof of **Theorem 3.**, we can show that $f_P^{-1}([0,t])$ $(t\in(0,1))$ retracts to its subset $X_{P,0}$. Then composing with the inclusion map $X_{P,t}\subset f_P^{-1}([0,t])$, we get an exact sequence

$$\mathrm{H}_{P,0} \;\to\; \mathrm{H}_1(X_{P,t},\mathbf{Z}) \;\xrightarrow{r}\; \mathrm{H}_1(X_{P,0},\mathbf{Z}) \;\to\; 0,$$

where the restriction of r to the submodule $H_{P,1}$ composed with the isomorphism (2.1.20) induces the identity on $H_{P,1}$. This implies that $\mathrm{H}_{P,1}$ is a factor of $\mathrm{H}_1(X_{P,t},\mathbf{Z})$.

iii) What remains to show is that $\mathrm{H}_{P,0}$ is injectively embedded in $\mathrm{H}_1(X_{P,t},\mathbf{Z})$. This can be partially shown by using the non-degeneracy of the intersection relations (2.1.18) as follows.

Let $\gamma\in\mathrm{H}_{P,0}$ be a non-zero element, whose image in $\mathrm{H}_1(X_{P,t},\mathbf{Z})$ is zero. Then solving the relation $I_P(\gamma,\gamma_{P,c})=0$ for $c=c_{P,1}^{(n)}\in C_{P,1}$ (see Notation in §2.2) from large enough $n\in\mathbf{Z}_{>0}$ back wards to 1, we see successive vanishings of the coefficients of γ, and finally see that γ, up to a constant factor, is equal to $\gamma^+_{D,0}-\gamma^-_{D,0}$ (see §2.2 for Notation $\gamma^+_{D,0}$ and $\gamma^-_{D,0}$). In order to show that this is not possible, we prepare a fact.

iv) **Fact.** *The function f_P of type P is invariant by the involution $\sigma:\mathbf{X}_P\to\mathbf{X}_P$, $(x,y)\mapsto(x,-y)$ on its domain, i.e. $f_P\circ\sigma=f_P$. The induced involution on the surface $X_{P,t}$, denoted again by σ, is equivariant with the covering map $\pi_{P,t}$ (1.3.11), i.e. $\pi_{P,t}\circ\sigma=\pi_{P,t}$. Then, one has $\sigma_*(\gamma_{P,c})=-\gamma_{P,c}$ for all $c\in C_P$, except for the following two cases*

$$\sigma_*(\gamma^+_{D,0})=-\gamma^-_{D,0} \;\text{ and }\; \sigma_*(\gamma^-_{D,0})=-\gamma^+_{D,0}.$$

Proof of **Fact.** Except for the cases $\gamma_{D,0}^+$ and $\gamma_{D,0}^-$, we can choose the coordinate in (2.1.13) in such manner that $\sigma(u,v)=(u,-v)$. $\square$

v) Assuming $\gamma_{D,0}^+=\gamma_{D,0}^-$, let us show a contradiction. Consider the homomorphism $(\pi_D)_*:\mathrm{H}_1(X_{D,t},\mathbf{Z})\to\mathrm{H}_1(\mathbf{C}_D,\mathbf{Z})\simeq\mathbf{Z}$. Above **Fact.** implies $(\pi_D)_*(\gamma_{D,0}^+)=(\pi_D\circ\sigma)_*(\gamma_{D,0}^+)=(\pi_D)_*\circ\sigma_*(\gamma_{D,0}^+)=-(\pi_D)_*(\gamma_{D,0}^-)$ which, by the assumption, is equal to $-(\pi_D)_*(\gamma_{D,0}^+)$. Thus, we get $(\pi_D)_*(\gamma_{D,0}^+)=0$. This contradicts to the fact that $(\pi_D)_*(\gamma_{D,0}^+)$ generates $\mathrm{H}_1(\mathbf{C}_D,\mathbf{Z})\simeq\mathbf{Z}$ (observed easily from the fact that the equation $x=0$ defines i) a branch of $X_{D,0,\mathbf{R}}$ at the nodal point $c_{D,0}^+$ and also ii) the puncture in $\mathbf{C}_D$, and from the description of $\gamma_{D,0}^+$ in (2.1.13)).

This completes a proof of Theorem 4. and 5. $\square$

Remark. In the step v) in above proof, we may use a σ-invariant form $\omega:=\mathrm{Res}\big[\frac{ydxdy}{f_D-t}\big]$. Since $\int_{\gamma_{D,0}^+}\omega=\int_{\gamma_{D,0}^+}\sigma^*(\omega)=\int_{\sigma_*(\gamma_{D,0}^+)}\omega=-\int_{\gamma_{D,0}^-}\omega$, the assumption $\gamma_{D,0}^+=\gamma_{D,0}^-$ implies $\int_{\gamma_{D,0}^+}\omega=0$. On the other hand, $\omega=\mathrm{Res}\big[\frac{ydxdy}{f_D-t}\big]=\frac{dx}{2x}|_{X_{D,t}}$, and hence $\int_{\gamma_{D,0}^+}\omega=\pm\sqrt{-1}\pi\neq 0$. A contradiction!

2.2. Quivers of type $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$. .

We encode homological data of vanishing cycles of f_P in a quiver Γ_P.

Definition. A quiver Γ_P of type $P\in\{\mathrm{A}_{\frac{1}{2}\infty},\mathrm{D}_{\frac{1}{2}\infty}\}$ is defined by

i) The set of vertices of Γ_P is bijective to $\{\gamma_{P,c}\mid c\in C_{P,0}\cup C_{P,1}\}$.

ii) We put an oriented edge from $\gamma_{P,c}$ to $\gamma_{P,c'}$ if and only if $c\in C_{P,0}$, $c'\in C_{P,1}$ and $c\in\overline{B}_{c'}$, that is, when $J_P(\gamma_{P,c},\gamma_{P,c'})=-1$.

Let us fix a numbering of elements in $C_{P,0}\cup C_{P,1}$ as follows.

$$\begin{aligned}
C_{A,0} &= \{c_{A,0}^{(n)}:=(n^2\pi^2,0)\}_{n\in\mathbf{Z}_{>0}}\\
C_{A,1} &= \{c_{A,1}^{(n)}:=((n-\frac{1}{2})^2\pi^2,0)\}_{n\in\mathbf{Z}_{>0}}\\
C_{D,0} &= \{c_{D,0}^{(n)}:=(n^2\pi^2,0)\}_{n\in\mathbf{Z}_{>0}}\ \cup\ \{c_{D,0}^+:=(0,1),c_{D,0}^-:=(0,-1)\}\\
C_{D,1} &= \{c_{D,1}^{(n)}:=((n-\frac{1}{2})^2\pi^2,0)\}_{n\in\mathbf{Z}_{>0}}.
\end{aligned}$$

According to them, the vertices of the quiver Γ_P are numbered as below.

$$\Gamma_{\mathrm{A}_{\frac{1}{2}\infty}}:\quad \gamma_{A,1}^{(1)}\longrightarrow\gamma_{A,0}^{(1)}\longleftarrow\gamma_{A,1}^{(2)}\longrightarrow\gamma_{A,0}^{(2)}\longleftarrow\gamma_{A,1}^{(3)}\longrightarrow\gamma_{A,0}^{(3)}\longleftarrow\quad\cdots$$

$$\Gamma_{\mathrm{D}_{\frac{1}{2}\infty}}:\quad \begin{tikzcd}[column sep=small] \gamma_{D,0}^+ & & & & & & \\ & \gamma_{D,1}^{(1)} \arrow[ul] \arrow[dl] \arrow[r] & \gamma_{D,0}^{(1)} & \gamma_{D,1}^{(2)} \arrow[l] \arrow[r] & \gamma_{D,0}^{(2)} & \gamma_{D,1}^{(3)} \arrow[l] \arrow[r] & \cdots \\ \gamma_{D,0}^- & & & & & & \end{tikzcd}$$

Note that the decomposition of the critical set C_P into $C_{P,0}\cup C_{P,1}$ gives arise the bi-partite (or principal) decomposition of the quiver Γ_P.

Remark. A real polynomial in one variable, such that 1) it has only non degenerate critical points with two critical values 0 and 1 and 2) vansihing cycles associated with its critical points form the bipartite decomposed Dykin diagram of type A_l, is (up to suspensions, see §2.3) well-known as the *Chebyshev polynomial*. Thus, the functions $f_{\mathrm{A}_{\frac{1}{2}\infty}}$ and $f_{\mathrm{D}_{\frac{1}{2}\infty}}$ may be regarded as transcendental analogues of Chebyshev poynomials.

More generally, for any Dynkin quiver of finite type P (i.e. $P \in \{\mathrm{A}_l\ (l\geq 1), \mathrm{B}_l\ (l\geq 2),\ C_l\ (l\geq 3), \mathrm{D}_l\ (l\geq 4), \mathrm{E}_l\ (l=6,7,8), \mathrm{F}_4, \mathrm{G}_2\}$), there are real polynomials $f_P(x,y)$ such that they have only non-degenerate critical points with only two critical values and 2) the vanishing cycles associated with the critical points give the bi-partite decomposition of the Dynkin quiver of type P. They form a (half) line, called the *real vertex orbit axis*, in the real deformation parameter space of real simple singularities (see [Sa2, §2.5]). Thus, the functions $f_{\mathrm{A}_{\frac{1}{2}\infty}}$ and $f_{\mathrm{D}_{\frac{1}{2}\infty}}$ in the present paper are their transcendental analogues for the quivers of types $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$, respectively. Theory of primitive forms for simple singularities is established [Sa1]. The present paper is a step towards construction of primitive forms of types $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$.

2.3. **Suspensions to higher dimensions.** .

In this subsection, we briefly describe the suspensions of the results in previous subsections to higher dimensional cases.

For a type $P\in\{\mathrm{A}_{\frac{1}{2}\infty}, \mathrm{D}_{\frac{1}{2}\infty}\}$ and $n\in\mathbf{Z}_{\geq 0}$, let us introduce the n-th *suspension* of f_P as the entire functions in $2+n$-variables x, y and $\underline{z}=(z_1,\cdots,z_n)$ defined by

$$f_P^{(n)}(x,y,\underline{z}) := f_P(x,y) - z_1^2 - \cdots - z_n^2. \tag{2.3.21}$$

Then, replacing the function f_P by $f_P^{(n)}$ and the domain $\mathbf{X}_P=\mathbf{C}^2$ by $\mathbf{X}_P^{(n)}=\mathbf{C}^2\times\mathbf{C}^n$, we obtain a holomorphic map $(1.3.5)^{(n)}$ whose fibers, denoted by $X_{P,t}^{(n)}$ $(t\in\mathbf{C})$, are Stein variety of complex dimension $n+1$.

Replacing, further, the real form $\mathbf{R}^2$ of $\mathbf{X}_P$ by the real form $\mathbf{R}^2\times\mathbf{R}^n$ of $\mathbf{X}_P^{(n)}$, **Theorem 1., 2., 3.** in §1.3 hold completely parallelly for $f_P^{(n)}$, where the set of critical points of $f_P^{(n)}$ is bijective to that of f_P by the natural embedding $\mathbf{X}_P\subset\mathbf{X}_P^{(n)}$ so that we identify them. Then the signature of Hessians of $f_P^{(n)}$ at points of $C_{P,0}$ is $(1,n+1)$ and that at points of $C_{P,1}$ is $(0, n{+}2)$. The suspended fibration shall be referred by $(1.3.10)^{(n)}$. The proof are reduced to the original case $n{=}0$.

Applying n-times suspension S on a homology class γ in $\mathrm{H}_1(X_{P,t}, \mathbf{Z})$, we obtain an element $S^n\gamma$ of the middle homology group $\mathrm{H}_{n+1}(X^{(n)}_{P,t}, \mathbf{Z})$ of the fiber $X^{(n)}_{P,t}$. In particular, the suspension $S^n\gamma_{P,c}$ of a vanishing cycle $\gamma_{P,c}$ of f_P at a critical point $c \in C_P$ is a vanishing cycle of $f^{(n)}_P$ at the same critical point, which, for simplicity, we shall denote again by $\gamma_{P,c}$. Then replacing $\mathrm{H}_1(X_{P,t}, \mathbf{Z})$ by the middle homology group $\mathrm{H}_{n+1}(X^{(n)}_{P,t}, \mathbf{Z})$, **Theorem 4.** in §2.1 holds completely parallely, where we keep notations (2.1.14) and (2.1.15).

The intersection form $I^{(n)}_P$ on the middle homology group is wellknown to be symmetric or skew-symmetric according as cycles are even or odd dimensional (i.e. according as $n-1$ is even or odd). It is also wellknown that $I^{(n)}_P(\gamma_{P,c}, \gamma_{P,c}) = (-1)^{\frac{n+1}{2}}2$ for even dimensional vanishing cycles (i.e. when n is odd). Therefore, the formula (2.1.17) of the intersection form in **Theorem 5.** need to be slightly modified as in the following theorem, where we keep the notation J_P and ${}^tJ_P$ together with the formulae (2.1.18) and (2.1.19).

Theorem 5$^{(n)}$. *Let $I^{(n)}_P : \mathrm{H}_{n+1}(X^{(n)}_{P,t}, \mathbf{Z}) \times \mathrm{H}_{n+1}(X^{(n)}_{P,t}, \mathbf{Z}) \to \mathbf{Z}$ be the intersection form on middle-homology groups of the fibers of the fibration $(1.3.10)^{(n)}$. Then we have the following 4-periodic expression.*

$$I^{(n)}_P = (-1)^{[\frac{n+1}{2}]} J_P - (-1)^{[\frac{n}{2}]}\, {}^tJ_P. \tag{2.3.22}$$

The proof of Theorem is standard, and is omitted. Actually, the form $I^{(n)}_P$ is symmetric for n odd and is skew symmetric for n even.

Remark. We may regard that the form J_P is an infinite rank analogue of a *Seifert matrix* with respect to a "suitable compactification" of the three-fold $f_P^{-1}(S^1)$, where S^1 is a circle in the base space $\mathbf{C}$ of (1.3.5) which encloses the two points 0 and 1. However, we do not pursue any further this analogy (see §1.3 Remark and the next subsection §2.4).

2.4. **Monodromy Transformations and Coxeter elements.** .

The fundamental group $\pi_1(\mathbf{C} \setminus \{0,1\}, t_0)$ with $t_0 \in (0,1)$ of the base space of the fibration $(1.3.10)^{(n)}$ has two generators g_0 and g_1 which are presented by circular paths in $\mathbf{C}\setminus\{0,1\}$ starting at t_0 and turning once around the point 0 and 1 counterclockwise, respectively. Let $\sigma^{(n)}_{P,0}$ (resp. $\sigma^{(n)}_{P,1}$) be the monodromy action of g_0 (resp. g_1) on the middle homology group $(2.1.13)^{(n)}$ of the fiber of the family $(1.3.10)^{(n)}$, which preserves the intersection form (2.3.22). Though the singular fibers $X^{(n)}_{P,0}$ and

$X^{(n)}_{P,1}$ have infinitely many critical points, we can apply Picard-Lefschetz formula. That is, for $u \in H_P := H_{P,0} \oplus H_{P,1}$

$$
\begin{aligned}
\sigma^{(n)}_{P,0}(u) &= u + (-1)^{[\frac{n}{2}]} \textstyle\sum_{c \in C_{P,0}} I^{(n)}_P(u, \gamma_{P,c})\gamma_{P,c} \\
&= u + \textstyle\sum_{c \in C_{P,0}} ((-1)^n J_P(u, \gamma_{P,c}) - J_P(\gamma_{P,c}, u))\gamma_{P,c} \\
&= \begin{cases} (-1)^n u & \text{if } u \in H_{P,0} \\ u - \sum_{c \in C_{P,0}} J_P(\gamma_{P,c}, u)\gamma_{P,c} & \text{if } u \in H_{P,1} \end{cases}
\end{aligned} \tag{2.4.23}
$$

$$
\begin{aligned}
\sigma^{(n)}_{P,1}(u) &= u + (-1)^{[\frac{n}{2}]} \textstyle\sum_{c \in C_{P,1}} I^{(n)}_P(u, \gamma_{P,c})\gamma_{P,c} \\
&= u + \textstyle\sum_{c \in C_{P,1}} ((-1)^n J_P(u, \gamma_{P,c}) - J_P(\gamma_{P,c}, u))\gamma_{P,c}, \\
&= \begin{cases} u + (-1)^n \sum_{c \in C_{P,1}} J_P(u, \gamma_{P,c})\gamma_{P,c} & \text{if } u \in H_{P,0} \\ (-1)^n u & \text{if } u \in H_{P,1}. \end{cases}
\end{aligned} \tag{2.4.24}
$$

Note that $\sigma^{(n)}_{P,0} = \sigma^{(n+2)}_{P,0}$ and $\sigma^{(n)}_{P,1} = \sigma^{(n+2)}_{P,1}$ for $n \in \mathbf{Z}_{\geq 0}$.

Note. From the definition immediately, we see the involutivity relations

$$
(\sigma^{(n)}_{P,0})^2 = (\sigma^{(n)}_{P,1})^2 = \mathrm{id}_{H_P} \quad \text{for odd } n \in \mathbf{Z}_{\geq 0} \tag{2.4.25}
$$

are satisfied. Using the fact that the type of the quiver Γ_P is either $A_{\frac{1}{2}\infty}$ or $D_{\frac{1}{2}\infty}$, i.e. the "inductive limit" of A_l or D_l for $l \to \infty$, we can show that there is no more relations among $\sigma^{(n)}_{P,0}$ and $\sigma^{(n)}_{P,1}$. Actually, we shall see in the next section that the eigenvalues in a suitable sense of the product $\sigma^{(n)}_{P,0} \circ \sigma^{(n)}_{P,1}$ is "dense" in the unit circle S^1 in $\mathbf{C}^\times$.

Definition. In analogy with the classical simple singularities, let us call the product of the two monodromy transformations $\sigma^{(n)}_{P,0}$ and $\sigma^{(n)}_{P,1}$ a *Coxeter element*. Two Coxeter elements depending on the order of the product are conjugate to each other. We fix one order as follows and call the product the Coxeter element.

(2.4.26)

$$
\begin{aligned}
Cox^{(n)}_P(u) &:= \sigma^{(n)}_{P,0} \circ \sigma^{(n)}_{P,1}\,(u) \\
&= \begin{cases} (-1)^n \big(u + \sum_{c \in C_{P,1}} J_P(u, \gamma_{P,c})\gamma_{P,c} \\ \quad - \sum_{c \in C_{P,1}} \sum_{d \in C_{P,0}} J_P(u, \gamma_{P,c}) J_P(\gamma_{P,d}, \gamma_{P,c})\gamma_{P,d}\big) & \text{if } u \in H_{P,0} \\ (-1)^n \big(u - \sum_{c \in C_{P,0}} J_P(\gamma_{P,c}, u)\gamma_{P,c}\big) & \text{if } u \in H_{P,1}. \end{cases}
\end{aligned}
$$

Observation. *The Coxeter element is, up to the sign factor $(-1)^n$, independent of the suspensions for $n \in \mathbf{Z}_{\geq 0}$* (2.3.21).

Remark. It is wellknown that a classical Coxeter element for a root system of finite type is semisimple of finite order, and $\frac{1}{2\pi i}\log$ of its eigenvalues, referred as *spectra*, play important role ([Bo]). The Coxeter elements of types $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$ are no longer of finite order. However, in the next section, we show that they are diagnalizable in suitable sense and the *spectra* for them are introduced, where the sign factor $(-1)^n$ of the Coxeter elements is lifted to the shift by $\frac{n}{2}$ of the spectra. The spectra should play a key role for primitive forms of type $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$ in a forth coming paper, where the shift of the spectra corresponds to the $\frac{n}{2}$-shift of the primitive forms in the semi-infinite Hodge filtration.

3. Spectra of Coxeter elements

We study spectra of the Coxeter element $Cox_P^{(n)}$ for $P \in \{\mathrm{A}_{\frac{1}{2}\infty}, \mathrm{D}_{\frac{1}{2}\infty}\}$. For the purpose, we extend the domain of the Coxeter element to the completion of $H_{P,C} := H_P \otimes_{\mathbf{Z}} \mathbf{C}$ with respect to the l^2-norm with the orthonormal basis $\{\gamma_{P,c}\}_{c\in \mathbf{C}_P}$. The Coxeter element action on this space is diagonalizable (in a suitable sense) where the eigenvalues take values in the unit circle $S^1 \subset \mathbf{C}^{\times}$. Then, we introduce the *spectra* of the Coxeter element as the $\frac{1}{2\pi\sqrt{-1}}$ log of the eigenvalues where the branch of the logarithm is normalized to the interval $(\frac{n-1}{2}, \frac{n+1}{2})$.

3.1. Hilbert space $\overline{H}_{P,\mathbf{C}}$. .

Consider $\mathbf{C}$-vector spaces obtained by the complexification of the $\mathbf{Z}$-lattices $H_{P,0}$, $H_{P,1}$ and H_P (recall (2.1.14),(2.1.15) and (2.1.16)):

$$H_{P,0,\mathbf{C}} := H_{P,0} \otimes_{\mathbf{Z}} \mathbf{C},\ H_{P,1,\mathbf{C}} := H_{P,1} \otimes_{\mathbf{Z}} \mathbf{C} \quad \text{and} \quad H_{P,\mathbf{C}} := H_P \otimes_{\mathbf{Z}} \mathbf{C}. \tag{3.1.27}$$

We equip them with a hermitian inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle \sum_{c\in C_P} a_c \gamma_{P,c}, \sum_{c\in C_P} b_c \gamma_{P,c}\rangle := \sum_{c\in C_P} a_c \bar{b}_c, \tag{3.1.28}$$

where a_c, b_c $(c \in C_P)$ are complex numbers. Then, the l^2-completions of the spaces with respect to this inner product are separable Hilbert spaces, denoted by $\overline{H}_{P,0,\mathbf{C}}$, $\overline{H}_{P,1,\mathbf{C}}$ and $\overline{H}_{P,\mathbf{C}}$, respectively. We have the orthogonal direct sum decomposition:

$$\overline{H}_{P,\mathbf{C}} = \overline{H}_{P,0,\mathbf{C}} \oplus \overline{H}_{P,1,\mathbf{C}}. \tag{3.1.29}$$

Let us denote by π_0 and π_1 the orthogonal projections of the space $\overline{H}_{P,\mathbf{C}}$ to the subspaces $\overline{H}_{P,0\mathbf{C}}$ and $\overline{H}_{P,1,\mathbf{C}}$, respectively, so that the sum

$$id_{\overline{H}_{P,\mathbf{C}}} = \pi_0 + \pi_1$$

is the identity map on $\overline{H}_{P,\mathbf{C}}$.

Remark that the lattice H_P is self-dual: $\mathrm{Hom}_{\mathbf{Z}}(H_{\mathbf{P}}, \mathbf{Z}) \cap \overline{H}_{P,\mathbf{C}} = H_P$.

Convention. In the sequel of the present paper, we freely identify a continuous bilinear form A on $\overline{H}_{P,\mathbf{C}}$ (resp. $H_{P,\mathbf{C}}$) and a continuous endomorphism $\dot{A}$ on $\overline{H}_{P,\mathbf{C}}$ (resp. $H_{P,\mathbf{C}}$) by the following relations:

$$A(\xi,\eta)=\langle \dot{A}(\xi),\eta\rangle \quad \text{and} \quad \sum_{c\in C_P} A(u,\gamma_{P,c})\gamma_{P,c}=\dot{A}(u).$$

Transposes ${}^tA$ of A and ${}^t(\dot{A})$ of $\dot{A}$ are defined by the relations ${}^tA(\xi,\eta)=A(\eta,\xi)$ and $\langle \dot{A}(u),v\rangle=\langle u,{}^t(\dot{A})(v)\rangle$, respectively. Then, ${}^t(\dot{A})=({}^tA)^{\cdot}$.

3.2. **Extendability of $I_P^{(n)}$ and $Cox_P^{(n)}$ on $\overline{H}_P$.** .

In order to calculate the eigenvalues of the intersection forms $I_P^{(n)}$ and the Coxeter elements $Cox_P^{(n)}$, we use the identification mentioned at the end of §3.1. Before we do this, we need to check that they are continuously extendable to the completion $\overline{H}_{P,\mathbf{C}}$. This is achieved by using the extendabilities of the endomorphisms $\dot{J}_P$, ${}^t\dot{J}_P$ associated with the bilinear forms (2.1.18) and (2.1.19). Put

$$\text{(3.2.30)} \qquad \begin{aligned} \dot{J}_P(u) &:= \textstyle\sum_{c\in C_P} J(u,\gamma_{P,c})\gamma_{P,c} \\ &= \begin{cases} u+\sum_{c\in C_{P,1}} J_P(u,\gamma_{P,c})\gamma_{P,c} & \text{if } u\in H_{P,0} \\ u & \text{if } u\in H_{P,1} \end{cases} \end{aligned}$$

$$\text{(3.2.31)} \qquad \begin{aligned} {}^t\dot{J}_P(u) &:= \textstyle\sum_{c\in C_P} {}^tJ(u,\gamma_{P,c})\gamma_{P,c} \\ &= \begin{cases} u & \text{if } u\in H_{P,0} \\ u+\sum_{c\in C_{P,0}} J_P(\gamma_{P,c},u)\gamma_{P,c} & \text{if } u\in H_{P,1} \end{cases} \end{aligned}$$

which are endomorphisms on $H_{P,\mathbf{C}}$, since the quiver Γ_P in §2.2 is locally finite, i.e. any vertex is connected with only finite number of other vertices. The inverse action of $\dot{J}_P$ (resp. ${}^t\dot{J}_P$) on $H_{P,\mathbf{C}}$ can be obtained by just replacing "+" by "−" in RHS of (3.2.30) (resp. (3.2.31)).

Assertion 1. *The endomorphisms $\dot{J}_P$, ${}^t\dot{J}_P$ and their inverses $\dot{J}_P^{-1}$, ${}^t\dot{J}_P^{-1}$ acting on $H_{P,\mathbf{C}}$ are extendable to bounded endomorphisms on $\overline{H}_{P,\mathbf{C}}$. The extensions are transpose to each other.*

Proof. We show only the extendability of the domain of endomorphisms $\dot{J}_P$, ${}^t\dot{J}_P$ and their inverses $\dot{J}_P^{-1}$, ${}^t\dot{J}_P^{-1}$ from $H_{P,\mathbf{C}}$ to $\overline{H}_{P,\mathbf{C}}$, where the extensions are denoted by the same notation. Then the relations ${}^t(\dot{J}_P)={}^t\dot{J}_P$, $\dot{J}_P\dot{J}_P^{-1}=\mathrm{id}_{H_P}, \dots$, etc. are automatically preserved for the extensions.

The quivers $\Gamma_{A_{\frac{1}{2}\infty}}$ and $\Gamma_{D_{\frac{1}{2}\infty}}$ show that any critical point $c\in C_{P,0}$ is adjacent to at most two bdd components. In view of (3.2.30), this implies the inequality $\|\dot{J}_P(u)-u\|\leq 2\|u\|$. Hence $\dot{J}_P$ is extendable to a bounded endomorphims on $\overline{H}_{P,\mathbf{C}}$, denoted by the same $\dot{J}_P$.

We observe also that, to any bdd component, at most 3 critical points in $C_{P,0}$ are adjacent (actually, 3 occurs only one bdd component for the critical point $c^{(1)}_{D,1}$ of type $\mathrm{D}_{\frac{1}{2}\infty}$). In view of (3.2.31), we get an inequality $\| \ {}^t\dot{J}_P(u) - u \| \leq 3 \| u \|$, implying again the extendability of ${}^t\dot{J}_P$ to a bounded endomorphism on $\overline{H}_{P,\mathbf{C}}$, denoted by the same ${}^t\dot{J}_P$.

Similar arguments shows the extendability of the inverses. $\square$

An immediate consequence of **Assertion 1** is that *the endomorphism*

$$(2.3.22)^\bullet \qquad \dot{I}^{(n)}_P := (-1)^{[\frac{n+1}{2}]}\dot{J}_P - (-1)^{[\frac{n}{2}]}\ {}^t\dot{J}_P$$

defined on $H_{P,\mathbf{C}}$ is extendable to a bounded endomorphism on $\overline{H}_{P,\mathbf{C}}$.

Another important consequence of **Assertion 1** is the following.

Corollary. *The Coxeter element $Cox^{(n)}_P$ ($n \in \mathbf{Z}_{\geq 0}$) defined on $H_{P,\mathbf{C}}$ is extendable to an invertible bounded automorphism on $\overline{H}_{P,\mathbf{C}}$.*

Proof. Let us, first, show a formula:

$$(3.2.32) \qquad Cox^{(n)}_P = (-1)^n\ ({}^t\dot{J}_P)^{-1}\dot{J}_P,$$

on H_P by a direct calculation using formulae (2.4.26), (3.2.30) and

$$(3.2.30)^{-1} \qquad ({}^t\dot{J}_P)^{-1}(u) \stackrel{!}{=} \begin{cases} u & \text{if } u \in H_{P,0} \\ u - \sum_{c\in C_{P,0}} J_P(\gamma_{P,c},u)\gamma_{P,c} & \text{if } u \in H_{P,1}. \end{cases} \qquad !$$

Then, RHS of (3.2.32) is extendable to a bounded operator on $\overline{H}_{P,\mathbf{C}}$.

Invertibility of $Cox^{(n)}_P$ follows from that of $\dot{J}_P$ and ${}^t\dot{J}_P$. $\square$

Remark. Let $\check{H}_{P,\mathbf{C}} := \mathrm{Hom}_{\mathbf{C}}(H_{P,\mathbf{C}},\mathbf{C})$ be the (formal) dual vector space of $H_{P,\mathbf{C}}$. The contragradient actions on $\check{H}_{P,\mathbf{C}}$ of the endomorphisms $\dot{J}_P$, ${}^t\dot{J}_P$, $\dot{I}^{(n)}_P$, ${}^t\dot{I}^{(n)}_P$, $Cox^{(n)}_P$ and ${}^tCox^{(n)}_P$ on $H_{P,\mathbf{C}}$ shall be denoted, as usual, by the super script " ^{t}(-) " such that " ^{tt}(-)=(-) ".

On the other hand, by regarding $\{\gamma_{P,c}\}_{c\in C_P}$ as the self-dual basis, $\check{H}_{P,\mathbf{C}}$ is identified with the direct product $\prod_{c\in C_P}\mathbf{C}\gamma_{P,c}$ so that we have natural inclusions of $\mathbf{C}$-vector spaces:

$$H_{P,\mathbf{C}} \subset \overline{H}_{P,\mathbf{C}} \subset \check{H}_{P,\mathbf{C}}.$$

Then it is easy to verify that the extensions of $\dot{J}_P$, ${}^t\dot{J}_P$, $\dot{I}^{(n)}_P$, ${}^t\dot{I}^{(n)}_P$, $Cox^{(n)}_P$ and ${}^tCox^{(n)}_P$ to the spaces $\overline{H}_{P,\mathbf{C}}$ and $\check{H}_{P,\mathbf{C}}$ are naturally compatible with respect to the above inclusions. The relationships between these extensions and the transpositions are given as follows:

$${}^t\dot{I}^{(n)}_P = (-1)^{n+1}\dot{I}^{(n)}_P \quad \text{and} \quad ({}^tCox^{(n)}_P)^{-1} = \dot{J}_P\ Cox^{(n)}_P\ \dot{J}_P^{-1}.$$

However, the bilinear form $I_{P,C}$ itself is no longer extandable to $\check{H}_{P,\mathbf{C}}$ and the endomorphism $\dot{I}_P$ on $\check{H}_{P,\mathbf{C}}$ has non-trivial kernel.

3.3. Spectral decomposition of $I_P^{(n)}$ for odd n. .

Using the fact (2.3.22), the bilinear form $I_P^{(n)}$ is symmetric for odd n. Let us consider the operator for the cases $n \in \mathbf{Z}_{\geq 0}$ with $n \equiv 3 \bmod 4$,[6]

$$\dot{I}_P \;:=\; \dot{I}_P^{(n)} \;=\; \dot{J}_P + {}^t\dot{J}_P. \tag{3.3.33}$$

We, first, determine the point spectrum of the symmetric operator $\dot{I}_P$ on $\overline{H}_{P,\mathbf{C}}$. Let us consider following two eigenspaces for $\lambda \in \mathbf{C}$:

$$\check{H}_{P,\lambda} := \{\xi \in \check{H}_{P,\mathbf{C}} \mid \dot{I}_P(\xi) = \lambda\xi\} \quad \text{and} \quad \overline{H}_{P,\lambda} := \check{H}_{P,\lambda} \cap \overline{H}_{P,\mathbf{C}}. \tag{3.3.34}$$

Assertion 2. *For each type $P \in \{\mathrm{A}_{\frac{1}{2}\infty}, \mathrm{D}_{\frac{1}{2}\infty}\}$ and all $\lambda \in \mathbf{C}$, we have*

$$\dim_{\mathbf{C}} \check{H}_{P,\lambda} = 1 \qquad \textit{and} \qquad \dim_{\mathbf{C}} \overline{H}_{P,\lambda} = 0, \tag{3.3.35}$$

except for the case $P = \mathrm{D}_{\frac{1}{2}\infty}$ and $\lambda = 2$, where we have

$$\dim_{\mathbf{C}} \check{H}_{\mathrm{D}_{\frac{1}{2}\infty},2} = 2 \qquad \textit{and} \qquad \dim_{\mathbf{C}} \overline{H}_{\mathrm{D}_{\frac{1}{2}\infty},2} = 1, \tag{3.3.36}$$

and $\overline{H}_{\mathrm{D}_{\frac{1}{2}\infty},2}$ is spanned by a vector $\eta_{\mathrm{D}_{\frac{1}{2}\infty},2} := \gamma_{D,0}^{+} - \gamma_{D,0}^{-}$.

Proof. This is shown by solving the equation $\dot{I}_P(\xi) = \lambda\xi$ for the coefficients of $\xi = \sum_{c \in C_P} a_c \gamma_{P,c} \in \check{H}_{P,\mathbf{C}}$ formally and inductively according to the following labeling and ordering of coefficients:

$$\Gamma_{\mathrm{A}_{\frac{1}{2}\infty}}: \qquad a_0 \longrightarrow a_1 \longleftarrow a_2 \longrightarrow a_3 \longleftarrow a_4 \longrightarrow a_5 \longleftarrow \quad \cdots$$

$$\Gamma_{\mathrm{D}_{\frac{1}{2}\infty}}: \qquad \begin{matrix} b_0^{+} \\ \nwarrow \\ \\ \swarrow \\ b_0^{-} \end{matrix} \!\!\! \rangle \; b_1 \longrightarrow b_2 \longleftarrow b_3 \longrightarrow b_4 \longleftarrow b_5 \longrightarrow \quad \cdots$$

Details of the calculation are omitted. Results are summerized as:

$\mathrm{A}_{\frac{1}{2}\infty}$: The space $\check{H}_{\mathrm{A}_{\frac{1}{2}\infty},\lambda}$ for any $\lambda \in \mathbf{C}$ is spanned by

$$\check{\xi}_{\mathrm{A}_{\frac{1}{2}\infty},\lambda}: \quad a_n = \frac{\exp((n+1)\sqrt{-1}\pi\theta) - \exp(-(n+1)\sqrt{-1}\pi\theta)}{\exp(\sqrt{-1}\pi\theta) - \exp(-\sqrt{-1}\pi\theta)} \quad (n \geq 0)$$

where θ is any complex number satisfying $\lambda = 4\sin^2(\frac{\pi}{2}\theta)$. In case $\lambda = 0$ or 4 (i.e. when $\theta \in \mathbf{Z}$), we interpret this formula as $a_n = \pm(n+1)$.

$\mathrm{D}_{\frac{1}{2}\infty}$: For all $\lambda \in \mathbf{C}$, let us introduce a vector

$$\check{\xi}_{\mathrm{D}_{\frac{1}{2}\infty},\lambda}: \quad b_0^{+} = 1,\; b_0^{-} = 1,\; b_n = \exp(n\sqrt{-1}\theta) + \exp(-n\sqrt{-1}\theta) \quad (n \geq 1)$$

where θ is any complex number satisfying the equation $\lambda = 4\sin^2(\frac{\pi}{2}\theta)$. Then, the space $\check{H}_{\mathrm{D}_{\frac{1}{2}\infty},\lambda}$ for any $\lambda \neq 2$ is spanned by $\check{\xi}_{\mathrm{D}_{\frac{1}{2}\infty},\lambda}$. The space $\check{H}_{\mathrm{D}_{\frac{1}{2}\infty},2}$ is spanned by $\check{\xi}_{\mathrm{D}_{\frac{1}{2}\infty},2}$ and

$$\eta_{\mathrm{D}_{\frac{1}{2}\infty}} := \gamma_{D,0}^{+} - \gamma_{D,0}^{-}: \quad b_0^{+} = 1,\; b_0^{-} = -1,\; b_n = 0 \;\; (n \geq 1).$$

The norm $\langle \check{\xi}_{P,\lambda}, \check{\xi}_{P,\lambda} \rangle$ (3.1.28) for all $P \in \{\mathrm{A}_{\frac{1}{2}\infty}, \mathrm{D}_{\frac{1}{2}\infty}\}$ and $\lambda \in \mathbf{C}$ is unbounded, whereas $\eta_{\mathrm{D}_{\frac{1}{2}\infty}}$ has the norm $\langle \eta_{\mathrm{D}_{\frac{1}{2}\infty}}, \eta_{\mathrm{D}_{\frac{1}{2}\infty}} \rangle = 2$. □

[6] We choose the form I_P for $n \equiv 3 \bmod 4$, since it is positive and symmetric, defining a "root lattice structure of infinite rank" on H_P (cf. Proof of Assertion 3.).

Corollary. *The point spectrum of the operator* $\dot{I}_{\mathrm{A}_{\frac{1}{2}\infty}}$ *on* $\overline{H}_{P,\mathbf{C}}$ *is empty, and that of* $\dot{I}_{\mathrm{D}_{\frac{1}{2}\infty}}$ *consists in the single eigenvalue* $\lambda = 2$ *with multiplicity* 1*. In particular, the operator* $\dot{I}_P$ *is non-degenerate on* $\overline{H}_{P,\mathbf{C}}$.

Remark. By introducing the double cover of the λ-plane by $\mu := \exp(\pi\sqrt{-1}\theta) \in \mathbf{C} \setminus \{0\}$ with the relation $2-\lambda = \mu+\mu^{-1}$, the base $\check{\xi}_{P,\lambda}$ in the proof of **Assertion 2** can be expressed in terms of Laurent polynomials in μ. Then, the reader may be puzzled in the above proof by the reason of introducing the parameter θ instead of μ. We used the parameter θ since it shall parametrize the spectra of Coxeter elements in the next paragraph. We remark also that $\lambda \in [0,4] \ \Leftrightarrow \theta \in \mathbf{R}$.

For a symmetric operator $\dot{I}_P$ on $\overline{H}_{P,\mathbf{C}}$, the *greatest lower bound* and the *least upper bound* are defined as the maximal real number m and the minimal real number M satisfying the following inequalities, respectively (see [R-N, §104]).

$$m\langle\xi,\xi\rangle \ \leq \ \langle\dot{I}_P(\xi),\xi\rangle = I_P(\xi,\xi) \ \leq \ M\langle\xi,\xi\rangle \quad \forall\xi \in \overline{H}_{P,\mathbf{C}} \tag{3.3.37}$$

Assertion 3. *The greatest lower bound* m *and the least upper bound* M *of* $\dot{I}_P$ *for both* $P \in \{\mathrm{A}_{\frac{1}{2}\infty}, \mathrm{D}_{\frac{1}{2}\infty}\}$ *is given by* $m = 0$ *and* $M = 4$.

Proof. For the definition of m and M, it is sufficient to run ξ only in H_P in the defining relation (3.3.37), since $H_{P,\mathbf{C}}$ is dense in $\overline{H}_{P,\mathbf{C}}$. Any $\xi \in H_P$ is contained in a sublattice L of H_P generated by the vertices of a finite (connected) subdiagram Γ of Γ_P (recall §2.2). Actually, Γ is a diagram of type either A_l or D_l for some $l \in \mathbf{Z}_{>0}$ and $I_P|_L$ gives a root lattice structure of that type on L. That is, $\{I_P(\gamma_{P,c},\gamma_{P,d})\}_{c,d\in\Gamma\subset C_P}$ is the Cartan matrix of type Γ. In particular, the eigenvalues of $\dot{I}_P|_L$ $(n\in\mathbf{Z}_{\geq 0})$ is given by $4\sin^2\big(\frac{\pi}{2}\frac{m_i}{h}\big)$ $(i=1,\cdots,l=\mathrm{rank}(L))$, where m_i are the exponents and h is the Coxeter number of the root system of type Γ (see e.g. [Bo]). Since the smallest and the largest exponent of the (finite) root system are 1 and $h-1$, respectively, the minimal and the maximal of the eigenvalues are $4\sin^2\big(\frac{\pi}{2}\frac{1}{h}\big)$ and $4\cos^2\big(\frac{\pi}{2}\frac{1}{h}\big)$, respectively. Since $h \to \infty$ according as Γ "exhaust" Γ_P, we obtain

$$\begin{aligned} m &= \inf_{\Gamma\subset\Gamma_P} 4\sin^2\Big(\frac{\pi}{2}\frac{1}{h}\Big) = \lim_{h\to\infty} 4\sin^2\Big(\frac{\pi}{2}\frac{1}{h}\Big) = 0. \\ M &= \sup_{\Gamma\subset\Gamma_P} 4\cos^2\Big(\frac{\pi}{2}\frac{1}{h}\Big) = \lim_{h\to\infty} 4\cos^2\Big(\frac{\pi}{2}\frac{1}{h}\Big) = 4. \end{aligned}$$

□

We apply the spectral decomposition theory of bounded symmetric operators (see [R-N, §107 Theorem]) to the operator $\dot{I}_P$. Let us reformulate the result in [ibid] by adjusting the notation to our setting.

Theorem 6. *For each type $P \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$, there exists a unique spectral family $\{E_{P,\lambda}\}_{\lambda\in\mathbf{R}}$ (i.e. a family of projection operators[7] on $\overline{H}_{P,\mathbf{C}}$ satisfying the following a), b), c)):*

a) For $\lambda \leq \mu$, one has $E_{P,\lambda} \leq E_{P,\mu}$ ($\underset{def}{\Leftrightarrow}$ $E_{P,\lambda}E_{P,\mu} = E_{P,\lambda}$).

b) The family is strongly continuous with respect to λ, i.e.

$$E_{P,\lambda+0}(:= \lim_{\mu\downarrow 0} E_{P,\lambda+\mu}) = E_{P,\lambda-0}(:= \lim_{\mu\uparrow 0} E_{P,\lambda-\mu}),$$

except at $\lambda = 2$ for type $P = D_{\frac{1}{2}\infty}$, where we have

$$E_{D_{\frac{1}{2}\infty},2+0} - E_{D_{\frac{1}{2}\infty},2-0} = \text{the projection: } \overline{H}_{D_{\frac{1}{2}\infty},\mathbf{C}} \to \overline{H}_{D_{\frac{1}{2}\infty},2}. \tag{3.3.38}$$

c) One has $E_{P,\lambda} = 0$ for $\lambda \leq 0$ and $E_{P,\lambda} = \mathrm{Id}_{\overline{H}_{P,\mathbf{C}}}$ for $\lambda \geq 4$.

so that following (3.3.39) holds.

$$(\dot{I}_P)^r = \int_0^4 \lambda^r dE_{P,\lambda} \qquad (\textit{for} \quad r = 0, 1, 2, \cdots). \tag{3.3.39}$$

where the integral is in the sense of Lebesgue-Stieltjes. [8]

3.4. **Spectra of Coxeter elements.** .

Recall that $\lambda \in [0, 4]$ in §3.3 Theorem 6 is the parameter for the spectra of the intersection form $I_P := I_P^{(n)}$ for $n \equiv 3 \bmod 4$. What is wonderful, is the fact that this parameter gives a clue to parametrize the spectra of the Coxeter elements $Cox_P^{(n)}$ for all $n \in \mathbf{Z}_{\geq 0}$. In order to achieve this, we introduce another parameter θ and re-parametrize λ by the relation (which we once observed in a proof of **Assertion 2**.)

$$\lambda \;=\; 4\sin^2\left(\theta\frac{\pi}{2}\right) \quad \text{for} \quad 0 \leq \theta \leq 1. \tag{3.4.40}$$

We state now the goal results of the present paper.

Theorem 7. *For each type $P \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$, by the coordinate transform (3.4.40), we introduce a Stieltjes measure on the interval $\theta \in [0, 1]$:*

$$\xi_{P,\theta} \;:=\; U_\theta \cdot dE_{P,\lambda} \cdot U_\theta^{-1} \tag{3.4.41}$$

---

[7] Here, we mean by a *projection operator* an orthogonal projection map from $\overline{H}_{P,\mathbf{C}}$ to its closed subspace such that the real form $\overline{H}_{P,\mathbf{R}}$ is mapped into itself. The fact that $E_{P,\lambda}$ is real, is not explicitly stated in the literature [R-N], but follows trivially from its construction and from the fact that $\dot{I}_P$ is real.

[8] More generally [R-N, §107 Theorem], for any complex valued continuous function $u(\lambda)$ on the interval $[0, 4]$, we have an equality $u(\dot{I}_P) = \int_0^4 u(\lambda)dE_{P,\lambda}$ between bounded operators, where LHS is defined by a (monotone decreasing) polynomial aproximation of u and RHS is given by the norm-limit of the Stieltjes type summation. Then, for any $\xi, \eta \in \overline{H}_{P,\mathbf{C}}$, we have $\langle u(\dot{I}_P)\xi, \eta\rangle = \int_0^4 u(\lambda)d\langle E_{P,\lambda}\xi, \eta\rangle$.

where U_θ $(0 \le \theta \le 1)$ *is a family of unitary operators on* $\overline{H}_{P,\mathbf{C}}$ *given by*

$$U_\theta := \exp\big(-\frac{\pi}{2}\sqrt{-1}\theta\big)\pi_0 - \exp\big(\frac{\pi}{2}\sqrt{-1}\theta\big)\pi_1, \tag{3.4.42}$$

and (i) $\{E_{P,\lambda}\}_{\lambda\in[0,4]}$ *is the spectral family in* §3.3 **Theorem 6**,
(ii) $\pi_i : \overline{H}_{P,\mathbf{C}} \to \overline{H}_{P,\mathbf{C},i}$ $(i = 0, 1)$ *are orthgonal projections.*

Then the following two formulae hold:

$$Cox_P^{(n)} \cdot \xi_{P,\theta} = \exp\left(2\pi\sqrt{-1}\big(\theta + \frac{n-1}{2}\big)\right)\xi_{P,\theta}, \tag{3.4.43}$$

and

$$\int_{\theta=0}^{\theta=1} \xi_{P,\theta} \ = \ \frac{1}{2}\ \dot{I}_P. \tag{3.4.44}$$

Proof. 1. Proof of (3.4.43).

Consider the infinitesimal form of the formula (3.3.39) for $r=1$:

$$\dot{I}_P \cdot dE_{P,\lambda} \ = \ \lambda dE_{P,\lambda}. \tag{3.4.45}$$

Substitute the decomposition $dE_{P,\lambda} = \pi_0 \cdot dE_{P,\lambda} + \pi_1 \cdot dE_{P,\lambda}$ in this formula. Then, using (3.3.33) , the LHS is equal to

$$\begin{aligned}
\dot{I}_P \cdot dE_{P,\lambda} &= (\dot{J}_P + {}^t\dot{J}_P)(\pi_0 \cdot dE_{P,\lambda} + \pi_1 \cdot dE_{P,\lambda}) \\
&= 2\ \pi_0 \cdot dE_{P,\lambda} + 2\ \pi_1 \cdot dE_{P,\lambda} \\
&\quad + (\dot{J}_P - id)(\pi_0 \cdot dE_{P,\lambda}) \ + \ (\dot{J}_P - id)(\pi_1 \cdot dE_{P,\lambda}) \\
&\quad + ({}^t\dot{J}_P - id)(\pi_0 \cdot dE_{P,\lambda}) \ + \ ({}^t\dot{J}_P - id)(\pi_1 \cdot dE_{P,\lambda}).
\end{aligned}$$

On the other hand, recalling (3.2.30) and (3.2.31), we know that

$$\begin{aligned}
&(\dot{J}_P - id)(\pi_1 \cdot dE_{P,\lambda}) = 0, \quad (\dot{J}_P - id)(\pi_0 \cdot dE_{P,\lambda}) \in \mathrm{Hom}(\overline{H}_{P,\mathbf{C}}, \overline{H}_{P,\mathbf{C},1}), \\
&({}^t\dot{J}_P - id)(\pi_0 \cdot dE_{P,\lambda}) = 0, \quad ({}^t\dot{J}_P - id)(\pi_1 \cdot dE_{P,\lambda}) \in \mathrm{Hom}(\overline{H}_{P,\mathbf{C}}, \overline{H}_{P,\mathbf{C},0}).
\end{aligned}$$

Equating this with $\lambda dE_{P,\lambda} = \lambda\pi_0 \cdot dE_{P,\lambda} + \lambda\pi_1 \cdot dE_{P,\lambda}$ (3.4.44), we obtain

$$({}^t\dot{J}_P - id)(\pi_1 dE_{P,\lambda}) = (\lambda-2)\pi_0 dE_{P,\lambda}, \ (\dot{J}_P - id)(\pi_0 dE_{P,\lambda}) = (\lambda-2)\pi_1 dE_{P,\lambda}.$$

Rewriting these together in matrix expressions, we obtain

$$\dot{J}_P \begin{pmatrix} \pi_0 \cdot dE_{P,\lambda} \\ \pi_1 \cdot dE_{P,\lambda} \end{pmatrix} = \begin{pmatrix} 1 & \lambda - 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \cdot dE_{P,\lambda} \\ \pi_1 \cdot dE_{P,\lambda} \end{pmatrix}. \tag{3.4.46}$$

$${}^t\dot{J}_P \begin{pmatrix} \pi_0 \cdot dE_{P,\lambda} \\ \pi_1 \cdot dE_{P,\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda - 2 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \cdot dE_{P,\lambda} \\ \pi_1 \cdot dE_{P,\lambda} \end{pmatrix}. \tag{3.4.47}$$

and, hence, also

$$({}^t\dot{J}_P)^{-1} \begin{pmatrix} \pi_0 \cdot dE_{P,\lambda} \\ \pi_1 \cdot dE_{P,\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 - \lambda & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \cdot dE_{P,\lambda} \\ \pi_1 \cdot dE_{P,\lambda} \end{pmatrix}.$$

Thus, combining these with the expression (3.2.32), we obtain

$$(3.4.48)\qquad Cox_P^{(n)}\begin{pmatrix}\pi_0\cdot dE_{P,\lambda}\\ \pi_1\cdot dE_{P,\lambda}\end{pmatrix}=(-1)^n\begin{pmatrix}1 & \lambda-2\\ 2-\lambda & 1-(\lambda-2)^2\end{pmatrix}\begin{pmatrix}\pi_0\cdot dE_{P,\lambda}\\ \pi_1\cdot dE_{P,\lambda}\end{pmatrix}.$$

Substitute λ in the RHS matrix by the expression (3.4.40) :

$$(-1)^n\begin{pmatrix}1 & \lambda-2\\ 2-\lambda & 1-(\lambda-2)^2\end{pmatrix}=(-1)^n\begin{pmatrix}1 & -2\cos(\pi\theta)\\ 2\cos(\pi\theta) & \sin^2(\pi\theta)-3\cos^2(\pi\theta)\end{pmatrix}.$$

We see that the matrix is semi-simple for any θ. The eigenvalues are

$$\exp\left(\pm 2\pi\sqrt{-1}(\ \theta+\frac{n-1}{2})\right),$$

and associated row eigenvectors (independent of n) are

$$\left(\exp\big(\mp\frac{\pi}{2}\sqrt{-1}\theta\big),-\exp\big(\pm\frac{\pi}{2}\sqrt{-1}\theta\big)\right).$$

Therefore, by introducing the unitary operators

$$(3.4.49)\qquad U_{\pm\theta}:=\exp\big(\mp\frac{\pi}{2}\sqrt{-1}\theta\big)\pi_0-\exp\big(\pm\frac{\pi}{2}\sqrt{-1}\theta\big)\pi_1$$

satisfying relations: ${}^tU_{\pm\theta}=U_{\pm\theta}=\overline{U_{\mp\theta}}$ and $U_{\pm\theta}\cdot U_{\mp\theta}=\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}$, we introduce a Stieltjes measure on $[0,4]:=\{\lambda\in\mathbf{R}\mid 0\le\lambda\le 4\}\simeq[0,1]:=\{\theta\in\mathbf{R}\mid 0\le\theta\le 1\}$:

$$(3.4.50)\qquad \xi_\theta^{\pm}:=U_{\pm\theta}\cdot dE_{P,\lambda}\cdot U_{\mp\theta}.$$

Then, from (3.4.48), we obtain

$$(3.4.51)\qquad Cox_P^{(n)}\cdot\xi_\theta^{\pm}=\exp\left(\pm 2\pi\sqrt{-1}(\theta+\frac{n-1}{2})\right)\xi_\theta^{\pm}.$$

Putting $\xi_{P,\theta}:=\xi_\theta^{+}$, we obtain (3.4.43).

2. Proof of (3.4.44).

Using (3.4.41) and (3.4.42), we decompose $\xi_{P,\theta}$ into 4 pieces:

$$\pi_0\cdot dE_{P,\theta}\cdot\pi_0+\pi_1\cdot dE_{P,\theta}\cdot\pi_1-\exp(\pi\sqrt{-1}\theta)\pi_1\cdot dE_{P,\theta}\cdot\pi_0-\exp(-\pi\sqrt{-1}\theta)\pi_0\cdot dE_{P,\theta}\cdot\pi_1.$$

The first two terms are integrated easily by

$$\int_{\theta=0}^{\theta=1}\pi_0\cdot dE_{P,\theta}\cdot\pi_0=\pi_0\cdot\left(\int_{\theta=0}^{\theta=1}dE_{P,\theta}\right)\cdot\pi_0=\pi_0\cdot\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}\cdot\pi_0=\pi_0,$$
$$\int_{\theta=0}^{\theta=1}\pi_1\cdot dE_{P,\theta}\cdot\pi_1=\pi_1\cdot\left(\int_{\theta=0}^{\theta=1}dE_{P,\theta}\right)\cdot\pi_1=\pi_1\cdot\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}\cdot\pi_1=\pi_1.$$

The third and fourth terms are integrated by the use of Footnote 8.

First, we introduce bounded nilpotent operators $\dot{K}_P:\overline{H}_{P,0,\mathbf{C}}\to\overline{H}_{P,1,\mathbf{C}}$ and ${}^t\dot{K}_P:\overline{H}_{P,1,\mathbf{C}}\to\overline{H}_{P,0,\mathbf{C}}$, by $\dot{K}_P:=\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}-\dot{J}_P$ and ${}^t\dot{K}_P:=\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}-{}^t\dot{J}_P$ so that we have $\dot{K}_P^2={}^t\dot{K}_P^2=0$ and $\dot{I}_P=2\ \mathrm{id}_{\overline{H}_{P,\mathbf{C}}}-\dot{K}_P-{}^t\dot{K}_P$. Then,

$$\begin{aligned}
&\int_{\theta=0}^{\theta=1} \exp(\pi\sqrt{-1}\theta)\pi_1 \cdot dE_{P,\theta}\cdot \pi_0\\
= \;& \pi_1 \left[\int_{\theta=0}^{\theta=1} \left(1-2\sin^2(\tfrac{\pi}{2}\theta)+\sqrt{-1}\; 2\sqrt{1-\sin^2(\tfrac{\pi}{2}\theta)}\sin(\tfrac{\pi}{2}\theta)\right) dE_{P,\lambda}\right]\pi_0\\
= \;& \pi_1 \left[\int_{\theta=0}^{\theta=1} \left(1-\tfrac{\lambda}{2}+\tfrac{\sqrt{-1}}{2}\sqrt{(4-\lambda)\lambda}\right) dE_{P,\lambda}\right]\pi_0\\
= \;& \pi_1 \left[\mathrm{id}_{\overline{H}_{P,\mathbf{C}}} - \tfrac{\dot I_P}{2} + \tfrac{\sqrt{-1}}{2}\sqrt{(4\,\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}-\dot I_P)\dot I_P}\right]\pi_0
\end{aligned}$$

After sandwitching by π_1 and π_0, the first and the second terms turn out to be $\pi_1 \cdot \mathrm{id}_{\overline{H}} \cdot \pi_0 = 0$ and $\pi_1 \cdot \frac{\dot I_P}{2}\cdot \pi_0 = -\frac{\dot K_P}{2}$, respectively. The third term turns out to be zero, since the operator

$$\begin{aligned}
\sqrt{(4\,\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}-\dot I_P)\dot I_P} &= \sqrt{(2\,\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}+\dot K_P + {}^t\dot K_P)(2\,\mathrm{id}_{\overline{H}_{P,\mathbf{C}}}-\dot K_P - {}^t\dot K_P)}\\
&= \sqrt{4\,\mathrm{id}_{\overline{H}_{P,\mathbf{C}}} - \dot K_P\cdot {}^t\dot K_P - {}^t\dot K_P\cdot \dot K_P}
\end{aligned}$$

preserves the decomposition (3.1.29) so that it does not have the "cross" term sandwitched by π_1 and π_0. Thus, we get

$$\int_{\theta=0}^{\theta=1} \exp(\pi\sqrt{-1}\theta)\pi_1 \cdot dE_{P,\lambda}\cdot\pi_0 = \frac{\dot K_P}{2}.$$

Similarly, we obtain also

$$\int_{\theta=0}^{\theta=1} \exp(-\pi\sqrt{-1}\theta)\pi_0 \cdot dE_{P,\lambda}\cdot\pi_1 = \frac{{}^t\dot K_P}{2}.$$

These altogether show the formula (3.4.44) □

Corollary. *Let $\varphi(\theta) = \sum_{m\in\mathbf{Z}} a_m \exp(2\pi\sqrt{-1}m(\theta+\frac{n-1}{2}))$ be an absolutely convergent Fourier expansion of a complex valued continuous function on the interval $\theta\in[0,1]$. Then, we have*

$$(3.4.52) \qquad 2\int_{\theta=0}^{\theta=1} \varphi(\theta)\cdot\xi_\theta = \sum_{m\in\mathbf{Z}} a_m (Cox_P^{(n)})^m \cdot \dot I_P.$$

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