OPPOSITE POWER SERIES

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Dedicated to Professor Antonio Machì
on the occasion of his 70th birthday.

Abstract. In order to analyze the singularities of a power series function $P(t)$ on the boundary of its convergent disc, we introduced the space $\Omega(P)$ of *opposite power series* in the opposite variable $s = 1/t$, where $P(t)$ was, mainly, the growth function (Poincaré series) for a finitely generated group or a monoid [S1]. In the present paper, forgetting about that geometric or combinatorial background, we study the space $\Omega(P)$ abstractly for any suitably tame power series $P(t) \in \mathbb{C}\{t\}$. For the case when $\Omega(P)$ is a finite set and $P(t)$ is meromorphic in a neighbourhood of the closure of its convergent disc, we show *a duality between* $\Omega(P)$ *and the highest order poles of* $P(t)$ *on the boundary of its convergent disc.*

Contents

1. Introduction

There seems a remarkable “resonance” between oscillation behavior[1] of a sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of complex numbers satisfying a tame condition (see equation (2.1.2)) and the singularities of its generating function $P(t)=\sum_{n=0}^{\infty}\gamma_n t^n$ on the boundary of the disc of convergence in $\mathbb{C}$. The idea was inspired by and strongly used in the study of growth functions (Poincaré series) for finitely generated groups and monoids [S1, §11].

Let us explain the “resonance” by a typical example due to Machì [M] (for details, see Examples in §3.3 and §5.4 of the present paper. Other simple examples are given in §3.4 (see [C, S2, S3]) and §3.5). By choosing generators of order 2 and 3 in $\mathrm{PSL}(2,\mathbb{Z})$, Machì has shown that the number γ_n of elements of $\mathrm{PSL}(2,\mathbb{Z})$ which are expressed in words of length less or equal than $n\in\mathbb{Z}_{\geq 0}$ w.r.t. the generators is given by $\gamma_{2k}=7\cdot 2^k-6$ and $\gamma_{2k+1}=10\cdot 2^k-6$ for $k\in\mathbb{Z}_{\geq 0}$. On one hand, this means that the sequence of ratios γ_{n-1}/γ_n $(n=1,2,\cdots)$ accumulates to *two distinct “oscillation” values* $\{\frac{5}{7},\frac{7}{10}\}$ according as n is even or odd. On the other hand, the generating function (or, so called, the growth function) can be expressed as a rational function $P(t)=\frac{(1+t)(1+2t)}{(1-2t^2)(1-t)}$, and it has *two poles* at $\{\pm\frac{1}{\sqrt{2}}\}$ on the boundary of its convergent disc of radius $\frac{1}{\sqrt{2}}$. *We see that there is a “resonance” between the set* $\{\frac{5}{7},\frac{7}{10}\}$ *of “oscillations” of the sequence* $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ *and the set* $\{\pm\frac{1}{\sqrt{2}}\}$ *of “poles” of the function* $P(t)$, in the way we shall explain in the present paper.

In order to analyze these phenomena, in [S1, §11], we introduced a space $\Omega(P)$ of *opposite power series* in the opposite variable $s=1/t$, as a compact subset of $\mathbb{C}[[s]]$, where each opposite series is defined by using “oscillations” of the sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ so that $\Omega(P)$ carries a comprehensive information of oscillations (see §2.2 Definition (2.2.2)). On the other hand, the space $\Omega(P)$ has duality with the singularities of the function $P(t)$ (§5 Theorem). Thus, $\Omega(P)$ becomes a bridge between the two subjects: oscillations of $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ and singularities of $P(t)$. Since the method is independent of the group theoretic background and is extendable to a wider class of series (see §2.1 Example 2), which we call *tame*, we separate the results and proofs in a self-contained way in the present paper. We study in details the case when $\Omega(P)$ is finite, where we have good understanding of the above mentioned resonance by a use of *rational subset* explained in the following paragraph, and Machì's example is understood in that frame.

One key concept in the present paper is a *rational subset* U (§3), which is a subset of the positive integers $\mathbb{Z}_{\geq 0}$ such that the sum $\sum_{n\in U} t^n$

[1]By an oscillation behavior, we mean that, for each fixed $k\in\mathbb{Z}_{\geq}0$ called a period, the sequence of the rate γ_{n-k}/γ_n $(n\in\mathbb{Z}_{\gg 0})$ has several different accumulation values.

is a rational function in t (i.e. U, up to finite, is a finite union of arithmetic progressions). The concept is used twice in the present paper. The first time it is used is in §3, where we show that, if the space of opposite series $\Omega(P)$ is finite, then there is a finite partition $\mathbb{Z}_{\geq 0} = \amalg_i U_i$ of $\mathbb{Z}_{\geq 0}$ into rational subsets so that there is no longer oscillation inside in each $\{\gamma_n : n \in U_i\}$. We call such phenomena "finite rational accumulation" (§3.2 Theorem) (such phenomena already appeared when we were studying the F-limit functions for monoids [S1, §11.5 Lemma]). The second time it is used is in §5, where we introduce a rational operator T_U acting on a power series $P(t) \in \mathbb{C}[[t]]$ by letting $T_U P(t) := \sum_{n\in U} \gamma_n t^n$. The rational operators form a machine that "manipulates" singularities of the power series $P(t)$. In this way, rational subsets combine the oscillation of a sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ and the singularities of the generating function $P(t) := \sum_{n=0}^{\infty} \gamma_n t^n$ for the case when $\Omega(P)$ is finite.

The contents of the present paper are as follows.

In §2, we introduce the space $\Omega(P)$ of opposite series as the accumulating subset in $\mathbb{C}[[s]]$ of the sequence $X_n(P) := \sum_{k=0}^{n} \frac{\gamma_{n-k}}{\gamma_n} s^k$ ($n = 0, 1, 2, \cdots$) with respect to the coefficient-wise convergence topology, where the kth coefficient describes an oscillation of period k. Dividing by period-one oscillation, we construct a shift action τ_Ω on the set $\Omega(P)$ to itself, which shifts k-period oscillations to $k-1$-period oscillations.

In 3.1, we introduce the key concept: *finite rational accumulation.* We show that if $\Omega(P)$ is a finite set, then $\Omega(P)$ is automatically a finite rational accumulation set and the τ_Ω-action becomes invertible and transitive. That is, τ_Ω is acting cyclically on $\Omega(P)$.

Starting with §4, we assume always finite rational accumulation for $\Omega(P)$. In §4, we analyze in details of the opposite series in $\Omega(P)$ and the module $\mathbb{C}\Omega(P)$ spanned by $\Omega(P)$, showing that the opposite series become rational functions with the common denominator $\Delta^{op}(s)$ in 4.1, and that the rank of $\mathbb{C}\Omega(P)$ is equal to $\deg(\Delta^{op}(s))$ in §4.4.

In §5, we assume that the series $P(t)$ defines a meromorphic function in a neighbourhood of the closed convergent disc. Then we show that $\Delta^{op}(s)$ is opposite to the polynomial $\Delta^{top}(t)$ of the highest order part of poles of $P(t)$ (Duality Theorem in §5.3), and, in particular, the rank of the space $\mathbb{C}\Omega(P)$ is equal to the number of poles of the highest order of $P(t)$ on the boundary of the convergent disc. We get an identification of some transition matrices obtained in s-side and in t-side, which plays a crucial role in the trace formula for limit F-function [S1, 11.5.6].

Problems. The space $\Omega(P)$ is new with respect to the study of the singularities of a power series function $P(t)$, and the author thinks the following directions of further study may be rewarding.

1. Generalize the space $\Omega(P)$ in order to capture lower order poles of $P(t)$ on the boundary of its convergent disc (c.f. [S1, §12, **2.**]).
2. Generalize the duality for the case when $\Omega(P)$ is infinite. Some probabilistic approach may be desirable (c.f. [S1, §12, **1.**]).

2. The space of opposite series.

In this section, we introduce the space $\Omega(P)$ of opposite series for a tame power series $P \in \mathbb{C}[[t]]$, and equip it with a τ_Ω-action.

2.1. **Tame power series.**

Let us call a complex coefficient power series in t

(2.1.1) $$P(t) \;=\; \sum_{n=0}^{\infty} \gamma_n t^n$$

to be *tame*, if there are positive real numbers $u, v \in \mathbb{R}_{>0}$ such that

(2.1.2) $$u \;\le\; |\gamma_{n-1}/\gamma_n| \;\le\; v$$

for sufficiently large integers n (i.e. for $n \ge N_P$ for some $N_P \in \mathbb{Z}_{\ge 0}$). This implies that there are positive constants c_1, c_2 with $c_1 \le c_2$ so that

(2.1.3) $$c_1 v^{-n} \le |\gamma_n| \le c_2 u^{-n}$$

for sufficiently large integer $n \in \mathbb{Z}_{\ge 0}$ (actually, put $c_1 = |\gamma_{N_P}| v^{N_P}$ and $c_2 = |\gamma_{N_P}| u^{N_P}$ for $n \ge N_P$). Let us consider two limit values:

(2.1.4) $$u \;\le\; r_P := 1/\overline{\lim_{n\to\infty}} |\gamma_n|^{1/n} \;\le\; R_P := 1/\underline{\lim_{n\to\infty}} |\gamma_n|^{1/n} \;\le\; v.$$

Cauchy-Hadamard Theorem says that P is convergent of radius r_P.

Example 1. Let Γ be a group or a monoid with a finite generator system G. Then the length $l(g)$ of an element $g \in \Gamma$ is the shortest length of words expressing g in the letter G. Set $\Gamma_n := \{g \in \Gamma \mid l(g) \le n\}$ and $\gamma_n := \#(\Gamma_n)$. Then the growth function (Poincaré series) for Γ with respect to G is defined by $P_{\Gamma,G}(t) := \sum_{n=0}^{\infty} \gamma_n t^n$. The sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\ge 0}}$ is increasing and semi-multiplicative $\gamma_{m+n} \le \gamma_m \gamma_n$. Therefore, by choosing $u = 1/\gamma_1$ and $v = 1$, the growth series is tame.

2. Ramsey's theorem says that, for any $n \in \mathbb{Z}_{>0}$, there exists a positive integer N such that if the edges of the complete graph on N vertices are colored either red or blue, then there exists n vertices such that all edges joining them have the same colour. The least such integer N is denoted by $R(n)$, and is called the nth *diagonal Ramsey number*, e.g. $R(1) = 1, R(2) = 2, R(3) = 6, R(4) = 18$ (c.f. [SR]). Then, the following estimates are known due to Erdös [E] and Szekeres:

$$2^{n/2} \le R(n) \le 2^{2n}.$$

So, $R(t) := \sum_{n=0}^{\infty} R(n) t^n$ (where put $R(0) = 1$) form a tame series.

2.2. **The space $\Omega(P)$ of opposite series.**

Let P be a tame power series. Then, there is a positive integer N_P such that γ_n is invertible for all $n \geq N_P$. Therefore, for $n \in \mathbb{Z}_{\geq N_P}$, we define the *opposite polynomial of degree* n by

$$X_n(P) \;:=\; \textstyle\sum_{k=0}^{n} \frac{\gamma_{n-k}}{\gamma_n}\; s^k. \tag{2.2.1}$$

Regarding $\{X_n(P)\}_{n \geq N_P}$ as a sequence in the space $\mathbb{C}[[s]]$ of formal power series, where $\mathbb{C}[[s]]$ is equipped with the classical topology, i.e. the product topology of coefficient-wise convergence in classical topology, we define *the space of opposite series* by

$$\Omega(P) := \begin{array}{l}\text{the set of accumulation points of the sequence}\\ \text{(2.2.1) with respect to the classical topology.}\end{array} \tag{2.2.2}$$

That is, an element of $\Omega(P)$ can be viewed as an equivalence class of infinite convergent subsequences $\{X_{n_m}(P)\}_m$ of opposite polynomials.

The first statement on $\Omega(P)$ is the following.

Assertion 1. *Let P be a tame series. Then $\Omega(P)$ is a non-empty compact closed subset of* $\mathbb{C}[[s]]$.

Proof. For each $k \in \mathbb{Z}_{\geq 0}$, the kth coefficient $\frac{\gamma_{n-k}}{\gamma_n}$ of the polynomial $X_n(P)$ for sufficiently large $n \in \mathbb{Z}_{\geq 0}$ with respect to P and k (i.e. for $n \geq N_P + k - 1$) has the approximation $u^k \leq |\frac{\gamma_{n-k}}{\gamma_n}| = |\frac{\gamma_{n-1}}{\gamma_n}||\frac{\gamma_{n-2}}{\gamma_{n-1}}| \cdots |\frac{\gamma_{n-k}}{\gamma_{n-k+1}}| \leq v^k$, i.e. it lies in the compact annulus

$$\bar{D}(0, u^k, v^k) := \{a \in \mathbb{C} \mid u^k \leq |a| \leq v^k\}.$$

Thus, for each fixed $m \in \mathbb{Z}_{\geq 0}$, the image of the sequence (2.2.1) under the truncation map $\pi_{\leq m} : \mathbb{C}[[s]] \to \mathbb{C}^{m+1}$, $\sum_{k=0}^{\infty} a_k s^k \mapsto (a_0, \cdots, a_m)$ accumulates to an non-empty compact subset of $\prod_{k=0}^{m} \bar{D}(0, u^k, v^k)$, say $\Omega_{\leq m}$. Then, we have:

$$\Omega(P) = \cap_{m=0}^{\infty} \big((\pi_{\leq m})^{-1} \Omega_{\leq m} \cap \textstyle\prod_{k=0}^{\infty} \bar{D}(0, u^k, v^k)\big),$$

where the RHS, as an intersection of decreasing sequence of compact sets, is non-empty and compact. □

An element $a(s) = \Sigma_{k=0}^{\infty} a_k s^k$ of $\Omega(P)$ is called an *opposite series*. Its kth coefficients a_k, i.e. an *oscillation value of period* k, belongs to $\bar{D}(0, u^k, v^k)$. Given an opposite series $a(s)$, the constant term a_0 is equal to 1. The coefficient a_1, i.e. oscillation value of period 1, is called the *initial* of the opposite series a, and denoted by $\iota(a)$.

For later use, let us introduce an auxiliary space of the initials:

$$\Omega_1(P) := \text{the accumulation set of the sequence } \Big\{\frac{\gamma_{n-1}}{\gamma_n}\Big\}_{n \gg 0}, \tag{2.2.3}$$

which is a compact subset in $\bar{D}(0,u,v)$. The projection map $\Omega(P) \rightarrow \Omega_1(P),\ a \mapsto \iota(a)$ is surjective but may not be injective (see §3.5 Ex.).

2.3. **The τ_Ω-action on $\Omega(P)$.**

We introduce a continuous map τ_Ω form $\Omega(P)$ to itself.

Assertion 2. a. *Let $\{n_m\}_{m\in\mathbb{Z}_{\geq 0}}$ be a subsequence of $\mathbb{Z}_{\geq 0}$ tending to ∞. If the sequence $\{X_{n_m}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ converges to an opposite series a, then the sequence $\{X_{n_m-1}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ also converges to an opposite series, whose limit depends only on a and is denoted by $\tau_\Omega(a)$. Then, we have*

$$\tau_\Omega(a) \ = \ (a-1)/\iota(a)s. \tag{2.3.1}$$

b. *Let $\mathbb{C}\Omega(P)$ be the $\mathbb{C}$-linear subspace of $\mathbb{C}[[s]]$ spanned by $\Omega(P)$. Then the map $\tau : \Omega(P) \longrightarrow \mathbb{C}\Omega(P), \quad a \mapsto \iota(a)\tau_\Omega(a)$ naturally extends to an endomorphism of $\mathbb{C}\Omega(P)$.*

$$\tau \in End_{\mathbb{C}}(\mathbb{C}\Omega(P)) \tag{2.3.2}$$

Proof. a. By definition, for any $k \in \mathbb{Z}_{\geq 0}$, the sequence $\frac{\gamma_{n_m-k}}{\gamma_{n_m}}$ converges to a constant $a_k \in \bar{D}(u^k, v^k)$. Then, $\frac{\gamma_{(n_m-1)-(k-1)}}{\gamma_{n_m-1}} = \frac{\gamma_{n_m-k}}{\gamma_{n_m}} / \frac{\gamma_{n_m-1}}{\gamma_{n_m}}$ converges to a_k/a_1. That is, the sequence $\{X_{n_m-1}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ converges to an opposite series, whose $(k-1)$th coefficient is equal to a_k/a_1.

b. This is trivial, since $a \mapsto \iota(a)\tau_\Omega(a)$ is a restriction on $\Omega(P)$ of an affine linear endomorphism $(a-1)/s$ on $\mathbb{C}[[s]]$. □

2.4. **Examples of τ_Ω-actions.**

At present, except for the trivial cases when $\#\Omega(P) = 1$ so that $\tau_\Omega = \mathrm{id}$, there are only few examples where the action $(\Omega(P_{\Gamma,G}), \tau_\Omega)$ is explicitly known: namely, the groups of the form $\Gamma = (\mathbb{Z}/p_1\mathbb{Z}) * \cdots * \mathbb{Z}/p_n\mathbb{Z}$ for some $p_1, \cdots, p_n \in \mathbb{Z}_{>1}$ $(n \geq 2)$ with the generator system $G = \{a_1, \cdots, a_n\}$ where a_i is the standard generator of $\mathbb{Z}/p_i\mathbb{Z}$ for $1 \leq i \leq n$, which include Machì's example (see §3.3-4).

For the tame series $R(t)$ in §2.1 Example 2, we know nothing about $(\Omega(R), \tau_\Omega)$. It is already a question whether $\#\Omega(R)$ is equal to 1, finite many (>1), or infinite? The author would like to expect $\#\Omega(R) = 1$.

2.5. **Stability of $\Omega(P)$.**

In the present subsection, we are (mainly) concerned with following type of questions, which we will call *stability questions concerning* $\Omega(P)$: for a given tame series P, under which assumptions on another power series Q, is $P+Q$ again tame and $\Omega(P) = \Omega(P+Q)$? Or, if $\Omega(P+Q)$ changes from $\Omega(P)$, how does it change?

We discuss some miscellaneous results related to stability questions, but we do not pursue full generalities. Except that Assertion 3 is used in the proof of Assertion 13, results in the present paragraph are not used in the present article. Therefore, the reader may choose to skip the part of this subsection after Assertion 3 without substantial loss.

Assertion 3. *Let* $Q=\sum_{n=0}^{\infty} q_n t^n$ *converge in the disc of radius* r_Q *such that* $r_Q > R_P$. *Then* $P+Q$ *is tame and* $\Omega(P)=\Omega(P+Q)$.

Proof. Let c be a real number satisfying $r_Q > c > R_P$. Then, one has $\lim_{n\to\infty} q_n c^n = 0$ and $c^n \geq 1/|\gamma_n|$ for sufficiently large n. This implies $\lim_{n\to\infty} \frac{\gamma_n+q_n}{\gamma_n} = 1+ \lim_{n\to\infty} \frac{q_n}{\gamma_n} = 1$. The required properties follow. □

Assertion 4. *Let* r *be a positive real number with* $r<R_P$. *If* $\Omega_1(P) \cap \{z\in\mathbb{C} : |z|=r\}=\emptyset$. *Then there exists a power series* $Q(t)$ *of radius of convergence* $r_Q=r$ *such that* $P+Q$ *is tame and* $\Omega(P+Q) \not\subset \Omega(P)$.

Proof. We define the coefficients of $Q(t)=\sum_{n=0}^{\infty} q_n t^n$ by the following conditions: $|q_n| = r^{-n}$ and $\arg(q_n) = \arg(\gamma_n)$. Then, for tameness of $P+Q$, we have to show some positive bounds $0<U\leq A_n \leq V$ for $A_n = |\frac{\gamma_{n-1}+q_{n-1}}{\gamma_n+q_n}|$. Since $|\gamma_n+q_n|=|\gamma_n|+r^{-n}$, we have $A_n = \frac{|\gamma_{n-1}/\gamma_n|+r/(|\gamma_n|r^n)}{1+1/(|\gamma_n|r^n)}$. Then, evaluating term-by-term in the numerator, one gets $A_n \leq v+r =: V$. On the other hand, according as $1 \geq 1/(|\gamma_n|r^n)$ or not, we have $A_n \geq u/2$ or $A_n \geq r/2$. Therefore, we may set $U:=\min\{u/2, r/2\}$.

Let us find a particular element $d \in \Omega(P+Q)$ such that $d \notin \Omega(P)$. For a small positive real number ε satisfying the inequality $(1-\varepsilon)/r > 1/R_P$, there exists an increasing infinite sequence of integers n_m ($m\in \mathbb{Z}_{\geq 0}$) such that $((1-\varepsilon)/r)^{n_m} > |\gamma_{n_m}|$ for $m \in \mathbb{Z}_{\geq 0}$. By choosing a suitable sub-sequence (denoted by the same n_m), we may assume that $X_{n_m}(P+Q)$ converges to an element, say d, in $\Omega(P+Q)$. Its kth coefficient d_k is equal to the limit of the sequence $(\gamma_{n_m-k}+q_{n_m-k})/(\gamma_{n_m}+q_{n_m})$ for $n_m \to \infty$. For each fixed n_m, dividing the numerator and the denominator by q_{n_m}, we get an expression $(X+r^kY)/(Z+1)$ where $|X| = |\gamma_{n_m-k}/\gamma_{n_m}| \cdot |\gamma_{n_m} r^{n_m}| \leq v^k \cdot (1-\varepsilon)^{n_m}$ (for $n >> k$), $Y \in S^1$, and $|Z| = |\gamma_{n_m} r^{n_m}| < (1-\varepsilon)^{n_m}$. Thus, taking the limit $n_m \to \infty$, we have $X \to 0$, $Y \to e^{i\theta_k}$ for some $\theta_k \in \mathbb{R}$ and $Z \to 0$ so that $d_k = r^k e^{i\theta_k}$. On the other hand, we see that $d \notin \Omega(P)$, since $\iota(d) = re^{i\theta_1} \notin \Omega_1(P)$ by assumption. □

We do not use following Assertion in the present paper, since we know more precise information for the cases $\#\Omega(P)<\infty$. However, it may have a significance when we study the general case with $\#\Omega(P)=\infty$.

Assertion 5. *An opposite series converges with radius* $1/\sup\{|a| : a\in \Omega_1(P)\} \leq 1/R_P$.

Proof. Let $a(s) = \lim_{m\to\infty} X_{n_m}(P)$ for an increasing sequence $\{n_m\}_{m\in\mathbb{Z}_{\geq 0}}$ be an opposite series. By the Cauchy-Hadmard theorem, the radius of convergence of a is given by

$$r_a = 1/\overline{\lim_{k\to\infty}}|a_k|^{1/k} = 1/\overline{\lim_{k\to\infty}}\ |\lim_{m\to\infty}\gamma_{n_m-k}/\gamma_{n_m}|^{1/k},$$

where the RHS is lower bounded by $1/\sup\{|a| : a \in \Omega_1(P)\}$ from below. □

It seems natural to ask when we can replace $\sup\{|a| : a \in \Omega_1(P)\}$ by R_P? Finally, we state a result, which is not related to the stability.

Assertion 6. *For any positive integer m, we have the equality*

$$\Omega(P) \ = \ \Omega\left(\tfrac{d^mP}{dt^m}\right) \tag{2.5.1}$$

which is equivariant with the action of τ_Ω

Proof. It is sufficient to show the case $m = 1$. We show a slightly stronger statement: *the subsequence $\{X_{n_m}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ converges to a series $a(s)$ if and only if $\{X_{n_m}\left(\frac{dP}{dt}\right)\}_{m\in\mathbb{Z}_{\geq 0}}$ also converges to $a(s)$.*

For an increasing sequence $\{n_m\}_{m\in\mathbb{Z}_{\geq 0}}$ and for any fixed $k\in\mathbb{Z}_{\geq 0}$, the convergence of the sequence $\frac{\gamma_{n_m-k}}{\gamma_{n_m}}$ to c is equivalent to the convergence of the sequence $\frac{(n_m-k)\gamma_{n_m-k}}{n_m\gamma_{n_m}} = (1-k/n_m)\frac{\gamma_{n_m-k}}{\gamma_{n_m}}$ to the same c. □

3. Finite rational accumulation

We show that, if $\Omega(P)$ is a finite set, then it has a strong structure, which we call the *finite rational accumulation* (§3.2 Theorem and its Corollary). The whole sequel of the present paper focuses on its study.

3.1. Finite rational accumulation.

We introduce the concept of *finite rational accumulation*. To this end, we start with a preliminary concept: a *rational subset* of $\mathbb{Z}_{\geq 0}$. The following fact is easy and well known, so we omit its proof.

Fact. *The following conditions for a subset $U \subset \mathbb{Z}_{\geq 0}$ are equivalent.*

i) *Put $U(t) := \sum_{n\in U} t^n$. Then, $U(t)$ is a rational function in t.*

ii) *There exists $h \in \mathbb{Z}_{>0}$ and a polynomial $V(t)$ such that $U(t) = \frac{V(t)}{1-t^h}$.*

iii) *There exists $h \in \mathbb{Z}_{>0}$ such that $n + h \in U$ iff $n \in U$ for $n >> 0$.*

iv) *There exists $h\in\mathbb{Z}_{>0}$, a subset $u\subset\mathbb{Z}/h\mathbb{Z}$ and a finite set $D\subset\mathbb{Z}_{\geq 0}$ such that $U\setminus D=\cup_{[e]\in u}U^{[e]}\setminus D$, where, for a class $[e]\in\mathbb{Z}/h\mathbb{Z}$ of e, put*

$$U^{[e]} := \{n\in\mathbb{Z}_{\geq 0} \mid n\equiv e \bmod h\}. \tag{3.1.1}$$

Further more, ii), iii) and iv) are equivalent for a pair (U, h). The least such h for a fixed U will be called the *period* of U.

Definition. **1.** A subset U of $\mathbb{Z}_{\geq 0}$ is called a *rational subset* if it satisfies one of the above four equivalent conditions.

2. A *finite rational partition* of $\mathbb{Z}_{\geq 0}$ is a finite collection $\{U_a\}_{a\in\Omega}$ of rational subsets $U_a \subset \mathbb{Z}_{\geq 0}$ indexed by a finite set Ω such that there is a finite subset D of $\mathbb{Z}_{\geq 0}$ so that one has the disjoint decomposition

$$\mathbb{Z}_{\geq 0} \setminus D = \amalg_{a\in\Omega}(U_a \setminus D).$$

In particular, for $h \in \mathbb{Z}_{>0}$, the partition $\mathcal{U}_h := \{U^{[e]}\}_{[e]\in\mathbb{Z}/h\mathbb{Z}}$ of $\mathbb{Z}_{\geq 0}$ is called the *standard partition of period* h.

3. For a finite rational partition $\{U_a\}_{a\in\Omega}$ of $\mathbb{Z}_{\geq 0}$, the period of a standard partition, which subdivide $\{U_a\}_{a\in\Omega}$, is called a *period* of $\{U_a\}_{a\in\Omega}$. The smallest period $(=\mathrm{lcm}\{\text{period of } U_a |\ a \in \Omega\})$ of a finite rational partition $\{U_a\}_{a\in\Omega}$ is called *the period* of $\{U_a\}_{a\in\Omega}$.

We, now, arrived at the key concept of the present paper.

Definition. A sequence $\{X_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of points in a Hausdorff space is *finite rationally accumulating* if the sequence accumulates to a finite set, say Ω, such that for a system of pairwise-disjoint open neighborhoods $\mathcal{V}_a$ for $a \in \Omega$, the system $\{U_a\}_{a\in\Omega}$ for $U_a := \{n \in \mathbb{Z}_{\geq 0} \mid X_n \in \mathcal{V}_a\}$ is a finite rational partition of $\mathbb{Z}_{\geq 0}$. The (resp. a) period of the partition is called the (resp. a) *period of the finite rational accumulation set* Ω.

3.2. **τ_Ω-periodic point in $\Omega(P)$.**

Generally speaking, finiteness of the accumulation set Ω of a sequence does not imply that it is finite rationally accumulating (see §3.5 Example a). Therefore, the following theorem describes a distinguished property of the accumulation set $\Omega(P)$. This justifies the introduction of the concept of "finite rational accumulation".

Theorem. *Let $P(t)$ be a tame power series in t. Suppose there exists an isolated point of $\Omega(P)$, say a, which is periodic with respect to the τ_Ω-action on $\Omega(P)$. Then $\Omega(P)$ is a finite rational accumulation set, whose period h_P is equal to $\#\Omega(P)$. Furthermore, we have a natural bijection that identifies $\Omega(P)$ with the τ_Ω-orbit of a:*

$$\begin{array}{rcl} \mathbb{Z}/h_P\mathbb{Z} & \simeq & \Omega(P) \\ e \bmod h_P & \mapsto & a^{[e]} := \lim\limits_{n\to\infty} X_{e+h_P\cdot n}(P), \end{array} \tag{3.2.1}$$

where the standard subdivision $\mathcal{U}_{h_P}$ of the partition of $\mathbb{Z}_{\geq 0}$ is the exact partition for the space $\Omega(P)$ of the opposite series of P. The shift action $[e] \mapsto [e-1]$ in the LHS is equivariant to the τ_Ω action in the RHS.

Proof. The assumption on a means:

i) There exists a positive integer $h \in \mathbb{Z}_{>0}$ such that

$$(\tau_\Omega)^h a = a \neq (\tau_\Omega)^{h'} a \quad \text{for} \quad 0 < h' < h.$$

ii) There exists an open neighbourhood $\mathcal{V}_a$ of a in $\mathbb{C}[[s]]$ such that

$$\Omega(P) \cap \mathcal{V}_a = \{a\}.$$

In particular, $\Omega(P)\setminus\{a\}$ is a closed set.

Since $\Omega(P)$ is a compact Hausdorff space, it is a regular space, so we may assume further that $\Omega(P) \cap \overline{\mathcal{V}_a} = \{a\}$. Then, by setting $U_a := \{n \in \mathbb{Z}_{\geq 0} \mid X_n(P) \in \mathcal{V}_a\}$, the sequence $\{X_n(P)\}_{n \in U_a}$ converges to the unique limit element a. By the definition of τ_Ω in §2, the relation $(\tau_\Omega)^h a = a$ implies that the sequence $\{X_{n-h}(P)\}_{n \in U_a}$ converges to a. That is, there exists a positive number N such that for any $n \in U_a$ with $n > N$, $X_{n-h}(P) \in \mathcal{V}_a$, and hence $n-h$ belongs to U_a.

Consider the set $A := \{[e] \in \mathbb{Z}/h\mathbb{Z} \mid$ there are infinitely many elements of U_a which are congruent to $[e]$ modulo h $\}$. By the defining property of N, if $[e] \in A$, then U_a contains $U^{[e]} \cap \mathbb{Z}_{\geq N}$ (*Proof.* For any $m \in \mathbb{Z}_{\geq N}$ with $m \bmod h \equiv [e]$, there exists an integer $m' \in U_a$ such that $m' > m$ and $m' \bmod h = [e]$ by the definition of the set A. Then, by the definition of N, $m' - h \in U_a$. Obviously, either $m' - h = m$ or $m' - h > m$ occurs. If $m' - h > m$ then we repeat the same argument to $m'' := m' - h$ so that $m'' - h = m' - 2h \in U_a$. Repeating, similar steps, after finite k-steps, we show that $m' - kh = m \in U_a$).

Thus, U_a is, up to a finite number of elements, equal to the rational subset $\cup_{[e] \in A} U^{[e]}$. This implies $A \neq \emptyset$. Consider the rational subset $U_{(\tau_\Omega)^i a} := \{n - i \mid n \in U_a\}$ for $i = 0, 1, \cdots, h-1$. Due to §2.3 Assertion 2, $\{X_n(P)\}_{n \in U_{(\tau_\Omega)^i a}}$ converges to $(\tau_\Omega)^i a$, so $U_{(\tau_\Omega)^i a}$ is, up to a finite number of elements, equal to the rational subset $\cup_{[e] \in A} U^{[e-i]}$. By the assumption $a \neq \tau_\Omega^i a$ for $0 \leq i < h$, any pair of rational subsets $U_{(\tau_\Omega)^i a}$ $(0 \leq i < h)$ have at most finite intersection, so A is a singleton of the form $A = \{[e_0]\}$ for some $e_0 \in \mathbb{Z}$ and $U_{(\tau_\Omega)^i a} = U^{[e_0 - i]}$ up to a finite number of elements. On the other hand, since the union $\cup_{i=0}^{h-1} U_{(\tau_\Omega)^i a}$ already covers $\mathbb{Z}_{\geq 0}$ up to finite elements and since each $\{X_n(P)\}_{n \in U_{(\tau_\Omega)^i a}}$ converges only to $(\tau_\Omega)^i a$, the opposite sequence (2.2.1) can have no other accumulating point than the set $\{a, \tau_\Omega a, \cdots, (\tau_\Omega)^{h-1} a\}$. That is, $\Omega(P)$ is a finite rational accumulation set with the transitive h_P-periodic action of τ_Ω. □

Corollary. *If the set of isolated points of* $\Omega(P)$ *is finite, then* $\Omega(P)$ *is a finite rational accumulation set with the presentation* (3.2.1).

Proof. Since the τ_Ω action preserves the set of isolated points of $\Omega(P)$, there should exists a periodic point. □

3.3. **Example by Machì** [M]**.**

Let $\Gamma := \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \simeq \mathrm{PSL}(2,\mathbb{Z})$ with the generator system $G := \{a, b^{\pm 1}\}$ where a, b are the generators of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, respectively. Then, the number $\#\Gamma_n$ of elements of Γ expressed by the words in the letters G of length less or equal than n for $n \in \mathbb{Z}_{\geq 0}$ is given by

$$\#\Gamma_{2k} = 7 \cdot 2^k - 6 \quad \text{and} \quad \#\Gamma_{2k+1} = 10 \cdot 2^k - 6 \quad \text{for } k \in \mathbb{Z}_{\geq 0}.$$

Therefore, we get the following expression of the growth function:

$$P_{\Gamma,G}(t) \quad := \quad \textstyle\sum_{k=0}^{\infty} \#\Gamma_k t^k \quad = \quad \frac{(1+t)(1+2t)}{(1-2t^2)(1-t)}.$$

Then, we see that $\Omega_1(P_{\Gamma,G})$ and, hence, $\Omega(P_{\Gamma,G})$ are finite rationally accumulating of period 2. Explicitly, they are given as follows.

$$\Omega_1(P_{\Gamma,G}) = \Big\{ a_1^{[0]} := \lim_{n\to\infty} \tfrac{\#\Gamma_{2n-1}}{\#\Gamma_{2n}} = \tfrac{5}{7},\ a_1^{[1]} := \lim_{n\to\infty} \tfrac{\#\Gamma_{2n}}{\#\Gamma_{2n+1}} = \tfrac{7}{10} \Big\}$$
$$\Omega(P_{\Gamma,G}) = \Big\{ \ a^{[0]}(s)\ ,\ a^{[1]}(s) \ \Big\}$$

where

$$a^{[0]}(s) := \sum_{k=0}^{\infty} 2^{-k} s^{2k} + \frac{5}{7} s \sum_{k=0}^{\infty} 2^{-k} s^{2k}$$

$$= \frac{(1+\frac{5}{7}s)}{(1-\frac{s^2}{2})} = \frac{1}{2} \cdot \frac{1+\frac{5}{7}\sqrt{2}}{1-\frac{s}{\sqrt{2}}} + \frac{1}{2} \cdot \frac{1-\frac{5}{7}\sqrt{2}}{1+\frac{s}{\sqrt{2}}},$$

$$a^{[1]}(s) := \sum_{k=0}^{\infty} 2^{-k} s^{2k} + \frac{7}{10} s \sum_{k=0}^{\infty} 2^{-k} s^{2k}$$

$$= \frac{(1+\frac{7}{10}s)}{(1-\frac{s^2}{2})} = \frac{1}{2} \cdot \frac{1+\frac{7}{5}\frac{1}{\sqrt{2}}}{1-\frac{s}{\sqrt{2}}} + \frac{1}{2} \cdot \frac{1-\frac{7}{5}\frac{1}{\sqrt{2}}}{1+\frac{s}{\sqrt{2}}}.$$

In §5.4, these coefficients of fractional expansions are recovered by a use of, so called, rational operators (see §5.3 Theorem ii)).

We calculate also $r_P^2 = R_P^2 = a_1^{[0]} a_1^{[1]} = \frac{5}{7}\frac{7}{10} = \frac{1}{2}$.

3.4. **Simply accumulating Examples.**

A tame power series $P(t)$ is called *simply accumulating* if $\#\Omega(P) = 1$. Growth functions $P_{\Gamma,G}(t)$ for surface groups and Artin monoids are simply accumulating, respectively (Cannon [C], [S2, S3]). This fact for Artin monoids enables one to determine their F-functions [S4].

3.5. **Miscellaneous Examples.**

Before going further, we use a simple model of oscillating sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\ge 0}}$ to give some examples of the power series $P(t)$ such that

a) $\Omega_1(P)$ is finite but is not finite rationally accumulating,

b) $\Omega_1(P)$ is finite rationally accumulating but $\#\Omega_1(P)<\#\Omega(P)$,

c) $\Omega(P)\neq\Omega(P+Q)$ for a power series $Q(t)$ for any $R_P>r_Q>r_P$.

We do not use these results in the sequel so that the readers may skip present subsection without substantial loss.

Given a triple $\mathfrak{U}:=(U,a,b)$, where $U\subset\mathbb{Z}_{\ge 1}$ is any infinite subset with infinite complement and $a,b\in\mathbb{C}\setminus\{0\}$, we associate a sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\ge 0}}$ defined by an induction on n: $\gamma_0:=1$ and $\gamma_n:=\gamma_{n-1}\cdot a$ if $n\in U$ and $\gamma_{n-1}\cdot b$ if $n\notin U$. Set $P_{\mathfrak{U}}(t):=\sum_{n=0}^{\infty}\gamma_n t^n$. Then:

Fact i) *The series $P_{\mathfrak{U}}(t)$ is tame and $\Omega_1(P_{\mathfrak{U}})=\{a^{-1},b^{-1}\}$.*

ii) *The series $P_{\mathfrak{U}}(t)$ is finite rationally accumulating if and only if U is a rational subset of $\mathbb{Z}_{\ge 0}$.*

Proof. i) The inequalities: $\min\{|a|,|b|\}\le|\gamma_n/\gamma_{n-1}|\le\max\{|a|,|b|\}$ imply the tameness of $P_{\mathfrak{U}}$. The latter half is trivial since the proportion γ_n/γ_{n-1} takes only the values a or b.

ii) This follows from: $P_{\mathfrak{U}}$ is rational $\Leftrightarrow$ The sets $\{n\in\mathbb{Z}_{\ge 1}\mid\gamma_n/\gamma_{n-1}=a\}=U$ and $\{n\in\mathbb{Z}_{\ge 1}\mid\gamma_n/\gamma_{n-1}=b\}=U^c$ are rational$\Leftrightarrow$ U is rational. □

a) By choosing a non-rational subset U, we obtain an example a).

b) Even if U (and, hence, U^c also) is a rational subset, if $\{U,U^c\}$ is not the standard partition of $\mathbb{Z}_{\ge 0}$ of period 2, then the period of the partition $\{U,U^c\}=\#\Omega(P_{\mathfrak{U}})>2=\#\Omega_1(P_{\mathfrak{U}})$. This gives an example b).

c) To get an example satisfying c), we need a bit more consideration. Define $p_U:=\overline{\lim\limits_{n\to\infty}}\frac{\#(U\cap[1,n])}{n}$ and $q_U:=\underline{\lim\limits_{n\to\infty}}\frac{\#(U\cap[1,n])}{n}$. If U is a rational subset, then $p_U=q_U$ is a rational number. In general, the pair (p_U,q_U) can be any of $\{(p,q)\in[0,1]^2\mid p\ge q\}$. Suppose $|a|\ge|b|$.

$$1/r_P:=\overline{\lim_{n\to\infty}}\,|a|^{\frac{\#(U\cap[1,n])}{n}}\cdot|b|^{1-\frac{\#(U\cap[1,n])}{n}}=|a|^{p_U}|b|^{1-p_U},$$
$$1/R_P:=\underline{\lim_{n\to\infty}}\,|a|^{\frac{\#(U\cap[1,n])}{n}}\cdot|b|^{1-\frac{\#(U\cap[1,n])}{n}}=|a|^{q_U}|b|^{1-q_U}.$$

Thus, r_P and R_P can take any values, satisfying: $|a|^{-1}\le r_P\le R_P\le|b|^{-1}$. If there is a gap $r_P<R_P$, then for any $r\in\mathbb{R}_{>0}$ such that $r_P<r<R_P$, $Q(t):=\sum_{n=0}^{\infty}e^{i\theta_n}(t/r)^n$ for $\theta_n=\#(U\cap\mathbb{Z}_{1\le\cdot\le n})\arg(a)+(n-\#(U\cap\mathbb{Z}_{1\le\cdot\le n}))\arg(b)$ gives example c) (since $\Omega_1(P_{\mathfrak{U}})\cap\{z\in\mathbb{C}:|z|=r\}=\emptyset$ and §2.4 Assertion4).

4. Rational expression of opposite series

From this section, we restrict our attention to a tame power series having the finite rational accumulation set $\Omega(P)$.

4.1. **Rational expression.**

We show that opposite series become rational functions of special form. We start with a characterization of a finite rational accumulation.

Assertion 7. *Let $P(t)$ be a tame power series in t. The set $\Omega(P)$ is a finite rational accumulation set of period $h_P \in \mathbb{Z}_{\geq 1}$ if and only if $\Omega_1(P)$ is so. We say P is finite rationally accumulating of period h_P.*

Proof. If $\Omega(P)$ is finite rationally accumulating, then, in particular, the sequence $\frac{\gamma_{n-1}}{\gamma_n}$ is finite rationally accumulating. To show the converse and to show the coincidence of the periods, assume that $\{\gamma_{n-1}/\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ accumulate finite rationally of period h_1. Then, for the standard subdivision $\mathcal{U}_{h_1} := \{U^{[e]}\}_{[e]\in\mathbb{Z}/h_1\mathbb{Z}}$, the subsequence $\{\gamma_{n-1}/\gamma_n\}_{n\in U^{[e]}}$ for each $[e]\in\mathbb{Z}/h_1\mathbb{Z}$ converges to some number, which we denote by $a_1^{[e]}\in\mathbb{C}$.

For any $k\in\mathbb{Z}_{\geq 0}$ and sufficiently large (depending on k) n, one has

$$\frac{\gamma_{n-k}}{\gamma_n} \;=\; \frac{\gamma_{n-1}}{\gamma_n}\frac{\gamma_{n-2}}{\gamma_{n-1}}\cdots\frac{\gamma_{n-k}}{\gamma_{n-k+1}}.$$

For $n \in U^{[e]}$ with $[e] \in \mathbb{Z}/h_1\mathbb{Z}$, we see that the RHS converges to $a_1^{[e]}a_1^{[e-1]}\dots a_1^{[e-k+1]}$. Then, for $[e]\in\mathbb{Z}/h_1\mathbb{Z}$ and $k \in\mathbb{Z}_{\geq 0}$, by putting

$$a_k^{[e]} \;:=\; a_1^{[e]}a_1^{[e-1]}\dots a_1^{[e-k+1]}, \tag{4.1.1}$$

the sequence $\{X_n(P)\}_{n\in U_{[e]}}$ converges to $a^{[e]} := \sum_{k=0}^{\infty} a_k^{[e]} s^k$ with $a_1^{[e]} = \iota(a^{[e]})$ so that $\Omega(P)$ is finite rationally accumulating. Its period h_P is a divisor of h_1, but it cannot be strictly smaller than h_1, since otherwise the sequence $\{\gamma_{n-1}/\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ gets a period shorter than h_1. $\square$

Remark. That the period of the finite rational accumulation of $\Omega_1(P)$ is equal to h_P does not imply $\#\Omega_1(P) = h_P$. That is, the map $a \in \Omega(P) \mapsto \iota(a)\in\Omega_1(P)$ is not necessarily injective (see §3.5 Example b).

Assertion 8. *Let P be finite rationally accumulating of period $h_P \in \mathbb{Z}_{\geq 1}$. Then the opposite series $a^{[e]} = \sum_{k=0}^{\infty} a_k^{[e]} s^k$ in $\Omega(P)$ associated with the rational subset $U^{[e]}$ converges to a rational function*

$$a^{[e]}(s) \;=\; \frac{A^{[e]}(s)}{1 - A_P s^{h_P}}, \tag{4.1.2}$$

where the numerator $A^{[e]}(s)$ is a polynomial in s of degree $h_P - 1$:

$$A^{[e]}(s) \;:=\; \sum_{j=0}^{h_P-1} \left(\prod_{i=1}^{j} a_1^{[e-i+1]}\right) s^j \tag{4.1.3}$$

and

$$A_P := \prod_{i=0}^{h_P-1} a_1^{[i]} = a_{h_P}^{[0]} = \cdots = a_{h_P}^{[h_P-1]}. \tag{4.1.4}$$

We have a relation

$$(r_P)^{h_P} \;=\; (R_P)^{h_P} \;=\; |A_P|, \tag{4.1.5}$$

where r_P is the radius of convergence of $P(t)$ and R_P is given by (2.1.4).

Proof. Due to the h_P-periodicity of the sequence $a_1^{[e]}$ ($e \in \mathbb{Z}$), formula (4.1.1) implies the "*semi-periodicity*" with respect to the factor (4.1.4):

$$a^{[e]}_{mh_P+k} = (A_P)^m a_k^{[e]} \quad \text{for } m \in \mathbb{Z}_{\geq 0},\ k = 0, \cdots, h_P - 1.$$

This implies a factorization $a^{[e]} = A^{[e]} \cdot \sum_{m=0}^{\infty} (A_P s^{h_P})^m$ and hence (4.1.2).

To show (4.1.5), it is sufficient to show the existence of positive real constants c_1 and c_2 such that for any $k \in \mathbb{Z}_{\geq 0}$ there exists $n(k) \in \mathbb{Z}_{\geq 0}$ and for any integer $n \geq n(k)$, one has $c_1 r^k \leq \left|\frac{\gamma_{n-k}}{\gamma_n}\right| \leq c_2 r^k$.

Proof. We may choose $c_1, c_2 \in \mathbb{R}_{>0}$ satisfying $c_1 < \min\{\left|\frac{a_i^{[e]}}{r^i}\right| \mid [e] \in \mathbb{Z}/h\mathbb{Z}, i \in \mathbb{Z} \cap [0, h-1]\}$ and $c_2 > \max\{\left|\frac{a_i^{[e]}}{r^i}\right| \mid [e] \in \mathbb{Z}/h\mathbb{Z}, i \in \mathbb{Z} \cap [0, h-1]\}$. □

This completes a proof of Assertion 8. □

Corollary. *Let $\Omega(P)$ be finite. For any power series $Q(t)$ of radius r_Q of convergence larger than r_P, $P+Q$ is tame and $\Omega(P) = \Omega(P+Q)$.*

4.2. Coefficient matrix M_h of numerator polynomials.

In this and the next section, we study the linearly dependent relations among the opposite series $a^{[e]}(s)$ for $[e] \in \mathbb{Z}/h_P\mathbb{Z}$.

For the purpose, let us consider the matrix

$$M_h := \left(\prod_{i=1}^{f} a_1^{[e-i+1]}\right)_{e,f \in \{0,1,\cdots,h-1\}} \tag{4.2.1}$$

of the coefficients of the numerator polynomials (4.1.3). Regarding $a_1^{[0]}, \cdots, a_1^{[h-1]}$ as variables, let us introduce the "discriminant" by

$$D_h(a_1^{[0]}, \cdots, a_1^{[h-1]}) := \det(M_h) \;\in\; \mathbb{Z}[a_1^{[0]}, \cdots, a_1^{[h-1]}]. \tag{4.2.2}$$

Actually, D_h is an irreducible homogeneous polynomial of degree $h(h-1)/2$. Under the cyclic permutation $\sigma = (0, 1, \cdots, h-1)$ of the variables,

$$D_h \circ \sigma = (-1)^{h-1} D_h. \tag{4.2.3}$$

Our next task in §4.3 is to stratify the zero-loci of D_h according to the rank of M_h. This is achieved by introducing the *opposite denominator polynomial* Δ^{op}, whose degree describes the rank of the matrix M_h (see (4.3.3)). Here the coefficient is an arbitrary field K. In particular, for the case of $K = \mathbb{R}$, we give a precise stratification of the positive real parameter space $(\mathbb{R}_{>0})^h$ of the parameter $(a_1^{[0]}, \cdots, a_1^{[h-1]})$, whose strata are labeled by cyclotomic polynomials i.e. an integral factor of $1-s^h$ which contains also the factor $1-s$ (see Assertion 9.iv).

4.3. **Linear dependence relations among opposite series.**

Assertion 9. *Fix* $h \in \mathbb{Z}_{>0}$. *For each* $[e] \in \mathbb{Z}/h\mathbb{Z}$ *and each* $A \in K^\times$, *let* $A^{[e]}(s)$ *be the polynomial defined in equations* (4.1.3) *and* (4.1.4) *associated with any* h*-tuple* $\bar{a} = (a_1^{[0]}, \cdots, a_1^{[h-1]}) \in (K^\times)^h$.

i) *In* $K[s]$, *we have the equality of the greatest common divisors:*

$$\begin{aligned} & \gcd(A^{[0]}(s), 1-As^h) &= \cdots &= \gcd(A^{[h-1]}(s), 1-As^h) \\ = & \gcd(A^{[0]}(s), A^{[1]}(s)) &= \cdots &= \gcd(A^{[h-1]}(s), A^{[h]}(s)) \end{aligned}$$

(whose constant term is normalized to 1), which we denote by $\delta_{\bar{a}}(s)$.

Let us introduce the opposite denominator polynomial by

$$\Delta_{\bar{a}}^{op}(s) := (1-As^h)/\delta_{\bar{a}}(s). \tag{4.3.1}$$

ii) *For* $[e] \in \mathbb{Z}/h\mathbb{Z}$, *put*

$$b^{[e]}(s) := A^{[e]}(s)/\delta_{\bar{a}}(s). \tag{4.3.2}$$

The polynomials $b^{[e]}(s)$ *for* $[e] \in \mathbb{Z}/h\mathbb{Z}$ *span the space* $K[s]_{<\deg(\Delta_{\bar{a}}^{op})}$ *of polynomials of degree less than* $\deg(\Delta_{\bar{a}}^{op})$. *Hence, one has the equality:*

$$\operatorname{rank}(M_h) = \deg(\Delta_{\bar{a}}^{op}). \tag{4.3.3}$$

iii) *For* $\varphi(s) \in K[s]$, $\varphi(s) \mid \Delta_{\bar{a}}^{op}$ *if and only if* $\varphi(s) \mid 1-As^h$ *and* $\gcd(\varphi(s), A^{[e]}(s)) = 1$. *In particular, if* $\bar{a} \in (\mathbb{R}_{>0})^h$, *then* $\Delta_{\bar{a}}^{op}$ *is always divisible by* $1-\sqrt[h]{A}s$.

iv) *Let* $h \in \mathbb{Z}_{>0}$. *There exists a stratification* $\mathbb{R}^h_{>0} = \amalg_{\Delta^{op}} C_{\Delta^{op}}$, *where the index set is equal to*

$$\{\Delta^{op} \in \mathbb{R}[s] : 1-s \mid \Delta^{op}(s) \mid 1-s^h \ \& \ \Delta^{op}(0) = 1\}, \tag{4.3.4}$$

and $C_{\Delta^{op}}$ *is a smooth semi-algebraic set of* $\mathbb{R}$*-dimension* $\deg(\Delta^{op})-1$, *such that* $\Delta_{\bar{a}}^{op}(s) = \Delta^{op}(\sqrt[h]{A}s)$ *for* $\forall \bar{a} \in C_{\Delta^{op}}$ *and* $\overline{C_{\Delta_1^{op}}} \supset C_{\Delta_2^{op}} \Leftrightarrow \Delta_1^{op} | \Delta_2^{op}$

Proof. i) By Definitions (4.1.3), (4.1.4) and (4.1.1), we have the following relations:

$$a_1^{[e+1]} s A^{[e]}(s) + (1-As^h) = A^{[e+1]}(s) \tag{4.3.5}$$

for $[e] \in \mathbb{Z}/h\mathbb{Z}$. This implies $\gcd(A^{[e]}(s), 1-As^h) \mid \gcd(A^{[e+1]}(s), 1-As^h)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$. Thus, one may conclude that all of the polynomials $\gcd(A^{[e]}(s), 1-As^h) = \gcd(A^{[e]}(s), A^{[e+1]}(s))$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ are the same up to a constant factor. It is obvious that a factor of $1-As^h$ contains a nontrivial constant term, which we shall normalize to 1.

ii) Let V be the subspace of $K[s]/(\Delta_{\bar{a}}^{op})$ spanned by the images of $b^{[e]}(s) := A^{[e]}(s)/\delta_{\bar{a}}(s)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$. Relation (4.3.5) implies that V is closed under multiplication by s. On the other hand, $b^{[e]}(s)$ and $\Delta_{\bar{a}}^{op}$ are relatively prime, so they generate 1 as a $K[s]$-module. That is, V contains the class $[1]$ of 1. Hence, $V = K[s] \cdot [1] = K[s]/(\Delta_{\bar{a}}^{op})$. Since $\deg(b^{[e]}(s)) = h-1-\deg(\delta_{\bar{a}}(s)) = \deg(\Delta_{\bar{a}}^{op})-1$, $V \cap K[s]\Delta_{\bar{a}}^{op} = 0$. This means that the polynomials $b^{[e]}(s)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ span the space of polynomials of degree less than $\deg(\Delta_{\bar{a}}^{op})$. In particular, one has $\mathrm{rank}(M_h) = \mathrm{rank}_K V = \deg(\Delta_{\bar{a}}^{op})$.

iii) The first half is a reformulation of the definition of $\delta_{\bar{a}}$ and (4.3.1). We see that if $1-rs \not| \Delta_{\bar{a}}^{op})$ then $1-rs \mid A^{[e]}(s)$ (4.3.2) so $A^{[e]}(1/r) = 0$. This is impossible, since all coefficients of $A^{[e]}$ and $1/r$ are positive reals.

iv) Let Δ^{op} be a polynomial as given in (4.3.4) and put $d = \deg(\Delta^{op})$. Consider the set $\overline{C}_{\Delta^{op}} := \{c(s) = 1 + c_1 s + \cdots + c_{d-1}s^d \in \mathbb{R}[s] \mid \exists r \in \mathbb{R}_{>0}$ s.t. all coefficients of $A_c^{[0]} := c(s)(1-r^h s^h)/\Delta^{op}(rs)$ are positive$\}$. Then $\overline{C}_{\Delta^{op}}$ is an open semi-algebraic set in $\mathbb{R}^d$, which is nonempty since $\Delta^{op}(rs)/(1-rs)$ belongs to $\overline{C}_{\Delta^{op}}$. In particular, it is pure dimensional of real dimension $d-1$. To any $c \in \overline{C}_{\Delta^{op}}$, one can associate a unique $\bar{a} \in (\mathbb{R}_{>0})^h$ such that the associated polynomial $A^{[0]}$ (4.1.3) is equal to $A_c^{[0]}$. We identify $\overline{C}_{\Delta^{op}}$ with the semi-algebraic subset $\{a \in (\mathbb{R}_{>0})^h \mid a \leftrightarrow c \in \overline{C}_{\Delta^{op}}\}$ of pure dimension $d-1$ embedded in $(\mathbb{R}_{>0})^h$. Similarly, for any factor Δ' of Δ^{op} (over $\mathbb{R}$) divisible by $1-s$, we consider the semi-algebraic subsets $\overline{C}_{\Delta'}$ in $\mathbb{R}_{>0}^h$ of pure dimension $\deg(\Delta')$. Then, the multiplication of Δ^{op}/Δ' induces the inclusion $\overline{C}_{\Delta'} \subset \overline{C}_{\Delta^{op}}$. Then we define the semi-algebraic set $C_{\Delta^{op}}$ inductively by $\overline{C}_{\Delta^{op}} \setminus \cup_{\Delta'} C_{\Delta'}$, where the index Δ' runs over all factors of Δ^{op} which are not equal to Δ^{op} and are divisible by $1-rs$. By the induction hypothesis, $d-1 > \dim_{\mathbb{R}}(C_{\Delta'})$ so that the difference $C_{\Delta^{op}}$ is a non-empty open semi-algebraic set with pure real dimension $d-1$.

This completes the proof of Assertion 9. □

Suppose $char(K) \not| \, h$, and let $\tilde{K}$ be the splitting field of $\Delta_{\bar{a}}^{op}$ with the decomposition $\Delta_{\bar{a}}^{op} = \prod_{i=1}^{d}(1-x_i s)$ in $\tilde{K}$ for $d := \deg(\Delta_{\bar{a}}^{op})$. Then, one has the partial fraction decomposition:

$$\frac{A^{[e]}(s)}{1-As^h} = \sum_{i=1}^{d} \frac{\mu_{x_i}^{[e]}}{1-x_i s} \tag{4.3.6}$$

for $[e] \in \mathbb{Z}/h\mathbb{Z}$, where $\mu_{x_i}^{[e]}$ is a constant in $\tilde{K}$ given by the residue:

$$\mu_{x_i}^{[e]} = \frac{A^{[e]}(s)(1-x_i s)}{1-As^h}\Big|_{s=(x_i)^{-1}} = \frac{1}{h}A^{[e]}(x_i^{-1}). \tag{4.3.7}$$

Corollary. *The matrix* $\big((\mu_{x_i}^{[e]})_{[e]\in\mathbb{Z}/h\mathbb{Z}, x_i^{-1}\in V(\Delta_{\bar a}^{op})}\big)$ *is of maximal rank* d.

Proof. The rational function on the LHS of (4.3.6) for $[e]\in\mathbb{Z}/h\mathbb{Z}$ span a vector space of rank $d:=\deg(\Delta_{\bar a}^{op})$. Therefore, the coefficient matrix on the RHS has rank equal to d. □

Remark. 1. One has the equivariance $\sigma(\mu_{x_i}^{[e]})=\mu_{\sigma(x_i)}^{[e]}$ with respect to the action $\sigma\in\mathrm{Gal}(\tilde K,K)$ of the Galois group of the splitting field.

2. The index x_i in (4.3.7) may run over all roots x of the equation $x^h-A=0$. However, if $x^{-1}\notin V(\Delta_{\bar a}^{op})$ (i.e. $\Delta_{\bar a}^{op}(x^{-1})\neq 0$), then $\mu_x^{[e]}=0$.

3. For the given $h\in\mathbb{Z}_{>0}$, to consider the space of finite parameters $(a_1^{[0]},\cdots,a_1^{[h-1]})$ is equivalent to consider the space of infinite parameters $(a_i)_{i\in\mathbb{Z}}$ with "quasi"-periodicity $a_{i+h}=Aa_i$. Then it was suggested by the referee to regard the latter space over $\mathbb{C}$ as a h-"quasi"-periodic representation of $\mathbb{Z}$ and to decompose it to the direct sum the sequence $(a_i=A^{i/h\chi(i))})$ for $\chi\in Z/h\mathbb{Z}\to\mathbb{C}^\times$.

4.4. **The module $\mathbb{C}\Omega(P)$.**

We return to a tame power series $P(t)$ (2.1.1). Suppose $P(t)$ is finite rationally accumulating of a period h_P. Let $a_1^{[e]}$ be the initial of the opposite series $a^{[e]}\in\Omega(P)$ for $[e]\in\mathbb{Z}/h_P\mathbb{Z}$. Since $\Delta_{\bar a}^{op}(s)$ (4.3.1) for $\bar a:=(a_1^{[0]},\cdots,a_1^{[h-1]})$ depends only on P but not on the choice of a period h_P, we shall denote it by $\Delta_P^{op}(s)$ and call it the *opposite denominator polynomial* of P. Then, §4.3 Assertion 9.ii) says that we have the $\mathbb{C}$-isomorphism:

$$\begin{array}{rcl}\mathbb{C}\Omega(P) & \simeq & \mathbb{C}[s]/(\Delta_P^{op}(s)),\\ a^{[e]} & \mapsto & b^{[e]}:=\Delta_P^{op}\cdot a^{[e]} \bmod \Delta_P^{op}.\end{array}\tag{4.4.1}$$

Let us rewrite equality (4.3.2) and introduce the key number:

$$d_P:=\mathrm{rank}_{\mathbb{C}}\big(\mathbb{C}\Omega(P)\big)=\deg(\Delta_P^{op}).\tag{4.4.2}$$

Define an endomorphism σ on $\mathbb{C}\Omega(P)$ by letting

$$\sigma(a^{[e]}):=\tau_\Omega^{-1}(a^{[e]})=\tfrac{1}{a_1^{[e+1]}}a^{[e+1]}.\tag{4.4.3}$$

Assertion 10. *The actions of σ on the LHS and the multiplication of s on the RHS of* (4.4.1) *are naturally identified. Hence, the linear dependence relations among the generators $a^{[e]}$ ($[e]\in\mathbb{Z}/h\mathbb{Z}$) are obtained by the linear dependence relations $\Delta_P^{op}(\sigma)a^{[e]}$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$.*

Proof. The first part of Assertion 10 is a matter of calculation.
$\sum_{[e]\in\mathbb{Z}/h\mathbb{Z}}c_{[e]}b^{[e]}\equiv 0 \bmod \Delta_P^{op}(\sigma)b^{[e]}=0$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$. □

Note that the σ-action on $\mathbb{C}\Omega(P)$ is not $s|_{\mathbb{C}\Omega(P)}$ in the ring $\mathbb{C}[[s]]$.

5. Duality theorem

In this section, we restrict the class of functions $P(t)$ to those that are analytically continuable to a meromorphic function in a neighbourhood of the closed disc of convergence.[2] Under this assumption, we show a duality between $\Omega(P)$ and poles of $P(t)$ on the boundary of the disc.

5.1. **Functions of class $\mathbb{C}\{t\}_r$.**

For $r \in \mathbb{R}_{>0}$, we introduce a class

$$(5.1.1)\qquad \mathbb{C}\{t\}_r := \left\{ P(t) \in \mathbb{C}[[t]] \;\middle|\; \begin{array}{c}\text{i) } P(t) \text{ converges on the open disc } D(0,r).\\ \text{ii) } P(t) \text{ is analytically continuable to a meromorphic}\\ \text{function on an open neighbourhood of } \overline{D(0,r)}.\end{array} \right\}$$

For an element $P(t)$ of $\mathbb{C}\{t\}_r$, let us introduce a monic polynomial $\Delta_P(t)$, called the *polar part polynomial* of $P(t)$, characterized by

i) $\Delta_P(t)P(t)$ is holomorphic in a neighbourhood of the circle $|t| = r$,

ii) $\Delta_P(t)$ has lowest degree among all polynomials satisfying i).

Next, we decompose

$$(5.1.2)\qquad \Delta_P(t) = \prod_{i=1}^{N} (t - x_i)^{d_i}$$

where x_i $(i=1,\cdots,N,\ N \in \mathbb{Z}_{\geq 0})$ are mutually distinct complex numbers with $|x_i| = r$ and $d_i \in \mathbb{Z}_{>0}$ $(i=1,\cdots,N)$.

Definition. The *top denominator polynomial* $\Delta_P^{top}(t)$ of $P(t)$ is

$$(5.1.3)\qquad \Delta_P^{top}(t) := \prod_{i,d_i=d_m} (t - x_i) \quad \text{where} \quad d_m := \max\{d_i\}_{i=1}^{N}.$$

Note that $\Delta_P(t)$ may be equal to 1, and then $\Delta_P^{top}(t) = 1$. The converse: *if* $\Delta_P(t) \neq 1$, *then* $\Delta_P^{top}(t) \neq 1$, is also true.

5.2. **The rational operator T_U.**

Associated with a rational subset U of $\mathbb{Z}_{\geq 0}$, we introduce a linear operator T_U acting on $\mathbb{C}\{t\}_r$ to itself, which we call a *rational operator* or a *rational action* of U.

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[2]This assumption is necessary, since *the finite rational accumulation of $P(t)$ does not imply that $P(t)$ is meromorphic on the boundary of its convergent disc.*

Example. Consider the function $P(t) := \sqrt{\frac{1+t}{1-t}} = \sum_{n=0}^{\infty} \frac{(n-1)!}{2^n [n/2]![(n-1)/2]!} t^n$ which is tame. We see that the sequence of the proportion γ_{n-1}/γ_n of its coefficients accumulates to the unique values 1, i.e. $\Omega_1(P) = \{1\}$ and $\Omega(P) = \{1/(1-s)\}$. On the other hand, we observe that the function $P(t)$ has two singular points on the boundary of the unit disc $D(0,1)$ which are not meromorphic but algebraic. Such algebraic branching cases shall be treated in a forthcoming paper.

Definition. The action T_U on $\mathbb{C}[[t]]$ of a rational subset U of $\mathbb{Z}_{\geq 0}$ is

$$(5.2.1) \qquad T_U \ : \ P = \textstyle\sum_{n\in\mathbb{Z}_{\geq 0}} \gamma_n t^n \quad \mapsto \quad T_U P := \textstyle\sum_{n\in U} \gamma_n t^n.$$

One may regard $T_U P$ as a product of P with the rational function $U(t)$ (§3.1 Definition) in the sense of Hadamard [H].

The action T_U is continuous w.r.t. the adic topology on $\mathbb{C}[[t]]$ since $T_U\big(t^k\mathbb{C}[[t]]\big) \subset t^k\mathbb{C}[[t]]$ for any $k\in\mathbb{Z}_{\geq 0}$. It is also clear that the radius of convergence of $T_U P$ is not less than that of P.

Assertion 11. *For $h \in \mathbb{Z}_{\geq 0}$ and $[e] \in \mathbb{Z}/h\mathbb{Z}$, let us define the rational operator $T^{[e]} := T_{U^{[e]}}$. Then, we have*

$$(5.2.2) \qquad \sum_{e=0}^{h-1} T^{[e]} \ = \ 1,$$

$$(5.2.3) \qquad T^{[e]}\cdot t \ = \ t\cdot T^{[e-1]} \quad \& \quad T^{[e]}\cdot \frac{d}{dt} \ = \ \frac{d}{dt}\cdot T^{[e+1]}.$$

Proof. The equation (5.2.2) is a consequence of $\mathbb{Z}_{\geq 0} = \sqcup_{e=0}^{h-1} U^{[e]}$. The (5.2.3): for any t^m $(m \in \mathbb{Z}_{\geq 0})$, both sides return the same $t^{m+1}\delta_{[e],[m+1]} = t^{m+1}\delta_{[e-1],[m]}$ and $mt^{m-1}\delta_{[e],[m-1]} = mt^{m-1}\delta_{[e+1],[m]}$, respectively. □

Corollary. *The action T_U for a rational subset $U \subset \mathbb{Z}_{\geq 0}$ preserves $\mathbb{C}\{t\}_r$ for any $r\in\mathbb{R}_{>0}$. The highest order of poles on $|t| = r$ of $T_U P$ does not exceed that of $P\in\mathbb{C}\{t\}_r$.*

Proof. By decomposing the subset U as in §3.1 **Fact** iv), we need to prove this only for the case $U = U^{[e]}$ for some $[e \in \mathbb{Z}/h\mathbb{Z}]$ with $0 \leq e < h$. Since (5.2.3) implies $T^{[e]} = t^{e-h}T^{[0]}t^{h-e}$, we have only to prove the case when $U = U^{[0]} = h\mathbb{Z}$. But, then, $T_{U^{[0]}}$, which maps $P(t)$ to $\frac{1}{h}\sum_\zeta P(\zeta t)$, has the required property. □

5.3. **Duality theorem.**

The following is the goal of the present paper.

Theorem. (Duality) *Let $P(t)$ be a tame power series belonging to $\mathbb{C}\{t\}_r$ for $r = r_P$ (= the radius of convergence of P). Suppose that $P(t)$ is finite rationally accumulating of period h_P. Then*

i) *The opposite denominator polynomial $\Delta_P^{op}(s)$ (4.3.1) and the top denominator polynomial $\Delta_P^{top}(t)$ (5.1.3) of $P(t)$ are opposite to each other. That is,*

$$(5.3.1) \qquad \deg_t(\Delta_P^{top}(t)) \ = \ d_P \ = \ \deg_s(\Delta_P^{op}(s)),$$

and

$$(5.3.2) \quad t^{d_P}\Delta_P^{op}(t^{-1}) = \Delta_P^{top}(t), \ \textit{equivalently} \quad s^{d_P}\Delta_P^{top}(s^{-1}) = \Delta_P^{op}(s).$$

ii) *We have an equality of transition matrices:*

(5.3.3)

$$\left(\frac{P(t)}{T^{[e]}P(t)}\Big|_{t=x_i}\right)_{[e]\in\mathbb{Z}/h_P\mathbb{Z},\ x_i\in V(\Delta_P^{top}(t))} = \left(A^{[e]}\big|_{s=x_i^{-1}}\right)_{[e]\in\mathbb{Z}/h_P\mathbb{Z},\ x_i^{-1}\in V(\Delta_P^{op}(s))}.$$

In particular, $\left(\frac{P(t)}{T^{[e]}P(t)}\Big|_{t=x_i}\right)_{[e]\in\mathbb{Z}/h_P\mathbb{Z},\ x_i\in V(\Delta_P^{top}(t))}$ *is of maximal rank* d_P.

Proof. We start with the following obvious remark.

Assertion 12. *Let* $c \in \mathbb{C}^\times$ *be any non-zero complex constant. Change the variable* t *to* $\tilde{t} := t/c$ *and the opposite variable* s *to* $\tilde{s} := cs$*, and, for any tame series* P*, define a new tame series* $\tilde{P} := P|_{t=c\tilde{t}}$.

Then we have,

$$\begin{array}{rcl} \Omega(\tilde{P}) &=& \Omega(P)|_{s=\tilde{s}/c} := \{a(\tilde{s}/c) \mid a(t) \in \Omega(P)\}\ , \\ \Omega_1(\tilde{P}) &=& \Omega_1(P)/c := \{a_1/c \mid a_1 \in \Omega_1(P)\}. \end{array}$$

Proof. The equalities follows immediately from direct calculations. □

According to Assertion 12, we prove the theorem by changing the variable t to $\tilde{t} = t/c$ for $c = \sqrt[h_P]{A_P}$ (recall (4.1.4)) so that the new tame series has the constant $A_{\tilde{P}}$ equal to 1. Therefore, from now on, in the present proof, we shall assume that P is a finite rationally accumulating tame series with $A_P = 1$. In particular, this implies that the radius r_P of convergence of P is equal to 1 (recall (4.1.5)).

We first prove the theorem for a special but the key case when $\#\Omega(P) = 1$.

Assertion 13. *If* $P(t)$ *is simply accumulating then* $\Delta_P^{top} = t - 1$.

Proof. Consider the partial fractional expansion of P:

$$P(t) = \sum_{i=1}^{N}\sum_{j=1}^{d_i} \frac{c_{i,j}}{(t-x_i)^j} + Q(t), \tag{5.3.4}$$

where x_i $(i = 1, \cdot, N)$ is the location of a pole of P of order d_i on the unit circle $|x_i| = 1$, $c_{i,j}$ $(j = 1, \cdots, d_i)$ is a constant in $\mathbb{C}$, and $Q(t)$ is a holomorphic function on a disc of radius > 1.

We apply stability (Assertion 3 in §2.5) to the partial fractional expansion (5.3.4), to obtain $\Omega(P) = \Omega(P-Q)$. That is, the principal part $P_0 := P - Q$ gives rise to a simply accumulating power series. That is, $X_n(P_0) = \sum_{k=0}^{n} \frac{\sum_{i=1}^{N}\sum_{1\le j\le d_m} c_{i,j}x_i^{k-n-1}(n-k;j)/(j-1)!}{\sum_{i=1}^{N}\sum_{1\le j\le d_m} c_{i,j}x_i^{-n-1}(n;j)/(j-1)!} s^k$ $(n = 0, 1, 2, \cdots)$ converges to $\frac{1}{1-s} = \sum_{k=0}^{\infty} s^k$. Then, under this assumption, we'll show that if $c_{i,d_m} \neq 0$ then $x_i = 1$.

For each fixed $k \in \mathbb{Z}_{\ge 0}$, the numerator and denominator of the coefficient of s^k in $X_n(P_0)$ are polynomials in n of degree $\le d_m$. Let $v_n :=$

$\sum_{i=1}^{N} c_{i,d_m} x_i^{-n-1}$ be the coefficients of the top-degree term $n^{d_m}/(d_m-1)!$ in the denominator. Since the range of v_n is bounded (i.e. $|v_n| \leq \sum_i |c_{i,d_m}|$ due to the assumption $|x_i| = 1$), the sequence for $n=0,1,2,\cdots$ accumulates to a non-empty compact set in $\mathbb{C}$.

First, consider the case when the sequence $\{v_n\}_{n\in\mathbb{Z}_{ge0}}$ has a unique accumulating value v_0. Let us show that v_0 is non-zero and the result of Assertion 13 is true. (*Proof.* The mean sequence: $\{(\sum_{n=0}^{M-1} v_n)/M\}_{M\in\mathbb{Z}_{>0}}$ also converges to $v_0 = \lim_{n\to\infty} v_n$. This means that $\sum_{i=1}^{N} c_{i,d_m} \frac{\sum_{n=0}^{M-1} x_i^{-n-1}}{M}$ converges to v_0. If $x_i \neq 1$, the mean sum $\frac{\sum_{n=0}^{M-1} x_i^{-n-1}}{M} = \frac{1-x_i^{-M}}{(x_i-1)M}$ tends to 0 as $M \to \infty$. That is, $v_0 = c_{1,d_m}$, where we assume $x_1 = 1$ (even if, possibly $c_{1,d_m} = 0$). That is, the sequence $v'_n := v_n - c_{1,d_m} = \sum_{i=2}^{N} c_{i,d_m} x_i^{-n-1}$ converges to 0. For a fixed $n_0 \in \mathbb{Z}_{>0}$, consider the relations: $v'_{n_0+k} = \sum_{i=2}^{N} (c_{i,d_m} x_i^{-n_0}) x_i^{-k+1}$ for $k = 1, \cdots, N-1$. Regarding $c_{i,d_m} x_i^{-n_0}$ $(i = 2,\cdots, N)$ as the unknown, we can solve the linear equation for them, since the Vandermonde determinant for the matrix $(x_i^{-k+1})_{i=2,\cdots,N,k=1,\cdots,N-1}$ does not vanish. So, we obtain a linear approximation: $|c_{i,d_m}| = |c_{i,d_m} x_i^{-n_0}| \leq c \cdot \max\{|v'_{n_0+k}|\}_{k=1}^{N-1}$ $(i = 2,\cdots, N)$ for a constant $c > 0$ which depends only on $x_i's$ and N but not on n_0. The RHS tend to zero as $n_0 \to \infty$, whereas the LHS are unchanged. This implies $|c_{i,d_m}| = 0$, i.e. $d_i < d_m$ for $i = 2,\cdots, N$. As we have already remarked $\Delta_P(t) \neq 1$ implies $\Delta_P^{top}(t) := \prod_{d_i=d_m} (t - x_i) \neq 1$, and hence c_{1,d_m} cannot be 0. So $\Delta_P^{top}(t) = t - 1$.

Next, consider the case when the sequence v_n has more than two accumulating values. Then, one of them is non-zero. Suppose the subsequence $\{v_{n_m}\}_{m\in\mathbb{Z}_{>0}}$ converges to a non-zero value, say c. Recall the assumption that the sequence γ_{n-1}/γ_n converges to 1. So, the subsequence $\frac{\gamma_{n_m-1}}{\gamma_{n_m}} = \frac{v_{n_m-1}+\text{lower terms}}{v_{n_m}+\text{lower terms}}$ should also converge to 1 as $m \to \infty$. In the denominator, the first term tends to $c\neq 0$ and the second term $(=$ (a polynomial in n of degree $d_m-1)/n^{d_m})$ tends to zero. Similarly, in the numerator, the second term tends to zero. This implies that the first term in the numerator also converges to $c \neq 0$. Repeating the same argument, we see that for any $k \in \mathbb{Z}_{\geq 0}$, the subsequence $\{v_{n_m-k}\}_{m\in\mathbb{Z}_{\gg 0}}$ converges to the same c. Then, for each fixed $M \in \mathbb{Z}_{>0}$, the average sequence $\{(\sum_{k=0}^{M-1} v_{n_m-k})/M\}_{m\in\mathbb{Z}_{\gg 0}}$ converges to c, whereas the values is given by $\sum_{i=2}^{N} c_{i,d_m} x_i^{-n_m} \frac{1-x_i^{-M}}{(1-x_i^{-1})M} + c_{1,d_m}$ which is close to c_{1,d_m} for sufficiently large M and $n_m \gg M$. This implies $c = c_{1,d_m}$. Thus, the sequences $\{v'_{n_m-k} = \sum_{i=2}^{N} c_{i,d_m} x_i^{n_m-k}\}_{m\in\mathbb{Z}_{\gg 0}}$ for any $k \geq 0$ converge to 0. Then, an argument similar to that of the previous case implies $|c_{i,d_m}|=0$, i.e. $d_i < d_m$ $(i=2,\cdots,N)$. Hence, we have $\Delta_P^{top}(t) = t - 1$.

The proof of Assertion 13 is complete. □

We return to the proof of the general case, where P is finite rationally accumulating of period h, but may no longer be simply accumulating.

Assertion 14. *Let $P \in \mathbb{C}\{t\}_1$ be finite rational accumulating and the top denominator polynomial of P is defined as in* (5.1.3)*. Then,*

i) The top denominator polynomial of P is a factor of $t^h - 1$.

ii) For any $0 \le f < h$, $T^{[f]}P$ as a power series in $\tau := t^h$ is simply accumulating, where top order of its denominator is equal to d_m.

Proof. Since P is rationally finite accumulating of period h with radius of convergence $r_P = 1$, we have $\lim_{m\to\infty} \gamma_{f+(m-1)h}/\gamma_{f+mh} = 1 (= r_P^h)$ for any $0 \le f < h$. Regarding $T^{[f]}P = t^f \sum_{m=0}^{\infty} \gamma_{f+mh}\tau^m$ as a power series in $\tau = t^h$ and t^f as a constant factor of the series, this implies that $\Omega_1(T^{[f]}P) = \{1\}$ and, hence, that $T^{[f]}P$ is simply accumulating. Then, Assertion 13 implies that the highest order poles of $T^{[f]}P$ (in the variable τ) is only at $\tau - 1 = 0$ for all $[f] \in \mathbb{Z}/h\mathbb{Z}$, and Corollary to Assertion 11 implies that the order of the pole at $\tau = 1$ is less or equal than d_m :=the highest order of poles of $P(t)$. Thus, we get an expression $T^{[f]}P = t^f \frac{g^{[f]}(\tau)}{(\tau-1)^{d_m}}$, where $g^{[f]} \in \mathbb{C}\{\tau\}_1$ such that orders of poles of $g^{[f]}$ is strictly less than d_m. In view of (5.3.4), we obtain

$$*) \qquad P = \sum_{f=0}^{h-1} T^{[f]}P = \frac{\sum_{f=0}^{h-1} t^f g^{[f]}(\tau)}{(\tau-1)^{d_m}}.$$

This means, in particular, the the location of poles of P of top order d_f is contained in the solutions of $t^h - 1 = 0$, i.e. $\Delta^{top}(t)|(t^h-1)$ and i) is proven. To show the the latter half of ii), we need to show that $g^{[f]}(1) \neq 0$ for all f. However, $*)$ says that $g^{[f_0]}(1) \neq 0$ for some f_0.

Assuming $g^{[f]}(1) = 0$ for some f, we show a contradiction. Consider the sequence $\{\gamma_{f+mh}/\gamma_{f_0+mh}\}_{m\in\mathbb{Z}_{\ge 0}}$. On one side, this converges to a non-zero number since P is finite rational accumulating of order h. On the other hand, since $g^{[f_0]}(\tau)/(\tau-1)^{d_m}$ has pole of order d_m only at $\tau = 1$, we have $\gamma_{f_0+mh} = g^{[f_0]}(1)m^{d_m} + O(m^{d_m-1})$ and order of poles of $g^{[f]}(\tau)/(\tau-1)^{d_m}$ are strictly less than d_m by assumption, we have $\gamma_{f+mh} = O(m^{d_m-1})$. Thus the sequence converges to 0, which contradicts to the non-zero limit! □

For $0 \le e, f < h$, let us calculate the value of the proportion $\frac{T^{[f]}P}{T^{[e]}P}(t)$ at a root x of the equation $t^h - 1$ (defined by cancelling the poles at the point as a meromorphic function).

$$*) \qquad \left.\frac{T^{[f]}P}{T^{[e]}P}(t)\right|_{t=x} = x^{f-e} \frac{g^{[f]}\big|_{\tau=1}}{g^{[e]}\big|_{\tau=1}}.$$

In order to calculate this value, we prepare an elementary Fact.

Fact. *Let* $A(\tau)=\sum_{m=0}^{\infty} a_m\tau^m$, $B(\tau)=\sum_{m=0}^{\infty} b_m\tau^m \in \mathbb{C}\{\tau\}_1$ *such that their highest order poles of the same order* d *exist only at* $\tau=1$*. Then,*

$$**) \qquad \frac{A(\tau)}{B(\tau)}\Big|_{\tau=1} = \lim_{m\to\infty} \frac{a_m}{b_m}.$$

Proof. Replacing t and c_{ij} in (5.3.4) with τ and a_{ij} or b_{ij}, respectively, the RHS of $**)$ is written as $\lim_{m\to\infty}\frac{\sum_{i=1}^N\sum_{j\le d} a_{i,j}x_i^{-m-1}(m;j)/(j-1)!}{\sum_{i=1}^N\sum_{j\le d} b_{i,j}x_i^{-m-1}(m;j)/(j-1)!}$, where x_i is a complex number with $|x_i|=1$ and $x_1=1$. Since $a_{1,d}=(\tau-1)^dA(\tau)\big|_{\tau=1}$ and $b_{1,d}=(\tau-1)^dB(\tau)\big|_{\tau=1}$ are non-zero but $a_{i,d}=b_{i,d}=0$ for $i\neq 1$, this is equal to $\lim_{m\to\infty}\frac{a_{1,d}(m;d)/(d-1)!+O(m^{d-1})}{b_{1,d}(m;d)/(d-1)!+O(m^{d-1})} = \frac{a_{1,d}}{b_{1,d}} = \frac{A(\tau)}{B(\tau)}\big|_{\tau=1}$. □

Applying this Fact, the RHS of $*)$ is equal to $x^{f-e}\lim_{m\to\infty}\frac{\gamma_{f+mh}}{\gamma_{e+mh}}$. Then, applying to this expression a similar argument for (4.1.1), we obtain:

$$(5.3.5) \qquad \frac{T^{[f]}P}{T^{[e]}P}(t)\Big|_{t=x} = \begin{cases} x^{f-e}/a_1^{[f]}a_1^{[f-1]}\cdots a_1^{[e+1]} & \text{if } e<f \\ 1 & \text{if } e=f \\ x^{f-e}a_1^{[e]}a_1^{[e-1]}\cdots a_1^{[f+1]} & \text{if } e>f. \end{cases}$$

Since the RHS are non-zero in all cases, the order of the poles of $T^{[e]}P(t)$ at a solution x of the equation t^h-1 is independent of $[e]\in\mathbb{Z}/h\mathbb{Z}$. Summing up both sides of (5.3.5) for $0\le f<h$, we obtain

$$(5.3.6) \qquad \frac{P}{T^{[e]}P}(t)\Big|_{t=x} = A^{[e]}(x^{-1}).$$

(recall the $A^{[e]}(s)$ (4.1.3)). Let x be a solution of $t^h-r^h=0$ but $\Delta_P^{op}(x^{-1})\neq 0$. Then $\delta_a(x^{-1})=0$ (see (4.3.1)) and $A^{[e]}(x^{-1})=0$ for all $[e]\in\mathbb{Z}/h\mathbb{Z}$ (see Assertion 9. i)). That is, $\frac{T^{[e]}P}{P}(t)$ has a pole at $t=x$. This implies that $P(t)$ cannot have a pole of order d_m at $t=x$ (otherwise, due to Corollary to Assertion 11, the pole at $t=x$ of $T^{[e]}P$ is at most of order d_m, which is cancelled in $\frac{T^{[e]}P}{P}(t)$ by dividing by P, yielding a contradiction!). That is, we get one division relation.

Assertion 15. $\Delta_P^{top}(t) \mid t^{d_P}\Delta_P^{op}(t^{-1})$ *and* $\deg(\Delta_P^{top})\le d_P$.

Finally, let us show the opposite division relation.

Assertion 16. *Let* $P(t)$ *be a tame power series belonging to* $\mathbb{C}\{t\}_r$, *which is finite rationally accumulating of period* h*. Then*

i) There exists a constant $c\in\mathbb{R}_{>0}$ *such that* $|\gamma_n|\ge cr^{-n}n^{d_m}$ *for* $n\gg 0$.

ii) $$t^d\Delta_P^{op}(t^{-1}) \mid \Delta_P^{top}(t) .$$

Proof. i) Consider the Taylor expansion of the partial fractional expansion eq:5.3.4. Using notation v_n in Assertion 13, we have $\gamma_n = -v_n\frac{r^{-n-1}(n;d_m)}{(d_m-1)!}$ +(terms coming from poles of order $<d_m$)+(terms coming from $Q(t)$),

where $v_n = \sum_i c_{i,d_m}(x_i/r)^{-n-1}$ depends only on n mod h since x_i is the root of the equation $t^h - r^h = 0$. They cannot all be zero (otherwise, by solving the equations $v_n = 0$ $(0 \le n < h)$, we get $c_{i,d_m} = 0$ for all i, which contradicts to the vanishing of d_m). Let us show that none of the v_n is zero. Suppose the contrary and $v_e = 0 \neq v_f$ for some integers $0 \le e, f < h$. Then, one observes easily that $\lim_{m\to\infty} \frac{\gamma_{e+mh}}{\gamma_{f+mh}} = 0$. This contradicts to formula (5.3.5) and the non-vanishing of $a_1^{[e]}$ $([e] \in \mathbb{Z}/h\mathbb{Z})$.

ii) Since Δ_P^{top} cancels all poles of maximal order, the fractional expansion of $\Delta_P^{top}(t)P(t)$ has poles of order at most $d_m - 1$. Set $\Delta_P^{top}(t) = t^l + \alpha_1 t^{l-1} + \cdots + \alpha_l$. Then, this means that the sequence $\{\gamma_N\}$ (Taylor coefficients of P) satisfies

$$***) \qquad \gamma_N \cdot \alpha_l + \gamma_{N-1} \cdot \alpha_{l-1} + \cdots + \gamma_{N-l} \cdot 1 \quad \sim \quad o(N^{d_m} r^{-N})$$

as $N \to \infty$. Let $\sum_k a_k s^k \in \Omega(P)$ be an opposite series given by a sequence $\{X_{n_m}(P)\}_{m \in \mathbb{Z}_{\geq 0}}$ (2.2.1). For each fixed $k \in \mathbb{Z}_{\geq l}$, substitute N by $n_m - k + l$ in $***)$ and divide it by γ_{n_m}. Then, taking the limit $m \to \infty$ using the part i), the RHS converges to 0, so that we get

$$a_{k-l}\alpha_l + a_{k-l+1}\alpha_{l-1} + \cdots + a_k \;=\; 0.$$

Thus $s^l \Delta_P^{top}(s^{-1}) a(s)$ is a polynomial of degree $< l$ and the denominator $\Delta_P^{op}(s)$ of $a(s)$ divides $s^l \Delta_P^{top}(s^{-1})$. So, $d_P \le l$ and ii) is proved.

This completes a proof of Assertion 16. □

The proof of the theorem: (5.3.1) and (5.3.2) are already shown by Assertions 15 and 16, and (5.3.3) is shown by (4.3.7) and (5.3.6). □

5.4. **Example by Machì (continued).**

Recall §3.3 Machì's example, where we learned that the growth function $P_{\Gamma,G}(t) = \sum_{n=0}^{\infty} \#\Gamma_n t^n$ for the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ with respect to certain generator system G is equal to $\frac{(1+t)(1+2t)}{(1-2t^2)(1-t)}$ and that it is finite rationally accumulating of period $h = 2$.

Using this data, we calculate further the rational actions on it.

$$T^{[0]} P_{\Gamma,G}(t) = \sum_{k=0}^{\infty} \#\Gamma_{2k} t^{2k} = \frac{1+5t^2}{(1-2t^2)(1-t^2)},$$

$$T^{[1]} P_{\Gamma,G}(t) = \sum_{k=0}^{\infty} \#\Gamma_{2k+1} t^{2k+1} = \frac{2t(2+t^2)}{(1-2t^2)(1-t^2)},$$

The opposite denominator polynomial of the series $a^{[e]}$ $([e] \in \mathbb{Z}/2\mathbb{Z})$ and the top denominator polynomial of $P_{\Gamma,G}(t)$ are given as follows.

$$\Delta^{op}_{P_{\Gamma,G}}(s) = 1 - \frac{1}{2}s^2 \qquad \& \qquad \Delta^{top}_{P_{\Gamma,G}}(t) = t^2 - \frac{1}{2}.$$

Then the transformation matrix is given by

$$\begin{bmatrix} \frac{P_{\Gamma,G}(t)}{T^{[0]}P(t)} = \frac{(1+t)^2(1+2t)}{1+5t^2}\Big|_{t=\frac{1}{\sqrt{2}}} & \frac{P_{\Gamma,G}(t)}{T^{[1]}P(t)} = \frac{(1+t)^2(1+2t)}{2t(2+t^2)}\Big|_{t=\frac{1}{\sqrt{2}}} \\ \frac{P_{\Gamma,G}(t)}{T^{[0]}P(t)} = \frac{(1+t)^2(1+2t)}{1+5t^2}\Big|_{t=\frac{-1}{\sqrt{2}}} & \frac{P_{\Gamma,G}(t)}{T^{[1]}P(t)} = \frac{(1+t)^2(1+2t)}{2t(2+t^2)}\Big|_{t=\frac{-1}{\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} 1+\frac{5}{7}\sqrt{2} & 1+\frac{7}{5}\frac{1}{\sqrt{2}} \\ 1-\frac{5}{7}\sqrt{2} & 1-\frac{7}{5}\frac{1}{\sqrt{2}} \end{bmatrix}.$$

In fact, *this matrix coincides with the matrix* $2 \cdot \left(\mu_{x_i}^{[e]}\right)_{[e]\in\mathbb{Z}/2\mathbb{Z}, x_i\in\{\pm\sqrt{2}^{-1}\}}$ (4.3.7), which was already calculated in §3.3 Example as the coefficient of fractional expansion of the opposite series $a^{[0]}$ and $a^{[1]}$. In particular, its determinant, equal to $\frac{\sqrt{2}}{35}$, is non-zero. The matrix is an essential ingredient of the trace formula for limit F-functions [S1, (11.5.6)]

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