ON THE FIRST AND SECOND K-GROUPS OF AN ELLIPTIC CURVE OVER A GLOBAL FIELD OF POSITIVE CHARACTERISTIC

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Abstract. It is shown that the maximal divisible subgroup of the K_1 and K_2 of an elliptic curve E over a function field are uniquely divisible, and those K-groups modulo the uniquely divisible subgroups are computed explicitly. We also calculate the motivic cohomology groups of the minimal regular model of E, which is an elliptic surface over a finite field.

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1. Introduction

The aim of this paper is to compute explicitly the K_1, K_2 and the motivic cohomology groups of an elliptic curve over a function field, and the motivic cohomology groups of an elliptic surface over a finite field. In this introduction, let us state the thoerems (Theorems 1.1, 1.2) concerning the K_1 and K_2 of an elliptic curve. We refer to Theorems 6.1, 6.2, 6.3, 7.1, 7.2 for other main results.

Let k be a global field of positive characteristic p and let E be an elliptic curve over $\operatorname{Spec} k$. Let C be the proper smooth irreducible curve over a finite field whose function field is k. We regard a place $\wp$ of k as a closed point of C and vice versa. We let $\kappa(\wp)$ denote the residue field at $\wp$ of C. Let $f : \mathcal{E} \to C$ denote the minimal regular model of the elliptic curve $E \to \operatorname{Spec} k$. This f is a proper, flat, generically smooth morphism such that for almost all closed points $\wp$ of C, the fiber $\mathcal{E} \times_C \operatorname{Spec} \kappa(\wp)$ at $\wp$ is a genus one curve, and such that the generic fiber is the elliptic curve $E \to \operatorname{Spec} k$.

Let us identify the K-theory and the G-theory of regular noetherian schemes. There is a localization sequence of G-theory:

$$K_i(\mathcal{E}) \to K_i(E) \xrightarrow{\oplus \partial^i_\wp} \bigoplus_{\wp} G_i(\mathcal{E}_\wp) \to K_i(\mathcal{E})$$

where $\wp$ runs over all primes of k. Let us denote by ∂_2 the boundary map $\oplus\partial^2_\wp$, by $\partial : K_2(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ the boundary map induced by ∂_2, and by $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ the boundary map induced by $\oplus\partial^1_\wp$. Here, for an

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abelian group M, we let M_{div} denote the maximal divisible subgroup of M, and we put $M^{\mathrm{red}} = M/M_{\mathrm{div}}$.

In Theorems 1.1, 1.2 below, we use the following notation. We use the subscript $-_{\mathbb{Q}}$ to mean $-\otimes_{\mathbb{Z}}\mathbb{Q}$. For a scheme X, let X_0 (resp. $\mathrm{Irr}(X)$) denote the set of the closed points (resp. the irreducible components) of X. Let $\mathbb{F}_q$ denote the field of constants of C. For a scheme X of finite type over $\mathrm{Spec}\,\mathbb{F}_q$ and for $i \in \mathbb{Z}$, choose a prime number $\ell \neq p$ and put $L(h^i(X), s) = \det(1 - \mathrm{Frob}\cdot q^{-s}; H^i_{\mathrm{et}}(X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,\overline{\mathbb{F}}_q, \mathbb{Q}_\ell))$ where $\overline{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$ and $\mathrm{Frob} \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the geometric Frobenius element. In all the cases considered in Theorems 1.1, 1.2, the function $L(h^i(X), s)$ does not depend on the choice of ℓ. We let $T'_{(1)}$ denote what we call the twisted Mordell-Weil group $T'_{(1)} = \bigoplus_{\ell\neq p}(E(k\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_q)_{\mathrm{tors}}\otimes_{\mathbb{Z}}\mathbb{Z}_\ell(1))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$. We write S_0 (resp. S_2) for the set of primes of k at which E has split multiplicative (resp. bad reduction); we also regard it as a closed subscheme of C with the reduced structure (The set of primes S_1 will be introduced later). For a set M, we denote its cardinality by $|M|$. We put $r = |S_0|$.

Theorem 1.1 (see Theorem 6.1(1)(2), Theorem 7.2(2))**.** *Suppose that S_2 is nonempty, or equivalently f is not smooth.*

1. *The dimension of the $\mathbb{Q}$-vector space $(K_2(E)^{\mathrm{red}})_{\mathbb{Q}}$ is r.*
2. *The cokernel of the boundary map $\partial_2 : K_2(E) \to \bigoplus_{\wp\in C_0} G_1(\mathcal{E}_\wp)$ is a finite group of order*
$$\frac{(q-1)^2|L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})), -1)|}{|T'_{(1)}|\cdot|L(h^0(S_2), -1)|}.$$
3. *The group $K_2(E)_{\mathrm{div}}$ is uniquely divisible, and the kernel of the boundary map $\partial : K_2(E)^{\mathrm{red}} \to \bigoplus_{\wp\in C_0} G_1(\mathcal{E}_\wp)$ is a finite group of order $|L(h^1(C), -1)|^2$*

Let $L(E, s)$ denote the L-function of E (see Section 6.3). We write $\mathrm{Jac}(C)$ for the Jacobian of C.

Theorem 1.2 (see Theorem 6.1(3)(4))**.** *Suppose that S_2 is nonempty, or equivalently f is not smooth.*

1. *The group $K_1(E)_{\mathrm{div}}$ is uniquely divisible.*
2. *The kernel of the boundary map $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp\in C_0} G_0(\mathcal{E}_\wp)$ is a finite group of order $(q-1)^2|T'_{(1)}|\cdot|L(E, 0)|$. The cokernel of ∂_1 is a finitely generated abelian group of rank $2 + |\mathrm{Irr}(\mathcal{E}_{S_2})| - |S_2|$ whose torsion subgroup is isomorphic to $\mathrm{Jac}(C)(\mathbb{F}_q)^{\oplus 2}$.*

Remark 1.1. Takao Yamazaki has pointed the authors to the similarity between Theorem 1.2(2) and the Birch-Tate conjecture (see [44, p.206-207]). The Birch-Tate conjecture concerns the K_2 of a global field in any characteristic. They study the boundary map, denoted λ, from K_2 to the direct sum of the K_1 of the residue fields. A part of the conjecture is that the order of the kernel of the boundary map λ is expressed using the special value $\zeta_F(-1)$ and an invariant w_F, which is expressed in terms of the number of roots of unity).

In our theorem, we consider K_1 instead of K_2, an elliptic curve over a function field instead of the function field itself (regarded as a zero-dimensional variety over a function field), the Hasse-Weil L-function $L(E, s)$ instead of the zeta function. The value $|T'_{(1)}|$ plays the role of w_F in our setting. There is no counterpart for the factor

$(q-1)^2$ in their conjecture. This is due to the fact that there are more than one degrees for which the cohomology groups are (conjecturally) nonzero in our setting because the variety is one-dimensional as opposed to being zero-dimensional.

It seems interesting to formulate a conjecture for curves of higher genus over a function field, and over a global field of characteristic zero, but we have not persued this point.

The principal ingredient in proving these theorems is the following theorem which was proved in our previous paper:

Theorem 1.3. [25, Theorem 1.1] *Let the notations be as above. Then for an arbitrary set S of closed points of C, the homomorphism induced by the boundary map ∂_2*

$$K_2(E)_{\mathbb{Q}} \xrightarrow{\oplus \partial_{\wp\mathbb{Q}}} \bigoplus_{\wp \in S} G_1(\mathcal{E}_\wp)_{\mathbb{Q}}$$

is surjective.

Let us give an outline of proof. First, given an elliptic surface $\mathcal{E}$, we relate the order of (a quotient by some divisible subgroup of) the motivic cohomology groups and the special values of the zeta function of $\mathcal{E}$. For many of the (bi-)degrees, this is made possible by the Beilinson-Lichtenbaum conjecture (now a theorem using the Bloch-Kato conjecture and a theorem of Geisser and Levine), but there are degrees where it is not a direct consequence of the conjecture. In such cases, we need some extra work. The details on this extra work are found in the beginning of Section 6.

Then, we recall in Section 6.3 that the main part of the zeta function of $\mathcal{E}$ is the L-function of E. We note that there are contributions from the L-function of the base curve C and also of the singular fibers of $\mathcal{E}$ (Lemma 6.3, Corollary 6.4). We relate the G-groups of the singular fibers to the motivic cohomology groups and then they are in turn related to the values of the L-function (Lemmas 6.8 and 6.9). The technical input for these lemmas are our result from our paper [24] and the classification of the singular fibers of an elliptic fibration.

The sections are organized as follows. In Sections 2–5, we consider the general case of a curve of arbitrary genus over a function field and a smooth surface over a finite field. Those sections are fairly independent of others. In Section 2, we compute the motivic cohomology groups of an arbitrary smooth surface X over finite fields. The difficult case is that of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$ and is treated in Section 2.2. In Section 3, we prove the compatibility of Chern characters and the localization sequence of motivic cohomology. In Section 4, we define Chern characters for singular curves over finite fields. The treatment is quite ad hoc. In Section 5, we give the relation via the Chern class map between the K_1 and K_2 of curves over function fields and the motivic cohomology groups. We present our main result in Section 6. Applying the results in Sections 2, 4, and 5, and using the special features of elliptic surfaces, including Theorem 1.3, we compute the orders of certain torsion groups explicitly. We treat the p-part separately in Appendix A. See its introduction for more technical details. In Section 7, we will use the Bloch-Kato conjecture (see Lemma 2.1) and generalize the results in Section 6.

2. Motivic cohomology groups of smooth surfaces

Aside from the uniquely divisible part, we understand the motivic cohomology groups of smooth surfaces over finite fields fairly well. The divisible part is conjecturally zero.

The main goal of this section is to prove Theorem 2.1. Let us give a brief description of the statement. Let X be a smooth surface over a finite field. Then we find that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is an extension of a finitely generated abelian group by a uniquely divisible group except when $(i, j) = (3, 2)$. Except for the cases $(i, j) = (0, 0), (1, 1), (3, 2), (4, 2)$, the finitely generated abelian group is a finite group, which is either 0 or is written in terms of étale cohomology groups of X. We refer to Theorem 2.1, the following table, and the following paragraph for the details.

For a prime number ℓ, we let $|\ |_\ell : \mathbb{Q}_\ell \to \mathbb{Q}$ denote the ℓ-adic absolute value normalized so that $|\ell|_\ell = \ell^{-1}$.

2.1. **Motivic cohomology of surfaces over a finite field.** Let $\mathbb{F}_q$ be the field of q elements of characteristic p. For a separated scheme X which is essentially of finite type over $\operatorname{Spec} \mathbb{F}_q$, we define the motivic cohomology group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ as the homology group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) = H_{2j-i}(z^j(X, \bullet))$ of Bloch's cycle complex $z^j(X, \bullet)$ ([7, Introduction, p. 267] see also [14, 2.5, p. 60] to remove the condition that X is quasi-projective). When X is essentially smooth over $\operatorname{Spec} \mathbb{F}_q$, it coincides with the motivic cohomology group defined in [26, Part I, Chapter I, 2.2.7, p. 21] or [47] (cf. [27, Theorem 1.2, p. 300], [46, Corollary 2, p. 351]). For a discrete abelian group M, we put $H^i_{\mathcal{M}}(X, M(j)) = H_{2j-i}(z^j(X, \bullet) \otimes_{\mathbb{Z}} M)$.

We apologize that this notation is not appropriate if X is not essentially smooth (for in that case it should be a Borel-Moore homology group). A reason for using this notation is that, in Section 4.2, we will define Chern classes for higher Chow groups of low degrees as if higher Chow groups form a cohomology theory.

First let us recall that the groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ have been known for $j \leq 1$. By definition, $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) = 0$ for $j \leq 0$ and $(i, j) \neq (0, 0)$, and $H^0_{\mathcal{M}}(X, \mathbb{Z}(0)) = H^0_{\mathrm{Zar}}(X, \mathbb{Z})$. We have $H^i_{\mathcal{M}}(X, \mathbb{Z}(1)) = 0$ for $i \neq 1, 2$. By [7, Theorem 6.1, p. 287], we have $H^1_{\mathcal{M}}(X, \mathbb{Z}(1)) = H^0_{\mathrm{Zar}}(X, \mathbb{G}_m)$, and $H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) = \operatorname{Pic}(X)$.

The following is a conjecture of Bloch-Kato ([22, §1, Conjecture 1, p. 608]). and now is proved by Rost, Voevodsky, Haesemeyer, and Weibel.

Lemma 2.1. *Let $j \geq 1$ be an integer. Then for any finitely generated field K over $\mathbb{F}_q$ and for any positive integer $\ell \neq p$, the symbol map $K^M_j(K) \to H^j_{\mathrm{et}}(\operatorname{Spec} K, \mathbb{Z}/\ell(j))$ is surjective.*

Definition 2.1. Let M be an abelian group. We say that M is *finite modulo a uniquely divisible subgroup* (resp. *finitely generated modulo a uniquely divisible subgroup*) if M_{div} is uniquely divisible and M^{red} is finite (resp. M_{div} is uniquely divisible and M^{red} is finitely generated).

We note that, if M is finite modulo a uniquely divisible subgroup, then M_{tors} is a finite group and $M = M_{\mathrm{div}} \oplus M_{\mathrm{tors}}$.

Recall that for a scheme X, we let $\operatorname{Irr}(X)$ denote the set of irreducible components of X. The aim of Section 2.1 is to prove the following theorem.

Theorem 2.1. *Let X be a smooth surface over $\mathbb{F}_q$. Let $R \subset \mathrm{Irr}(X)$ denote the subset of irreducible components of X which is projective over $\operatorname{Spec} \mathbb{F}_q$. For $X' \in \mathrm{Irr}(X)$, let $q_{X'}$ denote the cardinality of the field of constants of X'.*

1. *The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))$ is finitely generated modulo a uniquely divisible subgroup if $i \neq 3$ or if X is projective. More precisely,*
   - (a) *The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))$ is zero for $i \geq 5$.*
   - (b) *The group $H^4_{\mathcal{M}}(X, \mathbb{Z}(2))$ is a finitely generated abelian group of rank $|R|$.*
   - (c) *If $i \leq 1$ or if X is projective and $i \leq 3$, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))$ is finite modulo a uniquely divisible subgroup.*
   - (d) *The group $H^2_{\mathcal{M}}(X, \mathbb{Z}(2))$ is finitely generated modulo a uniquely divisible subgroup.*
   - (e) *For $i \leq 2$, the cohomology group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$ is canonically isomorphic to the direct sum $\bigoplus_{\ell \neq p} H^{i-1}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$. In particular, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))$ is uniquely divisible for $i \leq 0$.*
   - (f) *If X is projective, then the group $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$ is isomorphic to the direct sum of the group $\bigoplus_{\ell \neq p} H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ and a finite p-group of order $|\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)| \cdot |L(h^2(X), 0)|_p^{-1}$. Here we let $\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)$ denote the set of morphisms $\mathrm{Pic}^o_{X/\mathbb{F}_q} \to \mathbb{G}_m$ of group schemes over $\operatorname{Spec} \mathbb{F}_q$.*
2. *Let $j \geq 3$ be an integer. Then for any integer i, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo a uniquely divisible subgroup. More precisely,*
   - (a) *The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is zero for $i \geq \max(6, j+1)$, is isomorphic to $\bigoplus_{X' \in R} \mathbb{Z}/(q_{X'}^{j-2} - 1)$ for $(i,j) = (5,3)$, $(5,4)$, and is finite for $(i,j) = (4,3)$.*
   - (b) *The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ is canonically isomorphic to the direct sum $\bigoplus_{\ell \neq p} H^{i-1}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$. In particular, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is uniquely divisible for $i \leq 0$ or $6 \leq i \leq j$, and the group $H^1_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ is isomorphic to the direct sum $\bigoplus_{X' \in \mathrm{Irr}(X)} \mathbb{Z}/(q_{X'}^j - 1)$.*

In the following table, we summarize the description of the groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ stated in Theorem 2.1. Here we write u.d, f./u.d., f.g./u.d., f., f.g. for uniquely divisible, finite modulo a uniquely divisible subgroup, finite generated modulo a uniquely divisible subgroup, finite, finitely generated respectively.

<table>
<tr><th>$j \setminus i$</th><th>< 0</th><th>0</th><th>$0 < i < j$</th><th>$j(\neq 0)$</th><th>$j+1$</th><th>$j+2$</th><th>$\geq j+3$</th></tr>
<tr><td>0</td><td>0</td><td>$H^0(\mathbb{Z})$</td><td colspan="2">-</td><td colspan="3">0</td></tr>
<tr><td>1</td><td colspan="2">0</td><td>-</td><td>$H^0(\mathbb{G}_m)$</td><td>$\mathrm{Pic}(X)$</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">2</td><td colspan="2" rowspan="2">u. d.</td><td rowspan="2">f./u. d.</td><td>f. g./u. d.</td><td>?</td><td rowspan="2">f. g.</td><td rowspan="2">0</td></tr>
<tr><td colspan="2">f./u. d. if projective</td></tr>
<tr><td>3</td><td colspan="2">u. d.</td><td colspan="2">f./u. d.</td><td colspan="2">f.</td><td>0</td></tr>
<tr><td>4</td><td colspan="2">u. d.</td><td colspan="2">f./u. d.</td><td>f.</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">≥ 5</td><td colspan="2" rowspan="2">u. d.</td><td colspan="2">f./u. d.</td><td colspan="3" rowspan="2">0</td></tr>
<tr><td colspan="2">u. d. if $6 \leq i \leq j$</td></tr>
</table>

Lemma 2.2. *Let X be a separated scheme essentially of finite type over $\operatorname{Spec} \mathbb{F}_q$. Let i and j be integers. If both $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ and $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ are finite, then $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo a uniquely divisible subgroup and its torsion subgroup is isomorphic to $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$.*

Proof. Let us consider the exact sequence

$$(2.1)\qquad \begin{aligned} &0 \to H^{i-1}_{\mathcal{M}}(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j)) \\ &\to H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}} \to 0. \end{aligned}$$

Since $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ is a finite group, all the groups in the above exact sequence are finite groups. Then the group $H^{i-1}_{\mathcal{M}}(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ must be zero since it is finite and divisible. Hence we have a canonical isomorphism $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$. The finiteness of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ implies that the divisible group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{div}}$ is uniquely divisible and the canonical homomorphism

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(j))^{\mathrm{red}} \to \varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}(j))/m$$

is injective. The latter group $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}(j))/m$ is canonically embedded in the finite group $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$. Hence we conclude that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))^{\mathrm{red}}$ is finite. This proves the claim. □

Lemma 2.3. *Let X be a smooth projective surface over $\mathbb{F}_q$. Let j be an integer. Then the group $H^i_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ and the group $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ are finite if $i \neq 2j$ or $j \geq 3$.*

Proof. The claim for $j \leq 1$ is clear. Suppose that $j = 2$. Then the claim for $i \geq 5$ is clear. If $p \nmid m$, from [14, Corollary 1.2, p. 56. See also Corollary 1.4] and [30, (11.5), THEOREM, p. 328], it follows that the cycle class map $H^i_{\mathcal{M}}(X, \mathbb{Z}/m(2)) \to H^i_{\mathrm{et}}(X, \mathbb{Z}/m(2))$ is an isomorphism for $i \leq 2$ and is injective for $i = 3$. By [9, Théorème 2, p. 780] and the exact sequence [9, 2.1 (29) p. 781], for $i \leq 3$, the group $\varinjlim_{m,\, p\nmid m} H^i_{\mathrm{et}}(X, \mathbb{Z}/m(2))$ and the group $\varprojlim_{m,\, p\nmid m} H^i_{\mathrm{et}}(X, \mathbb{Z}/m(2))$ are finite.

Let $W_n\Omega^\bullet_{X,\log}$ denote the logarithmic de Rham-Witt sheaf (cf. [21, I, 5.7, p. 596]). This was introduced by Milne in [32]. There is an isomorphism $H^i_{\mathcal{M}}(X, \mathbb{Z}/p^n(2)) \cong H^{i-2}_{\mathrm{Zar}}(X, W_n\Omega^2_{X,\log})$ (cf. [13, Theorem 8.4, p. 491]). In particular, we have $H^i_{\mathcal{M}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$ for $i \leq 1$.

By [9, §2, Théorème 3, p. 782], $\varinjlim_n H^i_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$ is a finite group for $i = 0, 1$. By [9, Pas n^o 1, p. 783], the projective system $\{H^i(X, W_n\Omega^2_{X,\log})\}_n$ satisfies the Mittag-Leffler condition. Using the argument used in [9, Pas n^o 4, p. 784], we obtain the exact sequence

$$\begin{aligned} &0 \to \varprojlim_n H^i(X, W_n\Omega^2_{X,\log}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to \varinjlim_n H^i(X, W_n\Omega^2_{X,\log}) \\ &\to \varprojlim_n H^{i+1}(X, W_n\Omega^2_{X,\log})_{\mathrm{tors}} \to 0. \end{aligned}$$

We then see that $\varprojlim_n H^i_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$ is also finite for $i = 0, 1$ and is isomorphic to $\varinjlim_n H^{i-1}_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$. Since the homomorphism

$$H^i_{\mathrm{Zar}}(X, W_n\Omega^2_{X,\log}) \to H^i_{\mathrm{et}}(X, W_n\Omega^2_{X,\log}),$$

induced by the change of topology $\varepsilon : X_{\mathrm{et}} \to X_{\mathrm{Zar}}$, is an isomorphism for $i = 0$ and is injective for $i = 1$, we see that $\varprojlim_n H^2_{\mathcal{M}}(X, \mathbb{Z}/p^n(2))$ is zero, and that both $H^2_{\mathcal{M}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ and $\varprojlim_n H^3_{\mathcal{M}}(X, \mathbb{Z}/p^n(2))$ are finite groups. This proves the claim for $j = 2$.

Suppose $j \geq 3$. The claim for the case $i \geq 2j$ is clear. Since $j \geq 3$, we have $H^i_{\mathcal{M}}(X, \mathbb{Z}/p^n(j)) \cong H^{i-j}_{\mathrm{Zar}}(X, W_n\Omega^j_{X,\log}) = 0$. Using Lemma 2.1 for j, by [14, Theorem 1.1, p. 56], the group $H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ is isomorphic to the group $H^i_{\mathrm{Zar}}(X, \tau_{\leq j} R\varepsilon_* \mathbb{Z}/m(j))$ if $p \nmid m$. Since any affine surface over $\mathbb{F}_q$ has ℓ-cohomological dimension 3 for any $\ell \neq p$, it follows that $H^i_{\mathrm{Zar}}(X, \tau_{\leq j} R\varepsilon_* \mathbb{Z}/m(j)) \cong H^i_{\mathrm{et}}(X, \mathbb{Z}/m(j))$ for all i. Hence by [9, Théorème 2, p. 780] and the exact sequence [9, 2.1 (29) p. 781], for $i \leq 2j-1$, the groups $H^i_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ and $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ are finite. This proves the claim for $j \geq 3$. □

Lemma 2.4. *Let Y be a scheme of dimension $d \leq 1$ which is of finite type over* $\operatorname{Spec} \mathbb{F}_q$. *Then $H^i_{\mathcal{M}}(Y, \mathbb{Z}(j))$ is a torsion group unless $0 \leq j \leq d$ and $j \leq i \leq 2j$.*

Proof. By taking a smooth affine open subscheme of Y_{red} whose complement is of dimension zero, and using the localization sequence of motivic cohomology, we are reduced to the case where Y is connected, affine, and smooth over $\operatorname{Spec} \mathbb{F}_q$. When $d = 0$ (resp. $d = 1$), the claim follows from the result of Quillen [41, THEOREM 8(i), p. 583] (resp. Harder [19, Korollar 3.2.3, p. 175] (see [16, Theorem 0.5, p. 70] for the correct interpretation of his result)) on the structure of the K-groups of Y, combined with the Riemann-Roch theorem for higher Chow groups [7, Theorem 9.1, p. 296]. □

We use the following lemma, whose proof is easy and is left to the reader.

Lemma 2.5. *Let $\varphi : M \to M'$ be a homomorphism of abelian groups such that* $\operatorname{Ker} \varphi$ *is finite and* $(\operatorname{Coker} \varphi)_{\mathrm{div}} = 0$. *If M_{div} or M'_{div} is uniquely divisible, then φ induces an isomorphism $M_{\mathrm{div}} \xrightarrow{\cong} M'_{\mathrm{div}}$.* □

Proof of Theorem 2.1. Without loss of generality, we may assume that X is connected. We first prove the claims assuming X is projective. It is clear that the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is zero for $i \geq \min(j+3, 2j+1)$. It follows from [9, p. 787, Proposition 4] that the degree map $H^4_{\mathcal{M}}(X, \mathbb{Z}(2)) = \mathrm{CH}_0(X) \to \mathbb{Z}$ has finite kernel and cokernel. This proves the claim for $i \geq \min(j+3, 2j)$. Fix $j \geq 2$. For $i \leq 2j-1$, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo a uniquely divisible subgroup by Lemmas 2.2 and 2.3. The claim on the identification of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ with the étale cohomology follows immediately from the argument in the proof of Lemma 2.3 except for the p-primary part of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$, which follows from Proposition A.1.

To finish the proof, it remains to prove that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{div}}$ is zero for $j \geq 3$ and $i = j+1, j+2$. It suffices to prove that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is a torsion group for $j \geq 3$ and $i \geq j+1$. Consider the limit

$$\varinjlim_{Y} H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \to \varinjlim_{Y} H^i_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j))$$

of the localization sequence where Y runs over the reduced closed subschemes of X of pure codimension one. The group $H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1))$ is torsion by Lemma 2.4 and we have $\varinjlim_Y H^{i-2}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j-1)) = 0$ for dimension reasons. Hence the claim follows. This completes the proof in the case where X is projective.

For general connected surface X, take an embedding $X \hookrightarrow X'$ of X into a smooth projective surface X' over $\mathbb{F}_q$ such that $Y = X' \setminus X$ is of pure codimension one in X'. We can show that such an X' exists by using [35] and a resolution of singularities ([3, p. 111], [28, p. 151]). Then the claims, except for that on the identification of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ with the étale cohomology, easily follow from Lemma 2.5 and by using the localization sequence

$$\cdots \to H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1)) \to H^i_{\mathcal{M}}(X', \mathbb{Z}(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \to \cdots.$$

The claim on the identification of $H^i_{\mathcal{M}}(X,\mathbb{Z}(j))_{\mathrm{tors}}$ with the étale cohomology can be obtained in a way similar to that in the proof of Lemma 2.3. This completes the proof. □

2.2. **A criterion for the finiteness of $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$.**

Proposition 2.1. *Let X be a smooth surface over $\mathbb{F}_q$. Let $X \hookrightarrow X'$ be an open immersion such that X' is smooth projective over $\mathbb{F}_q$ and $Y = X' \setminus X$ is of pure codimension one in X'. Then the following conditions are equivalent.*

(1) *The group $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$ is finitely generated modulo a uniquely divisible subgroup.*
(2) *The group $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$ is finite.*
(3) *The pull-back map $H^3_{\mathcal{M}}(X',\mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$ induces an isomorphism $H^3_{\mathcal{M}}(X',\mathbb{Z}(2))_{\mathrm{div}} \xrightarrow{\cong} H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{div}}$.*
(4) *The kernel of the pull-back map $H^3_{\mathcal{M}}(X',\mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$ is finite.*
(5) *The cokernel of the boundary map $\partial : H^2_{\mathcal{M}}(X,\mathbb{Z}(2)) \to H^1_{\mathcal{M}}(Y,\mathbb{Z}(1))$ is finite.*

Moreover, if the above equivalent conditions are satisfied, then the torsion group $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$ is isomorphic to the direct sum of a finite group of p-power order and the group $\bigoplus_{\ell\neq p} H^2_{\mathrm{et}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$, and the localization sequence induces the long exact sequence

$$\cdots \to H^{i-2}_{\mathcal{M}}(Y,\mathbb{Z}(1)) \to H^i_{\mathcal{M}}(X',\mathbb{Z}(2))^{\mathrm{red}} \to H^i_{\mathcal{M}}(X,\mathbb{Z}(2))^{\mathrm{red}} \to \cdots \tag{2.2}$$

of finitely generated abelian groups.

Proof. The condition (1) clearly implies the condition (2). The localization sequence shows that the conditions (4) and (5) are equivalent and that the condition (3) implies the condition (1). By the localization sequence and Lemma 2.5, the condition (4) implies the condition (3).

We claim that the condition (2) implies the condition (4). Assume the condition (2) and suppose that the condition (4) is not satisfied. We put $M = \mathrm{Ker}[H^3_{\mathcal{M}}(X',\mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X,\mathbb{Z}(2))]$. The localization sequence shows M is finitely generated. By assumption, M is not torsion. Since $H^3_{\mathcal{M}}(X',\mathbb{Z}(2))$ is finite modulo a uniquely divisible subgroup, the intersection $H^3_{\mathcal{M}}(X',\mathbb{Z}(2))_{\mathrm{div}} \cap M$ is a non-trivial free abelian group of finite rank. Hence the group $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$ contains a group isomorphic to $H^3_{\mathcal{M}}(X',\mathbb{Z}(2))_{\mathrm{div}}/(H^3_{\mathcal{M}}(X',\mathbb{Z}(2))_{\mathrm{div}} \cap M)$, which contradicts the condition (2). Hence the condition (2) implies the condition (4). This completes the proof of the equivalence of the conditions (1)-(5).

Suppose that the conditions (1)-(5) are satisfied. The localization sequence shows that the kernel (resp. the cokernel) of the pull-back $H^i_{\mathcal{M}}(X',\mathbb{Z}(2)) \to H^i_{\mathcal{M}}(X,\mathbb{Z}(2))$ is a torsion group (resp. has no non-trivial divisible subgroup) for any $i \in \mathbb{Z}$. Hence, by Lemma 2.5, $H^i_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{div}}$ is uniquely divisible and the sequence (2.2) is exact. The condition (2) and the exact sequence (2.1) for $(i,j) = (3,2)$ give the isomorphism $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}} \cong H^2_{\mathcal{M}}(X,\mathbb{Q}/\mathbb{Z}(2))^{\mathrm{red}}$. Then the claim on the structure of $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$ follows from [14, Corollary 1.2, p. 56. See also Corollary 1.4] and [30, (11.5), THEOREM, p. 328]. This completes the proof. □

Let X be a smooth projective surface over $\mathbb{F}_q$. Suppose that X admits a flat, surjective and generically smooth morphism $f : X \to C$ to a connected, smooth projective curve C over $\mathbb{F}_q$. For each point $\wp \in C$, let $X_\wp = X \times_C \wp$ denote the fiber of f at $\wp$.

Corollary 2.1. *Let the notations be as above. Let $\eta \in C$ denote the generic point. Suppose that the cokernel of the homomorphism $\partial : H^2_{\mathcal{M}}(X_\eta, \mathbb{Z}(2)) \to \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(X_\wp, \mathbb{Z}(1))$, which is the inductive limit of the boundary maps of the localization sequences, is a torsion group. Then the group $H^i_{\mathcal{M}}(X_\eta, \mathbb{Z}(2))_{\mathrm{div}}$ is uniquely divisible for all $i \in \mathbb{Z}$ and the inductive limit of localization sequences induces the long exact sequence*

$$\cdots \to \bigoplus_{\wp \in C_0} H^{i-2}_{\mathcal{M}}(X_\wp, \mathbb{Z}(1)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(2))^{\mathrm{red}} \to H^i_{\mathcal{M}}(X_\eta, \mathbb{Z}(2))^{\mathrm{red}} \to \cdots .$$

Proof. Since the group $\bigoplus_{\wp \in C_0} H^i_{\mathcal{M}}(X_\wp, \mathbb{Z}(1))$ has no non-trivial divisible subgroup for all $i \in \mathbb{Z}$, and is torsion for $i \neq 1$ by Lemma 2.4, the claim follows from Lemma 2.5. □

3. The compatibility of Chern characters and the localization sequence

The aim of this section is to prove Lemma 3.1. What we will need in later sections is the statement of Lemma 3.1 and Remark 3.1. We will not use the details of the proof; the reader may skip them.

This lemma is known as the Riemann-Roch without denominators for higher K-theory. For example, it is known to hold for the setup given in [15]. Throughout, we use Bloch's higher Chow groups as motivic cohomology theory, but the Riemann-Roch without denominators is not known (at least not in the literature) for the Chern class as defined in [7].

Instead, we will use as the Chern class, the Chern class for motivic cohomology of Levine (see below) composed with the comparison isomorphism between Levine's motivic cohomology groups and higher Chow groups. Then since the Riemann-Roch theorem is known to hold for Levine's motivic cohomology, our task is to check the compatibility of the localization sequence and the comparison isomorphisms (Lemmas 3.2 and 3.3).

3.1. **Main statement.** Given an essentially smooth scheme X over $\operatorname{Spec} \mathbb{F}_q$ and integers $i, j \geq 0$, let $c_{i,j} : K_i(X) \to H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j))$ be the Chern class map. Several ways are proposed to construct the map $c_{i,j}$ ([7, p. 293], [26, Part I, Chapter III, 1.4.8. Examples. (i), p. 123], [40, DEFINITION 5, p. 315]). All of them are based on the method of Gillet [15, p. 228–229, Definition 2.22]. In this paper we adopt the definition of Levine [26, Part I, Chapter III, 1.4.8. Examples. (i), p. 123], where the map $c_{i,j}$ is denoted by $c^{j,2j-i}_X$. The definition, given in [26, Part I, Chapter I, 2.2.7, p. 21], of the target $H^{2j-i}(X, \mathbb{Z}(j)) = H^{2j-i}_X(X, \mathbb{Z}(j))$ of $c^{j,2j-i}_X$, which we will denote by $H^{2j-i}_{\mathcal{L}}(X, \mathbb{Z}(j))$, is different from the definition of the group $H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j))$. However [26, Part I, Chapter II, 3.6.6. THEOREM, p. 105] shows that there is a canonical isomorphism

$$\beta^i_j : H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \xrightarrow{\cong} H^i_{\mathcal{L}}(X, \mathbb{Z}(j)) \tag{3.1}$$

which is compatible with the product structures. The precise definition of our Chern class map $c_{i,j}$ is the composition $c_{i,j} = (\beta^i_j)^{-1} \circ c^{j,2j-i}_X$.

The map $c_{i,j}$ is a group homomorphism if $i \geq 1$ or $(i,j) = (0,1)$. We let $\mathrm{ch}_{0,0} : K_0(X) \to H^0_{\mathcal{M}}(X, \mathbb{Z}(0)) \cong H^0_{\mathrm{Zar}}(X, \mathbb{Z})$ denote the homomorphism which sends the class of a locally free $\mathcal{O}_X$-module $\mathcal{F}$ to the rank of $\mathcal{F}$. For $i \geq 1$ and $a \in K_i(X)$, we put formally $\mathrm{ch}_{i,0}(a) = 0$.

Lemma 3.1. *Let X be a scheme which is a localization of a smooth quasi-projective scheme over* $\operatorname{Spec} \mathbb{F}_q$*. Let $Y \subset X$ be a closed subscheme of pure codimension d which is essentially smooth over* $\operatorname{Spec} \mathbb{F}_q$*. Then for $i, j \geq 1$ or $(i,j) = (0,1)$, the diagram*

$$\begin{array}{ccc} K_i(Y) & \xrightarrow{\alpha_{i,j}} & H^{2j-i-2d}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) \\ \downarrow & & \downarrow \\ K_i(X) & \xrightarrow{c_{i,j}} & H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j)) \\ \downarrow & & \downarrow \\ K_i(X \setminus Y) & \xrightarrow{c_{i,j}} & H^{2j-i}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j)) \\ \downarrow & & \downarrow \\ K_{i-1}(Y) & \xrightarrow{\alpha_{i-1,j}} & H^{2j-i-2d+1}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) \end{array} \tag{3.2}$$

is commutative. Here the homomorphism $\alpha_{i,j}$ is defined as follows: for $a \in K_i(Y)$, the element $\alpha_{i,j}(a)$ equals

$$G_{d,j-d}(\mathrm{ch}_{i,0}(a), c_{i,1}(a), \ldots, c_{i,j-d}(a); c_{0,1}(\mathcal{N}), \ldots, c_{0,j-d}(\mathcal{N})),$$

where $G_{d,j-d}$ is the universal polynomial in [2, Exposé 0, Appendice, Proposition 1.5, p. 37] *and $\mathcal{N}$ is the conormal sheaf of Y in X, and the left (resp. the right) vertical sequence is the localization sequence of K-theory (resp. of higher Chow groups established in* [6, Corollary (0.2), p. 357]*).*

Proof. We may assume that X is quasi-projective and smooth over $\operatorname{Spec} \mathbb{F}_q$. It follows from [26, Part I, Chapter III, 1.5.2, p. 130] and the Riemann-Roch theorem without denominators [26, Part I, Chapter III, 3.4.7. THEOREM, p. 174] that the diagram (3.2) is commutative if we replace the right vertical sequence by the Gysin sequence

$$\begin{aligned} &H^{2j-i-2d}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) \to H^{2j-i}_{\mathcal{L}}(X, \mathbb{Z}(j)) \\ &\to H^{2j-i}_{\mathcal{L}}(X \setminus Y, \mathbb{Z}(j)) \to H^{2j-i-2d+1}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) \end{aligned} \tag{3.3}$$

in [26, Part I, Chapter III, 2.1, p. 132]. It suffices to show that the Gysin sequence (3.3) is identified with the localization sequence of higher Chow groups. We use the notations in [26, Part I, Chapter I, II]. Let $S = \operatorname{Spec} \mathbb{F}_q$ and let $\mathcal{V}$ denote the category of schemes which is essentially smooth over $\operatorname{Spec} \mathbb{F}_q$. Let $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ be the DG category defined in [26, Part I, Chapter I, 1.4.10 DEFINITION, p. 15]. For an object Z in $\mathcal{V}$ and a morphism $f : Z' \to Z$ in $\mathcal{V}$ which admits a smooth section, and for $j \in \mathbb{Z}$, we have an object $\mathbb{Z}_Z(j)_f$ in $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$. When $f = \mathrm{id}_Z$ is the identity, we abbreviate $\mathbb{Z}_Z(j)_{\mathrm{id}_Z}$ by $\mathbb{Z}_Z(j)$. For a closed subset $W \subset Z$, let $\mathbb{Z}_{Z,W}(j)$ be the object introduced in [26, Part I, Chapter I, (2.1.3.1), p. 17]; it is an object in the DG category $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$ of bounded complexes in $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$. The object $\mathbb{Z}_Z(j)_f$ belongs to the full subcategory $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})^*$ of $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ introduced in [26, Part I, Chapter I, 3.1.5, p. 38], and the object $\mathbb{Z}_{Z,W}(j)$ belongs to the DG category $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$ of bounded complexes in $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})^*$. For $i \in \mathbb{Z}$ we put $H^i_{\mathcal{L},W}(Z, \mathbb{Z}(j)) =$

$\mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \mathbb{Z}_{Z,W}(j)[i])$, where 1 denotes the object $\mathbb{Z}_{\mathrm{Spec}\,\mathbb{F}_q}(0)$ and $\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})$ denotes the category introduced in [26, Part I, Chapter I, 2.1.4 DEFINITION, p. 17–18].

Let X, Y be as in the statement of Lemma 3.1. Let $\mathbf{K}^b_{\mathrm{mot}}(\mathcal{V})$ be the homotopy category of $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$. We have a distinguished triangle

$$\mathbb{Z}_{X,Y}(j) \to \mathbb{Z}_X(j) \to \mathbb{Z}_{X\setminus Y}(j) \xrightarrow{+1}$$

in $\mathbf{K}^b_{\mathrm{mot}}(\mathcal{V})$. This distinguished triangle yields a long exact sequence

$$\begin{aligned} \cdots \to H^i_{\mathcal{L},Y}(X, \mathbb{Z}(j)) \to H^i_{\mathcal{L}}(X, \mathbb{Z}(j)) \\ \to H^i_{\mathcal{L}}(X \setminus Y, \mathbb{Z}(j)) \to H^{i+1}_{\mathcal{L},Y}(X, \mathbb{Z}(j)) \to \cdots . \end{aligned} \tag{3.4}$$

In [26, Part I, Chapter III, (2.1.2.2), p. 132], Levine constructs an isomorphism $\iota_* : \mathbb{Z}_Y(j-d)[-2d] \to \mathbb{Z}_{X,Y}(j)$ in $\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})$. This isomorphism induces an isomorphism $\iota_* : H^{i-2d}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) \xrightarrow{\cong} H^i_{\mathcal{L},Y}(X, \mathbb{Z}(j))$. This isomorphism, together with the long exact sequence (3.4) gives the Gysin sequence (3.3).

We put $z^j_Y(X, -\bullet) = \mathrm{Cone}(z^j(X, -\bullet) \to z^j(X \setminus Y, -\bullet))[-1]$ and define cohomology with support $H^i_{\mathcal{M},Y}(X, \mathbb{Z}(j)) = H^{i-2j}(z^j_Y(X, -\bullet))$. The distinguished triangle

$$z^j_Y(X, -\bullet) \to z^j(X, -\bullet) \to z^j(X \setminus Y, -\bullet) \xrightarrow{+1}$$

in the derived category of abelian groups induces a long exact sequence

$$\begin{aligned} \cdots \to H^i_{\mathcal{M},Y}(X, \mathbb{Z}(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \\ \to H^i_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j)) \to H^{i+1}_{\mathcal{M},Y}(X, \mathbb{Z}(j)) \to \cdots . \end{aligned} \tag{3.5}$$

The push-forward map $z^{j-d}(Y, -\bullet) \to z^j(X, -\bullet)$ of cycles gives a homomorphism $z^{j-d}(Y, -\bullet) \to z^j_Y(X, -\bullet)$ of complexes of abelian groups, which is known to be a quasi-isomorphism by [6, Theorem (0.1), p. 537]. Hence it induces an isomorphism $\iota_* : H^{2j-i-2d}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) \xrightarrow{\cong} H^{2j-i}_{\mathcal{M},Y}(X, \mathbb{Z}(j))$. Then the claim follows from Lemmas 3.2 and 3.3 below. □

3.2. **Compatibility of localization sequences.**

Lemma 3.2. *For each $i, j \in \mathbb{Z}$, there exists a canonical isomorphism*

$$\beta^i_{Y,j} : H^i_{\mathcal{M},Y}(X, \mathbb{Z}(j)) \xrightarrow{\cong} H^i_{\mathcal{L},Y}(X, \mathbb{Z}(j))$$

such that the long exact sequence (3.4) is identified with the long exact sequence (3.5) via this isomorphism and the isomorphism (3.1).

Proof. The functor $\mathcal{Z}_{\mathrm{mot}}$ ([26, Part I, Chapter I, (3.3.1.2), p. 40]) from the category $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$ to the category of bounded complexes of abelian groups is compatible with taking cones. Hence the DG functor $\mathcal{Z}_{\mathrm{mot}}(\ ,*)$ ([26, Part I, Chapter II, 2.2.4. DEFINITION, p. 68]) from the category $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$ to the category of complexes of abelian groups which are bounded from below is also compatible with taking cones. Since $\mathcal{Z}_{\mathrm{mot}}(\mathbb{Z}_X(j)_{\mathrm{id}_X}, *)$ is canonically isomorphic to the cycle complex $z^j(X, -\bullet)$, the complex $\mathcal{Z}_{\mathrm{mot}}(\mathbb{Z}_{X,Y}(j)_{\mathrm{id}_X}, *)$ is canonically isomorphic to the complex $z^j_Y(X, -\bullet)$. For an object Γ in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$, let $\mathcal{CH}(\Gamma, p)$ be the higher Chow group defined in [26, Part I, Chapter II, 2.5.2. DEFINITION, p. 76]. From the definition of $\mathcal{CH}(\Gamma, p)$, we obtain canonical homomorphisms $H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j)) \to \mathcal{CH}(\mathbb{Z}_X(j), i)$,

$H^{2j-i}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j)) \to \mathcal{CH}(\mathbb{Z}_{X\setminus Y}(j), i)$, and $H^{2j-i}_{\mathcal{M},Y}(X, \mathbb{Z}(j)) \to \mathcal{CH}(\mathbb{Z}_{X,Y}(j), i)$ such that the diagram

$$\begin{array}{ccc} H^{2j-i}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) & \longrightarrow & \mathcal{CH}(\mathbb{Z}_{X,Y}(j),i) \\ \downarrow & & \downarrow \\ H^{2j-i}_{\mathcal{M}}(X,\mathbb{Z}(j)) & \longrightarrow & \mathcal{CH}(\mathbb{Z}_{X}(j),i) \\ \downarrow & & \downarrow \\ H^{2j-i}_{\mathcal{M}}(X \setminus Y,\mathbb{Z}(j)) & \longrightarrow & \mathcal{CH}(\mathbb{Z}_{X\setminus Y}(j),i) \\ \downarrow & & \downarrow \\ H^{2j-i+1}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) & \longrightarrow & \mathcal{CH}(\mathbb{Z}_{X,Y}(j),i-1) \end{array} \tag{3.6}$$

is commutative.

We recall the definition of the cycle class map $\mathrm{cl}(\Gamma) : \mathcal{CH}(\Gamma) = \mathcal{CH}(\Gamma, 0) \to \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma)$ ([26, p. 76]) for an object $\Gamma \in \mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$. We recall that $\mathcal{CH}(\Gamma) = \varinjlim_{\Gamma \to \Gamma_{\widetilde{\mathcal{U}}}} H^0(\mathcal{Z}_{\mathrm{mot}}(\mathrm{Tot}\,\Gamma_{\widetilde{\mathcal{U}}}, *))$ where $\Gamma \to \Gamma_{\widetilde{\mathcal{U}}}$ runs over the hyper-resolutions of Γ ([26, Part I, Chapter II, 1.4.1. DEFINITION, p. 59]), and $\mathrm{Tot} : \mathbf{C}^b(\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*) \to \mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$ denotes the total complex functor in [26, Part I, Chapter II, 1.3.2, p. 58]. The homomorphism $\mathrm{cl}(\Gamma)$ is defined as the inductive limit of the composite

$$\begin{aligned} \mathrm{cl}_{\mathrm{naif}}(\mathrm{Tot}\,\Gamma_{\widetilde{\mathcal{U}}}) : \quad & H^0(\mathcal{Z}_{\mathrm{mot}}(\mathrm{Tot}\,\Gamma_{\widetilde{\mathcal{U}}}, *)) \to \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \mathrm{Tot}\,\Gamma_{\widetilde{\mathcal{U}}}) \\ & \xleftarrow{\cong} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma). \end{aligned}$$

For an object Γ in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$, the homomorphism $\mathrm{cl}_{\mathrm{naif}}(\Gamma)$ is, by definition ([26, Part I, Chapter II, (2.3.6.1), p. 71]), equal to the composite

$$\begin{aligned} H^0(\mathcal{Z}_{\mathrm{mot}}(\Gamma, *)) \quad & \xleftarrow{\cong} H^0(\mathcal{Z}_{\mathrm{mot}}(\Sigma^N(\Gamma)[N])) \\ & \xleftarrow{\cong} \mathrm{Hom}_{\mathbf{K}^b_{\mathrm{mot}}(\mathcal{V})}(\mathfrak{e}^{\otimes a} \otimes 1, \Sigma^N(\Gamma)[N]) \\ & \to \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(\mathfrak{e}^{\otimes a} \otimes 1, \Sigma^N(\Gamma)[N]) \\ & \xleftarrow{\cong} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma) \end{aligned}$$

for sufficiently large integers $N, a \geq 0$. Here Σ^N is the suspension functor in [26, Part I, Chapter II, 2.2.2. DEFINITION, p. 68], and $\mathfrak{e}$ is the object in [26, Part I, Chapter I, 1.4.5, p. 13] which we regard as an object in $\mathcal{A}_{\mathrm{mot}}(V)$. Let $\Gamma \to \Gamma'$ be a morphism in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$ and put $\Gamma'' = \mathrm{Cone}(\Gamma \to \Gamma')[-1]$. Since the functor Σ^N is compatible with cones, the diagram

$$\begin{array}{ccc}
\mathcal{CH}(\Gamma'', i) & \xrightarrow{\mathrm{cl}(\Gamma''[-i])} & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma''[-i]) \\
\downarrow & & \downarrow \\
\mathcal{CH}(\Gamma, i) & \xrightarrow{\mathrm{cl}(\Gamma[-i])} & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma[-i]) \\
\downarrow & & \downarrow \\
\mathcal{CH}(\Gamma', i) & \xrightarrow{\mathrm{cl}(\Gamma'[-i])} & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma'[-i]) \\
\downarrow & & \downarrow \\
\mathcal{CH}(\Gamma'', i-1) & \xrightarrow{\mathrm{cl}(\Gamma''[-i+1])} & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma''[-i+1])
\end{array} \tag{3.7}$$

is commutative.

The homomorphism $\beta^i_j : H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \to H^{2j-i}_{\mathcal{L}}(X, \mathbb{Z}(j))$ is, by definition, equal to the composite

$$\begin{aligned}
&H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \to \mathcal{CH}(\mathbb{Z}_X(j), 2j-i) \\
&\xrightarrow{\mathrm{cl}(\mathbb{Z}_X(j)[i-2j])} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \mathbb{Z}_X(j)[i]) = H^i_{\mathcal{L}}(X, \mathbb{Z}(j)).
\end{aligned}$$

We define the homomorphism $\beta^i_{Y,j} : H^i_{\mathcal{M},Y}(X, \mathbb{Z}(j)) \to H^{2j-i}_{\mathcal{L},Y}(X, \mathbb{Z}(j))$ to be the composite

$$\begin{aligned}
&H^i_{\mathcal{M},Y}(X, \mathbb{Z}(j)) \to \mathcal{CH}(\mathbb{Z}_{X,Y}(j), 2j-i) \\
&\xrightarrow{\mathrm{cl}(\mathbb{Z}_{X,Y}(j)[i-2j])} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \mathbb{Z}_{X,Y}(j)[i]) = H^i_{\mathcal{L},Y}(X, \mathbb{Z}(j)).
\end{aligned}$$

By (3.6) and (3.7), we have a commutative diagram

$$\begin{array}{ccc}
H^{2j-i}_{\mathcal{M},Y}(X, \mathbb{Z}(j)) & \xrightarrow{\beta^{2j-i}_{Y,j}} & H^{2j-i}_{\mathcal{L},Y}(X, \mathbb{Z}(j)) \\
\downarrow & & \downarrow \\
H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j)) & \underset{\cong}{\xrightarrow{\beta^{2j-i}_{j}}} & H^{2j-i}_{\mathcal{L}}(X, \mathbb{Z}(j)) \\
\downarrow & & \downarrow \\
H^{2j-i}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j)) & \underset{\cong}{\xrightarrow{\beta^{2j-i}_{j}}} & H^{2j-i}_{\mathcal{L}}(X \setminus Y, \mathbb{Z}(j)) \\
\downarrow & & \downarrow \\
H^{2j-i+1}_{\mathcal{M},Y}(X, \mathbb{Z}(j)) & \xrightarrow{\beta^{2j-i+1}_{j}} & H^{2j-i+1}_{\mathcal{L},Y}(X, \mathbb{Z}(j))
\end{array}$$

where the right vertical arrow is the long exact sequence (3.4). Hence $\beta^{2j-i}_{Y,j}$ is an isomorphism and the claim follows. □

3.3. **Compatibility of Gysin maps.**

Lemma 3.3. *The diagram*

$$\begin{array}{ccc} H^{2j-i-2d}_{\mathcal{M}}(Y,\mathbb{Z}(j-d)) & \xrightarrow[\cong]{\iota_*} & H^{2j-i}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \\ {\scriptstyle \beta^{2j-i-2d}_{j-d}}\downarrow\cong & & {\scriptstyle \beta^{2j-i}_{Y,j}}\downarrow\cong \\ H^{2j-i-2d}_{\mathcal{L}}(Y,\mathbb{Z}(j-d)) & \xrightarrow[\cong]{\iota_*} & H^{2j-i}_{\mathcal{L},Y}(X,\mathbb{Z}(j)) \end{array}$$

is commutative.

Proof. Let us recall the construction of the upper horizontal isomorphism ι_* in [26, Part I, Chapter III, (2.1.2.2), p. 132]. Let Z be the blow-up of $X \times_{\operatorname{Spec}\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$ along $Y \times_{\operatorname{Spec}\mathbb{F}_q} \{0\}$. Let W be the proper transform of $Y \times_{\operatorname{Spec}\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$ to Z. Then W is canonically isomorphic to $Y \times_{\operatorname{Spec}\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$. Let P be the inverse image of $Y \times_{\operatorname{Spec}\mathbb{F}_q} \{0\}$ under the map $Z \to X \times_{\operatorname{Spec}\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$ and let $Q = P \times_Z W$. We put $Z' = Z \amalg (X \times_{\operatorname{Spec}\mathbb{F}_q} \{1\}) \amalg P$ and let $f : Z' \to Z$ denote the canonical morphism. We have canonical morphisms

$$\mathbb{Z}_{P,Q}(j) \leftarrow \mathbb{Z}_{Z,W}(j)_f \to \mathbb{Z}_{X\times_{\operatorname{Spec}\mathbb{F}_q}\{1\},Y\times_{\operatorname{Spec}\mathbb{F}_q}\{1\}}(j) = \mathbb{Z}_{X,Y}(j)$$

in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$, which become isomorphisms in the category $\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})$.

Let $g : P \to Y \times_{\operatorname{Spec}\mathbb{F}_q} \{0\} \cong Y$ be the canonical morphism. The restriction of g to $Q \subset P$ is an isomorphism and hence gives a section $s : Y \to P$ to g. The cycle class $\mathrm{cl}^d_{P,Q}(Q) \in H^{2d}_Q(P,\mathbb{Z}(d))$ in [26, Part I, Chapter I, (3.5.2.7), p. 48] comes from the map $[Q]_Q : \mathfrak{e} \otimes 1 \to \mathbb{Z}_{P,Q}(d)[2d]$ in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$ defined in [26, Part I, Chapter I, (2.1.3.3), p. 17]. We have morphisms

$$(3.8)\qquad \begin{array}{rl} & \mathfrak{e} \otimes \mathbb{Z}_P(j-d)[-2d] \to \mathbb{Z}_{P,Q}(d) \otimes \mathbb{Z}_P(j-d) \\ \xrightarrow{\gamma} & \mathbb{Z}_{P\times_{\operatorname{Spec}\mathbb{F}_q}P,Q\times_{\operatorname{Spec}\mathbb{F}_q}P}(j) \\ \leftarrow & \mathbb{Z}_{P\times_{\operatorname{Spec}\mathbb{F}_q}P,Q\times_{\operatorname{Spec}\mathbb{F}_q}P}(j)_{f'} \xrightarrow{\Delta_P^*} \mathbb{Z}_{P,Q}(j). \end{array}$$

in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$. Here γ is the map induced from the external products $\boxtimes_{P,P} : \mathbb{Z}_P(d) \otimes \mathbb{Z}_P(j-d) \to \mathbb{Z}_{P\times_{\operatorname{Spec}\mathbb{F}_q}P}(j)$ and $\boxtimes_{Q,P} : \mathbb{Z}_Q(d) \otimes \mathbb{Z}_P(j-d) \to \mathbb{Z}_{Q\times_{\operatorname{Spec}\mathbb{F}_q}P}(j)$, the morphism $\Delta_P : P \to P \times_{\operatorname{Spec}\mathbb{F}_q} P$ denotes the diagonal embedding, and f' is the morphism

$$f' = \mathrm{id}_{P\times_{\operatorname{Spec}\mathbb{F}_q}P} \amalg \Delta_P : P \times_{\operatorname{Spec}\mathbb{F}_q} P \amalg P \to P \times_{\operatorname{Spec}\mathbb{F}_q} P.$$

The morphisms in (3.8) induce a morphism $\delta : \mathbb{Z}_P(j-d)[-2d] \to \mathbb{Z}_{P,Q}(j)$ in $\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})$. The composite morphism $\mathbb{Z}_Y(j-d)[-2d] \xrightarrow{q^*} \mathbb{Z}_P(j-d)[-2d] \xrightarrow{\delta} \mathbb{Z}_{P,Q}(j)$ induces a homomorphism $\delta_* : H^{i-2d}_{\mathcal{L}}(Y,\mathbb{Z}(j-d)) \to H^i_{\mathcal{L},Q}(P,\mathbb{Z}(j))$ for each $i \in \mathbb{Z}$.

From the construction of the morphism δ_*, we see that the diagram

$$\begin{array}{ccc} H^{i-2d}_{\mathcal{M}}(Y,\mathbb{Z}(j-d)) & \xrightarrow{s_*} & H^i_{\mathcal{M},Q}(P,\mathbb{Z}(j)) \\ {\scriptstyle \beta^{i-2d}_{j-d}}\downarrow & & {\scriptstyle \beta^i_{Q,j}}\downarrow \\ H^{i-2d}_{\mathcal{L}}(Y,\mathbb{Z}(j-d)) & \xrightarrow{\delta_*} & H^i_{\mathcal{L},Q}(P,\mathbb{Z}(j)) \end{array}$$

is commutative. Here the upper horizontal arrow s_* is the homomorphism which sends the class of a cycle $V \in z^{j-d}(Y, 2j-i)$ to the class in $H^i_{\mathcal{M},Q}(P,\mathbb{Z}(j))$ of the cycle $s(V)$ which belongs to the kernel of $z^j(P, 2j-i) \to z^j(P \setminus Q, 2j-i)$.

The isomorphism $\iota_* : H^{2j-i-2d}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) \xrightarrow{\cong} H^{2j-i}_{\mathcal{L},Y}(X, \mathbb{Z}(j))$ equals the composite

$$\begin{aligned} &H^{2j-i-2d}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) \xrightarrow{\delta_*} H^{2j-i}_{\mathcal{L},Q}(P, \mathbb{Z}(j)) \leftarrow H^{2j-i}_{\mathcal{L},W}(Z, \mathbb{Z}(j)) \\ &\to H^{2j-i}_{\mathcal{L},Y\times_{\operatorname{Spec}\mathbb{F}_q}\{1\}}(X \times_{\operatorname{Spec}\mathbb{F}_q} \{1\}, \mathbb{Z}(j)) = H^{2j-i}_{\mathcal{L},Y}(X, \mathbb{Z}(j)). \end{aligned}$$

It can be checked easily that the isomorphism $\iota_* : H^{2j-i-2d}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) \xrightarrow{\cong} H^{2j-i}_{\mathcal{M},Y}(X, \mathbb{Z}(j))$ equals the composite

$$\begin{aligned} &H^{2j-i-2d}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) \xrightarrow{s_*} H^{2j-i}_{\mathcal{M},Q}(P, \mathbb{Z}(j)) \leftarrow H^{2j-i}_{\mathcal{M},W}(Z, \mathbb{Z}(j)) \\ &\to H^{2j-i}_{\mathcal{M},Y\times_{\operatorname{Spec}\mathbb{F}_q}\{1\}}(X \times_{\operatorname{Spec}\mathbb{F}_q} \{1\}, \mathbb{Z}(j)) = H^{2j-i}_{\mathcal{M},Y}(X, \mathbb{Z}(j)). \end{aligned}$$

Hence the claim follows. □

Remark 3.1. For $j = d$, we have $\alpha_{i,d} = (-1)^{d-1}(d-1)! \cdot \mathrm{ch}_{i,0}$. For $i \geq 1$ and $j = d+1$, we have $\alpha_{i,d+1} = (-1)^d d! \cdot c_{i,1}$.

Suppose that $d = 1$ and $\mathcal{N} \cong \mathcal{O}_Y$. Then we have $\alpha_{i,1} = \mathrm{ch}_{i,0}$ and $\alpha_{i,j}(a) = (-1)^{j-1} Q_{j-1}(c_{i,1}(a), \dots, c_{i,j-1}(a))$ for $i \geq 0$, $j \geq 2$, where Q_{j-1} denotes the $(j-1)$-st Newton polynomial which expresses the $(j-1)$-st power sum polynomial in terms of the elementary symmetric polynomials. In particular, $\alpha_{i,2} = -c_{i,1}$ for $i \geq 0$, and $\alpha_{i,j} = -(j-1)c_{i,j-1}$ for $i \geq 1$, $j \geq 2$.

4. Motivic Chern characters for singular curves over finite fields

We construct Chern characters of low degrees for singular curves over finite fields with values in the higher Chow groups in an ad hoc manner. Bloch defines Chern characters with values in the higher Chow groups tensored with $\mathbb{Q}$ in [7, (7.4), p. 294]. We restrict ourselves to one dimensional varieties over finite fields but the target group is with coefficients in $\mathbb{Z}$.

4.1. Let us record a lemma to be used in this section, and later in Lemma 5.4. For a scheme X, we let $\mathcal{O}(X) = H^0(X, \mathcal{O}_X)$ denote the coordinate ring of X.

Lemma 4.1. *Let X be a connected scheme of pure dimension 1 which is separated and of finite type over $\operatorname{Spec}\mathbb{F}_q$. Then the push-forward map*

$$\alpha_X : H^3_{\mathcal{M}}(X, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\operatorname{Spec}\mathcal{O}(X), \mathbb{Z}(1))$$

is an isomorphism if X is proper, and the group $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$ is zero if X is not proper.

Proof. This follows from Thoerem 1.1 of [24]. □

4.2. Let Z be a scheme over $\operatorname{Spec}\mathbb{F}_q$ of pure dimension one which is separated of finite type over $\operatorname{Spec}\mathbb{F}_q$. We construct a canonical homomorphism $\mathrm{ch}'_{i,j} : G_i(Z) \to H^{2j-i}_{\mathcal{M}}(Z, \mathbb{Z}(j))$ for $(i,j) = (0,0), (0,1), (1,1)$, and $(1,2)$. Then we will show (Proposition 4.1) that the homomorphism

$$(\mathrm{ch}'_{i,i}, \mathrm{ch}'_{i,i+1}) : G_i(Z) \to H^i_{\mathcal{M}}(Z, \mathbb{Z}(i)) \oplus H^{i+2}_{\mathcal{M}}(Z, \mathbb{Z}(i+1)) \tag{4.1}$$

is an isomorphism for $i = 0, 1$. Since the G-theory of Z and the G-theory of Z_{red} are isomorphic, and the same holds for the motivic cohomology, it suffices to treat the case where Z is reduced.

Take a dense affine open smooth subscheme $Z_{(0)} \subset Z$, and let $Z_{(1)} = Z \setminus Z_{(0)}$ be the complement of $Z_{(0)}$ with the reduced scheme structure. We define $\mathrm{ch}'_{0,0}$ to be the composite

$$G_0(Z) \to K_0(Z_{(0)}) \xrightarrow{\mathrm{ch}_{0,0}} H^0_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(0)) \cong H^0_{\mathcal{M}}(Z, \mathbb{Z}(0)).$$

We use the following lemma.

Lemma 4.2. *For $i = 0$ (resp. $i = 1$), the diagram*

$$\begin{array}{ccc} K_{i+1}(Z_{(0)}) & \longrightarrow & K_i(Z_{(1)}) \\ {\scriptstyle c_{i+1,i+1}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_{0,0}\ (resp.\ c_{1,1})} \\ H^{i+1}_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(i+1)) & \longrightarrow & H^i_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(i)) \end{array}$$

where each horizontal arrow is a part of the localization sequence, is commutative.

Proof. Let $\widetilde{Z}$ denote the normalization of Z. We write $\widetilde{Z}_{(0)} = Z_{(0)} \times_Z \widetilde{Z} (\cong Z_{(0)})$ and $\widetilde{Z}_{(1)} = (Z_{(1)} \times_Z \widetilde{Z})_{\mathrm{red}}$. Comparing the diagrams

$$\begin{array}{ccc} K_{i+1}(\widetilde{Z}_{(0)}) & \to & K_i(\widetilde{Z}_{(1)}) \\ \downarrow & & \downarrow \\ K_{i+1}(Z_{(0)}) & \to & K_i(Z_{(1)}) \end{array} \quad \text{and} \quad \begin{array}{ccc} H^{i+1}_{\mathcal{M}}(\widetilde{Z}_{(0)}, \mathbb{Z}(i+1)) & \to & H^i_{\mathcal{M}}(\widetilde{Z}_{(1)}, \mathbb{Z}(i)) \\ \downarrow & & \downarrow \\ H^{i+1}_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(i+1)) & \to & H^i_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(i)) \end{array}$$

reduces us to proving the same claim for $\widetilde{Z}_{(0)}$ and $\widetilde{Z}_{(1)}$, This then follows from Lemma 3.1. $\square$

We define $\mathrm{ch}'_{1,1}$ to be the composite

$$\begin{aligned} &G_1(Z) \to \mathrm{Ker}[K_1(Z_{(0)}) \to K_0(Z_{(1)})] \\ &\xrightarrow{c_{1,1}} \mathrm{Ker}[H^1_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1)) \to H^0_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(0))] \cong H^1_{\mathcal{M}}(Z, \mathbb{Z}(1)). \end{aligned}$$

Next we define $\mathrm{ch}'_{1,2}$ when Z is connected. If Z is not proper, then $H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))$ is zero by Lemma 4.1. We put $\mathrm{ch}'_{1,2} = 0$ in this case. If Z is proper, then the push-forward map $H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathrm{Spec}\, H^0(Z, \mathcal{O}_Z), \mathbb{Z}(1)) \cong K_1(\mathrm{Spec}\, H^0(Z, \mathcal{O}_Z))$ is an isomorphism by Lemma 4.1. We define $\mathrm{ch}'_{1,2}$ to be (-1)-times the composite

$$G_1(Z) \to K_1(\mathrm{Spec}\, H^0(Z, \mathcal{O}_Z)) \cong H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)).$$

We define $\mathrm{ch}'_{1,2}$ for non-connected Z to be the direct sum of $\mathrm{ch}'_{1,2}$ for each connected component of Z.

Observe that the group $G_0(Z)$ is generated by the two subgroups

$$M_1 = \mathrm{Image}[K_0(Z_{(1)}) \to G_0(Z)] \text{ and } M_2 = \mathrm{Image}[K_0(\widetilde{Z}) \to G_0(Z)].$$

One can see by using Lemma 4.2 and the localization sequences that the isomorphism $\mathrm{ch}_{0,0} : K_0(Z_{(1)}) \xrightarrow{\cong} H^0_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(0))$ induces a homomorphism $\mathrm{ch}'_{0,1} : M_1 \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$. The kernel of $K_0(\widetilde{Z}) \to G_0(Z)$ is contained in the image of $K_0(\widetilde{Z}_{(1)}) \to K_0(\widetilde{Z})$. It can be checked easily that the composite

$$K_0(\widetilde{Z}_{(1)}) \to K_0(\widetilde{Z}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(\widetilde{Z}, \mathbb{Z}(1)) \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$$

equals the composite

$$K_0(\widetilde{Z}_{(1)}) \to K_0(Z_{(1)}) \twoheadrightarrow M_1 \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(Z, \mathbb{Z}(1)).$$

Hence the homomorphism $c_{0,1} : K_0(\widetilde{Z}) \to H^2_{\mathcal{M}}(\widetilde{Z}, \mathbb{Z}(1))$ induces a homomorphism $\mathrm{ch}'_{0,1} : M_2 \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$ such that the two homomorphisms $\mathrm{ch}'_{0,1} : M_i \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$, $i = 1, 2$, coincide on $M_1 \cap M_2$. Thus we obtain a homomorphism $\mathrm{ch}'_{0,1} : G_0(Z) \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$.

It is easily seen that the four homomorphisms $\mathrm{ch}'_{0,0}$, $\mathrm{ch}'_{0,1}$, $\mathrm{ch}'_{1,1}$, and $\mathrm{ch}'_{1,2}$ do not depend on the choice of $Z_{(0)}$.

Proposition 4.1. *The homomorphism (4.1) for $i = 0, 1$ is an isomorphism.*

Proof. It follows from [4, Corollary 4.3, p. 95] that the Chern class map $c_{1,1} : K_1(Z_{(0)}) \to H^1_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))$ is an isomorphism. Hence, by construction, $\mathrm{ch}'_{1,1}$ is surjective and its kernel equals the image of $K_1(Z_{(1)}) \to G_1(Z)$. It follows from the vanishing of K_2 groups of finite fields that the homomorphism $c_{2,2} : K_2(Z_{(0)}) \to H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(2))$ is an isomorphism. We then have isomorphisms

$$\begin{aligned} & \mathrm{Image}[K_1(Z_{(1)}) \to G_1(Z)] \\ \cong\ & \mathrm{Image}[H^1_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(1)) \to H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))] \cong H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)), \end{aligned}$$

the first of which is by Lemma 4.2, and the second is by [4, Corollary 4.3, p. 95]. Therefore the composite $\mathrm{Ker}\,\mathrm{ch}'_{1,1} \hookrightarrow G_1(Z) \xrightarrow{\mathrm{ch}'_{1,2}} H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))$ is an isomorphism. This proves the claim for $G_1(Z)$.

By the construction of the map $\mathrm{ch}'_{0,1}$, the image of $\mathrm{ch}'_{0,1}$ contains the image of $H^0_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$, and the composite $K_0(\widetilde{Z}) \to G_0(Z) \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(Z, \mathbb{Z}(1)) \to H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))$ equals the composite

$$K_0(\widetilde{Z}) \to K_0(\widetilde{Z}_{(0)}) \cong K_0(Z_{(0)}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1)).$$

This implies that $\mathrm{ch}'_{0,1}$ is surjective and the homomorphism

$$\mathrm{Ker}\,\mathrm{ch}'_{0,1} \to \mathrm{Ker}[K_0(Z_{(0)}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))]$$

is an isomorphism. This proves the claim for $G_0(Z)$. □

5. K-groups and motivic cohomology of curves over a function field

From the computations of the motivic cohomology of a surface with a fibration, we deduce some results concerning the K-groups of low degrees of the generic fiber. We relate the two using the Chern class maps and by taking the limit.

Let C be a smooth projective, geometrically connected curve over a finite field $\mathbb{F}_q$. Let k denote the function field of C. Let X be a smooth projective geometrically connected curve over k. Let $\mathcal{X}$ be a regular model of X which is proper and flat over C.

Lemma 5.1. *The map*

$$K_1(X) \xrightarrow{(c_{1,1}, c_{1,2})} k^\times \oplus H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$$

is an isomorphism. The group $H^4_{\mathcal{M}}(X, \mathbb{Z}(3))$ is a torsion group and there exists a canonical short exact sequence

$$0 \to H^4_{\mathcal{M}}(X, \mathbb{Z}(3)) \xrightarrow{\beta} K_2(X) \xrightarrow{c_{2,2}} H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \to 0$$

such that the composite $c_{2,3} \circ \beta$ equals the multiplication-by-2 map.

Proof. Let X_0 denote the set of closed points of X. We let $\kappa(x)$ denote the residue field at $x \in X_0$. Construct a commutative diagram by connecting the localization sequence

$$\begin{array}{rl} & \bigoplus_{x\in X_0} K_2(\kappa(x)) \to K_2(X) \to K_2(k(X)) \\ \to & \bigoplus_{x\in X_0} K_1(\kappa(x)) \to K_1(X) \to K_1(k(X)) \to \bigoplus_{x\in X_0} K_0(x) \end{array}$$

with the localization sequence

$$\begin{array}{rl} & \bigoplus_{x\in X_0} H^0_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(1)) \to H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \to H^2_{\mathcal{M}}(\operatorname{Spec} k(X), \mathbb{Z}(2)) \\ \to & \bigoplus_{x\in X_0} H^1_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(1)) \to H^3_{\mathcal{M}}(X, \mathbb{Z}(2)) \to H^3_{\mathcal{M}}(\operatorname{Spec} k(X), \mathbb{Z}(2)), \end{array}$$

using the Chern class maps. Since $H^0(\operatorname{Spec}\kappa(x), \mathbb{Z}(1)) = 0$ and the K-groups and the motivic cohomology groups of fields agree in low degrees, the claim for $K_1(X)$ follows from diagram chasing.

It also follows from diagram chasing that

$$K_3(k(X)) \to \bigoplus_{x\in X_0} K_2(\kappa(x)) \to K_2(X) \xrightarrow{c_{2,2}} H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \to 0$$

is exact. By [36, THEOREM 4.9, p. 143] and [45, THEOREM 1, p. 181], the groups $H^3_{\mathcal{M}}(\operatorname{Spec} k(X), \mathbb{Z}(3))$ and $H^2_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(2))$ for each $x \in X_0$ are isomorphic to the Milnor K-groups $K^M_3(k(X))$ and $K^M_2(\kappa(x))$ respectively. We easily see from the definition of these isomorphisms in [45] that the boundary map $H^3_{\mathcal{M}}(k(X), \mathbb{Z}(3)) \to H^2_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(2))$ is identified under these isomorphisms with the boundary map $K^M_3(k(X)) \to K^M_2(\kappa(x))$. Hence by [31, Proposition 11.11, p. 562], we have the first of the two isomorphisms:

$$\begin{array}{rl} & \operatorname{Coker}[K_3(k(X)) \to \bigoplus_{x\in X_0} K_2(\kappa(x))] \\ \xrightarrow{\cong} & \operatorname{Coker}[H^3_{\mathcal{M}}(k(X), \mathbb{Z}(3)) \to \bigoplus_{x\in X_0} H^2_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(2))] \\ \xrightarrow{\cong} & H^4_{\mathcal{M}}(X, \mathbb{Z}(3)). \end{array}$$

This gives the desired short exact sequence. The identity $c_{2,3} \circ \beta = 2$ follows from Remark 3.1. Since $H^2_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(2))$ is a torsion group for each $x \in X_0$, the group $H^4_{\mathcal{M}}(X, \mathbb{Z}(3))$ is a torsion group. This completes the proof. □

Lemma 5.2. *Let $U \subset C$ be a non-empty open subscheme. We denote by $\mathcal{X}^U$ the complement $\mathcal{X} \setminus \mathcal{X} \times_C U$ with the reduced scheme structure. Then for $(i, j) = (0, 0)$, $(0, 1)$ or $(1, 1)$, the diagram*

$$\begin{array}{ccc} K_{i+1}(X) & \longrightarrow & G_i(\mathcal{X}^U) \\ {\scriptstyle c_{i+1,j+1}}\downarrow & & \downarrow{\scriptstyle (-1)^j \operatorname{ch}'_{i,j}} \\ H^{2j-i+1}_{\mathcal{M}}(X, \mathbb{Z}(j+1)) & \longrightarrow & H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(j)) \end{array} \tag{5.1}$$

where each horizontal arrow is a part of the localization sequence, is commutative.

Proof. Let $\mathcal{X}^U_{\mathrm{sm}} \subset \mathcal{X}^U$ denote the smooth locus. The commutativity of the diagram (5.1) for $(i, j) = (1, 1)$ (resp. for $(i, j) = (0, 0)$) follows from the commutativity, which follows from Lemma 3.1, of the diagram

$$\begin{array}{ccc} K_{i+1}(X) & \longrightarrow & K_i(\mathcal{X}^U_{\mathrm{sm}}) \\ {\scriptstyle c_{i+1,j+1}}\downarrow & & \downarrow{\scriptstyle -c_{1,1} \text{ (resp. } \operatorname{ch}_{0,0})} \\ H^{2j-i+1}_{\mathcal{M}}(X, \mathbb{Z}(j+1)) & \longrightarrow & H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U_{\mathrm{sm}}, \mathbb{Z}(j)) \end{array}$$

and the injectivity of $H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(j)) \to H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U_{\mathrm{sm}}, \mathbb{Z}(j))$.

By Lemma 5.1, the group $K_1(X)$ is generated by the image of the push-forward $\bigoplus_{x\in X_0} K_1(\kappa(x)) \to K_1(X)$ and the image of the pull-back $K_1(k) \to K_1(X)$. Then the commutativity of the diagram (5.1) for $(i,j) = (0,1)$ follows from the commutativity of the diagram

$$\begin{array}{ccc} K_0(Y) & \longrightarrow & G_0(\mathcal{X}^U) \\ {\scriptstyle \mathrm{ch}_{0,0}}\downarrow & & {\scriptstyle \mathrm{ch}'_{0,1}}\downarrow \\ H^0_{\mathcal{M}}(Y, \mathbb{Z}(0)) & \longrightarrow & H^2_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(1)) \end{array}$$

for any reduced closed subscheme $Y \subset \mathcal{X}^U$ of dimension zero, where the horizontal arrows are the push-forward maps by the closed immersion, and the fact that the composite $K_0(C \setminus U) \xrightarrow{f^{U*}} G_0(\mathcal{X}^U) \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(1))$ is zero. Here $f^U : \mathcal{X}^U \to C \setminus U$ denotes the morphism induced by the morphism $\mathcal{X} \to C$. □

Lemma 5.3. *The diagram*

(5.2)
$$\begin{array}{ccccccccc} 0 \to & H^4_{\mathcal{M}}(X, \mathbb{Z}(3)) & \to & K_2(X) & \xrightarrow{c_{2,2}} & H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \bigoplus_{\wp\in C_0} H^3_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(2)) & \to & \bigoplus_{\wp\in C_0} G_1(\mathcal{X}_\wp) & \xrightarrow{-\mathrm{ch}'_{1,1}} & \bigoplus_{\wp\in C_0} H^1_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(1)) & \to 0 \end{array}$$

where the first row is as in Lemma 5.1, the second row is obtained from Proposition 4.1, and the vertical maps are the boundary maps in the localization sequences, is commutative.

Proof. It follows from Lemma 5.2 that the right square is commutative. For each closed point $x \in X_0$, let D_x denote the closure of x in $\mathcal{X}$ and write $D_{x,\wp} = D_x \times_C \operatorname{Spec} \kappa(\wp)$ for $\wp \in C_0$. Then the commutativity of the left square in (5.2) follows from the commutativity of the diagram

$$\begin{array}{ccccccc} H^4_{\mathcal{M}}(X, \mathbb{Z}(3)) & \leftarrow & H^2_{\mathcal{M}}(\operatorname{Spec} \kappa(x), \mathbb{Z}(2)) & \cong & K_2(\kappa(x)) & \to & K_2(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{\wp\in C_0} H^3_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(2)) & \leftarrow & \bigoplus_{\wp\in C_0} H^1_{\mathcal{M}}(D_{x,\wp}, \mathbb{Z}(1)) & \cong & \bigoplus_{\wp\in C_0} G_1(D_{x,\wp}) & \to & \bigoplus_{\wp\in C_0} G_1(\mathcal{X}_\wp) \end{array}$$

Here, each vertical map is a boundary map, the middle horizontal arrows are Chern classes, and the left and right horizontal arrows are the push-forward maps by the closed immersion. □

Lemma 5.4. *Let U be an open subscheme of C such that $U \neq C$. Let $\partial : H^4_{\mathcal{M}}(\mathcal{X} \times_C U, \mathbb{Z}(3)) \to H^3_{\mathcal{M}}(\mathcal{X}_U, \mathbb{Z}(2))$ denote the boundary map of the localization sequence. Then the following composite is an isomorphism:*

$$\alpha : \operatorname{Coker} \partial \hookrightarrow H^5_{\mathcal{M}}(\mathcal{X}, \mathbb{Z}(3)) \to H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(1)) \cong \mathbb{F}_q^\times.$$

Here the first map is induced by the push-forward map by the closed immersion, and the second map is the push-forward map by the structure morphism.

Proof. For each closed point $x \in X_0$, let D_x denote the closure of x in $\mathcal{X}$. We put $D_{x,U} = D_x \times_C U$. Let D^U_x denote the complement $D_x \setminus D_{x,U}$ with the reduced scheme structure. Let $\iota_x : D_x \hookrightarrow \mathcal{X}$, $\iota_{x,U} : D_{x,U} \hookrightarrow \mathcal{X}_U$, $\iota_x^U : D_x^U \hookrightarrow \mathcal{X}^U$ denote the canonical inclusions. Let us consider the commutative diagram

$$\begin{array}{ccccc} H^2_{\mathcal{M}}(D_{x,U}, \mathbb{Z}(2)) & \longrightarrow & H^1_{\mathcal{M}}(D_x^U, \mathbb{Z}(1)) & \xrightarrow{\beta} & H^3_{\mathcal{M}}(D_x, \mathbb{Z}(2)) \\ \downarrow {\scriptstyle \iota_{x,U*}} & & \downarrow {\scriptstyle \iota^U_{x*}} & & \\ H^4_{\mathcal{M}}(\mathcal{X} \times_C U, \mathbb{Z}(3)) & \xrightarrow{\partial} & H^3_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(2)) & \longrightarrow & \operatorname{Coker} \partial \to 0 \end{array}$$

where the first row is the localization sequence. Since X is geometrically connected, it follows from [18, Corollaire (4.3.12), p. 134] that each fiber of $\mathcal{X} \to C$ is connected. In particular D_x intersects every connected component of $\mathcal{X}^U$. This implies that the homomorphism ι^U_{x*} in the above diagram is surjective. Hence we have a surjective homomorphism $\operatorname{Image} \beta \twoheadrightarrow \operatorname{Coker} \partial$. Let $\mathbb{F}(x)$ denote the finite field $H^0(D_x, \mathcal{O}_{D_x})$. Then the isomorphism $H^3_{\mathcal{M}}(D_x, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}(x), \mathbb{Z}(1)) \cong \mathbb{F}(x)^\times$ (see Lemma 4.1) gives an isomorphism $\operatorname{Image} \beta \cong \mathbb{F}(x)^\times$. Hence $|\operatorname{Coker} \partial|$ divides $\gcd_{x \in X_0}(|\mathbb{F}(x)^\times|) = q-1$, where the equality follows from [43, 1.5.3 Lemme 1, p. 325]. It can be checked easily that the composite

$$\mathbb{F}(x)^\times \cong \operatorname{Image} \beta \twoheadrightarrow \operatorname{Coker} \partial \xrightarrow{\alpha} \mathbb{F}_q^\times$$

equals the norm map $\mathbb{F}(x)^\times \to \mathbb{F}_q^\times$, which implies $|\operatorname{Coker} \partial| \geq q-1$. Hence $|\operatorname{Coker} \partial| = q-1$ and the homomorphism α is an isomorphism. The claim is proved. □

6. Main results for $j \leq 2$

The objective is to prove Theorems 6.1, 6.2, 6.3. The statements give some information on the structures of K-groups and motivic cohomology groups of elliptic curves over global fields and of the (open) complements of some fibers of an elliptic surface over finite fields. Milne [33] expresses the special values of zeta functions in terms of the order of arithmetic étale cohomology groups. We compute the orders of some torsion groups, in terms of the special values of L-functions, the torsion subgroup of (twisted) Mordell-Weil groups, and some invariants of the base curve.

Let us list the ingredients of the proof. Using Theorem 1.3, we deduce that the torsion subgroups we are interested in are actually finite. Then the theorem of Geisser and Levine and the theorem of Merkurjev and Suslin relate the motivic cohomology groups modulo their uniquely divisible parts and the étale cohomology and the cohomology of de Rham-Witt complexes. We use the arguments which appear in [32], [9], [17] to compute such cohomology groups. The computation of the exact orders of the torsion may be new. One geometric property of an elliptic surface which makes this explicit calculation possible is that the (abelian) fundamental group is isomorphic to that of the base curve. This follows from a theorem in [42] for the prime-to-p part. The use of the class field theory of Kato and Saito for surfaces over finite fields ([23]) is somewhat indirect but we then know that the groups of zero-cycles on the elliptic surface and on the base curve are isomorphic.

6.1. Notations. Let k, E, S_0, S_2, r, C, and $\mathcal{E}$ be as in Section 1. We also let S_1 denote the set of primes of k at which E has multiplicative reduction. Thus we have $S_0 \subset S_1 \subset S_2$. Let p denote the characteristic of k. The closure of the origin of E in $\mathcal{E}$ gives a section to $\mathcal{E} \to C$, which we denote by $\iota : C \to \mathcal{E}$. Throughout this section, we assume that the structure morphism $f : \mathcal{E} \to C$ is not smooth. For any scheme X over C, let $\mathcal{E}_X$ denote the base change $\mathcal{E} \times_C X$. For any non-empty open subscheme $U \subset C$, we denote by $\mathcal{E}^U$ the complement $\mathcal{E} \setminus \mathcal{E}_U$ with the reduced scheme structure.

Let $\mathbb{F}_q$ denote the field of constants of C. We take an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let $\mathrm{Frob} \in G_{\mathbb{F}_q} = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ denote the geometric Frobenius. For a scheme X over $\mathrm{Spec}\,\mathbb{F}_q$, we denote by $\overline{X}$ its base change $\overline{X} = X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,\overline{\mathbb{F}}_q$ to $\overline{\mathbb{F}}_q$. We often regard the set $\mathrm{Irr}(X)$ of irreducible components of X as a finite étale scheme over $\mathrm{Spec}\,\mathbb{F}_q$ corresponding to the $G_{\mathbb{F}_q}$-set $\mathrm{Irr}(\overline{X})$.

6.2. Results. We put $T = E(k \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)_{\mathrm{tors}}$. For each integer $j \in \mathbb{Z}$, we let $T'_{(j)} = \bigoplus_{\ell \neq p} (T \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}}$.

Theorem 6.1. *Let the notations and the assumptions be as above. Let $L(E, s)$ denote the L-function of the elliptic curve E over the global field k (see Section 6.3).*

1. *The $\mathbb{Q}$-vector space $(K_2(E)^{\mathrm{red}})_{\mathbb{Q}}$ is of dimension r.*
2. *The cokernel of the boundary map $\partial_2 : K_2(E) \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ is a finite group of order*
$$\frac{(q-1)^2 |L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})), -1)|}{|T'_{(1)}| \cdot |L(h^0(S_2), -1)|}.$$
3. *The group $K_1(E)_{\mathrm{div}}$ is uniquely divisible.*
4. *The kernel of the boundary map $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ is a finite group of order $(q-1)^2 |T'_{(1)}| \cdot |L(E, 0)|$. The cokernel of ∂_1 is a finitely generated abelian group of rank $2 + |\mathrm{Irr}(\mathcal{E}_{S_2})| - |S_2|$ whose torsion subgroup is isomorphic to $\mathrm{Jac}(C)(\mathbb{F}_q)^{\oplus 2}$, where $\mathrm{Jac}(C)$ denotes the Jacobian of C (when the genus of C is 0, we understand it to be a point) .*

Let X be a scheme of finite type over $\mathrm{Spec}\,\mathbb{F}_q$. For an integer $i \in \mathbb{Z}$ and a prime number $\ell \neq p$, we put $L(h^i(X), s) = \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell))$. In all the cases considered in this paper, the function $L(h^i(X), s)$ does not depend on the choice of ℓ.

For each non-empty open subscheme $U \subset C$, let T_U denote the torsion subgroup of the group $\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}$ of divisors on $\overline{\mathcal{E}}_U$ modulo algebraic equivalence. For each integer $j \in \mathbb{Z}$, we put $T'_{U,(j)} = \bigoplus_{\ell \neq p} (T_U \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}}$. One can deduce from [42, THEOREM 1.3, p. 214] that the canonical homomorphism $\varinjlim_{U'} T_{U'} \to T$, where the limit is taken over the open subschemes of C, is an isomorphism. It can be checked easily that the canonical homomorphism $T_U \to \varinjlim_{U'} T_{U'}$ is injective and is an isomorphism if $\mathcal{E}_U \to U$ is smooth. In particular, we have an injection $T'_{U,(j)} \hookrightarrow T'_{(j)}$ which is an isomorphism if $\mathcal{E}_U \to U$ is smooth.

In Section 6.7, we deduce Theorem 6.1 from the following two theorems.

Theorem 6.2. *Let the notations and the assumptions be as above. Let $\partial^i_{\mathcal{M},j} : H^i_{\mathcal{M}}(E, \mathbb{Z}(j))^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} H^{i-1}_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(j-1))$ denote the homomorphism induced by the boundary map of the localization sequence established in* [6, Corollary (0.2), p. 537].

1. *For any $i \in \mathbb{Z}$, the group $H^i_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$ is uniquely divisible.*

(2) *For $i \le 0$, the cohomology group $H^i_{\mathcal{M}}(E, \mathbb{Z}(2))$ is uniquely divisible. The group $H^1_{\mathcal{M}}(E, \mathbb{Z}(2))$ is finite modulo a uniquely divisible subgroup and the group $H^1_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{tors}}$ is cyclic of order $q^2 - 1$.*

(3) *The kernel (resp. cokernel) of the homomorphism $\partial^2_{\mathcal{M},2}$ is a finite group of order $|L(h^1(C), -1)|$ (resp. of order*
$$\frac{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})), -1)|}{|T'_{(1)}| \cdot |L(h^0(S_2), -1)|}).$$

(4) *The kernel (resp. cokernel) of the homomorphism $\partial^3_{\mathcal{M},2}$ is a finite group of order $(q-1)|T'_{(1)}| \cdot |L(E, 0)|$ (resp. is isomorphic to $\mathrm{Pic}(C)$).*

(5) *For $i \ge 4$, the group $H^i_{\mathcal{M}}(E, \mathbb{Z}(2))$ is zero.*

(6) *The group $H^4_{\mathcal{M}}(E, \mathbb{Z}(3))$ is a torsion group, and the cokernel of the homomorphism $\partial^4_{\mathcal{M},3}$ is a finite cyclic group of order $q - 1$.*

Theorem 6.3. *Let $U \subset C$ be a non-empty open subscheme. Then*

(1) *For any $i \in \mathbb{Z}$, the group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ is finitely generated modulo a uniquely divisible subgroup.*

(2) *For $i \le 0$, the cohomology group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ is uniquely divisible. The group $H^1_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ is finite modulo a uniquely divisible subgroup and the group $H^1_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}$ is cyclic of order $q^2 - 1$.*

(3) *The rank of $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ is $|S_0 \setminus U|$. If $U = C$ (resp. $U \ne C$), the torsion subgroup of $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ is of order $|L(h^1(C), -1)|$ (resp. of order $|T'_{U,(1)}| \cdot |L(h^1(C), -1)L(h^0(C \setminus U), -1)|/(q-1)$).*

(4) *If $U = C$ (resp. $U \ne C$), the cokernel of the boundary homomorphism $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is zero (resp. is finite of order*
$$\frac{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}{|T'_{U,(1)}| \cdot |L(h^0(C \setminus U), -1)|}).$$

(5) *The rank of $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ is $\max(|C \setminus U| - 1, 0)$. If $U = C$ (resp. $U \ne C$), the torsion subgroup $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}$ is finite of order $|L(h^2(\mathcal{E}), 0)|$ (resp. of order*
$$\frac{|T'_{U,(1)}| \cdot |L(h^2(\mathcal{E}), 0)L^*(h^1(\mathcal{E}^U), 0)L(h^0(C \setminus U), -1)|}{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}).$$
Here
$$L^*(h^1(\mathcal{E}^U), 0) = \lim_{s \to 0} (s \log q)^{-|S_0 \setminus U|} L(h^1(\mathcal{E}^U), s)$$
is the leading coefficient of $L(h^1(\mathcal{E}^U), s)$.

(6) *The group $H^4_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ is canonically isomorphic to $\mathrm{Pic}(U)$. For $i \ge 5$, the group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ is zero.*

6.3. Relation between $L(E, s)$ and the congruence zeta function of $\mathcal{E}$. Let $\ell \ne p$ be a prime number. By the Grothendieck-Lefschetz trace formula, we have
$$L(E, s) = \prod_{i=0}^{2} \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell))^{(-1)^{i-1}}.$$

Lemma 6.1. *Let D be a scheme of dimension ≤ 1 which is proper over $\mathrm{Spec}\, \mathbb{F}_q$. Let $\ell \ne p$ be an integer. Then the group $H^i_{\mathrm{et}}(\overline{D}, \mathbb{Z}_\ell)$ is torsion free for any $i \in \mathbb{Z}$, and is zero for $i \ne 0, 1, 2$. The group $H^i_{\mathrm{et}}(\overline{D}, \mathbb{Q}_\ell)$ is pure of weight i for $i \ne 1$, and* *is mixed of weight $\{0,1\}$ for $i = 1$. The group $H^1_{\mathrm{et}}(\overline{D}, \mathbb{Q}_\ell)$ is pure of weight one (resp. pure of weight zero) if D is smooth (resp. every irreducible component of $\overline{D}$ is rational).*

Proof. We may assume that D is reduced. Let D' be the normalization of D. Let $\pi : D' \to D$ denote the canonical morphism. Let $\mathcal{F}_n$ denote the cokernel of the homomorphism $\mathbb{Z}/\ell^n \to \pi_*(\mathbb{Z}/\ell^n)$ of étale sheaves. The sheaf $\mathcal{F}_n$ is supported on the singular locus D_{sing} of D and is isomorphic to $i_*(\mathrm{Coker}[\mathbb{Z}/\ell^n \to \pi_{\mathrm{sing}*}(\mathbb{Z}/\ell^n)])$, where $i : D_{\mathrm{sing}} \hookrightarrow D$ is the canonical inclusion and $\pi_{\mathrm{sing}} : D' \times_D D_{\mathrm{sing}} \to D_{\mathrm{sing}}$ is the base change of π. Then the claim follows from the long exact sequence

$$\cdots \to H^i_{\mathrm{et}}(\overline{D}, \mathbb{Z}/\ell^n) \to H^i_{\mathrm{et}}(\overline{D}', \mathbb{Z}/\ell^n) \to H^i_{\mathrm{et}}(\overline{D}, \mathcal{F}_n) \to \cdots .$$

□

Lemma 6.2. *If $i \neq 1$, then $H^i_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) = 0$.*

Proof. For any point $x \in \overline{C}(\overline{\mathbb{F}}_q)$ lying over a closed point $\wp \in C$, the canonical homomorphism $H^0_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}_x, \mathbb{Q}_\ell)$ is injective since $H^0_{\mathrm{et},c}(\overline{C} \backslash \{x\}, R^1 f_* \mathbb{Q}_\ell) = 0$. By Lemma 6.1, the module $H^1_{\mathrm{et}}(\mathcal{E}_x, \mathbb{Q}_\ell)$ is pure of weight 1 (resp. of weight 0) if $\mathcal{E}_x$ is smooth (resp. is not smooth). Since we have assumed that $f : \mathcal{E} \to C$ is not smooth, $H^0_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) = 0$.

Take a non-empty open subscheme $U \subset C$ such that $f_U : \mathcal{E}_U \to U$ is smooth. The group $H^2_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) \cong H^2_{\mathrm{et},c}(\overline{U}, R^1 f_{U*} \mathbb{Q}_\ell)$ is the dual of $H^0_{\mathrm{et}}(\overline{U}, R^1 f_{U*} \mathbb{Q}_\ell(1))$ by Poincare duality. Assume that $H^0_{\mathrm{et}}(\overline{U}, R^1 f_{U*} \mathbb{Q}_\ell(1)) \neq 0$. Let $T_\ell(E)$ denote the ℓ-adic Tate module of E. The étale fundamental group $\pi_1(U)$ acts on $T_\ell(E)$. By the assumption, the $\pi_1(\overline{U})$-invariant part $V = (T_\ell(E) \otimes \mathbb{Q}_\ell)^{\pi_1(\overline{U})}$ is non-zero. Since f is not smooth, V is one dimensional. Hence we have a non-zero homomorphism $\pi_1(\overline{U})^{\mathrm{ab}} \to \mathrm{Hom}(T_\ell(E) \otimes \mathbb{Q}_\ell / V, V)$ of $G_{\mathbb{F}_q}$-modules. By the weight argument, we see that this is impossible. Hence $H^2_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell(1)) = 0$. □

As an immediate consequence, we obtain the following corollary.

Corollary 6.1. *The spectral sequence*

$$E_2^{i,j} = H^i_{\mathrm{et}}(\overline{C}, R^j f_* \mathbb{Q}_\ell) \Rightarrow H^{i+j}_{\mathrm{et}}(\overline{\mathcal{E}}, R^j f_* \mathbb{Q}_\ell)$$

is E_2-degenerate. □

Lemma 6.3. *Let $U \subset C$ be a non-empty open subscheme such that $f_U : \mathcal{E}_U \to U$ is smooth. Let $\mathrm{Irr}^0(\mathcal{E}^U) \subset \mathrm{Irr}(\mathcal{E}^U)$ denote the subset of the irreducible components of $\mathcal{E}^U$ which does not intersect $\iota(C)$. We regard $\mathrm{Irr}^0(\mathcal{E}^U)$ as a closed subscheme of $\mathrm{Irr}(\mathcal{E}^U)$ (recall the convention 6.1 on $\mathrm{Irr}(\mathcal{E}^U)$). Then*

$$L(h^i(\mathcal{E}), s) = \begin{cases} (1 - q^{-s}), & \text{if } i = 0, \\ L(h^1(C), s), & \text{if } i = 1, \\ (1 - q^{1-s})^2 L(E, s) L(h^0(\mathrm{Irr}^0(\mathcal{E}^U)), s - 1), & \text{if } i = 2, \\ L(h^1(C), s - 1), & \text{if } i = 3, \\ (1 - q^{2-s}), & \text{if } i = 4. \end{cases}$$

Proof. We prove the lemma for $i = 2$; other cases are easy. Since $R^2 f_{U*} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell(-1)$, there exists an exact sequence

$$0 \to H^0_{\mathrm{et}}(\overline{C}, R^2 f_* \mathbb{Q}_\ell) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1)).$$

The map $H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1))$ decomposes as

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1)) \to H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1)).$$

Hence $H^0_{\mathrm{et}}(\overline{C}, R^2 f_* \mathbb{Q}_\ell)$ is isomorphic to the inverse image of the image of the homomorphism $H^0_{\mathrm{et}}(\overline{C}, \mathbb{Q}_\ell(-1)) \to H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1))$ under the surjective homomorphism $H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1))$. This proves the claim. □

6.4. The fundamental group of $\mathcal{E}$.

Lemma 6.4. *For $i = 0, 1$, the pull-back $H^i(C, \mathcal{O}_C) \to H^i(\mathcal{E}, \mathcal{O}_\mathcal{E})$ is an isomorphism.*

Proof. The claim for $i = 0$ is clear. We prove the claim for $i = 1$. Let us write $\mathcal{L} = R^1 f_* \mathcal{O}_\mathcal{E}$. It suffices to prove $H^0(C, \mathcal{L}) = 0$. We note that $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module since the morphism $\mathcal{E} \to C$ has no multiple fiber. The Leray spectral sequence $E_2^{i,j} = H^i(C, R^j f_* \mathcal{O}_\mathcal{E}) \Rightarrow H^{i+j}(\mathcal{E}, \mathcal{O}_\mathcal{E})$ shows that the Euler-Poincare characteristic $\chi(\mathcal{O}_\mathcal{E})$ equals $\chi(\mathcal{O}_C) - \chi(\mathcal{L}) = -\deg \mathcal{L}$. By the well-known inequality $\chi(\mathcal{O}_\mathcal{E}) > 0$ (cf. [38], [12], or [39, Theorem 2, p. 81]), we have $\deg \mathcal{L} < 0$. This proves $H^0(C, \mathcal{L}) = 0$. □

Lemma 6.5. (1) *The canonical homomorphism $\pi_1^{\mathrm{ab}}(\mathcal{E}) \to \pi_1^{\mathrm{ab}}(C)$ between the abelian (étale) fundamental groups is an isomorphism.*
(2) *The canonical morphism $\mathrm{Pic}^o_{C/\mathbb{F}_q} \to \mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}$ between the identity components of the Picard schemes is an isomorphism.*

Proof. The homomorphism $\mathrm{Pic}^o_{C/\mathbb{F}_q} \to \mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q,\mathrm{red}}$ is an isomorphism by [42, Theorem 4.1, p. 219]. This, combined with the cohomology long exact sequence of the Kummer sequence, implies that, if $p \nmid m$, then $H^1_{\mathrm{et}}(C, \mathbb{Z}/m) \to H^1_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}/m)$ is an isomorphism. Hence, to prove (1), we are reduced to showing that $H^1_{\mathrm{et}}(C, \mathbb{Z}/p^n) \to H^1_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}/p^n)$ is an isomorphism for all $n \geq 1$.

For any scheme X which is proper over $\mathrm{Spec}\, \mathbb{F}_q$, there exists an exact sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to W_n \mathcal{O}_X \xrightarrow{1-\sigma} W_n \mathcal{O}_X \to 0$$

of étale sheaves, where $W_n \mathcal{O}_X$ is the sheaf of Witt vectors and $\sigma : W_n \mathcal{O}_X \to W_n \mathcal{O}_X$ is the Frobenius endomorphism. This gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \cdots \xrightarrow{1-\sigma} & H^0(C, W_n\mathcal{O}_C) & \to & H^1_{\mathrm{et}}(C, \mathbb{Z}/p^n) & \to & H^1(C, W_n\mathcal{O}_C) & \xrightarrow{1-\sigma} \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \xrightarrow{1-\sigma} & H^0(\mathcal{E}, W_n\mathcal{O}_\mathcal{E}) & \to & H^1_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}/p^n) & \to & H^1(\mathcal{E}, W_n\mathcal{O}_\mathcal{E}) & \xrightarrow{1-\sigma} \cdots . \end{array}$$

By Lemma 6.4 and induction on n, we see that the homomorphism $H^i(C, W_n\mathcal{O}_C) \to H^i(\mathcal{E}, W_n\mathcal{O}_\mathcal{E})$ is an isomorphism for $i = 0, 1$. Thus the map $H^1_{\mathrm{et}}(C, \mathbb{Z}/p^n) \to H^1_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}/p^n)$ is an isomorphism. This proves the claim (1).

For (2), it suffices to prove that the homomorphism $\mathrm{Lie}\,\mathrm{Pic}_{C/\mathbb{F}_q} \to \mathrm{Lie}\,\mathrm{Pic}_{\mathcal{E}/\mathbb{F}_q}$ between the tangent spaces is an isomorphism. Since this homomorphism is identified with the homomorphism $H^1(C, \mathcal{O}_C) \to H^1(\mathcal{E}, \mathcal{O}_\mathcal{E})$, the claim (2) follows from Lemma 6.4. □

Remark 6.1. Using Lemma 6.5 (1), we can prove that the homomorphism $\pi_1(\mathcal{E}) \to \pi_1(C)$ is an isomorphism. Since it is not used in this paper, let us only sketch the proof.

Let $x \to C$ be a geometric point. Since the morphism $f : \mathcal{E} \to C$ has a section, the fiber $\mathcal{E}_x$ of f at x has a reduced irreducible component. This, together with the regularity of $\mathcal{E}$ and C, shows that the canonical ring homomorphism $H^0(x, \mathcal{O}_x) \to H^0(Y, \mathcal{O}_Y)$ is an isomorphism for any connected finite etale covering Y of $\mathcal{E}_x$. Hence, by the same argument as in the proof of [1, X, Proposition 1.2, Théorème 1.3, p. 262], we have an exact sequence

$$\pi_1(\mathcal{E}_x) \to \pi_1(\mathcal{E}) \to \pi_1(C) \to 1.$$

In particular, the kernel of $\pi_1(\mathcal{E}) \to \pi_1(C)$ is abelian. Applying Lemma 6.5(1) to $\mathcal{E} \times_C C' \to C'$ for each finite connected étale cover $C' \to C$, we obtain the bijectivity of $\pi_1(\mathcal{E}) \to \pi_1(C)$.

The statements in Lemma 6.5 and the statement above that the fundamental groups are isomorphic are also valid for $\mathcal{E}$, a regular, proper, non-smooth, minimal elliptic fibration with a section over C, a proper smooth curve over an arbitrary perfect base field.

Corollary 6.2. *For any prime number $\ell \neq p$ and for any $i \in \mathbb{Z}$, the group $H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible.*

Proof. The claim for $i \neq 1, 2$ is obvious. By Lemma 6.5, we have $H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H^1_{\mathrm{et}}(\overline{C}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Hence $H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible. The group $H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible since $H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\mathrm{red}}$ is isomorphic to the Pontryagin dual of $H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$. □

Corollary 6.3. *For $i \in \mathbb{Z}$, we put $M^i_j = \bigoplus_{\ell \neq p} H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$. For a rational number a, we write $|a|^{(p')} = |a| \cdot |a|_p$.*

1. *For $i \leq -1$ or $i \geq 6$, the group M^i_j is zero.*
2. *For $j \neq 2$ (resp. $j = 2$), the group M^5_j is zero (resp. is isomorphic to $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$).*
3. *For $j \neq 0$, the group M^0_j is cyclic of order $q^{|j|} - 1$. The group M^0_0 is isomorphic to $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$.*
4. *For $j \neq 0$, the group M^1_j is finite of order $|L(h^1(C), 1 - j)|^{(p')}$.*
5. *For $j \neq 1$, the group M^2_j is finite of order $|L(h^2(\mathcal{E}), 2 - j)|^{(p')}$.*
6. *For $j \neq 1$, the group M^3_j is finite of order $|L(h^1(C), 2 - j)|^{(p')}$.*
7. *For $j \neq 2$, the group M^4_j is cyclic of order $q^{|2-j|} - 1$. The group M^4_2 is isomorphic to $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$.*

Proof. By Corollary 6.2, if $i \neq 2j + 1$ and $\ell \neq p$, the group $H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is isomorphic to $H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}}$. Then we have

$$|H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))| = |L(h^{4-i}(\mathcal{E}), 2 - j)|^{-1}_\ell$$

by Poincare duality for $i \neq 2j, 2j + 1$. Hence the claim follows from Lemma 6.3. □

6.5. Torsion in the étale cohomology of open elliptic surfaces. We fix a non-empty open subscheme $U \subset C$.

Lemma 6.6. *Let $\ell \neq p$ be a prime number. For $i \in \mathbb{Z}$, let γ_i denote the pull-back $\gamma_i : H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^i_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell)$.*

1. *For $i \neq 0, 2$, the homomorphism γ_i is zero.*

(2) *The cokernel* $(\operatorname{Coker}\gamma_2)_{\mathbb{Q}_\ell}$ *is isomorphic to the kernel of* $H^0_{\mathrm{et}}(\overline{C}\backslash\overline{U},\mathbb{Q}_\ell(-1)) \to H^0_{\mathrm{et}}(\operatorname{Spec}\overline{\mathbb{F}}_q,\mathbb{Q}_\ell(-1))$.

(3) *There is a canonical isomorphism*

$$\operatorname{Hom}_{\mathbb{Z}}(T_U,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \cong (\operatorname{Coker}\gamma_2)_{\mathrm{tors}}.$$

Proof. By Lemma 6.5, the pull-back $H^1_{\mathrm{et}}(\overline{C},\mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell)$ is an isomorphism. Hence the homomorphism $H^1_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\overline{\mathcal{E}^U},\mathbb{Z}_\ell)$ is zero. The claim (1) follows.

Let $\mathrm{NS}(\overline{\mathcal{E}}) = \operatorname{Pic}_{\mathcal{E}/\mathbb{F}_q}(\overline{\mathbb{F}}_q)/\operatorname{Pic}^o_{\mathcal{E}/\mathbb{F}_q}(\overline{\mathbb{F}}_q)$ denote the Neron-Severi group of $\overline{\mathcal{E}}$. For a prime number ℓ, we put $T_\ell M = \operatorname{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M)$.

We have an exact sequence from the Kummer theory:

$$0 \to \mathrm{NS}(\overline{\mathcal{E}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}_\ell} H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1)) \to T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{G}_m) \to 0.$$

We note that $T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{G}_m)$ is torsion free. For $D \in \operatorname{Irr}(\overline{\mathcal{E}^U})$, let $[D] \in \mathrm{NS}(\overline{\mathcal{E}})$ denote the class of the Weil divisor D_{red} on $\overline{\mathcal{E}}$. By [10, Cycle, Definition 2.3.2, p. 145], the D-component of the homomorphism $\gamma_2 : H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}^U},\mathbb{Z}_\ell) \cong \operatorname{Map}(\operatorname{Irr}(\overline{\mathcal{E}^U}),\mathbb{Z}_\ell(-1))$ is identified with the homomorphism

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell) \xrightarrow{\cup \mathrm{cl}_\ell([D])} H^4_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1)) \cong \mathbb{Z}_\ell(-1).$$

Let $M \subset \mathrm{NS}(\overline{\mathcal{E}})$ denote the subgroup generated by $\{[D] \mid D \in \operatorname{Irr}(\overline{\mathcal{E}^U})\}$. By Corollary 6.2, the cup-product

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1)) \times H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1)) \to H^4_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(2)) \cong \mathbb{Z}_\ell$$

is a perfect pairing. Hence the image of γ_2 is identified with the image of the composite

$$\begin{aligned}&\operatorname{Hom}_{\mathbb{Z}_\ell}(H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1)),\mathbb{Z}_\ell(-1)) \xrightarrow{\alpha^*} \operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z}_\ell(-1))\\ &\xrightarrow{\beta^*} \operatorname{Map}(\operatorname{Irr}(\overline{\mathcal{E}^U}),\mathbb{Z}_\ell(-1)) \cong H^2_{\mathrm{et}}(\overline{\mathcal{E}^U},\mathbb{Z}_\ell),\end{aligned}$$

where the homomorphism α^* is induced by the restriction $\alpha : M\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \hookrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1))$ of the cycle class map cl_ℓ to M, and the homomorphism β^* is induced by the canonical surjection $\beta : \bigoplus_{D\in\operatorname{Irr}(\overline{\mathcal{E}^U})} \mathbb{Z} \twoheadrightarrow M$. Since β is surjective, the homomorphism β^* is injective and we have an exact sequence

$$0 \to \operatorname{Coker}\alpha^* \to \operatorname{Coker}\gamma_2 \to \operatorname{Coker}\beta^* \to 0.$$

Since α is a homomorphism of finitely generated $\mathbb{Z}_\ell$-modules which is injective, the cokernel $\operatorname{Coker}\alpha^*$ is a finite group. The group M is a free abelian group with basis $\operatorname{Irr}^0(\overline{\mathcal{E}^U}) \cup \{D'\}$, where D' is an arbitrary element in $\operatorname{Irr}(\overline{\mathcal{E}^U}) \setminus \operatorname{Irr}^0(\overline{\mathcal{E}^U})$. Hence the cokernel $\operatorname{Coker}\beta^*$ is isomorphic to the group $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Ker}\beta,\mathbb{Z}_\ell(-1))$. This proves (2).

The torsion part of $\operatorname{Coker}\gamma_2$ is identified with the group $\operatorname{Coker}\alpha^*$. The homomorphism α^* is the composite of the two homomorphisms

$$\begin{aligned}&\operatorname{Hom}_{\mathbb{Z}_\ell}(H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1)),\mathbb{Z}_\ell(-1))\\ &\xrightarrow{\mathrm{cl}_\ell^*} \operatorname{Hom}_{\mathbb{Z}}(\mathrm{NS}(\overline{\mathcal{E}}),\mathbb{Z}_\ell(-1)) \xrightarrow{\iota^*} \operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z}_\ell(-1))\end{aligned}$$

where the first (resp. the second) homomorphism is induced by the cycle class map $\mathrm{cl}_\ell : \mathrm{NS}(\overline{\mathcal{E}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \hookrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1))$ (resp. the inclusion $M \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \hookrightarrow \mathrm{NS}(\overline{\mathcal{E}})$). Since

Coker cl_ℓ is torsion free as we noted, the homomorphism cl_ℓ^* is surjective. Hence we have isomorphisms

$$\begin{aligned}&\mathrm{Coker}\,\alpha^* \cong \mathrm{Coker}\,\iota^* \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{NS}(\overline{\mathcal{E}})/M, \mathbb{Z}_\ell(-1))\\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{NS}(\overline{\mathcal{E}})/M, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)).\end{aligned}$$

Thus we have the claim (3). □

Corollary 6.4. *For $i \neq 3$, the group $H^i_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell)$ is torsion free, and the group $H^3_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell)_{\mathrm{tors}}$ is canonically isomorphic to the group $\mathrm{Hom}_{\mathbb{Z}}(T_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))$. We put*

$$L(h^i_{c,\ell}(\mathcal{E}_U), s) = \det(1 - \mathrm{Frob}\cdot q^{-s}; H^i_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Q}_\ell)).$$

Then if $U \neq C$, we have

$$L(h^i_{c,\ell}(\mathcal{E}_U), s) = \begin{cases} 1, & \text{if } i \le 0 \text{ or } i \ge 5,\\ \frac{L(h^1(C),s)L(h^0(C\setminus U),s)}{1-q^{-s}} & \text{if } i = 1,\\ \frac{L(h^2(\mathcal{E}),s)L(h^1(\mathcal{E}^U),s)L(h^0(C\setminus U),s-1)}{(1-q^{1-s})L(h^2(\mathcal{E}^U),s)} & \text{if } i = 2,\\ \frac{L(h^1(C),s-1)L(h^0(C\setminus U),s-1)}{1-q^{1-s}} & \text{if } i = 3,\\ 1-q^{2-s}, & \text{if } i = 4.\end{cases}$$

Proof. This follows from Lemmas 6.3 and 6.6, and the long exact sequence

$$\cdots \to H^i_{\mathrm{et},c}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell) \to H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^i_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell) \to \cdots .$$

□

Remark 6.2. Corollary 6.4 in particular shows that the function $L(h^i_{c,\ell}(\mathcal{E}_U), s)$ is independent of $\ell \neq p$. We can show the ℓ-independence of $L(h^i_{c,\ell}(X), s) = \det(1 - \mathrm{Frob}\cdot q^{-s}; H^i_{c,\mathrm{et}}(X, \mathbb{Q}_\ell))$ for any normal surface X over $\mathbb{F}_q$ which is not necessarily proper. Since we will not need it, let us only give a sketch. There is a proper smooth surface X' and a closed subset $D \subset X'$ of pure codimension one such that $X = X' \setminus D$. One can express the cokernel and kernel of the restriction map $H^1_{\mathrm{et}}(\overline{X}', \mathbb{Q}_l) \to H^1_{\mathrm{et}}(\overline{D}, \mathbb{Q}_l)$ in terms of $\mathrm{Pic}_{X'/\mathbb{F}_q}$ and the Jacobian of the normalization of each irreducible component of D. Then we apply the same method as above to obtain the result.

Corollary 6.5. *Suppose that $U \neq C$. Then*

1. *The group $H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is zero for $i \le -1$ or $i \ge 5$.*
2. *For $j \neq 0$, the group $H^0_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is isomorphic to $\mathbb{Z}_\ell/(q^j - 1)$, and $H^0_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) = \mathbb{Q}_\ell/\mathbb{Z}_\ell$.*
3. *For $j \neq 0, 1$, the group $H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|T'_{U,(j-1)}|_\ell^{-1}\cdot|L(h^1(C), 1-j)L(h^0(C\setminus U), 1-j)|_\ell^{-1}}{|q^{j-1}-1|_\ell^{-1}}.$$

The group $H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$ is isomorphic to the direct sum of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and a finite group of order

$$\frac{|T'_{U,(-1)}|_\ell^{-1}\cdot|L(h^1(C), 1)L(h^0(C\setminus U), 1)|_\ell^{-1}}{|q-1|_\ell^{-1}}.$$

(4) *For $j \neq 1, 2$, the cohomology group $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|T'_{U,(j-1)}|_\ell^{-1} \cdot |L(h^2(\mathcal{E}), 2-j)L(h^1(\mathcal{E}^U), 2-j)L(h^0(C \setminus U), 1-j)|_\ell^{-1}}{|(q^{j-1}-1)L(h^2(\mathcal{E}^U), 2-j)|_\ell^{-1}}.$$

(5) *For $j \neq 1, 2$, the group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|L(h^1(C), 2-j)L(h^0(C \setminus U), 2-j)|_\ell^{-1}}{|q^{j-2}-1|_\ell^{-1}}.$$

The cohomology group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ is isomorphic to the direct sum of $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus |C \setminus U| - 1}$ and a finite group of order

$$\frac{|L(h^1(C), 1)L(h^0(C \setminus U), 1)|_\ell^{-1}}{|q-1|_\ell^{-1}}.$$

(6) *For $j \neq 2$ (resp. $j = 2$), the group $H^4_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is zero (resp. is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus |C \setminus U| - 1}$).*

Proof. The cohomology group $H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is the Pontryagin dual of the group $H^{5-i}_{\mathrm{et},c}(\mathcal{E}_U, \mathbb{Z}_\ell(2-j))$. The claims follow from Corollary 6.4 and the short exact sequence

$$\begin{aligned} 0 &\to H^{4-i}_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2-j))_{G_{\mathbb{F}_q}} \to H^{5-i}_{c,\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2-j)) \\ &\to H^{5-i}_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2-j))^{G_{\mathbb{F}_q}} \to 0. \end{aligned}$$

□

Lemma 6.7. *Suppose that $U \neq C$. Then $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$ is finite of order*

$$\frac{|T'_{U,(1)}|_\ell^{-1} \cdot |L(h^2(\mathcal{E}), 0)L^*(h^1(\mathcal{E}^U), 0)L(h^0(C \setminus U), -1)|_\ell^{-1}}{|(q-1)L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|_\ell^{-1}}.$$

Proof. We note that the group $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$ is canonically isomorphic to the group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))_{\mathrm{tors}}$. Let us consider the long exact sequence

$$\cdots \to H^i_{\mathcal{E}^U,\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \xrightarrow{\mu_i} H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \to H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2)) \to \cdots$$

The group $\operatorname{Ker} \mu_4$ is isomorphic to the Pontryagin dual of the cokernel of the homomorphism $H^1_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}^U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. By Lemma 6.5, this homomorphism factors through $H^1_{\mathrm{et}}(C \setminus U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}^U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. In particular, the group $(\operatorname{Ker} \mu_4)_{\mathrm{tors}}$ is isomorphic to the Pontryagin dual of $(H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_{\mathbb{F}_q}})^{\mathrm{red}}$. By the weight argument we see that $\operatorname{Coker} \mu_3$ is a finite group. It follows that

$$|H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))_{\mathrm{tors}}| = |L^*(h^1(\mathcal{E}^U), 0)| \cdot |\operatorname{Coker} \mu_3|.$$

Let μ' denote the homomorphism $H^2_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))$. We have the exact sequence

$$\operatorname{Ker} \mu_3 \to H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} \to (\operatorname{Coker} \mu')_{G_{\mathbb{F}_q}} \to \operatorname{Coker} \mu_3 \to 0. \tag{6.1}$$

Since $\operatorname{Ker} \mu_3 \cong \operatorname{Coker}[H^2_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))]$, the cokernel of $\operatorname{Ker} \mu_3 \to H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}}$ is isomorphic to the cokernel of the homomorphism

$$\nu' : H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} \to H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}}.$$

Let us consider the diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Coker}\mu' & \longrightarrow & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) & \xrightarrow{\nu} & H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \\ & & \downarrow{\scriptstyle 1-\mathrm{Frob}} & & \downarrow{\scriptstyle 1-\mathrm{Frob}} & & \downarrow{\scriptstyle 1-\mathrm{Frob}} \\ 0 & \longrightarrow & \operatorname{Coker}\mu' & \longrightarrow & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) & \xrightarrow{\nu} & H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)). \end{array}$$

Since $(\operatorname{Coker}\nu)^{G_{\mathbb{F}_q}} \subset H^3_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} = 0$, we have that $\operatorname{Coker}\nu'$ is isomorphic to the kernel of $(\operatorname{Coker}\mu')_{G_{\mathbb{F}_q}} \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}$. Hence by (6.1), $|\operatorname{Coker}\mu_3|$ equals the order of

$$\begin{aligned} M'' &= \operatorname{Image}[(\operatorname{Coker}\mu')_{G_{\mathbb{F}_q}} \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}] \\ &= \operatorname{Image}[H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}} \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}]. \end{aligned}$$

We put $M' = \operatorname{Image}[H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))]$. From the commutative diagram with exact rows

$$(6.2)\quad \begin{array}{ccccccc} 0 \to & \mathrm{NS}(\overline{\mathcal{E}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \to & H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) & \to & T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & (\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \to & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(1)) & \to & T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{G}_m) & \to 0 \end{array}$$

and the exact sequence

$$0 \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{G}_m) \to H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}/\mathbb{Z}),$$

we obtain an exact sequence

$$0 \to M' \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) \to T_\ell H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)).$$

By the weight argument, we have $(T_\ell H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)))^{G_{\mathbb{F}_q}} = 0$. Hence the canonical surjection $M'_{G_{\mathbb{F}_q}} \to M''$ is an isomorphism. From (6.2) we have an exact sequence

$$0 \to (\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1) \to M' \to T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1) \to 0.$$

By the weight argument, we have $(T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1))^{G_{\mathbb{F}_q}} = 0$. Hence

$$\begin{aligned} 0 &\to ((\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1))_{G_{\mathbb{F}_q}} \to M'_{G_{\mathbb{F}_q}} \\ &\to (T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1))_{G_{\mathbb{F}_q}} \to 0 \end{aligned}$$

is exact. Therefore $|\operatorname{Coker}\mu_3| = |M'_{G_{\mathbb{F}_q}}|$ equals

$$\frac{|(T_U \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1))_{G_{\mathbb{F}_q}}| \cdot |\det(1-\mathrm{Frob};\, H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell(2)))|_\ell^{-1}}{|\det(1-\mathrm{Frob};\, \operatorname{Ker}[\mathrm{NS}(\overline{\mathcal{E}}) \to \mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}] \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(1))|_\ell^{-1}}.$$

This proves the claim. □

6.6. Fix a non-empty open subscheme $U \subset C$. Let $f^U : \mathcal{E}^U \to C \setminus U$ denote the structure morphism and let $\iota^U : C \setminus U \to \mathcal{E}^U$ denote the morphism induced from $\iota : C \to \mathcal{E}$.

Lemma 6.8. *The homomorphism*

$$(\mathrm{ch}'_{1,1}, f^U_*) : G_1(\mathcal{E}^U) \to H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \oplus K_1(C \setminus U)$$

is an isomorphism.

Proof. The morphism $f^U : \mathcal{E}^U \to C \setminus U$ has connected fibers. Hence the claim follows from Proposition 4.1 and the construction of $\mathrm{ch}'_{1,2}$. □

Lemma 6.9. *The group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is finitely generated of rank $|C \setminus U|$. Moreover, $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ is of order $|L^*(h^1(\mathcal{E}^U), 0)|$.*

Proof. It suffices to prove the following claim: if E has good reduction (resp. non-split multiplicative reduction, resp. split multiplicative or additive reduction) at $\wp \in C_0$, then $H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))$ is a finitely generated abelian group of rank one, and $|H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))_{\mathrm{tors}}|$ equals $|\mathcal{E}_\wp(\kappa(\wp))|$ (resp. 2, resp. 1). We put $\mathcal{E}_{\wp,(0)} = (\mathcal{E}_{\wp,\mathrm{red}})_{\mathrm{sm}} \setminus \iota(\wp)$ and $\mathcal{E}_{\wp,(1)} = \mathcal{E}^U \setminus \mathcal{E}_{\wp,(0)}$. We have an exact sequence

$$\begin{aligned} &H^1_{\mathcal{M}}(\mathcal{E}_{\wp,(0)}, \mathbb{Z}(1)) \to H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \\ &\to H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \to \mathrm{Pic}(\mathcal{E}_{\wp,(0)}) \to 0. \end{aligned}$$

First suppose that E does not have non-split multiplicative reduction at $\wp$, or E has non-split multiplicative reduction at $\wp$ and $\mathcal{E}_\wp \otimes_{\kappa(\wp)} \overline{\mathbb{F}}_q$ has an even number of irreducible components. Then, using the classification due to Kodaira, Neron and Tate (cf. [29, 10.2.1, p. 484–489]) of singular fibers of $\mathcal{E} \to C$, we can verify the equality

$$\begin{aligned} &\mathrm{Image}[H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))] \\ = \ &\mathrm{Image}[\iota_* : H^0_{\mathcal{M}}(\operatorname{Spec} \kappa(\wp), \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))]. \end{aligned}$$

This shows that the group $H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))$ is isomorphic to the direct sum of the Picard group $\mathrm{Pic}(\mathcal{E}_{\wp,(0)})$ and $H^0_{\mathcal{M}}(\operatorname{Spec} \kappa(\wp), \mathbb{Z}(0)) \cong \mathbb{Z}$. In particular, we have $H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))_{\mathrm{tors}} \cong \mathrm{Pic}(\mathcal{E}_{\wp,(0)})$, from which we easily deduce the claim.

Now suppose that E has non-split multiplicative reduction at $\wp$ and $\mathcal{E}_\wp \otimes_{\kappa(\wp)} \overline{\mathbb{F}}_q$ has an odd number of irreducible components. In this case, we can verify directly that the image of $H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$ and $\mathrm{Pic}(\mathcal{E}_{\wp,(0)}) = 0$. The claim in this case follows. □

Lemma 6.10. *The diagram*

$$\begin{array}{ccc} K_1(E) & \longrightarrow & G_0(\mathcal{E}^U) \\ {\scriptstyle \iota^*}\big\downarrow & & {\scriptstyle \iota^{U*}}\big\downarrow \\ K_1(k) & \longrightarrow & K_0(C \setminus U) \end{array}$$

is commutative.

Proof. The group $K_1(E)$ is generated by the image of $f^* : K_1(k) \to K_1(E)$ and the image of $\bigoplus_{x \in E_0} K_1(\kappa(x)) \to K_1(E)$. The claim follows from the fact that the localization sequence in G-theory commutes with flat pull-backs and finite push-forwards. □

6.7. Proofs of Theorems 6.1, 6.2 and 6.3.

Lemma 6.11. *For any non-empty open subscheme $U \subset C$, the cokernel of the boundary map $\partial_U : H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is finite.*

Proof. If suffices to prove the claim for sufficiently small U. Hence we may assume that $\mathcal{E}_U \to U$ is smooth. Since $K_2(\mathcal{E}_U)_{\mathbb{Q}} \to K_2(E)_{\mathbb{Q}}$ is an isomorphism in this case, the claim follows from Theorem 1.3 and Lemma 5.2. □

Proof of Theorem 6.3. The claims (1) and (2) follow from Theorem 2.1, Proposition 2.1 and Lemma 6.11. Proposition 2.1 gives the exact sequence

$$\begin{aligned} & 0 \to H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \to H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}} \\ \xrightarrow{\partial^2_U} \ & H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \to H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}} \\ \xrightarrow{\partial^3_U} \ & H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to \mathrm{CH}_0(\mathcal{E}) \to \mathrm{CH}_0(\mathcal{E}_U) \to 0. \end{aligned}$$

From Lemma 6.11, it follows that $\operatorname{Coker} \partial^2_U$ is a finite group, which implies that the group $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ is of rank $|S_0 \setminus U|$. By Theorem 2.1, $|H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}|$ equals

$$\prod_{\ell \neq p} |H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))|.$$

By Corollaries 6.3 and 6.5, it equals

$$|T'_{U,(1)}| \cdot |L(h^1(C), -1) L(h^0(C \setminus U, -1)/(q-1)|.$$

This proves the claim (3).

As we have noted in the proof of Theorem 2.1 (1), the group $\mathrm{CH}_0(\mathcal{E})$ is a finitely generated abelian group of rank one and $\mathrm{CH}_0(\mathcal{E}_U)$ is finite if $U \neq C$. By Lemma 6.9, $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is a finitely generated abelian group of rank $|C \setminus U|$. Hence the rank of $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ equals $\max(|C \setminus U| - 1, 0)$.

From the class field theory of varieties over finite fields ([23, Theorem 1, p. 242], see also [8, p. 283–284]) and Lemma 6.5, it follows that the push-forward map $\mathrm{CH}_0(\mathcal{E}) \to \mathrm{Pic}(C)$ is an isomorphism. Hence the homomorphism $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to \mathrm{CH}_0(\mathcal{E}) \cong \mathrm{Pic}(C)$ factors though the push-forward map $f^U_* : H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0))$. By the surjectivity of f^U_*, we have isomorphisms

$$\mathrm{CH}_0(\mathcal{E}^U) \cong \operatorname{Coker}[H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0)) \to \mathrm{Pic}(C)] \cong \mathrm{Pic}(U),$$

which proves the claim (6). Since the group $H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0))$ is torsion free, the image of $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ in $\mathrm{CH}_0(\mathcal{E})$ is zero. Thus we have the exact sequence

$$\begin{aligned} & 0 \to \operatorname{Coker} \partial^2_U \to H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \\ & \to H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}} \to H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}} \to 0. \end{aligned}$$

By Proposition 2.1 and Lemma 6.7, the group $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}$ is finite of order

$$\frac{p^m |T'_{U,(1)}| \cdot |L(h^2(\mathcal{E}), 0) L^*(h^1(\mathcal{E}^U), 0) L(h^0(C \setminus U), -1)|}{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}$$

for some $m \in \mathbb{Z}$. By Lemma 6.9, the group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ is finite of order $|L^*(h^1(\mathcal{E}^U), 0)|$. By Lemma 6.5, the Picard scheme $\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}$ is an abelian variety and in particular $\mathrm{Hom}(\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}, \mathbb{G}_m) = 0$. Hence by Theorem 2.1 and Corollary 6.3, the group $H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}$ is of order $|L(h^2(\mathcal{E}), 0)|$. Therefore,

$$\begin{aligned} |\operatorname{Coker} \partial^2_U| &= \frac{|H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}| \cdot |H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}|}{|H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}|} \\ &= \frac{p^{-m}(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}{|T'_{U,(1)}| \cdot |L(h^0(C \setminus U), -1)|}. \end{aligned}$$

Since $|\operatorname{Coker} \partial^2_U|$ is prime to p, we have $m = 0$. This proves the claims (4) and (5). This completes the proof. □

Proof of Theorem 6.2. The claim (5) is clear. The claim (1) follows from Corollary 2.1 and Theorem 1.3. It can be checked easily that $H^i_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is zero for $i \le 0$. By the localization sequence of higher Chow groups (cf. [6, Corollary (0.2), p. 537]), we have $H^i_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2)) \cong H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ for $i \le 1$. Taking the inductive limit with respect to U, we obtain the claim (2).

By Corollary 2.1, we have an exact sequence

$$
\begin{aligned}
(6.3)\qquad & 0 \to H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \xrightarrow{\alpha} H^2_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}} \\
& \xrightarrow{\partial^2_{\mathcal{M},2}} \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \to H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}. \to H^3_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}} \\
& \xrightarrow{\partial^3_{\mathcal{M},2}} \bigoplus_{\wp \in C_0} H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \to \mathrm{Pic}(C) \to 0.
\end{aligned}
$$

Hence by Theorem 2.1 and Corollary 6.3, the group $\mathrm{Ker}\, \partial^2_{\mathcal{M},2}$ is finite of order $|L(h^1(C), -1)|$. For a non-empty open subscheme $U \subset C$, let us consider the group $\mathrm{Coker}\, \partial^2_U$ in the proof of Theorem 6.3. For two non-empty open subschemes $U', U \subset C$ with $U' \subset U$, the homomorphism $\mathrm{Coker}\, \partial^2_U \to \mathrm{Coker}\, \partial^2_{U'}$ is injective since both $\mathrm{Coker}\, \partial^2_U$ and $\mathrm{Coker}\, \partial^2_{U'}$ canonically inject into $H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}$. The claim (3) follows from the claim (4) of Theorem 6.3 by taking the inductive limit. The claim (4) follows from the exact sequence (6.3) and Lemma 6.3.

From the localization sequence, it follows that the push-forward homomorphism $\bigoplus_{x \in E_0} H^2_{\mathcal{M}}(\mathrm{Spec}\, \kappa(x), \mathbb{Z}(2)) \to H^4_{\mathcal{M}}(E, \mathbb{Z}(3))$ is surjective. Hence $H^4_{\mathcal{M}}(E, \mathbb{Z}(3))$ is a torsion group and the claim (6) follows from Lemma 5.4. This completes the proof. □

Proof of Theorem 6.1. Let us consider the restriction $\gamma : \mathrm{Ker}\, c_{2,3} \to H^2_{\mathcal{M}}(E, \mathbb{Z}(2))$ of $c_{2,2}$ to $\mathrm{Ker}\, c_{2,3}$. By Lemma 5.1, both $\mathrm{Ker}\, \gamma$ and $\mathrm{Coker}\, \gamma$ are annihilated by the multiplication-by-2 map. This implies that the image of γ contains $H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$ and that $\mathrm{Ext}^1_{\mathbb{Z}}(H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}, \mathrm{Ker}\, \gamma)$ is zero. From this it follows that the homomorphism γ induces an isomorphism $(\mathrm{Ker}\, c_{2,3})_{\mathrm{div}} \xrightarrow{\cong} H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$. This shows that the homomorphism $K_2(E)^{\mathrm{red}} \to H^2_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}}$ induced by $c_{2,2}$ is surjective with torsion kernel. Thus the claim (1) follows from Theorem 6.2 (3).

The claim (3) follows from Theorem 6.2 (1) and Lemma 5.1.

For $\wp \in C_0$, let $\iota_\wp : \mathrm{Spec}\, \kappa(\wp) \to \mathcal{E}_\wp$ denote the reduction at $\wp$ of the morphism $\iota : C \to \mathcal{E}$. The diagram (5.2) gives an exact sequence

$$\mathrm{Coker}\, \partial^4_{\mathcal{M},3} \to \mathrm{Coker}\, \partial_2 \to \mathrm{Coker}\, \partial^2_{\mathcal{M},2} \to 0.$$

By Lemma 5.4, we have an isomorphism $\mathrm{Coker}\, \partial^4_{\mathcal{M},3} \cong \mathbb{F}_q^\times$. By the construction of this isomorphism, we see that the composite

$$\mathbb{F}_q^\times \cong \mathrm{Coker}\, \partial^4_{\mathcal{M},3} \to \mathrm{Coker}\, \partial_2 \hookrightarrow K_1(\mathcal{E}) \to K_1(\mathrm{Spec}\, \mathbb{F}_q) \cong \mathbb{F}_q^\times$$

equals the identity. Hence the map $\mathrm{Coker}\, \partial^4_{\mathcal{M},3} \to \mathrm{Coker}\, \partial_2$ is injective. Then the claim (2) follows from Theorem 6.2 (3).

From Proposition 4.1 and Lemmas 5.1, 5.2, and 6.10, it follows that the homomorphism $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ is identified with the direct sum of the map $\partial'_1 : k^\times \to \bigoplus_{\wp \in C_0} H^0_{\mathcal{M}}(\mathrm{Spec}\, \kappa(\wp), \mathbb{Z}(0)) \to \bigoplus_{\wp} H^0_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(0))$ and the map

$\partial^3_{\mathcal{M},2} : H^3_{\mathcal{M}}(E,\mathbb{Z}(2))^{\mathrm{red}} \to \bigoplus_{\wp} H^2_{\mathcal{M}}(\mathcal{E}_\wp,\mathbb{Z}(1))$. We have isomorphisms

$$\operatorname{Ker}\partial'_1 \cong \mathbb{F}_q^\times,\ \operatorname{Coker}\partial'_1 \cong \operatorname{Pic}(C)\oplus\bigoplus_{\wp}\mathbb{Z}^{|\operatorname{Irr}(\mathcal{E}_\wp)|-1},$$
$$\operatorname{Ker}\partial^3_{\mathcal{M},2} \cong H^3_{\mathcal{M}}(\mathcal{E},\mathbb{Z}(2))_{\mathrm{tors}}/\operatorname{Coker}\partial^2_{\mathcal{M},2},\ \operatorname{Coker}\partial^3_{\mathcal{M},2} \cong \operatorname{Pic}(C).$$

The claim (4) follows. This completes the proof of Theorem 6.1. □

7. Results for $j \geq 3$

We obtain results for $j \geq 3$, generalizing the theorems in Section 6, but the proofs of the results use neither class field theory nor Drinfeld modular curves. The notations are as in Section 6.

For integers i, j, let us consider the boundary map

$$\partial^i_{\mathcal{M},j} : H^i_{\mathcal{M}}(E,\mathbb{Z}(j))^{\mathrm{red}} \to \bigoplus_{\wp\in C_0} H^{i-1}_{\mathcal{M}}(\mathcal{E}_\wp,\mathbb{Z}(j-1)).$$

Theorem 7.1. *Let $j \geq 3$ be an integer.*

(1) *For any $i\in\mathbb{Z}$, both $\operatorname{Ker}\partial^i_{\mathcal{M},j}$ and $\operatorname{Coker}\partial^i_{\mathcal{M},j}$ are finite groups.*

(2) *We have*

$$|\operatorname{Ker}\partial^i_{\mathcal{M},j}| = \begin{cases} 1, & \text{if } i\leq 0 \text{ or } i\geq 5,\\ q^j-1, & \text{if } i=1,\\ |L(h^1(C),1-j)|, & \text{if } i=2,\\ \frac{|T'_{(j-1)}|\cdot|L(h^2(\mathcal{E}),2-j)|}{q^{j-1}-1}, & \text{if } i=3,\\ |L(h^1(C),2-j)|, & \text{if } i=4.\end{cases}$$

Moreover, the group $\operatorname{Ker}\partial^1_{\mathcal{M},j}$ is cyclic of order q^j-1.

(3) *We have*

$$|\operatorname{Coker}\partial^i_{\mathcal{M},j}| = \begin{cases} 1, & \text{if } i\leq 1,\ i=3,\ \text{or } i\geq 5,\\ \frac{q^{j-1}-1}{|T'_{(j-1)}|}, & \text{if } i=2,\\ q^{j-2}-1 & \text{if } i=4.\end{cases}$$

(4) *Let $U\subset C$ be a non-empty open subscheme. Then the group $H^i_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(j))$ is finite modulo a uniquely divisible subgroup for any $i\in\mathbb{Z}$. The group $H^i_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(j))$ is zero if $i\geq\max(6,j)$, and is finite for $(i,j)=(4,3),(5,3),(4,4),(5,4)$, or $(5,5)$.*

(5) *The group $H^i_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(j))$ is uniquely divisible for $i\leq 0$ or $6\leq i\leq j$, and the group $H^1_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(j))_{\mathrm{tors}}$ is cyclic of order q^j-1.*

(6) *Suppose that $U=C$ (resp. $U\neq C$). The group $H^2_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^1(C),1-j)|$ (resp. of order*

$$\frac{|T'_{U,(j-1)}|\cdot|L(h^1(C),1-j)L(h^0(C\setminus U),1-j)|}{q^{j-1}-1}).$$

The group $H^3_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^2(\mathcal{E}),2-j)|$ (resp. of order

$$\frac{|T'_{U,(j-1)}|\cdot|L(h^2(\mathcal{E}),2-j)L(h^1(\mathcal{E}^U),2-j)L(h^0(C\setminus U),1-j)|}{(q^{j-1}-1)|L(h^0(\operatorname{Irr}(\mathcal{E}^U)),1-j)|}).$$

The group $H^4_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^1(C),2-j)|$ (resp. of order

$$\frac{|L(h^1(C),2-j)L(h^0(C\setminus U),2-j)|}{q^{j-2}-1}).$$

The group $H^5_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ *is cyclic of order* $q^{j-2}-1$ *(resp. is zero).*

Theorem 7.2. *The following statements hold.*

(1) *The group* $K_2(E)_{\mathrm{div}}$ *is uniquely divisible and the map* $c_{2,2}$ *induces an isomorphism* $K_2(E)_{\mathrm{div}} \cong H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$.

(2) *The kernel of the boundary map* $\partial : K_2(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ *is a finite group of order* $|L(h^1(C), -1)|^2$.

Lemma 7.1. *Let* X *be a smooth projective geometrically connected curve over a global field* k'. *Let* $k'(X)$ *denote the function field of* X. *Then the Milnor* K*-group* $K^M_n(k'(X))$ *is torsion for* $n \geq 2 + \mathrm{gon}(X)$, *and is of exponent* 2 *(resp. is zero) for* $n \geq 3 + \mathrm{gon}(X)$ *if* $\mathrm{char}(k') = 0$ *(resp.* $\mathrm{char}(k') > 0$*). Here* $\mathrm{gon}(X)$ *denotes the gonality of* X, *namely, the minimal degree of morphisms from* X *to* $\mathbb{P}^1_{k'}$.

Proof. The field $k'(X)$ is an extension of degree $\mathrm{gon}(X)$ of a subfield K of the form $K = k'(t)$. Looking at the split exact sequence

$$0 \to K^M_n(k') \to K^M_n(K) \to \bigoplus_P K^M_{n-1}(k'[t]/P) \to 0$$

in [34, Theorem 2.3, p. 325] (where P runs over the irreducible monic polynomials in $k'[t]$), and using [5, Chapter II, (2.1), p. 396], we see that $K^M_n(K)$ is torsion for $n \geq 3$, and is of exponent 2 (resp. is zero) for $n \geq 4$ if $\mathrm{char}(k') = 0$ (resp. $\mathrm{char}(k') > 0$).

Take a flag $K = V_1 \subset V_2 \subset \cdots \subset V_{\mathrm{gon}(X)} = k'(X)$ of K-subspaces of $k'(X)$ with $\dim_K V_i = i$. For each i we put $V_i^* = V_i \setminus \{0\}$. Suppose $i \geq 2$ and take two elements $\alpha, \beta \in V_i \setminus V_{i-1}$. Then there exist $a, b \in K^\times$ such that $\gamma = a\alpha + b\beta \in V_{i-1}$. If $\gamma = 0$ (resp. $\gamma \neq 0$), then $\{a\alpha, b\beta\} = 0$ (resp. $\{a\alpha/\gamma, b\beta/\gamma\} = 0$) in $K^M_2(k'(X))$. Expanding this equality, we see that $\{\beta, \gamma\}$ belongs to the subgroup of $K^M_2(k'(X))$ generated by $\{V_i^*, V_{i-1}^*\}$. Hence for $n \geq \mathrm{gon}(X) - 1$, the group $K^M_n(k'(X))$ is generated by the image of $\{V^*_{\mathrm{gon}(X)}, \ldots, V_2^*\} \times K^M_{n-\mathrm{gon}(X)+1}(K)$. This proves the claim. ∎

Lemma 7.2. *Then the push-forward homomorphism* $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1)) \to H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))$ *is zero.*

Proof. Let us consider the composite

$$H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1)) \to H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j)) \xrightarrow{f_*} H^2_{\mathcal{M}}(C, \mathbb{Z}(j-1))$$

of push-forwards. This is the zero map since this factors through the group $H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(j-2))$ which is zero by [14, Corollary 1.2, p. 56]. By Lemma 2.4, the group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1))$ is torsion. Hence it suffices to show that the homomorphism $f_{*,\mathrm{tors}} : H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{tors}} \to H^2_{\mathcal{M}}(C, \mathbb{Z}(j-1))_{\mathrm{tors}}$ induced by f_* is an isomorphism.

Let us consider the commutative diagram

$$\begin{array}{ccc} H^4_{\mathcal{M}}(\mathcal{E},\mathbb{Z}(j))_{\mathrm{tors}} & \xrightarrow{f_{*,\mathrm{tors}}} & H^2_{\mathcal{M}}(C,\mathbb{Z}(j-1))_{\mathrm{tors}} \\ \cong\uparrow & & \cong\uparrow \\ H^3_{\mathcal{M}}(\mathcal{E},\mathbb{Q}/\mathbb{Z}(j)) & \longrightarrow & H^1_{\mathcal{M}}(C,\mathbb{Q}/\mathbb{Z}(j-1)) \\ \cong\downarrow & & \cong\downarrow \\ \bigoplus_{\ell\neq p} H^3_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}} & \longrightarrow & \bigoplus_{\ell\neq p} H^1_{\mathrm{et}}(\overline{C},\mathbb{Q}_\ell/\mathbb{Z}_\ell(j-1))^{G_{\mathbb{F}_q}}. \end{array}$$

Here the horizontal arrows are push-forward maps, the upper vertical arrows are the boundary maps obtained from the exact sequence $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$, and the lower vertical arrows are obtained from Theorem 2.1(2)(b) (the same argument also applies to curves) further using the weight argument. The homomorphism at the bottom is an isomorphism by Lemma 6.5. Hence $f_{*,\mathrm{tors}}$ is an isomorphism, as desired. □

Proof of Theorem 7.1. From Theorem 2.1(2) and Lemma 7.1, the claims (4) and (5) follow. The claim (6) follows from Theorem 2.1(2) and Corollary 6.5. In a manner similar to that in the proof of Corollary 2.1, we are able to show that the pull-back map induces an isomorphism $H^i_{\mathcal{M}}(\mathcal{E},\mathbb{Z}(j))_{\mathrm{div}}\cong H^i_{\mathcal{M}}(E,\mathbb{Z}(j))_{\mathrm{div}}$ for all $i\in\mathbb{Z}$, and that the localization sequence induces the long exact sequence

$$(7.1)\qquad \begin{aligned} \cdots &\to \bigoplus_{\wp\in C_0} H^{i-2}_{\mathcal{M}}(\mathcal{E}_\wp,\mathbb{Z}(j-1)) \\ &\to H^i_{\mathcal{M}}(\mathcal{E},\mathbb{Z}(j))_{\mathrm{tors}}\to H^i_{\mathcal{M}}(E,\mathbb{Z}(j))_{\mathrm{tors}}\to\cdots. \end{aligned}$$

Using Lemma 2.1, we easily see that for any $\wp\in C_0$, even if $\mathcal{E}_\wp$ is singular, the group $H^i_{\mathcal{M}}(\mathcal{E}_\wp,\mathbb{Z}(j-1))$ is finite for all i, is zero for $i\le 0$ or $i\ge 4$, and is cyclic of order $q_\wp^{j-1}-1$, where $q_\wp=|\kappa(\wp)|$ is the cardinality of the residue field at $\wp$, for $i=1$. By looking at the exact sequence (7.1) and using Lemma 7.2, we can deduce the claims (1), (2) and (3) from the claims (4), (5) and (6). This completes the proof. □

Proof of Theorem 7.2. Let $U\neq C$. Then by Lemmas 5.4 and 7.2, the following sequence is exact:

$$0\to H^4_{\mathcal{M}}(\mathcal{E},\mathbb{Z}(3))\to H^4_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(3))\xrightarrow{\partial} H^3_{\mathcal{M}}(\mathcal{E}^U,\mathbb{Z}(2))\xrightarrow{\alpha}\mathbb{F}_q^\times\to 1.$$

Here ∂ and α are as in Lemma 5.4, and the second map is the pull-back. By taking the inductive limit, we obtain the exact sequence

$$(7.2)\qquad \begin{aligned} 0 &\to H^4_{\mathcal{M}}(\mathcal{E},\mathbb{Z}(3))\to H^4_{\mathcal{M}}(E,\mathbb{Z}(3)) \\ &\xrightarrow{\partial^4_{\mathcal{M},3}} \textstyle\bigoplus_{\wp\in C_0} H^3_{\mathcal{M}}(\mathcal{E}_\wp,\mathbb{Z}(2))\to\mathbb{F}_q^\times\to 1. \end{aligned}$$

By Theorem 2.1 and Corollary 2.1, the group $H^4_{\mathcal{M}}(E,\mathbb{Z}(3))_{\mathrm{div}}$ is zero. Hence, using Lemma 5.1, we have $K_2(E)_{\mathrm{div}}\subset\operatorname{Ker}c_{2,3}$. We saw that the map $c_{2,2}$ induces an isomorphism $(\operatorname{Ker}c_{2,3})_{\mathrm{div}}\xrightarrow{\cong}H^2_{\mathcal{M}}(E,\mathbb{Z}(2))_{\mathrm{div}}$ in the proof of Theorem 6.1. Hence $c_{2,2}$ induces an isomorphism $K_2(E)_{\mathrm{div}}\cong H^2_{\mathcal{M}}(E,\mathbb{Z}(2))_{\mathrm{div}}$, which proves the claim (1). The claim (2) follows from Theorems 6.1 and 6.2, the commutative diagram (5.2), and the exact sequence (7.2). This completes the proof. □

Appendix A. A proposition on the p-part

The aim of this Appendix is to give a proof of Proposition A.1 below. It is used in the proof of Theorem 2.1. Nothing in this Appendix is new except possibly the definition of the Frobenius map on the inductive limit (not on the inverse limit) given in Section A.3. A similar situation has already appeared in the work of Milne ([32]) and Nygaard ([37]).

Proposition A.1. *Let X be a smooth projective geometrically connected surface over a finite field $\mathbb{F}_q$ of q elements of characteristic p. Let $W_n\Omega^i_{X,\log}$ denote the logarithmic de Rham-Witt sheaf (cf. [21, I, 5.7, p. 596]). Then the inductive limit $\varinjlim_n H^0_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$ with respect to multiplication-by-p is finite of order $|\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)|_p^{-1} \cdot |L(h^2(X), 0)|_p^{-1}$. Here $\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)$ denotes the set of homomorphisms $\mathrm{Pic}^o_{X/\mathbb{F}_q} \to \mathbb{G}_m$ of $\mathbb{F}_q$-group schemes, and $L(h^2(X), s)$ is the (Hasse-Weil) L-function of $h^2(X)$.*

A.1. The de Rham-Witt complex. In this Appendix, let k be a perfect field of characteristic p. Let X be a scheme of dimension δ which is proper over $\mathrm{Spec}\, k$. For $i, n \in \mathbb{Z}$, let $W_n\Omega^\bullet_X$ denote the de Rham-Witt complex (cf. [21, I, 1.12, p. 548]) of the ringed topos of schemes over X with Zariski topology. We let $R : W_n\Omega^i_X \to W_{n-1}\Omega^i_X$, $F : W_n\Omega^i_X \to W_{n-1}\Omega^i_X$, and $V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$ denote the restriction, the Frobenius, and the Verschiebung, respectively. For each $i \in \mathbb{Z}$, the sheaf $W_n\Omega^i_X$ has a canonical structure of coherent $W_n\mathcal{O}_X$-modules, which enables us to regard $W_n\Omega^i_X$ as an étale sheaf. From now on until the end of this section, we work on the category of étale sheaves on schemes over X.

A.2. Logarithmic de Rham-Witt sheaves. For $n \in \mathbb{Z}$, let $W_n\Omega^i_{X,\log} \subset W_n\Omega^i_X$ denote the logarithmic de Rham-Witt sheaf (cf. [21, I, 5.7, p .596]).

Lemma A.1. *The homomorphism $V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$ sends $W_n\Omega^i_{X,\log}$ into $W_{n+1}\Omega^i_{X,\log}$.*

Proof. Let $x \in W_n\Omega^i_{X,\log}$ be an étale local section. By the definition of $W_n\Omega^i_{X,\log}$, there exists an étale local section $y \in W_{n+1}\Omega^i_{X,\log}$ such that $x = Ry$. We easily see that $Ry = Fy$. Hence $Vx = VRy = VFy = py \in W_{n+1}\Omega^i_{X,\log}$. □

Let $CW\Omega^i_X$ denote the inductive limit $CW\Omega^i_X = \varinjlim_{n,V} W_n\Omega^i_X$ with respect to V. The above lemma enables us to define the inductive limit $CW\Omega^i_{X,\log} = \varinjlim_{n,V} W_n\Omega^i_{X,\log}$.

A.3. Modified Frobenius operator. In this section we define an operator $F' : CW\Omega^i_X \to CW\Omega^i_X$ such that the sequence

$$0 \to CW\Omega^i_{X,\log} \to CW\Omega^i_X \xrightarrow{1-F'} CW\Omega^i_X \to 0 \tag{A.1}$$

is exact.

For $n \geq 0$, let $\widetilde{W}_n\Omega^i_X$ denote the cokernel of the homomorphism $V^n : \Omega^i_X = W_1\Omega^i_X \to W_{n+1}\Omega^i_X$. The homomorphisms R, F and V on $W_{n+1}\Omega^i_X$ induce homomorphisms on $\widetilde{W}_n\Omega^i_X$ which we denote by the same notations. If $n \geq 1$, the homomorphisms $R, F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$ factor through the canonical surjection $W_{n+1}\Omega^i_X \to \widetilde{W}_n\Omega^i_X$. We let $\widetilde{R}, \widetilde{F} : \widetilde{W}_n\Omega^i_X \to W_n\Omega^i_X$ denote the induced homomorphisms. Then both $\widetilde{R}$ and $\widetilde{F}$ commute with R, F and V. For $n \geq 0$, we let $\widetilde{W}_n\Omega^i_{X,\log}$

denote the image of $W_{n+1}\Omega^i_{X,\log}$ by the canonical surjection $W_{n+1}\Omega^i_X \to \widetilde{W}_n\Omega^i_X$. The restriction of $\widetilde{R} : \widetilde{W}_n\Omega^i_X \to W_n\Omega^i_X$ to $\widetilde{W}_n\Omega^i_{X,\log}$ gives a surjective homomorphism $\widetilde{R}_{\log} : \widetilde{W}_n\Omega^i_{X,\log} \to W_n\Omega^i_{X,\log}$.

Lemma A.2. *The homomorphisms $\widetilde{R}$, $\widetilde{R}_{\log}$ induce isomorphisms*

$$\varinjlim_{n\geq 0,\, V} \widetilde{W}_n\Omega^i_X \cong CW\Omega^i_X, \quad \varinjlim_{n\geq 0,\, V} \widetilde{W}_n\Omega^i_{X,\log} \cong CW\Omega^i_{X,\log}.$$

Proof. The surjectivity is clear. From [21, I, PROPOSITION 3.2, p. 568], it follows that the kernel of $\widetilde{R}$ equals the image of the composite $W_1\Omega^i_X \xrightarrow{dV^n} W_{n+1}\Omega^i_X \twoheadrightarrow \widetilde{W}_n\Omega^i_X$. Since $Vd = pdV$, we have $V(\operatorname{Ker}\widetilde{R}) = 0$. This proves the injectivity. □

We easily see that $\widetilde{W}_n\Omega^i_{X,\log}$ is contained in the kernel of $\widetilde{R} - \widetilde{F} : \widetilde{W}_n\Omega^i_X \to W_n\Omega^i_X$. Hence

$$0 \to \widetilde{W}_n\Omega^i_{X,\log} \to \widetilde{W}_n\Omega^i_X \xrightarrow{\widetilde{R}-\widetilde{F}} W_n\Omega^i_X \to 0 \tag{A.2}$$

is a complex.

Lemma A.3. *The inductive limit*

$$0 \to \varinjlim_{n\geq 0,\, V} \widetilde{W}_n\Omega^i_{X,\log} \to \varinjlim_{n\geq 0,\, V} \widetilde{W}_n\Omega^i_X \to CW\Omega^i_X \to 0$$

of (A.2) with respect to V is exact.

Proof. The argument in the proof of [21, I, THÉORÈME 5.7.2, p. 597] shows that the kernel of $R - F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$ is contained in $W_{n+1}\Omega^i_{X,\log} + \operatorname{Ker} R$. Hence the claim follows from Lemma A.2. □

The inductive limit of $\widetilde{F} : \widetilde{W}_n\Omega^i_X \to \widetilde{W}_{n+1}\Omega^i_X$ gives the endomorphism $F' : CW\Omega^i_X \cong \varprojlim_{n\geq 1,\, V} \widetilde{W}_n\Omega^i_X \to CW\Omega^i_X$. By Lemma A.2 and Lemma A.3, we have a canonical exact sequence (A.1).

A.4. The duality. Let $H^*(X, W_n\Omega^i_X)$ denote the cohomology groups of $W_n\Omega^i_X$ with respect to the Zariski topology.

The trace map $\operatorname{Tr} : H^\delta(X, W_n\Omega^\delta_X) \cong W_n(\mathbb{F}_q)$ is defined in [20, 4.1.3, p. 49]. This commutes with the homomorphisms R, F and V. For $0 \leq i, j \leq \delta$, the product $m : W_n\Omega^i_X \times W_n\Omega^{\delta-i}_X \to W_n\Omega^\delta_X$ gives a $W_n(k)$-bilinear paring

$$(\ ,\ ) : H^j(X, W_n\Omega^i_X) \times H^{\delta-j}(X, W_n\Omega^{\delta-i}_X) \to H^\delta(X, W_n\Omega^\delta_X) \xrightarrow{\operatorname{Tr}} W_n(k).$$

By [20, COROLLARY 4.2.2, p. 51], this pairing is perfect.

Since $m \circ (\mathrm{id} \otimes V) = V \circ m \circ (F \otimes \mathrm{id})$, the diagram

$$\begin{array}{ccc} W_{n+1}\Omega^i_X \times W_{n+1}\Omega^{\delta-i}_X & \longrightarrow & W_{n+1}(k) \\ F\big\downarrow \qquad\quad V\big\uparrow & & V\big\uparrow \\ W_n\Omega^i_X \ \times\ W_n\Omega^{\delta-i}_X & \longrightarrow & W_n(k) \end{array}$$

is commutative. Hence this induces an isomorphism

$$H^{\delta-j}(X, CW\Omega^{\delta-i}_X) \cong \varinjlim_{n} \operatorname{Hom}_{W_n(k)}(H^j(X, W_n\Omega^i_X), W_n(k)) \tag{A.3}$$

where the transition map in the inductive limit of the right hand side is given by $f \mapsto V \circ f \circ F$. We endow each $H^j(X, W_n\Omega^i_X)$ with the discrete topology. We put $H^j(X, W'\Omega^i_X) = \varprojlim_{n,F} H^j(X, W_n\Omega^i_X)$ and endow it with the induced topology. We turn $H^j(X, W'\Omega^i_X)$ into a $W(k)$-module by letting $a \cdot (b_n) = (\sigma^{-n}(a)b_n)$ for $a \in W(k)$, $b_n \in H^j(X, W_n\Omega^i_X)$. We put $D = \varinjlim_{n,V} W_n(k)$ and endow it with the discrete topology. We turn D into a $W(k)$-module by letting $a \cdot c_n = \sigma^{-n}(a)c_n$ for $a \in W(k)$, $c_n \in W_n(k)$. Then the right hand side of (A.3) equals $\mathrm{Hom}_{W(k),\mathrm{cont}}(H^j(X, W'\Omega^i_X), D)$. The homomorphism $R : H^j(X, W_n\Omega^i_X) \to H^j(X, W_{n-1}\Omega^i_X)$ induces the endomorphism $R' : H^j(X, W'\Omega^i_X) \to H^j(X, W'\Omega^i_X)$. The Frobenius endomorphism $\sigma : W_n(k) \to W_n(k)$ induces the endomorphism $\sigma : D \to D$.

Lemma A.4. *Under the isomorphism (A.3), the endomorphism*

$$F' : H^{\delta-j}(X, CW\Omega^{\delta-i}_X) \to H^{\delta-j}(X, CW\Omega^{\delta-i}_X)$$

is identified with the endomorphism of $\mathrm{Hom}_{W(k),\mathrm{cont}}(H^j(X, W'\Omega^i_X), D)$ *which sends a homomorphism* $f : H^j(X, W'\Omega^i_X) \to D$ *to the homomorphism* $\sigma \circ f \circ R'$.

Proof. Immediate from the definition of the isomorphism (A.3) and the module D. □

A.5. We are mainly concerned with the case where $i = 0$. We denote $H^j(X, W'\Omega^0_X)$ by $H^j(X, W'\mathcal{O}_X)$. Recall that $F : W_n\Omega^0_X \to W_{n-1}\Omega^0_X$ equals the composite $W_n\mathcal{O}_X \xrightarrow{\sigma} W_n\mathcal{O}_X \xrightarrow{R} W_{n-1}\mathcal{O}_X$. From [21, II, PROPOSITION 2.1, p. 607], it follows that $H^j(X, W\Omega^i_X) \to \varprojlim_{n,R} H^j(X, W_n\Omega^i_X)$ is an isomorphism. Hence $H^j(X, W'\mathcal{O}_X)$ is isomorphic to the projective limit

$$\widetilde{H}^j(X, W\mathcal{O}_X) = \varprojlim[\cdots \xrightarrow{\sigma} H^j(X, W\mathcal{O}_X) \xrightarrow{\sigma} H^j(X, W\mathcal{O}_X)].$$

The endomorphism $\sigma : H^j(X, W\mathcal{O}_X) \to H^j(X, W\mathcal{O}_X)$ induces an automorphism $\sigma : \widetilde{H}^j(X, W\mathcal{O}_X) \xrightarrow{\cong} \widetilde{H}^j(X, W\mathcal{O}_X)$. We easily see that the endomorphism R' on $H^j(X, W'\mathcal{O}_X)$ corresponds to the endomorphism σ^{-1} on $\widetilde{H}^j(X, W\mathcal{O}_X)$.

Let $K = \mathrm{Frac}\, W(k)$ denote the field of fractions of $W(k)$. The homomorphism $\sigma^n/p^n : W_n(k) \to K/W(k)$ for each $n \geq 1$ induces a canonical isomorphism $D \cong K/W(k)$ of $W(k)$-modules which commutes with the action of σ.

A.6. Proof of Proposition A.1. Suppose that $k = \mathbb{F}_q$. Then by Lemma A.4, $H^0(X, CW\Omega^d_X)$ is isomorphic to the Pontryagin dual of $\widetilde{H}^\delta(X, W\mathcal{O}_X)$. Hence the group

$$H^0(X, CW\Omega^\delta_{X,\log}) \cong \mathrm{Ker}[H^0(X, CW\Omega^\delta_X) \xrightarrow{1-F'} H^0(X, CW\Omega^\delta_X)]$$

is isomorphic to the Pontryagin dual of the cokernel of $1 - \sigma^{-1}$ on $\widetilde{H}^\delta(X, W\mathcal{O}_X)$.

Proposition A.2. *Let* $k = \mathbb{F}_q$ *be a finite field and* X *be a scheme of dimension* δ *which is smooth and projective over* $\mathrm{Spec}\, k$. *Suppose that the* V*-torsion part* T *of* $H^\delta(X, W\mathcal{O}_X)$ *is finite. Then* $H^0(X, CW\Omega^\delta_X)$ *is a finite group of order* $|T^\sigma| \cdot |L(h^\delta(X), 0)|_p^{-1}$. *Here* T^σ *denotes the* σ*-invariant part of* T.

Proof. By the argument above, the order of $H^0(X, CW\Omega^\delta_X)$ equals the order of the cokernel of $1 - \sigma$ on $\widetilde{H}^\delta(X, W\mathcal{O}_X)$ if it is finite. The torsion subgroup of $\widetilde{H}^\delta(X, W\mathcal{O}_X)$ is finite since it injects into T. By [21, II, COROLLAIRE 3.5, p. 616],

$\widetilde{H}^\delta(X, W\mathcal{O}_X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to the slope zero part of $H^\delta_{\mathrm{crys}}(X/W(k)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Hence the claim follows. □

Proof of Proposition A.1. Let the notations be as above, and suppose that $\delta = 2$. Then by [21, II, Remarque 6.4, p. 641], the module T in the above proposition is canonically isomorphic to the group

$$\mathrm{Hom}_{W(\mathbb{F}_q)}(M(\mathrm{Pic}^o_{X/\mathbb{F}_q}/\mathrm{Pic}^o_{X/\mathbb{F}_q,\mathrm{red}}), K/W(\mathbb{F}_q))$$

where $M(\ )$ denotes the contravariant Dieudonne module functor. In particular T is a finite group. Let T_σ denote the σ-coinvariants of T. Then by the Dieudonne theory (cf. [11]), $\mathrm{Hom}_{W(\mathbb{F}_q)}(T_\sigma, K/W(\mathbb{F}_q))$ is canonically isomorphic to $\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)$. Hence the claim follows from Proposition A.2. □

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