

String theory of the Omega deformation

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ABSTRACT: In this article, we construct a supersymmetric real mass deformation for the adjoint chiral multiplets in the gauge theory describing the dynamics of a stack of D2-branes in type II string theory. We do so by placing the D2-branes into the T-dual of a supersymmetric NS fluxbrane background. We furthermore note that this background is the string theoretic realization of an Ω deformation of flat space in the directions transverse to the branes where the deformation parameters satisfy $\varepsilon_1 = -\varepsilon_2$. This Ω deformation therefore serves to give supersymmetric real masses to the chiral multiplets of the 3D gauge theory. To illustrate the physical effect of the real mass term, we derive BPS-saturated classical solutions for the branes rotating in this background. Finally, we reproduce all the same structure in the presence of NS fivebranes and comment on the relationship to the gauge theory/spin-chain correspondence of Nekrasov and Shatashvili.

KEYWORDS: Superstring Vacua, Global Symmetries, Duality in Gauge Field Theories, Field Theories in Lower Dimensions

*This paper is dedicated to the memory of the victims
of the Tohoku earthquake and tsunami of March 2011.*

Contents

1	Introduction	1
2	Supersymmetric real masses in field theory and string theory	3
2.1	Real mass deformation of the D2-brane theory	4
2.2	Lift to $\mathcal{N} = 1$ theories in 3+1 dimensions on a circle	6
2.3	String embedding of the twisted mass for $\mathcal{N} = 8$ SYM in $D = 3$	8
3	Closed string fluxbrane and fluxtrap backgrounds	10
3.1	Bulk fields and T-duality transformations	10
3.2	Supersymmetry of the closed string background	13
3.3	Fluxtrap solution as the string theory of the Ω -background	15
4	Open strings	16
4.1	Action and fermionic symmetries	16
4.2	BPS bound for the DBI action	19
4.3	Low energy effective gauge theory	23
5	Relation to the gauge/Bethe correspondence	24
5.1	Brane configuration without twisted masses	24
5.2	Fluxtrap deformation of the brane configuration	25
5.3	Quantum nonabelian symmetry from the brane construction	26
6	Conclusions	29
A	The fluxtrap deformation of a set of NS-fivebranes	30
A.1	Bulk fields	30
A.2	Open strings	32
B	Conventions	35

1 Introduction

In this article, we study the low energy effective gauge theory describing the motions of a stack of D2-branes extended in the x^0, x^1, x^2 directions. Our aim is to give SUSY-preserving real masses to the fields describing the motions of the D2-branes in the directions x^4, \dots, x^7 . We will do so by placing the D2-branes into a closed string background corresponding to the T-dual of a supersymmetric NS fluxbrane [1–7]. We will point out that the fluxbrane is the string theory realization of an Ω -deformation [8–13] of flat space in the directions x^4, \dots, x^7 , where the deformation parameters fulfill $\varepsilon_1 = -\varepsilon_2$.

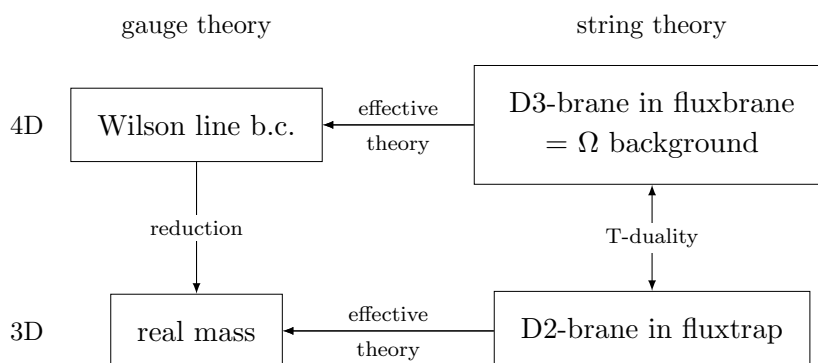


Figure 1. Gauge and string theory interpretation of the real mass in three and four dimensions. The reduction in gauge theory is realized as a T-duality in string theory.

direction	0	1	2	3	4	5	6	7	8	9
fluxtrap	×	×	×	×						×
D2	×	×	×							

Table 1. Embedding of the D2-brane with respect to the fluxtrap. The \times indicates a direction parallel to the brane (for the D2) or in which the bulk geometry is flat (for the fluxtrap).

Our strategy is as follows. The $(2+1)$ -dimensional gauge theory with *real mass* terms that we consider can be understood as coming from the reduction of $(3+1)$ -dimensional gauge theories with Wilson line boundary conditions for a global symmetry. This boundary condition in turn has a natural string theory interpretation in terms of D3-branes embedded in flat space with discrete identifications. Such backgrounds have been rediscovered a number of times, starting from the work of Melvin [1], and have taken different names, such as *fluxbranes* or Ω -*deformed* flat space. Since the string theory realization of the reduction from $3+1$ to $2+1$ dimensions can be achieved via a T-duality in a direction parallel to the D3-brane, we can give a string theory construction of the real mass in terms of D2-branes living in the T-dual of the fluxbrane background, that we will refer to as a *fluxtrap*. The different interpretations are summarized in figure 1. The setup we will be using is summarized in table 1.

The fluxtrap background described in this paper serves to give twisted masses to the chiral multiplets in a brane construction realizing the two-dimensional gauge theories in the gauge/Bethe correspondence of Nekrasov and Shatashvili [14, 15]. The full construction including NS5-branes and D4-branes in the fluxtrap background will be discussed briefly here, leaving more detailed elaboration to future work.¹

¹In an earlier paper [16], a brane construction was discussed based on the Hanany-Hori type of configuration [17], which reproduced certain aspects of the gauge theories but omitted the twisted masses. That construction differs in certain important ways from the one of relevance here, which reproduces all terms in the action of the gauge theories of [14, 15] precisely, twisted masses included.

The plan of this note is as follows. In section 2, we give a detailed introduction to the real mass deformations of $(2+1)$ -dimensional gauge theory, with emphasis on points useful to our construction in particular.

In section 3, we introduce the fluxbrane solution, which is equivalent to turning on an Omega background, and its T-dual (fluxtrap) that will give real masses to the adjoint fields living on a stack of D2-branes. We give the explicit supergravity solution in section 3.1. The Killing spinors preserved by the fluxtrap solution are given in section 3.2, and the relationship to the Ω -deformation of 4D gauge theory is discussed in section 3.3.

In section 4, we describe the three dimensional gauge theory living on the worldvolume of D2-branes extended in the (x_1, x_2) directions. The action, fermionic symmetries and preserved Killing spinors are detailed in section 4.1. In section 4.2, the BPS condition is derived and it is shown that the real mass parameter stemming from the fluxbrane background gives exactly the physical mass of the BPS states. In section 4.3, the low energy effective action is derived to quadratic order in the fields, where the real mass terms appear as expected. In section 5, the relation of our construction to the gauge/Bethe correspondence of Nekrasov and Shatashvili is discussed. Section 6 gives the conclusions. In appendix A we show how to incorporate a set of parallel of NS5 branes into the fluxtrap solution. Finally, in appendix B, some notations and conventions are collected.

2 Supersymmetric real masses in field theory and string theory

In this section we will give an exposition of the supersymmetric real mass terms for chiral superfields in $(2+1)$ and $(1+1)$ -dimensional gauge theories. The $(2+1)$ -dimensional real mass terms were first written down and used in [18], and their formal properties were developed systematically in [19, 20].

The real mass term is a mass term that cannot be thought of as a superpotential term, but comes rather from a deformation of the SUSY algebra itself. A real mass implies a deformation of the SUSY algebra and also vice versa: the real mass term is not supersymmetric under the undeformed SUSY algebra, and the deformation of the algebra automatically imparts a mass to the fields with non-vanishing eigenvalues under the central charge Z by which the algebra is deformed. This connection gives a clue to the construction of the real mass in string theory. In order to make our string theory embedding maximally clear, we will give a construction of real mass terms in a physical and superspace-free form, that can be thought of as involving a lift to one higher dimension.

The “twisted masses” in $1+1$ dimensions in the sense of [17] and the “real masses” in $2+1$ dimensions [18–20] are related. The principle of their relationship is that “twisted” mass terms in $1+1$ dimensions descend from “real” mass in $(2+1)$ dimensions upon dimensional reduction on a circle. Even though it is in general possible to switch on a second, imaginary component for the twisted mass term, in the system of [14, 15], only one real component of the complex mass term is ever activated, and that real component can be taken to be the one corresponding to a local deformation in $2+1$ dimensions. With this restriction of the twisted mass parameter to a real value, the real mass term in $2+1$ dimensions and twisted mass term in $1+1$ dimensions correspond canonically, and we will not distinguish them.

2.1 Real mass deformation of the D2-brane theory

For most of this paper we will focus on a real-mass deformation of the string theory embedding of maximally supersymmetric Yang-Mills theory in 2+1 dimensions, which corresponds to a D2-brane in flat space. Our treatment will closely follow that of [20], though we will keep our discussion independent of superspace, and we will emphasize the interpretation of the real mass term via dimensional oxidation on a circle.

A D2-brane in flat space preserves 16 supercharges among the 32 of Type IIA string theory. This is evident — the ambient space is flat and there is only one type of D-brane, which is a BPS state in flat space [21]. This configuration is invariant under $SO(2, 1)$ rotations in the 012 directions and $SO(7)$ rotations in the 3456789 directions. These are inherited as the Lorentz invariance and as the global R-symmetry group $SO(7)_{3456789}$ of the D2-brane theory. This is the theory that at low energies flows to the ABJM model [22], but the strong coupling behavior however is not relevant for the present article — we will always work in the limit $g_s \rightarrow 0$ of weak three-dimensional gauge coupling.

The D2-brane theory has many mass deformations. For instance there are superpotential terms as well as FI parameters one can add while preserving some amount of supersymmetry. These are a different type of mass terms than the twisted masses we will consider, in that the FI terms and superpotential terms leave the SUSY algebra undeformed. The twisted mass in 1+1 dimensions and the real mass in 2+1 dimensions have the property that their introduction always requires a (central) deformation of the SUSY algebra itself. This is mentioned in [17] for twisted masses in 1+1 dimensions and in [18–20] for real masses in 2+1 dimensions. The presence of the central extension is a logical necessity — the mass term amounts to a half-superspace integral of an integrand that would not be invariant under the other two supersymmetries if the SUSY representations of the chiral multiplets were undeformed. The deformation of the SUSY algebra by a central charge carried by the chiral multiplets deforms the SUSY representations of the chiral multiplets and allows half-superspace terms whose integrand could not otherwise be invariant under the complementary half of the SUSY.

How should we think about real masses in 3D? A real mass deformation is defined by a particular set of ingredients:

- A set of continuous Abelian global symmetries $U(1)_i$, whose Hermitean generators are q_i . These symmetries should *not* be R-symmetries — they should all act trivially on the supercharges themselves.
- More generally, if there is extended supersymmetry beyond $\mathcal{N} = 2$, the abelian symmetries should leave invariant at least one complex supercharge $Q_\alpha \neq Q_\alpha^\dagger$.
- A choice of real mass parameters m^i , one corresponding to each of the Abelian global symmetries.

If the symmetry generators \hat{q}_i are indeed exact symmetries of the dynamics, then the mass parameters m^i can be chosen arbitrarily. It may at times be useful to consider an enlarged set of approximate symmetries $U(1)_i$ that are not exact symmetries of the dynamics but

are broken by specific terms, e.g. by the superpotential. If only some linear combinations of the \hat{q}_i are exactly preserved, then this imposes a consistency condition relating the superpotential to the m^i : they must be chosen such that a certain linear combination $Z \equiv m^i \hat{q}_i$ of $U(1)$ symmetries (summation over i is implied) leaves the superpotential invariant. This Z is identical with the central term that deforms of the SUSY algebra.

The invariance of the action, including the superpotential, under Z is a necessary and sufficient condition for consistently combining superpotential terms with a twisted mass deformation. In the gauge theories of Nekrasov and Shatashvili, which we consider in section 5, this principle constrains the matter in the fundamental and antifundamental representations to carry exactly $-\frac{1}{2}$ the Z -charge of the matter in the adjoint representation, in order to accommodate the superpotential

$$W = \tilde{Q} \phi Q, \quad (2.1)$$

where Q, ϕ and \tilde{Q} are the fundamental, adjoint and antifundamental chiral multiplets, respectively.

Given these two ingredients — the symmetries \hat{q}_i and the mass parameters m^i satisfying the consistency condition — we can define a “real mass” deformation for a $(2+1)$ -dimensional SUSY theory with $\mathcal{N} = 2$ SUSY, in a canonical way. To describe the deformation, introduce a set of spurious, non-dynamical $\mathcal{N} = 2$ Abelian vector multiplets, one for each of the global symmetries $U(1)_i$. Then minimally couple these non-dynamical vector multiplets to the rest of the theory as dictated by gauge invariance and $\mathcal{N} = 2$ supersymmetry. Note that the “complex masses” (i.e. the quadratic terms in the superpotential) need not vanish in order for the supersymmetric minimal coupling to be well defined, nor even for the quadratic terms nor the superpotential as a whole to respect the symmetries $U(1)^i$ separately. All that is needed is for the superpotential (and the rest of the action) to be invariant under the combination $Z = m^i \hat{q}_i$, which is a weaker condition.

Now we give the prescription for defining the full deformation of the action. In three dimensions, an $\mathcal{N} = 2$ vector multiplet contains a single real scalar $\sigma = \sigma^*$, as well as a gauge field and a Dirac fermion. Let them be normalized such that the kinetic term for the gauge field would be

$$-\frac{1}{4g_3^2} F_{\mu\nu} F^{\mu\nu} \quad (2.2)$$

and the kinetic term for the scalar would be

$$-\frac{1}{2g_3^2} (\partial_\mu \sigma)(\partial^\mu \sigma) \quad (2.3)$$

With these normalizations, the gauge field and the scalar have mass dimension 1, and the SUSY transformations are coupling independent. Keeping only the space- and time-independent vevs $\langle \sigma^i \rangle$ of the real scalars σ^i and setting them equal to the mass parameters,

$$\langle \sigma^i \rangle = m^i, \quad (2.4)$$

we obtain a deformation of the action for the dynamical degrees of freedom.

Define the normalizations of the charges q_i to be coupling-independent. That is to say, under a constant gauge transformation $\chi = \theta = \text{const.}$, normalized such that $\theta = 2\pi$ is the smallest nonzero value of θ that defines a trivial gauge transformation, each chiral multiplet with charges gets a phase of $\exp[iq_i\theta^i]$. Then the real mass terms are such that each chiral multiplet with charges q_i gets a mass $q_i m^i$.

In order for this to be consistent, the SUSY algebra must be deformed by a real central charge $Z = m^i q_i$. Suppose the undeformed SUSY algebra is

$$\{Q_a, Q_b^\dagger\} = -2(\Gamma_\mu \Gamma_0)_{ab} P^\mu, \quad (2.5)$$

where we use the standard sign convention $0 < H = +P^0 = -P_0$, and μ runs from 0 to 2. Then when the masses are turned on, the central charge Z , normalized as defined above, enters as

$$\{Q_a, Q_b^\dagger\} = -2(\Gamma_\mu \Gamma_0)_{ab} P^\mu - 2i Z (\Gamma_0)_{ab}. \quad (2.6)$$

This description of the real mass deformation is completely equivalent to the description in [17, 20]. The exact same construction applies to construct “twisted mass” deformations in 2+1 dimensions, the difference being that the vev of the spurious vector multiplet scalars σ^i are now complex $\sigma \neq \sigma^\dagger$, and so the twisted mass parameters m^i can be complex instead of real.

2.2 Lift to $\mathcal{N} = 1$ theories in 3+1 dimensions on a circle

For $(2+1)$ -dimensional $\mathcal{N} = 2$ theories that lift to $(3+1)$ -dimensional $\mathcal{N} = 1$ theories by dimensional oxidation on a circle, there is a simpler way of understanding the real mass deformation, including the normalizations. For the construction of the real mass deformation to lift correctly, it’s important that the $U(1)$ symmetries \hat{q}_i that enter the central charge Z should lift to exact symmetries in four dimensions, rather than just emerging as accidental symmetries upon compactification to 2+1 dimensions and integrating out of Kaluza-Klein modes.

Generic $\mathcal{N} = 2$ theories in 2+1 dimensions do not have a lift to four dimensions, but many do, including the theories of present interest to us, namely maximally supersymmetric gauge theory. From the point of view of string theory, this dimensional oxidation to 3+1 dimensions is a T-duality on a coordinate x^8 transverse to the D2-brane, to a T-dual coordinate \tilde{x}^8 longitudinal to the a D3-brane. The size of the coordinate is of course fixed by consistency of the relation between gauge couplings. If the radius of the circle of compactification is \tilde{R} , and the four dimensional gauge coupling is g_4 , then the relationship is

$$\frac{2\pi\tilde{R}}{g_4^2} = \frac{1}{g_3^2}. \quad (2.7)$$

How do we think of the real mass deformation in 4-dimensional terms? We simply lift the three-dimensional $\mathcal{N} = 2$ vector multiplet in the obvious way to a four-dimensional $\mathcal{N} = 1$ vector multiplet, which contains a gauge field $A_{0,1,2,\tilde{8}}$, a Weyl gluino, and no scalars. With the normalizations in equation (2.3) and (2.2) for the real scalar σ and gauge field in the non-dynamical three-dimensional vector multiplet, the field σ is identified with the

zero mode piece of the $x^{\tilde{8}}$ component $A_{\tilde{8}}$ of the non-dynamical four-dimensional gauge field, with unit coefficient: taking $A_{\tilde{8}}$ to be constant in the $x^{\tilde{8}}$ direction, then

$$\sigma = 1 \cdot A_{\tilde{8}}. \quad (2.8)$$

The coefficient of proportionality can be determined from the relative normalizations of the kinetic terms for the (spurious, non-dynamical) fields σ and A_μ that we have coupled minimally to the $(2+1)$ -dimensional theory. So when we set the fictional vector multiplet scalar σ^i equal to the corresponding mass parameter m^i , this is the same thing as setting $A_{\tilde{8}}^i$ to m^i in the four-dimensional lift. In other words, this is a compactification with a Wilson line boundary condition for fields charged under the symmetries $U(1)_i$ such that, when parallel transported around the circle, every field transforms to itself up to the action of the monodromy

$$\hat{U}_8 \equiv \exp \left[i \sum_i \alpha^i \hat{g}_i \right], \quad (2.9)$$

where

$$\alpha^i \equiv \oint_0^{2\pi\tilde{R}} dx^{\tilde{8}} A_{\tilde{8}}^i = 2\pi\tilde{R} \sigma^i. \quad (2.10)$$

We are setting σ^i to m^i , which means

$$\alpha^i = 2\pi\tilde{R} m^i. \quad (2.11)$$

Therefore in the cases where the three-dimensional theory lifts to four dimensions (with the appropriate symmetries intact), the real mass term in the three-dimensional theory can be obtained by starting with the undeformed four-dimensional theory and compactifying down to three dimensions on a circle of radius \tilde{R} with monodromy

$$\hat{U}_8 \equiv \exp \left[2\pi i \tilde{R} \sum_i m^i \hat{g}_i \right] \quad (2.12)$$

around the $x^{\tilde{8}}$ direction, in the limit where $\tilde{R} \rightarrow 0$.

In this language the consistency conditions for twisted mass deformations are particularly transparent. It's clear that one can pick any symmetries \hat{g}_i and mass parameters m^i that one likes, as long as the Wilson line compactification preserves at least $\mathcal{N} = 2$ SUSY in 2+1 dimensions, the criterion for which is that $Z \equiv \sum m_i \hat{q}^i$ is a non-R symmetry with respect to at least one four-dimensional Weyl doublet of supercharges Q_α . So if the four dimensional theory has only $\mathcal{N} = 1$ and no extended supersymmetry, then this just means the combination Z must be a non-R global symmetry. If there is extended SUSY in four dimensions, then the condition is that the action of Z on supercharges must have at least one element in its kernel.

In the case of interest, the four dimensional theory is $\mathcal{N} = 4$ super-Yang-Mills in 3+1 dimensions. Its only continuous global symmetries are the $SO(6) \simeq SU(4)$ R-symmetry group. This group acts on $(3+1)$ -dimensional Weyl supercharges Q_α^A in the fundamental representation **4** of $SU(4)$. We are only interested in symmetries that preserve at least one

of the four Weyl doublets, say the fourth one $A = 4$. Then we will restrict the generators \hat{g} to lie in an $SU(3)$ subgroup that acts nontrivially on the first three elements of the **4** only. So our Wilson line compactification is defined by some mass parameters m^i , one for each appropriately normalized generator \hat{g}_i of the $SU(3) \subset SO(6)$ inside the R-symmetry group of $\mathcal{N} = 4$ of super-Yang-Mills theory in 3+1 dimensions. Then each massless four-dimensional chiral multiplet with eigenvalues q_i under the generators \hat{g}_i gets a mass in three dimensions that is equal to $|Z| = |\sum q_i m^i|$. For chiral multiplets that are not massless in four dimensions — if for instance they have D-term or F-term masses given by M_4 in four dimensions — then the construction makes it quite clear what their masses must be in three dimensions, since Z is a generalized momentum in the x^8 -direction. The mass formula at tree-level is

$$M_3 = \sqrt{M_4^2 + Z^2}, \quad Z \equiv \sum_i m^i q_i. \quad (2.13)$$

The mass M_4 is the same as the three-dimensional mass that comes from F-term and D-term potentials. So in strictly three-dimensional terms we can write

$$M_{\text{full, tree-level}} = \sqrt{M_{F+D}^2 + Z^2}, \quad Z \equiv \sum_i m^i q_i. \quad (2.14)$$

The full mass was computed from a classical four-dimensional dispersion relation so of course it will be modified by perturbative quantum corrections in general when $M_4 = M_{F+D}$ is nonzero. However when $M_4 = M_{F+D}$ vanishes, then the state is massless from the four-dimensional perspective and BPS from a three-dimensional perspective, and the quantum corrections to its mass should be under control — vanishing perturbatively and perhaps calculable nonperturbatively, as in [17].

So the data specifying a real mass in 2+1 dimensions in the 16-supercharge D2-brane theory are clear — for each $SU(3)$ generator \hat{g}_a pick a parameter m^a , and the real masses in the three-dimensional sense are equal to eigenvalues of the operator $Z \equiv m^a \hat{g}_a$ acting on chiral multiplets. The deformed theory can be thought of as coming from the compactification on the D3-brane theory on a circle of radius \tilde{R} , with a monodromy given by $\hat{U}_8 \equiv \exp[2\pi i \tilde{R} Z]$. Having noticed that the real mass can be realized by dimensional reduction with monodromy, let us use that description to find a string embedding of the D2-brane theory with a twisted mass.

2.3 String embedding of the twisted mass for $\mathcal{N} = 8$ SYM in $D = 3$

Consider an isolated D2-brane (we could equally well consider a set of N D2-branes) whose gauge coupling is g_3 . We want to lift to a D3-brane theory on a circle of radius \tilde{R} . The relation between gauge couplings is simply

$$\frac{2\pi\tilde{R}}{g_4^2} = \frac{1}{g_3^2}, \quad (2.15)$$

so

$$g_4 = g_3 \cdot \sqrt{2\pi\tilde{R}}. \quad (2.16)$$

So now let us consider a D3-brane extended in directions 012 $\tilde{8}$ in flat spacetime with line element

$$\tilde{ds}^2 = d\tilde{x}_{0\dots 2}^2 + d\tilde{x}_8^2 + d\tilde{y}_{1\dots 6}^2, \quad (2.17)$$

where

$$d\tilde{x}_{0\dots m}^2 = -dx_0^2 + dx_1^2 + \dots + dx_m^2 \quad \text{and} \quad d\tilde{y}_{1\dots m}^2 = dy_1^2 + \dots + dy_m^2. \quad (2.18)$$

The tilde denotes that the direction \tilde{x}^8 is going to be the T-dual of the x^8 direction transverse to the twobrane that will be the object of our ultimate interest. The threebrane in type IIB string theory is a BPS state that preserves sixteen supercharges. We wish to compactify the \tilde{x}^8 direction with radius \tilde{R} . However a straightforward identification $\tilde{x}^8 \simeq \tilde{x}^8 + 2\pi\tilde{R}$ would leave all sixteen supercharges unbroken and would not generate a mass term. It also would impose periodic boundary conditions on the fields living on the D3-brane, whereas we want to impose boundary conditions twisted by the monodromy \hat{U}_8 given in equation (2.12).

The only consistent way to do that in string theory is just to impose that same monodromy on the compactification of spacetime as a whole. From the point of view of the spacetime as a whole, the SO(6) generators of the D3-brane gauge theory are rotations of the six directions transverse to the threebrane, which in this case are y_1, \dots, y_6 . So the SO(6) of the gauge theory just acts on the coordinates y_i in the vector representation in an obvious way. We are interested in preserving at least $\mathcal{N} = 2$ supersymmetry in three dimensions, which forces us to restrict ourselves to an SU(3) subgroup of SO(6), which imposes a choice of complex structure on y_i -space. To focus on the more constrained case of $\mathcal{N} = 4$ supersymmetry in three dimensions, we can restrict the rotation to an SU(2) subgroup, in which case there are a triplet of such complex structures, but we will just focus on one for simplicity.

Either way, we choose a complex structure on y_i -space. So define

$$w_1 \equiv y_1 + i y_2, \quad w_2 \equiv y_3 + i y_4, \quad w_3 \equiv y_5 + i y_6, \quad (2.19)$$

then the condition to preserve at least $\mathcal{N} = 2$ SUSY in 3D is that the generators \hat{g} of the monodromy lie in a subgroup that acts as traceless Hermitean matrices on the three complex coordinates w_p . The condition to preserve $\mathcal{N} = 4$ in 3D is that the Hermitean matrices \hat{g} additionally lie in an SU(2) subgroup — that is, they have a common zero eigenvalue. It is now to that most supersymmetric case to which we would like to turn our attention.

We take the rotation matrices to lie in the SU(2) subgroup that acts only on the directions $w_{1,2}$, and leaves w_3 alone. Since we are compactifying only one dimension, we have only one linear combination of generators to worry about, so we pick $m^a g_a$ to be $m\sigma_3$, where σ_3 is the Pauli matrix acting on the directions $w_{1,2}$ and leaving w_3 invariant.

According to our prescription, we should impose $\hat{U}_8 = \exp[2\pi i m \tilde{R} \sigma^3]$ as a monodromy around the \tilde{x}^8 direction, which we compactify with radius \tilde{R} . This is equivalent to identifying the flat, ten-dimensional space by the combined identification (as opposed to two independent identifications) as follows:

$$\tilde{x}^8 \simeq \tilde{x}^8 + 2\pi\tilde{R}, \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \simeq \hat{U}_8 \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \quad (2.20)$$

This space, with these identifications, defines a purely closed-string background in which the D3-brane can be thought of as a probe. So for now, let us focus on the description of the closed string background itself.

3 Closed string fluxbrane and fluxtrap backgrounds

This type of space, obtained by taking a quotient by identifications of the form (2.20), has been studied already very well. Spaces of this kind, defined by a simultaneous identification by a shift of one direction and a rotation of some other directions, go by the name of “fluxbranes” and have been studied for quite some time, starting with work in the pure general relativity context by Melvin [1]. The “flux” in “fluxbrane” refers to the idea of starting in 5D general relativity with one circle compactified *à la* Kaluza-Klein.

In the case of an S^1 compactification with a monodromy \hat{U} around the S^1 acting on some other space \mathbf{X} , it is natural to think of \mathbf{X} as fibered over the circle, with the circle as the base and the fibration data defined as gluing maps on the fiber, specified by the monodromy. But the space carries a *second* fibration structure in which the S^1 is the fiber and \mathbf{X} the base of the fibration.

In the picture where the S^1 shift-circle is the fiber direction, the fibration structure is nontrivial even locally, in the sense of there being a local curvature of the connection of the bundle. In other words, there is Kaluza-Klein flux. This is not true in the original picture, where the S^1 is the base and the fibration of \mathbf{X} over it is described by a connection that is locally flat since the base is one-dimensional. The “fluxbrane” picture — in which the space \mathbf{X} is the base and the shift-circle S^1 the fiber — is the more natural in one in the Kaluza-Klein theory of 4D general relativity and electromagnetism, or any theory in which the S^1 is taken to be small. That is the picture in which these spaces are thought of in e.g. [1–7], whence the name “fluxbrane”. For us, the shift-circle S^1 is the direction \tilde{x}^8 in the type IIB string theory, and the Euclidean directions $w_{1,2}$ are the space \mathbf{X} , and indeed when we think of \mathbf{X} as the base, there really is Kaluza-Klein flux, as we shall now see.

3.1 Bulk fields and T-duality transformations

In the following, we specify the fluxbrane background in cylindrical coordinates. We will then perform a T-duality along the direction $x_{\tilde{8}}$ and derive the expressions for the metric, vielbein, B -field and the dilaton for the fluxtrap. This resulting geometry will provide the closed string background for the D2-branes in the following sections.

Fluxbrane. In cylindrical coordinates, defined by

$$\rho_1 e^{i\theta_1} \equiv w_1 = y_1 + i y_2 \quad \rho_2 e^{i\theta_2} \equiv w_2 = y_3 + i y_4 \quad x_3 + i x_9 \equiv y_5 + i y_6, \quad \tilde{x}_8 = \tilde{R} \tilde{u}, \quad (3.1)$$

our fields have the following simple form:

$$\tilde{g}_{\mu\nu} d\tilde{X}^\mu d\tilde{X}^\nu = d\tilde{x}_{0\dots 3}^2 + d\rho_1^2 + \rho_1^2 d\theta_1^2 + d\rho_2^2 + \rho_2^2 d\theta_2^2 + \tilde{R}^2 d\tilde{u}^2 + dx_9^2, \quad (3.2)$$

$$\tilde{B}_{\mu\nu} d\tilde{X}^\mu d\tilde{X}^\nu = 0, \quad (3.3)$$

$$\tilde{\Phi} = \log(\tilde{R} g_3^2), \quad (3.4)$$

where $\tilde{X}^\mu = (x_0, \dots, x_3, \rho_1, \theta_1, \rho_2, \theta_2, \tilde{u}, x_9)$. The reason for our choice of the constant value (3.4) for the dilaton $\tilde{\Phi}$ will become clear later on: g_3 will be the gauge coupling for the effective quantum field theory living on D2-branes at the origin.

The space \mathbb{R}^5/Γ is obtained by imposing the identifications in equation (2.20):

$$\begin{cases} \tilde{u} \simeq \tilde{u} + 2\pi k_1, \\ \theta_1 \simeq \theta_1 + 2\pi m\tilde{R}k_1, \\ \theta_2 \simeq \theta_2 - 2\pi m\tilde{R}k_1, \end{cases} \quad k_1 \in \mathbb{Z}, \quad (3.5)$$

in addition to the standard identifications for cylindrical coordinates,

$$\theta_1 \simeq \theta_1 + 2\pi k_2, \quad \theta_2 \simeq \theta_2 + 2\pi k_3, \quad k_2, k_3 \in \mathbb{Z}. \quad (3.6)$$

It is convenient to disentangle the periodicities. For this reason we introduce the new angular variables

$$\begin{cases} \phi_1 = \theta_1 - m\tilde{R}\tilde{u}, \\ \phi_2 = \theta_2 + m\tilde{R}\tilde{u}, \end{cases} \quad (3.7)$$

to rewrite the metric in the form

$$\begin{aligned} \widetilde{ds^2} = & d\tilde{x}_{0\dots 3}^2 + d\rho_1^2 + \rho_1^2 d\phi_1^2 + d\rho_2^2 + \rho_2^2 d\phi_2^2 \\ & + 2m\tilde{R}(\rho_1^2 d\phi_1 - \rho_2^2 d\phi_2) d\tilde{u} + \tilde{R}^2(1 + m^2(\rho_1^2 + \rho_2^2)) d\tilde{u}^2 + dx_9^2, \end{aligned} \quad (3.8)$$

with the three *independent* sets of identifications:

$$(\tilde{u}, \phi_1, \phi_2) \mapsto (\tilde{u} + 2\pi n_1, \phi_1 + 2\pi n_2, \phi_2 + 2\pi n_3), \quad n_1, n_2, n_3 \in \mathbb{Z}. \quad (3.9)$$

The space is of course still locally flat, but in this coordinate system one can see immediately the S^1 fibration structure where the fiber is described by \tilde{u} . This can be interpreted in terms of a non-flat Kaluza-Klein gauge connection ($\rho_1^2 d\phi_1 - \rho_2^2 d\phi_2$), which explains the origin of the name *fluxbrane*. The natural vielbein is given by

$$\begin{aligned} \tilde{e}^n &= dx^n, \quad n = 0, 1, 2, 3, 9 \\ \tilde{e}^4 &= d\rho_1, \quad \tilde{e}^5 = \rho_1(d\phi_1 + m\tilde{R}d\tilde{u}), \quad \tilde{e}^6 = d\rho_2, \quad \tilde{e}^7 = \rho_2(d\phi_2 - m\tilde{R}d\tilde{u}), \\ \tilde{e}^8 &= \tilde{R}d\tilde{u}. \end{aligned} \quad (3.10)$$

An alternative description for the same space can be obtained by passing to rectilinear coordinates:

$$z_1 \equiv x_4 + i x_5 \equiv \rho_1 e^{i\phi_1}, \quad z_2 \equiv x_6 + i x_7 \equiv \rho_2 e^{i\phi_2}, \quad x_8 \equiv \tilde{R}\tilde{u}, \quad (3.11)$$

which are related to the previous coordinate by a rotation in \tilde{u} :

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \cos(m\tilde{R}\tilde{u}) & \sin(m\tilde{R}\tilde{u}) \\ -\sin(m\tilde{R}\tilde{u}) & \cos(m\tilde{R}\tilde{u}) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} \cos(m\tilde{R}\tilde{u}) & \sin(m\tilde{R}\tilde{u}) \\ -\sin(m\tilde{R}\tilde{u}) & \cos(m\tilde{R}\tilde{u}) \end{pmatrix} \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}. \quad (3.12)$$

The metric can be recast in the form

$$\tilde{g}_{\mu\nu} d\tilde{X}^\mu d\tilde{X}^\nu = d\tilde{x}_{0\dots 3}^2 + \sum_{i=4}^7 (dx_i + mV^i dx_8)^2 + dx_8^2 + dx_9^2, \quad (3.13)$$

where $V^i \partial_i$ is the Killing vector

$$V^i \partial_i = -x^5 \partial_{x_4} + x^4 \partial_{x_5} + x^7 \partial_{x_6} - x^6 \partial_{x_7} = \partial_{\phi_1} - \partial_{\phi_2}, \quad (3.14)$$

with norm

$$\|V\|^2 = x_4^2 + x_5^2 + x_6^2 + x_7^2 = \rho_1^2 + \rho_2^2. \quad (3.15)$$

This is precisely the form of the Ω deformation of flat space in the directions z_1 and z_2 with parameters $\varepsilon_1 = -\varepsilon_2 = m$ as described in [11].

Fluxtrap. In order to make contact with the $(2+1)$ -dimensional theory living on the D2-branes we want to describe, we now perform a T-duality in the x_8 -direction. Since the three identifications in equation (3.9) are independent we can make use of Buscher's rules [23] for the T-duality to exchange the coordinate \tilde{u} for a new coordinate u , also with periodicity 2π . The metric and B -field become:

$$\begin{aligned} g_{\sigma\rho} &= \tilde{g}_{\sigma\rho} + \frac{\tilde{B}_{\tilde{u}\sigma} \tilde{B}_{\tilde{u}\rho} - \tilde{g}_{\tilde{u}\sigma} \tilde{g}_{\tilde{u}\rho}}{\tilde{g}_{\tilde{u}\tilde{u}}}, & g_{uu} &= \frac{(\alpha')^2}{\tilde{g}_{\tilde{u}\tilde{u}}}, & g_{u\sigma} &= \alpha' \frac{\tilde{B}_{\tilde{u}\sigma}}{\tilde{g}_{\tilde{u}\tilde{u}}}, \\ B_{\sigma\rho} &= \tilde{B}_{\sigma\rho} + \frac{\tilde{B}_{\tilde{u}\sigma} \tilde{g}_{\tilde{u}\rho} - \tilde{B}_{\tilde{u}\rho} \tilde{g}_{\tilde{u}\sigma}}{\tilde{g}_{\tilde{u}\tilde{u}}}, & B_{u\sigma} &= \alpha' \frac{\tilde{g}_{\tilde{u}\sigma}}{\tilde{g}_{\tilde{u}\tilde{u}}}, & \Phi &= \tilde{\Phi} - \frac{1}{2} \log \left(\frac{\tilde{g}_{\tilde{u}\tilde{u}}}{\alpha'} \right), \end{aligned} \quad (3.16)$$

where (σ, ρ) run over all coordinates except \tilde{u} or the dual coordinate u . In terms of the dual radius

$$R = \frac{\alpha'}{\tilde{R}}, \quad (3.17)$$

the new dimensionful coordinate is

$$x_8 = Ru, \quad (3.18)$$

such that the metric, B -field and dilation after T-duality are given by

$$ds^2 = d\tilde{x}_{0\dots 3}^2 + d\rho_1^2 + d\rho_2^2 + \rho_1^2 d\phi_1^2 + \rho_2^2 d\phi_2^2 + \frac{-m^2(\rho_1^2 d\phi_1 - \rho_2^2 d\phi_2)^2 + dx_8^2}{1 + m^2(\rho_1^2 + \rho_2^2)} + dx_9^2, \quad (3.19)$$

$$B = m \frac{\rho_1^2 d\phi_1 - \rho_2^2 d\phi_2}{1 + m^2(\rho_1^2 + \rho_2^2)} \wedge dx_8, \quad (3.20)$$

$$e^{-\Phi} = \frac{\sqrt{1 + m^2(\rho_1^2 + \rho_2^2)}}{g_3^2 \sqrt{\alpha'}}. \quad (3.21)$$

Observe that the complex coordinates $z_1 \equiv x_4 + ix_5$ and $z_2 \equiv x_6 + ix_7$ are left untouched by T-duality in the direction \tilde{u} since the three identifications in equation (3.9) are independent. Moreover, $V^\mu \partial_\mu$ remains a Killing vector for the geometry.

It is convenient to introduce a “natural” vielbein for the T-dual geometry. This is obtained by imposing

$$\tilde{e}^m{}_\mu \partial \tilde{X}^\mu = e^m{}_\mu \partial X^\mu, \quad (3.22)$$

where ∂X is the worldsheet derivative. Under T-duality, ∂X transforms as (see e.g. [24]):

$$\partial \tilde{u} \rightarrow \frac{1}{\tilde{g}_{\tilde{u}\tilde{u}}} (\alpha' \partial u - (\tilde{g}_{\sigma\tilde{u}} + \tilde{B}_{\sigma\tilde{u}}) \partial X^\sigma), \quad (3.23)$$

$$\partial \tilde{X}^\sigma \rightarrow \partial X^\sigma, \quad (3.24)$$

where X^σ runs again over all the coordinates other than u . The invariance of $e^m{}_\mu \partial X^\mu$ results in

$$\begin{cases} e^m{}_u = \frac{\alpha'}{\tilde{g}_{\tilde{u}\tilde{u}}} \tilde{e}^m{}_{\tilde{u}}, \\ e^m{}_\sigma = \tilde{e}^m{}_\sigma - \frac{\tilde{g}_{\sigma\tilde{u}} + \tilde{B}_{\sigma\tilde{u}}}{\tilde{g}_{\tilde{u}\tilde{u}}} \tilde{e}^m{}_{\tilde{u}} \quad \text{for } X^\sigma \neq u. \end{cases} \quad (3.25)$$

The inverse of these transformations is given by

$$\begin{cases} e_m{}^u = \frac{\tilde{g}_{\tilde{u}\tilde{u}}}{\alpha'} \tilde{e}_m{}^{\tilde{u}} + \frac{\tilde{g}_{\sigma\tilde{u}} + \tilde{B}_{\sigma\tilde{u}}}{\alpha'} \tilde{e}_m{}^\sigma, \\ e_m{}^\sigma = \tilde{e}_m{}^\sigma \end{cases} \quad \text{for } X^\sigma \neq u. \quad (3.26)$$

Starting from the vielbein for flat space in equation (3.10), we obtain

$$\begin{aligned} e^n &= dx^n, & n &= 0, 1, 2, 3, 9 \\ e^4 &= d\rho_1, & e^5 &= \frac{\rho_1}{\Delta^2} (d\phi_1 + m^2 \rho_2^2 (d\phi_1 + d\phi_2) + m dx_8), \\ e^6 &= d\rho_2, & e^7 &= \frac{\rho_2}{\Delta^2} (d\phi_2 + m^2 \rho_1^2 (d\phi_1 + d\phi_2) - m dx_8), \\ e^8 &= \frac{1}{\Delta^2} (-m \rho_1^2 d\phi_1 + m \rho_2^2 d\phi_2 + dx_8), \end{aligned} \quad (3.27)$$

where

$$\Delta^2 = 1 + m^2 (\rho_1^2 + \rho_2^2). \quad (3.28)$$

With this, we have collected all the necessary expressions for the fluxtrap geometry.

3.2 Supersymmetry of the closed string background

After having derived the form of the metric, B -field, dilaton and the vielbein in the fluxtrap background, we will now investigate the number of supersymmetries that are preserved by this background and explicitly give the preserved Killing spinors. It is convenient first to study the supersymmetries preserved by the fluxbrane background and then apply the T-duality to transform the Killing spinors.

In our choice of coordinates, the Killing spinors in the *flat* background are given by

$$K^{IIB} = \exp \left[\frac{1}{2} \theta_1 \Gamma_{45} + \frac{1}{2} \theta_2 \Gamma_{67} \right] \epsilon_0, \quad (3.29)$$

where ϵ_0 is a complex Weyl spinor. Introducing ϕ_1 and ϕ_2 , this becomes

$$K^{IIB} = \exp \left[\frac{1}{2} \phi_1 \Gamma_{45} + \frac{1}{2} \phi_2 \Gamma_{67} \right] \exp \left[\frac{m\tilde{R}\tilde{u}}{2} (\Gamma_{45} - \Gamma_{67}) \right] \epsilon_0. \quad (3.30)$$

Observe that all the variables are 2π -periodic. The matrix $\Gamma_{45} - \Gamma_{67}$ has eigenvalues $\pm 2i$ and 0. There are thus two possibilities for the Killing spinor to have the right periodicity [4]:

1. $m\tilde{R}$ is an integer. In this case the original variables θ_1 and θ_2 are only 2π -periodic and the only non-trivial identification in equation (3.5) is $\tilde{u} \simeq \tilde{u} + 2\pi k_1$. In other words, the spacetime is the standard flat space, preserving 32 real supercharges.
2. For generic values of $m\tilde{R}$, the second exponential is not periodic in \tilde{u} unless $\Gamma_{45} - \Gamma_{67}$ vanishes, in which case the dependence on \tilde{u} drops out of the spinor.

The first case is simply flat space without any deformation; in the following we will pursue the second alternative, which cuts down the number of Killing spinors by half. The orthogonal projectors

$$\Pi_{\pm}^{\text{flux}} = \frac{1}{2} (\mathbb{1} \pm \Gamma_{4567}), \quad (3.31)$$

satisfy

$$\Gamma_i \Pi_{\pm}^{\text{flux}} = \begin{cases} \Pi_{\mp}^{\text{flux}} \Gamma_i & \text{if } i = 4, 5, 6, 7, \\ \Pi_{\pm}^{\text{flux}} \Gamma_i & \text{otherwise.} \end{cases} \quad (3.32)$$

Using the fact that

$$\Pi_{-}^{\text{flux}} (\Gamma_{45} - \Gamma_{67}) = \Pi_{-}^{\text{flux}} (\mathbb{1} + \Gamma_{4567}) \Gamma_{45} = \Pi_{-}^{\text{flux}} \Pi_{+}^{\text{flux}} \Gamma_{45} = 0, \quad (3.33)$$

we find the expression for the 16 type IIB Killing spinors of the fluxbrane background:

$$K^{IIB} = \Pi_{-}^{\text{flux}} \exp \left[\frac{1}{2} \phi_1 \Gamma_{45} + \frac{1}{2} \phi_2 \Gamma_{67} \right] \epsilon_0. \quad (3.34)$$

Having obtained the expressions for the Killing spinors in type IIB, we can translate them into the T-dual type IIA fluxtrap picture (see e.g. [25]): $K^{IIA} = \epsilon_L + \epsilon_R$, where

$$\begin{cases} \epsilon_L = e^{-\Phi/8} (\mathbb{1} + \Gamma_{11}) \Pi_{-}^{\text{flux}} \exp \left[\frac{1}{2} \phi_1 \Gamma_{45} + \frac{1}{2} \phi_2 \Gamma_{67} \right] \epsilon_0, \\ \epsilon_R = e^{-\Phi/8} (\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_{-}^{\text{flux}} \exp \left[\frac{1}{2} \phi_1 \Gamma_{45} + \frac{1}{2} \phi_2 \Gamma_{67} \right] \epsilon_1, \end{cases} \quad (3.35)$$

with ϵ_0 and ϵ_1 constant Majorana spinors, and

$$\Gamma_u = R \Delta e^a_u \Gamma_a = \frac{m\rho_1}{\Delta} \Gamma_5 - \frac{m\rho_2}{\Delta} \Gamma_7 + \frac{1}{\Delta} \Gamma_8 \quad (3.36)$$

is the Γ matrix in the u direction, normalized to square to the identity, $(\Gamma_u)^2 = \mathbb{1}$. These spinors are such that the corresponding variations of the dilatino and gravitino (equations (B.9) and (B.10)) vanish.

Both ϵ_0 and ϵ_1 have 32 real components. The projectors Π_{-}^{flux} and $(\mathbb{1} \pm \Gamma_{11})$ each reduce the preserved supercharges by a factor of one half. In our fluxtrap background, we are therefore left with *16 preserved real supercharges*.

3.3 Fluxtrap solution as the string theory of the Ω -background

The fluxtrap solution (3.19)–(3.21) is the string theory of the Ω -deformation, and we would like to understand the meaning of that. So far we have considered branes — specifically D2-branes — embedded transverse to the $z_{1,2}$ directions. To understand the relationship with the Ω -deformation of four-dimensional gauge theory, let us consider a different type of D-brane, which we shall embed to fill the $z_{1,2}$ directions and denote with a prime.

The Ω -deformation of maximally supersymmetric gauge theory [11] can be defined by starting with five-dimensional gauge theory and dimensionally reducing on a circle with twisted boundary conditions defined by the identifications (2.20), where the rotation (2.20) acts on directions in the gauge theory itself, rather than on scalar fields. That is to say, the directions of the 5D gauge theory, in our coordinates, would be $x_{4,5,6,7,\tilde{8}}$. It is natural to interpret this gauge theory as the dynamical theory of Euclidean D4'-branes spanning the $x_{4,5,6,7,\tilde{8}}$ directions.

Dimensionally reducing to 4-dimensional gauge theory, as in [11] amounts to performing a T-duality along the \tilde{x}_8 direction, leaving us with the flux-trap solution (3.19)–(3.21). The D4'-branes have now been transformed into D3'-branes spanning the $z_{1,2}$ directions. The Ω -deformation of four-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory can be thought of precisely as the $\alpha' \rightarrow 0$ limit of a set of D3'-branes embedded in the fluxtrap solution (3.19)–(3.21), spanning the directions $z_{1,2}$.

The prime on the D3'-branes emphasizes that the four-dimensional gauge theory here is *not* to be identified with the gauge theory from which we constructed our three-dimensional theory with twisted masses. The two types of branes are entirely separate, and not to be included simultaneously in the same dynamical system. (Indeed, the primed branes live in type IIB string theory and the unprimed branes in type IIA, although this is not significant — a T-duality along a trivial direction such as $x_{1,2,3,9}$ transforms a IIA brane into a IIB brane and leaves the fluxtrap solution unaffected.)

The primed D3-branes, on which the Ω -deformed gauge theory lives, are Euclidean and space-filling in the $z_{1,2}$ directions. The unprimed D2-branes, on which the gauge theory of [14] lives, are transverse to the $z_{1,2}$ directions. The Ω -deformation appears in the former as a position-dependent gauge coupling, and in the latter as a twisted mass term.

The relationship between the twisted mass deformation of the unprimed D2-branes and the Ω -deformation of the primed D3-branes is that both arise from the same deformation of the closed string background in which each type of brane is embedded in its own way.

We expect that embedding the Ω -deformation in string theory via the fluxtrap solution clarifies and simplifies certain aspects of the Ω -deformation that otherwise appear somewhat technical and opaque. Let us take an easy example: one particularly salient feature of the Ω -deformation is its localization of instantons to the origin of the four spacetime dimensions of the gauge theory. Even from the gauge theory perspective, a moment's thought makes it clear that such a localization can only come about through a position-dependent gauge coupling with a maximum at the origin. A small instanton is pointlike, and cannot therefore couple to a metric or B -field; its only interaction with background fields is through the

gauge coupling. Since its action is inversely proportional to $g_4'^2$, the instanton's action is minimized where the gauge coupling attains its maximum value.

For a D3'-brane in the flux-trap solution, the four-dimensional gauge coupling is

$$g_{4\Omega}'^2 = \frac{g_{4(0)}'^2}{\sqrt{1 + \bar{\varepsilon}^2 r^2}}, \quad (3.37)$$

where

$$g_{4(0)}' \equiv (2\pi)^{1/2} (\alpha')^{1/4} g_3 \quad (3.38)$$

is the local gauge coupling of the four-dimensional gauge theory near the origin, and

$$\bar{\varepsilon} \equiv m. \quad (3.39)$$

We have introduced the primed branes only to clarify the relationship between the mass-deformed 3D theory and the Ω -deformed 4D Euclidean theory. Hereafter we shall leave the primed branes and not return to them in the present article. However we anticipate that the string theory embedding of the Ω -deformation may be useful for analyzing the Ω -deformed theory on the primed D3'-branes.

4 Open strings

In the following section, we describe the three-dimensional gauge theory that lives on the worldvolume of a single D2-brane extended in the directions x_1 and x_2 in the fluxtrap background (see table 1). After briefly discussing the kappa-symmetry-fixed action in the *static gauge*, we derive the expressions for the eight Killing spinors preserved by the D2-brane located at $\rho_1 = \rho_2 = 0$. We then derive the supersymmetry generators Q and find a *rotating brane solution* which saturates the BPS bound and preserves four supercharges. Finally we show how the low energy description of the D2-brane dynamics contains the expected form of a real mass term. The dynamics of N identical D2-branes in the same background can then be inferred up to commutator terms from the single-trace form of the D-brane action, since we are working at string tree level.

4.1 Action and fermionic symmetries

We would like to describe the dynamics of a D2-brane extended in the (x_1, x_2) directions. Since the background is symmetric under translations in the x_1 and x_2 directions, we can choose a consistent truncation of the theory where:

- the two-form on the D2-brane is vanishing,

$$B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta} = 0; \quad (4.1)$$

- the position of the D2-brane in the transverse direction only depends on time.
- The coordinates $x_{3,8,9}$ are constant and the gauge field is flat.

This truncation can be realized as the restriction to the subset of configuration space invariant under a set of discrete symmetries and translational invariances. Under this truncation, the relevant part of the bosonic action and the kinetic term for the fermions [26] are in our conventions (see appendix B):

$$S = -\mu_2 \int d^3\zeta \, e^{-\Phi} \sqrt{-\det g_{\alpha\beta}} \left(1 - \frac{e^{\Phi/4}}{2} \bar{\theta} (\mathbb{1} - \Gamma_{D2}) g^{\alpha\beta} \Gamma_\beta \partial_\alpha \theta \right) + O(\theta^4), \quad (4.2)$$

where $\mu_2 = (2\pi)^{-2}(\alpha')^{-3/2}$, $\theta = \theta_L + \theta_R$ is a Majorana spinor, $g_{\alpha\beta}$ is the pullback of the metric on the D2-brane,

$$g_{\alpha\beta} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \zeta^\alpha} \frac{\partial X^\nu}{\partial \zeta^\beta} \quad \alpha, \beta = 0, 1, 2, \quad (4.3)$$

ζ^α are the intrinsic coordinates on the worldvolume of the D2-brane and Γ_{D2} is given by [27],

$$\Gamma_{D2} = \frac{1}{\sqrt{-\det g_{ab}}} \frac{\epsilon^{\alpha\beta\gamma}}{3!} \Gamma_\alpha \Gamma_\beta \Gamma_\gamma, \quad (4.4)$$

where Γ_α is the pullback of the gamma matrices on the brane:²

$$\Gamma_\alpha = \frac{\partial X^\mu}{\partial \zeta^\alpha} e^m{}_\mu \Gamma_m. \quad (4.5)$$

Since ∂_{x^0} , ∂_{x^1} and ∂_{x^2} are Killing vectors for the bulk metric in equation (3.19), it is easy to fix reparametrization symmetry of the intrinsic coordinates ζ by choosing a static gauge for the embedding:

$$x^0 = \zeta^0, \quad x^1 = \zeta^1, \quad x^2 = \zeta^2. \quad (4.6)$$

The corresponding pullback of the metric is simply

$$g_{\alpha\beta} d\zeta^\alpha d\zeta^\beta = \hat{g}_{00} (d\zeta^0)^2 + (d\zeta^1)^2 + (d\zeta^2)^2, \quad (4.7)$$

where $\hat{g}_{00} = -1 + \partial_0 X^\rho \partial_0 X^\sigma g_{\rho\sigma}$ and X^ρ and X^σ run over the transverse coordinates,

$$X^\rho, X^\sigma = \{x_3, \rho_1, \phi_1, \rho_2, \phi_2, x_8, x_9\}. \quad (4.8)$$

The fact that the B field does not contribute can be understood by observing that ∂_{x^1} and ∂_{x^2} are Killing vectors and a double T-duality in these directions maps our D2-brane to a D0 brane.

The action is invariant under kappa-symmetry and under the susy transformations induced by the bulk Killing spinors ϵ . On the fermionic variable θ they act as follows:

$$\delta_\kappa \theta = (\mathbb{1} + \Gamma_{D2}) \kappa, \quad \delta_\epsilon^{\text{susy}} \theta = \epsilon, \quad (4.9)$$

where κ is a Majorana spinor. The transformation δ_κ can be used to impose a covariant gauge fixing,

$$\Gamma_{11} \theta = \theta \Rightarrow \theta_R = 0, \quad (4.10)$$

²The normalization factors are chosen such that Γ_{D2} squares to the identity: $(\Gamma_{D2})^2 = \mathbb{1}$.

in order to obtain the same number of bosonic and fermionic degrees of freedom. After gauge fixing, the kinetic term of the fermionic action takes the form

$$S_f = \frac{\mu_2}{2} \int d^3\zeta \, e^{-3\Phi/4} \sqrt{-\det g_{\alpha\beta}} \bar{\psi} g^{\alpha\beta} \Gamma_\beta \partial_\alpha \psi, \quad (4.11)$$

and using the form of the pullback in equation (4.7):

$$S_f = -\frac{\mu_2}{2} \int d^3\zeta \, \frac{e^{-3\Phi/4}}{\sqrt{-\det g_{\alpha\beta}}} \bar{\psi} \hat{\Gamma}_0 \dot{\psi}, \quad (4.12)$$

where ψ is the Majorana-Weyl spinor $\psi = \theta_L$ and $\hat{\Gamma}_0$ is the pullback of the gamma matrices in the direction ζ^0 :

$$\hat{\Gamma}_0 = \Gamma_\alpha|_{\alpha=0} = \frac{\partial X^\sigma}{\partial \zeta^0} e^m{}_\sigma \Gamma_m. \quad (4.13)$$

The action is invariant under the transformation

$$\delta_\epsilon \psi = (\delta_\epsilon^{\text{susy}} - \delta_\kappa|_{\kappa=\epsilon_R}) \psi = \epsilon_L - \Gamma_{D2} \epsilon_R, \quad (4.14)$$

which leaves θ_R invariant, consistently with the gauge choice $\theta_R = 0$.

Supersymmetries preserved by the static embedding. We say that a Killing spinor $\epsilon = \epsilon_L + \epsilon_R$ is *preserved* by the D2-brane if the associated transformation leaves ψ invariant:

$$\delta_\epsilon \psi = \epsilon_L - \Gamma_{D2} \epsilon_R = 0. \quad (4.15)$$

If we choose the *static embedding* in which $\hat{g}_{00} = 1$, the expression of Γ_{D2} is simply

$$\Gamma_{D2} = \Gamma_{012}. \quad (4.16)$$

In the previous section (equation (3.35)) we have found that the Killing spinors in the bulk are

$$\begin{cases} \epsilon_L = e^{-\Phi/8} (\mathbb{1} + \Gamma_{11}) \Pi_-^{\text{flux}} \exp \left[\frac{1}{2} \phi_1 \Gamma_{45} + \frac{1}{2} \phi_2 \Gamma_{67} \right] \epsilon_0, \\ \epsilon_R = e^{-\Phi/8} (\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_-^{\text{flux}} \exp \left[\frac{1}{2} \phi_1 \Gamma_{45} + \frac{1}{2} \phi_2 \Gamma_{67} \right] \epsilon_1. \end{cases} \quad (4.17)$$

Using the commutation relations of Π_-^{flux} with the components of Γ_u , we find that the conservation of supersymmetry requires

$$\Pi_-^{\text{flux}} \left(\epsilon_0 - \frac{1}{\Delta} \Gamma_{1208} \epsilon_1 \right) + \Pi_+^{\text{flux}} \left(\frac{m}{\Delta} \Gamma_{120} (\rho_1 \Gamma_5 - \rho_2 \Gamma_7) \right) \epsilon_1 = 0. \quad (4.18)$$

The two parts must vanish separately since the two projectors are orthogonal.³ We obtain the conditions

$$\rho_1 = \rho_2 = 0 \quad \text{and} \quad \epsilon_0 = \Gamma_{1208} \epsilon_1. \quad (4.19)$$

³The contribution of the directions x_1 and x_2 can be factored out, which is consistent with the fact that our configuration is T-dual to a D0-brane.

The first condition fixes the transverse position of the D2-brane, the second one breaks half of the 16 supersymmetries, resulting in a total of 8 *preserved supercharges*:

$$\begin{cases} \epsilon_L = e^{-\Phi/8} \Gamma_{1208} (\mathbb{1} + \Gamma_{11}) \Pi_-^{\text{flux}} \exp \left[\frac{1}{2} (\phi_1 + \phi_2) \Gamma_{67} \right] \epsilon_1, \\ \epsilon_R = e^{-\Phi/8} (\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_-^{\text{flux}} \exp \left[\frac{1}{2} (\phi_1 + \phi_2) \Gamma_{67} \right] \epsilon_1. \end{cases} \quad (4.20)$$

It is convenient to introduce the Majorana-Weyl spinor $\tilde{\epsilon}$,

$$\tilde{\epsilon} = e^{-\Phi/8} (1 + \Gamma_{11}) \Pi_-^{\text{flux}} \exp \left[\frac{1}{2} (\phi_1 + \phi_2) \Gamma_{67} \right] \epsilon_1, \quad (4.21)$$

and write the spinors conserved by the static embedding of the D2-brane as

$$\epsilon_L = \Gamma_{1208} \tilde{\epsilon}, \quad \epsilon_R = \Gamma_u \tilde{\epsilon}. \quad (4.22)$$

The spinor $\tilde{\epsilon}$ is normalized such that it can be written as the sum of 8 orthogonal components:⁴

$$\tilde{\epsilon} = \sum_{A=1}^8 \tilde{\epsilon}^A, \quad (\tilde{\epsilon}^A)^T \tilde{\epsilon}^B = \delta^{AB}, \quad A, B = 1, \dots, 8, \quad (4.23)$$

where ϵ^T is the transposed spinor.

4.2 BPS bound for the DBI action

We have seen that the static embedding of the D2-brane into the fluxtrap breaks half of the 16 supersymmetries of the bulk. Now we would like to describe a different BPS embedding that preserves only 1/4 of the bulk supersymmetries.

Hamiltonian formalism. In order better to understand the BPS condition we pass to the Hamiltonian formalism. The conjugate momentum to the bosonic variable X^ρ is given by

$$P_\rho \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\rho} = \mu_2 e^{-\Phi} \frac{\dot{X}^\sigma g_{\rho\sigma}}{\sqrt{-\det g_{\alpha\beta}}}, \quad (4.24)$$

where \dot{X} is the derivative with respect to ζ^0 :

$$\dot{X}^\rho \equiv \frac{\partial X^\rho}{\partial \zeta^0}. \quad (4.25)$$

The Hamiltonian is therefore given by

$$\mathcal{H} = P_\rho \dot{X}^\rho - \mathcal{L} = \mu_2 e^{-\Phi} \frac{\dot{X}^\rho \dot{X}^\sigma g_{\rho\sigma}}{\sqrt{-\det g_{\alpha\beta}}} + \mu_2 e^{-\Phi} \sqrt{-\det g_{\alpha\beta}} = \mu_2 \frac{e^{-\Phi}}{\sqrt{-\det g_{\alpha\beta}}}, \quad (4.26)$$

and in particular the energy of the static embedding configuration ($\hat{g}_{00} = -1$, $\rho_1 = \rho_2 = 0$) is

$$\mathcal{H}^{\text{static}} = \frac{\mu_2}{g_3^2 \sqrt{\alpha'}} = \frac{1}{4\pi^2 g_3^2 (\alpha')^2}. \quad (4.27)$$

⁴This is possible because $(\tilde{\epsilon})^T \tilde{\epsilon}$ does not depend on ϕ_1 or ϕ_2 .

The last quantity we want to derive from the bosonic action is the *angular momentum* for a rotation in the direction of the Killing vector $V^\rho \partial_\rho = \partial_{\phi_1} - \partial_{\phi_2}$:

$$\mathcal{J} = V^\rho P_\rho = \mu_2 e^{-\Phi} \frac{V^\rho \dot{X}^\sigma g_{\rho\sigma}}{\sqrt{-\det g_{\alpha\beta}}} = \mu_2 \frac{e^{-\Phi}}{\sqrt{-\det g_{\alpha\beta}}} \frac{\rho_1^2 \dot{\phi}_1 - \rho_2^2 \dot{\phi}_2}{1 + m^2(\rho_1^2 + \rho_2^2)}. \quad (4.28)$$

Clearly, the angular momentum is vanishing for the static configuration:

$$\mathcal{J}^{\text{static}} = 0. \quad (4.29)$$

In order to canonically quantize the *fermionic* part of the action we introduce the conjugate momentum

$$\Pi_a \equiv \frac{\delta \mathcal{L}}{\delta \dot{\psi}^a} = i \frac{\mu_2 e^{-3\Phi/4}}{2\sqrt{-\det g_{\alpha\beta}}} \psi^b (\Gamma_0 \hat{\Gamma}_0)_{ba}, \quad a, b = 1, \dots, 16, \quad (4.30)$$

which by definition satisfies the canonical anticommutation relation

$$\{\Pi_a, \psi^b\} \equiv i \delta_a^b, \quad a, b = 1, \dots, 16 \quad (4.31)$$

whence

$$\{\Pi_a, \Pi_b\} = -\frac{\mu_2}{2} \frac{e^{-3\Phi/4}}{\sqrt{-\det g_{\alpha\beta}}} (\Gamma_0 \hat{\Gamma}_0)_{ab}. \quad (4.32)$$

Using the conjugate momentum one can directly write down the supercharges that generate the supersymmetry transformations in equation (4.14):

$$Q_\epsilon = i \Pi_a \delta_\epsilon \psi^a + \mathcal{O}((\text{fermions})^3), \quad (4.33)$$

which satisfy the anticommutation relation

$$\{Q^A, Q^B\} = \delta_{\epsilon^A} \psi^a \{\Pi_a, \Pi_b\} \delta_{\epsilon^B} \psi^b + \mathcal{O}((\text{fermions})^2). \quad (4.34)$$

At this point we have all the ingredients to calculate the explicit expression for the anticommutator, up to fermion bilinear terms. Since we want to consider compare with the energy of the static embedding, we plug in the expressions for the preserved Killing spinors in equation (4.21):

$$\begin{aligned} \{Q^A, Q^B\} &= -\frac{\mu_2 e^{-3\Phi/4}}{2\sqrt{-\det g_{\alpha\beta}}} (\epsilon_L^A - \Gamma_{D2} \epsilon_R^A)^a (\Gamma_0 \hat{\Gamma}_0)_{ab} (\epsilon_L^B - \Gamma_{D2} \epsilon_R^B)^b \\ &= -\frac{\mu_2 e^{-3\Phi/4}}{2\sqrt{-\det g_{\alpha\beta}}} \left[\Gamma_{1208} \tilde{\epsilon}^A - \frac{\hat{\Gamma}_{012} \Gamma_u}{\sqrt{-\det g_{\alpha\beta}}} \tilde{\epsilon}^A \right]^a (\Gamma_0 \hat{\Gamma}_0)_{ab} \left[\Gamma_{1208} \tilde{\epsilon}^B - \frac{\hat{\Gamma}_{012} \Gamma_u}{\sqrt{-\det g_{\alpha\beta}}} \tilde{\epsilon}^B \right]^b, \end{aligned} \quad (4.35)$$

where we have dropped terms on the right-hand side containing two or more fermions, leaving only the purely bosonic terms.

After a straightforward calculation we obtain a simple expression for the anticommutator:

$$\{Q^A, Q^B\} = \left(\frac{\mu_2 e^{-\Phi}}{\sqrt{-\det g_{\alpha\beta}}} - \frac{\mu_2}{g_3^2 \sqrt{\alpha'}} \right) \delta^{AB} - \frac{\mu_2 e^{-\Phi}}{2\sqrt{-\det g_{\alpha\beta}}} \left\{ \Gamma_8 - \frac{1}{\Delta} \Gamma_u, \hat{\Gamma}_0 \right\} (\tilde{\epsilon}^A)^T \Gamma_{08} \tilde{\epsilon}^B. \quad (4.36)$$

Using the explicit expression for Γ_u in equation (3.36), and the pullback $\hat{\Gamma}_0$ in equation (4.13), we find that:

$$\left\{ \Gamma_8 - \frac{1}{\Delta} \Gamma_u, \hat{\Gamma}_0 \right\} = -\frac{2m}{\Delta^2} (\rho_1^2 \dot{\phi}_1 - \rho_2^2 \dot{\phi}_2), \quad (4.37)$$

which allows us to compare the anticommutator with the expressions for the Hamiltonian and angular momentum that we have found from the bosonic action in equations (4.26) and (4.28). The final result is:

$$\{Q^A, Q^B\} = (\mathcal{H} - \mathcal{H}^{\text{static}}) \delta^{AB} + m \mathcal{J} (\tilde{\epsilon}^A)^T \Gamma_{08} \tilde{\epsilon}^B. \quad (4.38)$$

The anticommutator vanishes for the static embedding since we are discussing the supercharges preserved by this configuration. This is also the case for the BPS-states that we construct in the following.

Rotating branes. The expression of the angular momentum suggests the following ansatz for a rotating D2-brane:

$$\phi_1 = \omega \zeta^0, \quad \phi_2 = -\omega \zeta^0, \quad (4.39)$$

where ω is constant and all the other transverse coordinates have a fixed value independent of ζ^0 . The non-trivial pullbacks of metric and gamma matrices are given by

$$\hat{g}_{00} = -\frac{1 + (\rho_1^2 + \rho_2^2)(m^2 - \omega^2)}{1 + m^2(\rho_1^2 + \rho_2^2)}, \quad (4.40)$$

$$\hat{\Gamma}_0 = \Gamma_0 - \frac{\omega}{m} \Gamma_8 + \frac{\omega}{m\Delta} \Gamma_u. \quad (4.41)$$

The bosonic part of the Lagrangian is then given by

$$\mathcal{L}_b = -\frac{1}{4\pi^2 g_3^2 (\alpha')^2} \sqrt{1 + (\rho_1^2 + \rho_2^2)(m^2 - \omega^2)}. \quad (4.42)$$

BPS states are extrema of the action. The non-trivial BPS equations are:

$$\rho_1(m^2 - \omega^2) = \rho_2(m^2 - \omega^2) = 0. \quad (4.43)$$

These are satisfied either if $\rho_1 = \rho_2 = 0$, which is the static embedding, or if

$$\omega = \pm m. \quad (4.44)$$

This is the *rotating D2-brane embedding*. Note that we have *not* restricted to small fluctuations about the static brane: even if we are not in a linear approximation, the frequency is independent of the amplitude and no conditions are imposed on the position of the D2-brane in $\rho_{1,2}$ or the other transverse directions.

By substituting the condition in equation (4.44) into the general expressions for energy and angular momentum we find

$$\mathcal{H}_{\pm}^{\text{rot}} = \frac{1}{4\pi^2 g_3^2 (\alpha')^2} (1 + m^2(\rho_1^2 + \rho_2^2)) \quad \mathcal{J}_{\pm}^{\text{rot}} = \pm \frac{m}{4\pi^2 g_3^2 (\alpha')^2} (\rho_1^2 + \rho_2^2), \quad (4.45)$$

where the \pm refers to the two branches of the solution $\omega = \pm m$. After subtracting the energy of the static configuration we obtain the BPS condition

$$\frac{\mathcal{H}_{\pm}^{\text{rot}} - \mathcal{H}^{\text{static}}}{\mathcal{J}_{\pm}^{\text{rot}}} = \pm m. \quad (4.46)$$

In order to verify that this is indeed a bound, we have to turn to the fermionic action. A Killing spinor ϵ is conserved iff:

$$\epsilon_L = \Gamma_{D2} \epsilon_R. \quad (4.47)$$

Using the expression for the bulk Killing spinors in equation (3.35), and the pullback of the Γ matrices on the rotating brane ansatz, we obtain the equation

$$\Pi_{-}^{\text{flux}}(\epsilon_0 - \Gamma_{1208} \epsilon_1) + \Pi_{+}^{\text{flux}}(m \Gamma_{120}(\mathbb{1} \pm \Gamma_{08})(\rho_1 \Gamma_5 - \rho_2 \Gamma_7)) \epsilon_1 = 0. \quad (4.48)$$

The two parts must vanish separately since the two projectors are orthogonal. This implies

$$\epsilon_0 = \Gamma_{1208} \epsilon_1, \quad \text{and} \quad (\mathbb{1} \pm \Gamma_{08}) \epsilon_1 = 0. \quad (4.49)$$

The two conditions together preserve a total of 4 *supercharges*. Explicitly:

$$\begin{cases} \epsilon_L = e^{-\Phi/8} \Gamma_{1208}(\mathbb{1} + \Gamma_{11}) \Pi_{-}^{\text{flux}} \exp \left[\frac{1}{2}(\phi_1 + \phi_2) \Gamma_{67} \right] (\mathbb{1} \mp \Gamma_{08}) \epsilon_2, \\ \epsilon_R = e^{-\Phi/8} (\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_{-}^{\text{flux}} \exp \left[\frac{1}{2}(\phi_1 + \phi_2) \Gamma_{67} \right] (\mathbb{1} \mp \Gamma_{08}) \epsilon_2, \end{cases} \quad (4.50)$$

where ϵ_2 is a constant Majorana spinor. This is precisely the same form that we have found for the static embedding in equation (4.21), with an extra projector $\frac{1}{2}(\mathbb{1} \mp \Gamma_{08})$. To be precise, a given Killing spinor $\epsilon = (\Gamma_{1208} + \Gamma_u) \tilde{\epsilon}$ which is preserved in the static embedding is also preserved by the rotating brane if

$$\Gamma_{08} \tilde{\epsilon}_{\text{pres}} = \mp \tilde{\epsilon}_{\text{pres}}, \quad (4.51)$$

and not preserved otherwise: $\Gamma_{08} \tilde{\epsilon}_{\text{pres}} = \pm \tilde{\epsilon}_{\text{pres}}$.

Using the expression for the anticommutator of the supercharges in equation (4.38) we find that the supercharges corresponding to Killing spinors preserved and not preserved by the rotating D2-brane satisfy

$$\{Q_{\text{pres}}^A, Q_{\text{pres}}^B\} = (\mathcal{H} - \mathcal{H}^{\text{static}} \mp m\mathcal{J}) \delta^{AB} \quad (4.52)$$

$$\{Q_{\text{pres}}^A, Q_{\text{pres}}^B\} = 0 \quad (4.53)$$

$$\{Q_{\text{pres}}^A, Q_{\text{pres}}^B\} = (\mathcal{H} - \mathcal{H}^{\text{static}} \pm m\mathcal{J}) \delta^{AB}. \quad (4.54)$$

This implies that

$$(\mathcal{H} - \mathcal{H}^{\text{static}}) + m\mathcal{J} \geq 0, \quad (\mathcal{H} - \mathcal{H}^{\text{static}}) - m\mathcal{J} \geq 0, \quad (4.55)$$

where one of the two conditions is trivial depending on the sign of $m\mathcal{J}$. The bound is saturated if $\omega = \pm m$:

$$\mathcal{H}_{\pm}^{\text{rot}} - \mathcal{H}^{\text{static}} \mp m\mathcal{J}_{\pm}^{\text{rot}} = 0. \quad (4.56)$$

4.3 Low energy effective gauge theory

In this section we derive the low energy action describing the dynamics of the D2-brane in the fluxtrap background. The parameter m that we have introduced in the identifications in section 3 will appear explicitly as a real mass term for the fields describing the motion of the D2-brane in the directions $x_4 \dots x_7$.

Let us start with the kappa-symmetry-fixed DBI action at second order in the fermions. In order to get the canonical normalization for the fermionic term it is convenient to pass to the democratic formulation (see [26]) in which the action is written as:

$$S = -\mu_2 \int d^3\zeta \, e^{-\Phi} \sqrt{-\det(g_{\alpha\beta} + B_{\alpha\beta})} \left[1 - \frac{1}{2} \bar{\psi} ((g + B)^{\alpha\beta} \Gamma_\beta D_\alpha + \Delta^{(1)}) \psi \right], \quad (4.57)$$

where

$$D_\alpha = \partial_\alpha X^\mu \left(\nabla_\mu + \frac{1}{8} H_{\mu mn} \Gamma^{mn} \right), \quad (4.58)$$

$$\Delta^{(1)} = \frac{1}{2} \Gamma^m \partial_m \Phi - \frac{1}{24} H_{mnp} \Gamma^{mnp}. \quad (4.59)$$

Since we are only interested in the low energy dynamics we expand all the terms at the respective leading order. The bulk fields are

$$g_{\mu\nu} dX^\mu dX^\nu = d\vec{x}_{0\dots 9}^2 + \mathcal{O}(X^4), \quad (4.60)$$

$$H_{\mu\nu\rho} dX^\mu \wedge dX^\nu \wedge dX^\rho = 2m(\rho_1 d\rho_1 \wedge d\phi_1 - \rho_2 d\rho_2 \wedge d\phi_2) \wedge dx_8 + \mathcal{O}(X^5), \quad (4.61)$$

$$e^{-\Phi} = \frac{1}{g_3^2 \sqrt{\alpha'}} \left(1 + \frac{m^2}{2} (\rho_1^2 + \rho_2^2) \right) + \mathcal{O}(X^4). \quad (4.62)$$

In our consistent truncation the dynamics only depends on ζ^0 , hence

$$\sqrt{-\det(g_{\alpha\beta} + B_{\alpha\beta})} = 1 - \frac{1}{2} \dot{X}^\sigma \dot{X}_\sigma + \mathcal{O}(X^4); \quad (4.63)$$

moreover the only relevant covariant derivative, at leading order reduces to

$$\nabla_0 = \partial_0. \quad (4.64)$$

A straightforward calculation shows that

$$g^{\alpha\beta} \Gamma_\beta D_\alpha = -\Gamma_0 \partial_0 + \mathcal{O}(X^2), \quad (4.65)$$

$$\Delta^{(1)} = -\frac{m}{2} (\Gamma_{45} - \Gamma_{67}) \Gamma_8 + \mathcal{O}(X^2). \quad (4.66)$$

And substituting the expansion into the action we obtain:

$$S = -\frac{1}{8\pi^2 g_3^2 (\alpha')^2} \int d^3\zeta \left[-\dot{X}^\sigma \dot{X}_\sigma + m^2 (\rho_1^2 + \rho_2^2) + \bar{\psi} \Gamma_0 \dot{\psi} + \frac{m}{2} \bar{\psi} (\Gamma_{45} - \Gamma_{67}) \Gamma_8 \psi \right] + \dots \quad (4.67)$$

The result is more transparent in rectilinear coordinates,

$$z_1 = x_4 + i x_5 = \rho_1 e^{i\phi_1}, \quad z_2 = x_6 + i x_7 = \rho_2 e^{i\phi_2}, \quad (4.68)$$

in which the relevant part of the action becomes

$$S = \frac{1}{8\pi^2 g_3^2 (\alpha')^2} \int d^3\zeta [\dot{z}_1 \dot{\bar{z}}_1 + \dot{z}_2 \dot{\bar{z}}_2 - m^2 (z_1 \bar{z}_1 + z_2 \bar{z}_2) - \bar{\psi} \Gamma_0 \dot{\psi} - i m \bar{\psi} (\Pi_-^{z_1} - \Pi_-^{z_2}) \Gamma_8 \psi], \quad (4.69)$$

where the projectors Π^{z_1} and Π^{z_2} are defined by

$$\Pi_-^{z_1} = \frac{1}{2}(\mathbb{1} - i \Gamma_{45}), \quad \text{and} \quad \Pi_-^{z_2} = \frac{1}{2}(\mathbb{1} - i \Gamma_{67}). \quad (4.70)$$

With gauge fields included, the quadratic action contains the additional term

$$S_{\text{gauge}} = -\frac{1}{4g_3^2} \int d^3\zeta F_{\alpha\beta} F^{\alpha\beta}. \quad (4.71)$$

For N identical D2-branes in the fluxtrap geometry, the $U(1)$ gauge connection is promoted to a $U(N)$ gauge connection, the scalar fields and fermions are promoted to matrices in the adjoint representation of $U(N)$, and the action is replaced by a single-trace version of itself, which is uniquely determined up to commutator terms. Since commutator terms involve at least three fields and we are only working to quadratic order, such terms do not affect the properties of the twisted mass in the gauge theory.

5 Relation to the gauge/Bethe correspondence

5.1 Brane configuration without twisted masses

We will now make some brief comments on the connection of our work to the gauge/Bethe correspondence of [14, 15]. The string theory embedding of the gauge theories there involve D2-branes suspended between NS-fivebranes, with D4-branes added to the background, as shown in table 2. This brane configuration is a subset of the one described in [16], with the NS5'-brane removed. After the removal of the orthogonal NS5' and the exchange $x^2 \leftrightarrow x^6$ and $x^{3,4,5} \leftrightarrow x^{7,8,9}$, the D2, D4 and NS5 branes of [16] becomes the branes we consider here. The D4-branes here are located at an arbitrary position between the two NS fivebranes.

Prior to the addition of the twisted mass deformation, the configuration in table 2 preserves $(4, 4)$ supersymmetry in 1+1 dimensions [28], unlike the configuration of [16] which preserves only $(2, 2)$, due to the presence of the orthogonal NS5'-brane that was used to give an infinite mass in the superpotential for the adjoint chiral multiplet degrees of freedom, following [17, 29]. In the configuration of [16], the presence of the NS5'-brane as one of the two boundaries for the D2 leaves only two massless adjoint scalars, enough to fill out a $(2, 2)$ vector multiplet. In our current configuration, by contrast, prior to the fluxtrap deformation, there are four massless scalars in the adjoint, enough to fill out a vector multiplet of $(4, 4)$ supersymmetry. The A_2 component of the D2-brane gauge field obeys Dirichlet boundary conditions at the NS-fivebrane [30], so there are no additional massless bosonic 2-dimensional degrees of freedom in the adjoint, beyond those of the $(4, 4)$ vector multiplet, consisting of the 2D gauge connection, and motions in the $x_{6,7,8,9}$ directions.

The two NS-fivebranes are separated in the x_2 direction by a distance δ_2 , and there is a set of L D4-branes touching the N D2s. For any such configuration, there is a set of

massless hypermultiplets in the fundamental and in the antifundamental representation of $SU(N)$, consisting of open strings connecting the D2-branes and the D4-branes [17, 28]. The $(4, 4)$ supersymmetry forces an interaction (2.1) which in $\mathcal{N} = (2, 2)$ language is a cubic superpotential involving the fundamental chiral multiplets, the antifundamental chiral multiplets, and the adjoint chiral multiplet:

$$W = \tilde{Q} \phi Q. \quad (5.1)$$

5.2 Fluxtrap deformation of the brane configuration

Thus the configuration we have described reproduces exactly the gauge theory of [14, 15], with precisely one exception: the sole missing ingredient is the twisted mass deformation for chiral multiplets in the fundamental and adjoint representations. These mass terms are present in [14, 15], playing a key role in the infrared dynamics. So we now deform our brane configuration by the fluxtrap deformation of the closed string background, which adds a twisted mass deformation for the adjoint and chiral multiplets of the D2-brane gauge theory, breaking SUSY to $\mathcal{N} = (2, 2)$.

The twisted mass for the adjoint in 1+1 dimensions simply descends from a local term in 2+1 dimensions, the real mass for the adjoint chiral multiplet. To see that the twisted mass term for the fundamental and antifundamental chiral multiplets must be present, one need only verify that the deformation preserves $\mathcal{N} = (2, 2)$ supersymmetry, and note that the superpotential must be neutral under the symmetry operator Z defining the central charge. Since the adjoint chiral multiplet is not neutral, the fundamental and antifundamental chiral multiplets must be non-neutral as well. Together they must cancel the Z -charge of the adjoint chiral multiplet, and so each must have a Z -charge equal to $-\frac{1}{2}$ the Z -charge of the adjoint chiral multiplet. Thus the fundamental matter is forced to have a mass equal to exactly half that of the adjoint matter [14].

It remains to demonstrate that the fluxtrap deformation can be combined consistently with the presence of these other branes, the NS5s and D4s. Once this is shown, our string realization of the gauge theories of [14, 15] will be complete.

One might have questioned whether the ingredient added here — the twisted mass deformation for the chiral multiplets *via* the fluxtrap solution — can in fact be combined consistently with the other ingredients of [16], the NS5-branes on which the D2's terminate, and the D4-branes providing the matter in the fundamental representation. The answer to that question is affirmative. The D4-branes can be added unproblematically to the solutions described in this article; the string coupling $\exp[\Phi]$ is bounded above by an arbitrarily small value in all the solutions we consider, the backreaction of the D4-branes on the rest of the geometry can be made arbitrarily small.

At first sight, combining the NS5's with the fluxtrap deformation may appear to be a trickier issue. Both the NS5's and the fluxtrap are solutions of nonlinear equations of motion of the massless modes of closed string theory. There is no principle that guarantees that such solutions need superpose with one another. However we show by explicit construction, in appendix A, that the solutions do in fact combine; there is a combined fivebrane-fluxtrap solution that reduces to the pure fluxtrap when the fivebranes are moved to infinity, and

reduces to a solution of arbitrarily positioned parallel NS-fivebranes when the fluxtrap deformation is turned off.

In appendix A we have written the full solution for the fluxtrap deformation of the geometry of parallel (but not necessarily coincident) NS-fivebranes, which break the supersymmetry of the fluxbranes again by half. This demonstrates the consistency of combining the NS-fivebranes with the fluxbrane deformation, establishing our construction as an exact string solution reducing to the Nekrasov-Shatashvili gauge/Bethe system at low energies.

In the appendix we have for completeness also included solutions of the Dirac-Born-Infeld action of the D2-branes in the fivebrane-fluxbrane background, which shows the persistence of the exact BPS formula for rotating trajectories of the adjoint fields with twisted masses, in the presence of the NS-fivebranes. We construct static and rotating BPS solutions of the D2-brane DBI action in the NS5-fluxbrane background, which preserve 4 and 2 supercharges, respectively, and again satisfy the relation $E - E_0 = |mJ|$. These static and rotating solutions are classically BPS-saturated embeddings of D2-branes into the fivebrane-fluxtrap geometry. The embeddings exactly saturating the BPS bound $E - E_0 = |mJ|$, can be thought of as representing a condensate of BPS-saturated particles in $(2+1)$ -dimensional gauge theory on the interval.

5.3 Quantum nonabelian symmetry from the brane construction

The existence of this brane construction has the potential to teach us many interesting things about the remarkable relationships among two-dimensional gauge theories. To take one immediate example, we will examine the emergence of nonabelian global symmetries relating the Nekrasov-Shatashvili gauge theories with different ranks $N = \# \text{D2}$. Take the case of two NS-fivebranes, parallel and separated by a distance δ_2 , with N D2-branes suspended between them and a twisted mass parameter m characterizing the strength of the fluxtrap, which gives the adjoint chiral multiplets a twisted mass m and the fundamental and antifundamental chiral multiplets a twisted mass $\frac{m}{2}$.

As the separation δ_2 between the fivebranes is taken to zero with g_3 held fixed, the two-dimensional gauge coupling becomes infinitely strong and quantum effects dominate the system; there is a rich set of quantum vacuum states depending on N and L , which have been shown [14, 15] to be in one-to-one correspondence with the *full* Hilbert space of the N -magnon sector of a spin chain with L spin sites. The set of vacuum states unexpectedly arranges itself to respect a global $SU(2)$ symmetry [31] organizing the states into $SU(2)$ representations with irreducible components of dimension at most $L + 1$. As $\delta_2 \rightarrow 0$ this symmetry becomes an exact symmetry of the supersymmetric vacuum states of the gauge theories.

From the spin chain point of view, the $SU(2)$ is immediately apparent: each spin variable transforms in a two-dimensional representation and the full state of the system is trivially a tensor product of those. From the gauge theory point of view, on the other hand, the quantum $SU(2)$ has been mysterious. Particularly striking is the nature of the action of the $SU(2)$ generators on the quantum number N , the number of D2-branes. In the correspondence of [14, 15], the number N is $\frac{L}{2}$ plus the Cartan generator of the $SU(2)$.

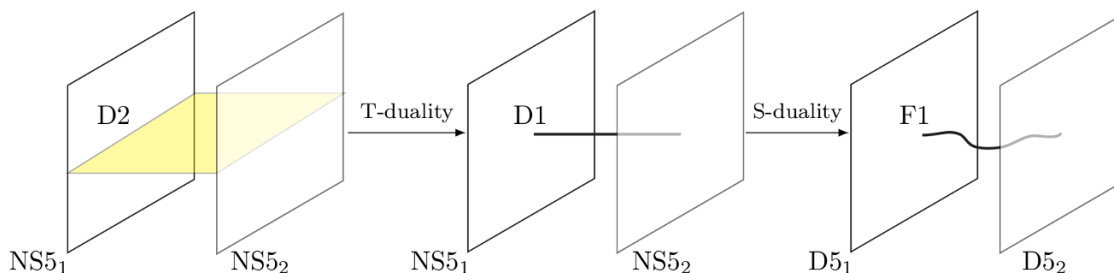


Figure 2. Realization of the $SU(2)$ symmetry via dualities in string theory.

The raising and lowering operators of the $SU(2)$ therefore raise and lower the rank of the $(1+1)$ -dimensional gauge theory itself. This peculiar $SU(2)$ is clearly a powerful and unfamiliar type of symmetry — it acts not on the SUSY vacuum sector of a particular gauge theory, but on the set of SUSY vacua of an *ensemble* of gauge theories of different ranks N , mapping vacua of theories of different rank to one another.

The string embedding sheds some light on the origin of this mysterious $SU(2)$ symmetry. Upon compactification of the spatial direction of the $(1+1)$ -dimensional gauge theory, the brane configuration becomes equivalent under T-duality to a system of D1-branes suspended between NS-fivebranes in type IIB string theory. It is well known that this configuration supports a dynamical $SU(2)$ gauge symmetry propagating on the system of NS-fivebranes that is broken when the NS5s are separated and restored when the NS5s become coincident. This gauge symmetry has the property that the number of D1-branes suspended between NS5s does indeed play the role of the Cartan generator, with the raising and lowering operators literally creating and destroying D1-branes. This seemingly exotic action of the gauge symmetry can be understood most simply through the S-duality of type IIB string theory, under which the NS5-branes become D5-branes and the D1-branes suspended between them become open fundamental strings, transforming in the adjoint of the $SU(2)$ (see figure 2).

Though a gauge symmetry from the point of view of the fivebranes themselves, the $SU(2)$ appears to the D2-branes as a global symmetry, because the gauge bosons of the $SU(2)$ propagate on the fivebranes and not on the two-branes. This $SU(2)$ also becomes unbroken when the NS5s become coincident. It appears to pass all the most obvious tests to play the role of the $SU(2)$ symmetry organizing the ground states of the supersymmetric gauge theories of [14, 15]. Indeed, the string-theoretic embedding opens an even more surprising possibility: if the $SU(2)$ is an exact dynamical symmetry of the system, then it ought to act on non-vacuum as well as vacuum states. This suggests an even more remarkable set of relationships among two-dimensional gauge theories and their quantum states than that contemplated in [14, 15].

There are several facts that may somewhat temper one's hopes in this regard. Among them:

- It is not clear that there is a decoupling limit in which the dynamics of non-vacuum states of the two-branes decouple from the degrees of freedom of the bulk and from

the fivebranes. The $SU(2)$ may be an exact symmetry of the system of string theory on the fivebranes, but that symmetry may not act on states that can be understood as excitations of the D2-branes alone; it may be necessary to add states in the bulk or attached to the fivebranes, in order to fill out complete $SU(2)$ representations.

- The $SU(2)$ may be destroyed at the quantum level by the fluxtrap deformation. The fluxtrap deformation does not carry $SU(2)$ quantum numbers and cannot break the gauge symmetry explicitly; nor can it trigger spontaneous breaking in the usual sense, as the solution with coincident fivebranes is still a valid supersymmetric solution even after the fluxtrap deformation (as shown in appendix A). Rather, the danger is that the fluxtrap deformation may trigger quantum dynamics on the fivebranes that would give rise to confinement of the $SU(2)$. The undeformed six-dimensional theory certainly does *not* confine, and indeed confinement would be impossible in a fully Poincaré-invariant six-dimensional gauge theory. However the fluxtrap breaks some of the Poincaré symmetry of the six-dimensional theory: in order to preserve supersymmetry and allow the D2-branes to be suspended between them while preserving supersymmetry, the NS5s must be oriented in the $x^{0,1,6,7,8,9}$ directions. The fluxtrap deformation therefore breaks the Poincaré symmetry on the NS-fivebranes down to $SO(2,1)_{019} \times SO(2)_{67}$, times translational symmetries in the 0, 1, 2, and 8 directions. The supersymmetry of the fivebrane theory is also partially broken by the fluxtrap down to eight supercharges, which is a low enough amount to allow a gauge coupling running to strong at long distances. This peculiar, Lorentz-breaking six-dimensional gauge theory is sufficiently unfamiliar that we cannot rule out the possibility that the deformed theory may confine in the infrared. It is a logical possibility that strong coupling dynamics deforms the moduli space such that there is no point at which the $SU(2)$ is restored, perhaps similarly to [32]. Our attitude in the present discussion is to take the non-confinement of the $SU(2)$ at the quantum level as a working hypothesis, but by no means a proven fact.
- Even if our string embedding explains the emergence of an $SU(2)$ after compactification of x_1 , we still have not explained the apparent existence of an $SU(2)$ prior to compactification, which still appears to be valid. The uncompactified type IIA NS-fivebrane does not support a gauge symmetry in the usual sense.

The points above make clear that the string embedding offers a plausible framework for explaining the $SU(2)$ symmetry but not a full explanation in the absence of further refinement.

All comments above apply, on the gauge theory side, the spin-chain side, and the string theory side, to the case of k fivebranes, where the gauge theory becomes a more complicated quiver [14, 15], the brane construction has k parallel fivebranes with two-branes suspended between them, and the mysterious quantum symmetry of the gauge/spin-chain is enhanced to $SU(k)$, as reproduced by the dynamics of the k fivebranes.

6 Conclusions

In this paper we have constructed a simple solution of type II string theory, the fluxtrap solution, realized as a T-dual of a free quotient of flat space preserving half the supersymmetry of the flat covering space. This fluxtrap can be viewed as a lift of the Ω -deformation to string theory. This background unifies the Ω -deformed 4-dimensional gauge theories of [11] and the Lorentz-invariant 3-dimensional gauge theories with twisted masses [14, 15]; each gauge theory is realized on a type of D-brane in the fluxtrap background, with the former oriented longitudinal to the $z_{1,2}$ directions of the fluxtrap geometry, and the latter transverse to $z_{1,2}$. The coupling to the curved metric, B -field and dilaton gradient of the closed string background provide simple ways of understanding the deformed dynamics of each type of theory. In particular we saw explicitly that the same deformation of the closed string background that produces the twisted masses on a set of D2-branes transverse to the fluxtrap geometry, can also produce the Omega deformation of the gauge theory on a set of Euclidean D3'-branes longitudinal the fluxtrap geometry.

We have constructed the BPS-saturated classical solutions of the D2-branes rotating in the fluxtrap background. These states are half-supersymmetric states of the branes in the fluxtrap background, preserving 4 of the 8 dynamical supercharges preserved by the static brane and satisfying the exact relation $E - E_0 = |m J|$ where m is the twisted/real mass parameter and J is the angular momentum generator that rotates z_1 and z_2 with opposite phases. The translationally invariant classical solutions are simply Bose-Einstein condensates of BPS oscillators that have zero momentum in the $x_{1,2}$ direction. We have further shown in the appendix that these classical solutions have analogous BPS solutions when NS-fivebranes are added to the background, with the D2s suspended between the NS-fivebranes, and either rotating or not.

We have discussed (without much detail) the addition of D4-branes together with NS-fivebranes to the solution, in order to make contact with the gauge/Bethe correspondence of Nekrasov and Shatashvili. By doing this, we have found a partial explanation of the mysterious quantum mechanical $SU(k)$ symmetry that acts on the quantum ground states of the system when the two-dimensional gauge coupling goes to infinity. Certain gaps, however, remain in this explanation.

The emergence of (not necessarily normalizable) classical BPS states, consisting of excitations of the z_1 and \bar{z}_2 degrees of freedom, and their superpartners, is intriguing. The BPS formula for these states suggests that their energies do not become infinite even when the two-dimensional gauge coupling goes to infinity. It would appear that the quantum vacuum states of the gauge theories are augmented by a set of non-vacuum BPS states that survive and should organize themselves into $SU(k)$ representations (in the presence of k fivebranes) in the strong coupling limit. It would be interesting indeed to understand how the spin chain picture could be enlarged to understand these states.

It may seem puzzling why such a simple deformation as a dimensional reduction on a twisted circle should need to be understood in terms of a complicated-looking supergravity solution involving curved metrics, B -fields and dilaton gradients. And yet already we have seen that some of these dynamical elements have allowed us to see aspects of the

Ω -deformation of 4D gauge theory, and twisted mass deformation of 3D and 2D gauge theory, with a certain clarity. Universal principles counsel that it is always better to use a description where irrelevant heavy degrees of freedom have been removed from the system. The irrelevant degrees of freedom, which were the momentum modes on the x^8 circle, have been transformed in the T-dual picture into infinitely heavy winding string modes, which play no role in the dynamics. Finally we would like to note that the fluxtrap background represents an integrable string theory on general grounds, as it is equivalent under a T-duality to a free quotient of flat space. Any such background is solvable by a generally applicable recipe [7] and indeed this particular background has already been to some extent solved, in its description as a fluxbrane, in [6].

We consider it likely that the fluxtrap description of the Ω -background will prove efficient for computations where the description as a twisted compactification is unwieldy. There is hope that this solution will further the investigation of the remarkable relationships among gauge theories first noted in [14, 15].

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A The fluxtrap deformation of a set of NS-fivebranes

A.1 Bulk fields

In order precisely to specify the brane configuration described in outline in section 5 of this paper, we need to consider D2-branes stretched between parallel NS5-branes. Consider a stack of parallel NS5-branes in flat space, extended in the directions x^1, x^6, x^7, x^8, x^9 (see table 2). Since the configuration preserves rotations in the 45 and 67 planes it is possible to repeat the same fluxbrane construction as in section 3. The fields in the bulk in the

direction	0	1	2	3	4	5	6	7	8	9
NS5	×	×					×	×	×	×
fluxtrap	×	×	×	×						×
D2	×	×	×							
D4	×	×		×	×	×				

Table 2. Embedding of the D2-brane with respect to the NS5 fluxtrap.

non-trivial directions read:

$$\widetilde{ds}^2 = U \left[dx_2^2 + dx_3^2 + d\rho_1^2 + \rho_1^2 (d\phi_1 + m\widetilde{R}d\tilde{u})^2 \right] + d\rho_2^2 + \rho_2^2 (d\phi_2 - m\widetilde{R}d\tilde{u})^2 + \widetilde{R}^2 d\tilde{u}^2, \quad (\text{A.1})$$

$$B = b_i dx^i \wedge (d\phi_1 + m\widetilde{R}d\tilde{u}), \quad (\text{A.2})$$

$$\Phi = \log(\widetilde{R}g_3^2) + \frac{1}{2} \log U, \quad (\text{A.3})$$

where

$$U = 1 + \frac{N_5 \alpha'}{x_2^2 + x_3^2 + \rho_1^2}, \quad b_i dx^i = \frac{dU}{dx_3} \left(-(x_3^2 + \rho_1^2) dx_2 + x_2 x_3 dx_3 + x_2 \rho_1 d\rho_1 \right), \quad (\text{A.4})$$

so that

$$d(b_i dx^i \wedge d\theta_1) = *dU, \quad (\text{A.5})$$

where the Hodge star is understood in the four-dimensional space $(x_2, x_3, \rho_1, \theta_1)$.

In rectilinear coordinates $(x_4 + i x_5 = \rho_1 e^{i\phi_1}, x_6 + i x_7 = \rho_2 e^{i\phi_2})$:

$$\widetilde{ds}^2 = U \left[dx_2^2 + dx_3^2 + \sum_{i=4}^5 (dx_i + mV^i dx_8)^2 \right] + \sum_{i=6}^7 (dx_i + mV^i dx_8)^2 + dx_8^2, \quad (\text{A.6})$$

where $V^i \partial_i$ is the same vector as in equation (3.14):

$$V^i \partial_i = -x^5 \partial_{x_4} + x^4 \partial_{x_5} + x^7 \partial_{x_6} - x^6 \partial_{x_7} = \partial_{\phi_1} - \partial_{\phi_2}. \quad (\text{A.7})$$

This provides the Ω -deformation of the NS5 background.

Following the same procedure as in section 3 we can T-dualize in the direction \tilde{u} and get the NS5 fluxtrap background:

$$ds^2 = d\tilde{x}_{0\dots 1}^2 + U \left[dx_2^2 + dx_3^2 + d\rho_1^2 + \rho_1^2 d\phi_1^2 \right] + d\rho_2^2 + \rho_2^2 d\phi_2^2 + dx_9^2 + \frac{1}{\Delta^2} \left[(m b_i dx^i + dx_8)^2 - m^2 (U \rho_1^2 d\phi_1 - \rho_2^2 d\phi_2)^2 \right] \quad (\text{A.8})$$

$$B = \frac{1}{\Delta^2} \left[b_i dx^i \wedge (d\phi_1 + m^2 \rho_2^2 (d\phi_1 + d\phi_2)) + m (U \rho_1^2 d\phi_1 - \rho_2^2 d\phi_2) \wedge dx_8 \right], \quad (\text{A.9})$$

$$e^{-\Phi} = \frac{1}{g_3^2 \sqrt{\alpha'}} \frac{\Delta}{\sqrt{U}}, \quad (\text{A.10})$$

where

$$\Delta^2 = 1 + m^2 (U \rho_1^2 + \rho_2^2). \quad (\text{A.11})$$

This configuration preserves *8 real supercharges* and the Killing spinors have the following explicit expression:

$$\begin{cases} \epsilon_L = e^{-\Phi/8}(\mathbb{1} + \Gamma_{11}) \Pi_-^{\text{NS5}} \Pi_-^{\text{flux}} \exp \left[\frac{1}{2} \phi_1 \Gamma_{45} + \frac{1}{2} \phi_2 \Gamma_{67} \right] \epsilon_0 \\ \epsilon_R = e^{-\Phi/8}(\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_+^{\text{NS5}} \Pi_-^{\text{flux}} \exp \left[\frac{1}{2} \phi_1 \Gamma_{45} + \frac{1}{2} \phi_2 \Gamma_{67} \right] \epsilon_1 \end{cases} \quad (\text{A.12})$$

where ϵ_0 and ϵ_1 are constant Majorana spinors,

$$\Pi_{\pm}^{\text{NS5}} = \frac{1}{2}(\mathbb{1} \pm \Gamma_{2345}), \quad (\text{A.13})$$

and

$$\Gamma_u = \frac{m\rho_1\sqrt{U}}{\Delta}\Gamma_5 - \frac{m\rho_2}{\Delta}\Gamma_7 + \frac{1}{\Delta}\Gamma_8. \quad (\text{A.14})$$

A.2 Open strings

D2-brane ansatz. The dynamics of a D2-brane extended in the (x_1, x_2) can be studied following the parallel computation in section 4 in absence of NS5-branes.

In order to construct BPS solutions, we start with the ansatz

$$F_{\alpha\beta} = 0, \quad x_0 = \zeta^0, \quad x_1 = \zeta^1, \quad x_2 = x_2(\zeta^0, \zeta^1, \zeta^2), \quad \phi_1 = \omega \zeta^0, \quad \phi_2 = -\omega \zeta^0. \quad (\text{A.15})$$

All the other coordinates are independent of ζ^a . This ansatz does not completely fix the reparametrization invariance ($\zeta \mapsto \tilde{\zeta}(\zeta)$) of the D2-brane. The *e.o.m.* for x_2 ,

$$\frac{\partial}{\partial \zeta^\alpha} \frac{\delta \mathcal{L}}{\delta(\partial_\alpha x_2)} = \frac{\delta \mathcal{L}}{\delta x_2} \quad (\text{A.16})$$

is satisfied for any choice of x_2 . This means that we can fix the *Diff* invariance by choosing the static gauge,

$$x_2 = \zeta^2. \quad (\text{A.17})$$

Note that the consistency of this choice is automatic because of the reparametrization invariance of the DBI action, but still appears nontrivial due to the fact that ∂_{x_2} is not a Killing vector.

The pullbacks of metric and B field are

$$g_{\alpha\beta} d\zeta^\alpha d\zeta^\beta = -\frac{\omega^2 + \Delta^2(m^2 - \omega^2)}{m^2 \Delta^2} (d\zeta^0)^2 + (d\zeta^1)^2 + (U + \Lambda^2)(d\zeta^2)^2, \quad (\text{A.18})$$

$$B_{\alpha\beta} d\zeta^\alpha \wedge d\zeta^\beta = \frac{\Lambda}{\Delta} \frac{\omega}{m} d\zeta^0 \wedge d\zeta^2, \quad (\text{A.19})$$

where

$$\Lambda = \frac{m}{\Delta} (x_3^2 + \rho_1^2) \frac{dU}{dx_3}. \quad (\text{A.20})$$

The bosonic part of the DBI action reads

$$S = -\mu_2 \int d^3 \zeta \sqrt{1 - \frac{\Delta^2(1 + \Lambda^2/U)}{m^2} (m^2 - \omega^2)}. \quad (\text{A.21})$$

Introducing

$$\Xi^2 = U(\zeta_2, x_3, \rho_1) \rho_1^2 + \rho_2^2 + (x_3^2 + \rho_1^2)^2 \frac{U_{,3}(\zeta_2, x_3, \rho_1)^2}{U(\zeta_2, x_3, \rho_1)} = \frac{\Delta^2(U + \Lambda^2)}{m^2 U} - \frac{1}{m^2} \quad (\text{A.22})$$

the equations of motion reduce to:

$$\frac{\delta \mathcal{L}}{\delta X^\sigma} = \frac{1}{4\pi^2 g_3^2 (\alpha')^2} \frac{\Xi(m^2 - \omega^2) \partial_\sigma \Xi}{\sqrt{1 + \Xi^2(m^2 - \omega^2)}} = 0, \quad \text{where } X^\sigma = \{x_3, \rho_1, \rho_2, x_8, x_9\}. \quad (\text{A.23})$$

There are two possibilities to satisfy these equations:

1. If we require

$$\Xi = 0, \quad (\text{A.24})$$

this is equivalent to

$$\Delta^2 + \frac{\Delta^2 \Lambda^2}{U} = 1. \quad (\text{A.25})$$

Since $\Delta \geq 1$ and U is non-negative, the condition can only be satisfied if

$$\Delta^2 = 1 \Rightarrow \rho_1 = \rho_2 = 0, \quad (\text{A.26})$$

$$\Lambda = 0. \quad (\text{A.27})$$

We will refer to this solution where the D2-brane is localized at $\rho_1 = \rho_2 = x_3 = 0$ as the *static embedding*.

2. If

$$\omega = \pm m. \quad (\text{A.28})$$

These are the two branches of the *rotating D2-brane embedding*. Note that just like it was in the absence of NS5-branes, even if we are not in a linear approximation the frequency is constant and no conditions are imposed on the position of the D2-brane in the other transverse directions.

Hamiltonian formalism. Let us now verify that the rotating solution satisfies has exactly the BPS energy $\mathcal{H} - \mathcal{H}^{\text{static}} = |m\mathcal{J}|$. The angular momentum density associated to the rotation in the direction of the Killing vector $V^i \partial_i$ is:

$$\mathcal{J} = V^\rho P_\rho = \frac{\delta \mathcal{L}}{\delta \omega} = -\frac{1}{4\pi^2 g_3^2 (\alpha')^2} \frac{\Xi^2 \omega}{\sqrt{1 + \Xi^2(m^2 - \omega^2)}}, \quad (\text{A.29})$$

and the Hamiltonian density reads:

$$\mathcal{H} = P_\rho \dot{X}^\rho - \mathcal{L} = -\frac{1}{4\pi^2 g_3^2 (\alpha')^2} \frac{1 + \Xi^2 m^2}{\sqrt{1 + \Xi^2(m^2 - \omega^2)}}. \quad (\text{A.30})$$

It follows that on-shell the relation

$$\left. \frac{\mathcal{H} - \mathcal{H}^{\text{static}}}{\mathcal{J}} \right|_{\omega=\pm m} = \pm m \quad (\text{A.31})$$

is satisfied without any extra consistency conditions.

Supersymmetry. We can now turn to the construction of the Killing spinors preserved by the D2-brane embeddings we have found above.

The gamma matrices pulled back to the D2-brane are

$$\hat{\Gamma}_0 = \Gamma_0 - \Gamma_8 + \frac{1}{\Delta} \Gamma_u, \quad (\text{A.32})$$

$$\hat{\Gamma}_1 = \Gamma_1, \quad (\text{A.33})$$

$$\hat{\Gamma}_2 = \sqrt{U} \Gamma_2 + \Lambda \Gamma_u. \quad (\text{A.34})$$

The gamma matrix appearing in the kappa symmetry transformation is modified by the presence of the B field:

$$\Gamma_{D2} = \frac{1}{\sqrt{-\det(g+B)}} (\mathbb{1} + B^{\alpha\beta} \Gamma_{\alpha\beta} \Gamma_{11}) \hat{\Gamma}_{012} = \frac{\Delta}{\sqrt{U}} \left(-\hat{\Gamma}_{02} + \frac{\Lambda}{\Delta} \Gamma_{11} \right) \hat{\Gamma}_1. \quad (\text{A.35})$$

The condition for preserving supersymmetry is again $\epsilon_L = \Gamma_{D2} \epsilon_R$, explicitly:

$$(\mathbb{1} + \Gamma_{11}) \Pi_-^{\text{NS5}} \Pi_-^{\text{flux}} \epsilon_0 = \Gamma_{D2} (\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_+^{\text{NS5}} \Pi_-^{\text{flux}} \epsilon_1. \quad (\text{A.36})$$

Plugging in the explicit expression for Γ_u , and using the fact that:

$$\Gamma_i \Pi_{\pm}^{\text{flux}} = \begin{cases} \Pi_{\mp}^{\text{flux}} \Gamma_i & \text{if } i = 4, 5, 6, 7 \\ \Pi_{\pm}^{\text{flux}} \Gamma_i & \text{otherwise} \end{cases} \quad \Gamma_i \Pi_{\pm}^{\text{NS5}} = \begin{cases} \Pi_{\mp}^{\text{NS5}} \Gamma_i & \text{if } i = 2, 3, 4, 5 \\ \Pi_{\pm}^{\text{NS5}} \Gamma_i & \text{otherwise} \end{cases} \quad (\text{A.37})$$

we find that the conditions for the preservation of supersymmetry become:

$$\epsilon_0 - \Gamma_{0128} \epsilon_1 = 0, \quad (\text{A.38})$$

$$m \rho_1 U \Gamma_{25} (\Gamma_0 \mp \Gamma_8) \epsilon_1 = 0, \quad (\text{A.39})$$

$$m \rho_2 \sqrt{U} \Gamma_{27} (\Gamma_0 \mp \Gamma_8) \epsilon_1 = 0, \quad (\text{A.40})$$

$$\Lambda \Gamma_1 (\Gamma_0 \mp \Gamma_8) \epsilon_1 = 0. \quad (\text{A.41})$$

Again we have two possibilities:

1. In the static embedding case we have $\rho_1 = \rho_2 = \Lambda = 0$, so we only need to impose the condition

$$\epsilon_0 = \Gamma_{0128} \epsilon_1. \quad (\text{A.42})$$

We find that the BPS static brane embedding preserves *4 real supercharges*:

$$\begin{cases} \epsilon_L = e^{-\Phi/8} (\mathbb{1} + \Gamma_{11}) \Pi_-^{\text{NS5}} \Pi_-^{\text{flux}} \Gamma_{1208} \exp \left[\frac{1}{2} (\phi_1 + \phi_2) \Gamma_{67} \right] \epsilon_2, \\ \epsilon_R = e^{-\Phi/8} (\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_+^{\text{NS5}} \Pi_-^{\text{flux}} \exp \left[\frac{1}{2} (\phi_1 + \phi_2) \Gamma_{67} \right] \epsilon_2. \end{cases} \quad (\text{A.43})$$

2. For the rotating embedding $\omega = \pm m$, together with $\epsilon_0 = \Gamma_{1208} \epsilon_1$ we need to impose the extra condition

$$\epsilon_1 = (\mathbb{1} \mp \Gamma_{08}) \epsilon_2, \quad (\text{A.44})$$

description	index	range
spacetime flat	m, n, \dots	$0, \dots, 9$
spacetime curved	μ, ν, \dots	$0, \dots, 9$
D2-brane worldvolume	α, β, \dots	$0, 1, 2$
transverse to the D2-brane	ρ, σ, \dots	$3, \dots, 9$
spinor components	a, b, \dots	$1, \dots, 16$
D2-brane-preserved Killing spinors	A, B, \dots	$1, \dots, 8$

Table 3. Conventions for the choice of indices.

where ϵ_2 is a constant Majorana spinor. This breaks another half of the supersymmetries so that the BPS rotating brane embedding preserves a total of *2 real supercharges*:

$$\begin{cases} \epsilon_L = e^{-\Phi/8}(\mathbb{1} + \Gamma_{11}) \Pi_-^{\text{NS5}} \Pi_-^{\text{flux}} \Gamma_{1208} \exp \left[\frac{1}{2}(\phi_1 + \phi_2) \Gamma_{67} \right] (\mathbb{1} \mp \Gamma_{08}) \epsilon_2, \\ \epsilon_R = e^{-\Phi/8}(\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_+^{\text{NS5}} \Pi_-^{\text{flux}} \exp \left[\frac{1}{2}(\phi_1 + \phi_2) \Gamma_{67} \right] (\mathbb{1} \mp \Gamma_{08}) \epsilon_2. \end{cases} \quad (\text{A.45})$$

B Conventions

In this appendix we collect the conventions used in the paper.

The indices are used according to table 3. The signature of the metric is $(-, +, \dots, +)$. Hence the flat Gamma matrices Γ_m satisfy the Clifford algebra:

$$\{\Gamma_m, \Gamma_n\} = 2 \eta_{mn} = 2 \text{diag}\{-1, 1, \dots, 1\}. \quad (\text{B.1})$$

The chirality matrix Γ_{11} is given by

$$\Gamma_{11} = \Gamma_0 \Gamma_1 \cdots \Gamma_9. \quad (\text{B.2})$$

The antisymmetric product of N gamma matrices is normalized as follows:

$$\Gamma_{m_1 \dots m_N} = \frac{1}{N!} (\Gamma_{m_1} \cdots \Gamma_{m_N} \pm \text{permutations}). \quad (\text{B.3})$$

The gamma matrices in the bulk are

$$\Gamma_\mu = e^m{}_\mu \Gamma_m, \quad \{\Gamma_\mu, \Gamma_\nu\} = g_{\mu\nu}, \quad (\text{B.4})$$

and their pullbacks on the D-brane are given by

$$\Gamma_\alpha = \frac{\partial X^\mu}{\partial \zeta^\alpha} e^m{}_\mu \Gamma_m, \quad \{\Gamma_\alpha, \Gamma_\beta\} = g_{\alpha\beta}. \quad (\text{B.5})$$

In order to avoid confusion, the pullback of the gamma matrices in the ζ^0 direction is denoted by $\hat{\Gamma}_0$:

$$\hat{\Gamma}_0 = \Gamma_\alpha|_{\alpha=0} = \frac{\partial X^\mu}{\partial \zeta^0} e^m{}_\mu \Gamma_m. \quad (\text{B.6})$$

In type IIA the spinors are Majorana and are decomposed into the sum of two chiral components:

$$\epsilon = \epsilon_L + \epsilon_R, \quad \Gamma_{11}\epsilon_L = \epsilon_L, \quad \Gamma_{11}\epsilon_R = -\epsilon_R. \quad (\text{B.7})$$

The conjugate is defined as:

$$\bar{\epsilon} = i\epsilon^T\Gamma^0 = -i\epsilon^T\Gamma_0. \quad (\text{B.8})$$

The supersymmetry transformations of the dilatino λ and the gravitino Ψ_m are given e.g. in [34]:

$$\delta_\epsilon\lambda = \frac{e^{\Phi/4}}{\sqrt{2}} \left[-\frac{1}{2} \partial_m \Phi \Gamma^m \Gamma_{11} + \frac{1}{24} H_{mnp} \Gamma^{mnp} \right] \epsilon, \quad (\text{B.9})$$

$$\delta_\epsilon\Psi_m = e^{\Phi/4} \left[\nabla_m + \frac{1}{8} \partial_n \Phi \Gamma^n{}_m + \frac{1}{96} H_{npq} (\Gamma_m{}^{npq} - 9 \delta_m{}^n \Gamma^{pq}) \Gamma_{11} \right] \epsilon, \quad (\text{B.10})$$

where the action of the covariant derivative on a spinor is given by

$$\nabla_m \epsilon = \partial_m \epsilon + \frac{1}{4} \omega_m{}^{np} \Gamma_{np} \epsilon, \quad (\text{B.11})$$

and ω is the spin connection. A spinor ϵ is a Killing spinor if

$$\delta_\epsilon\lambda = 0 \quad \text{and} \quad \delta_\epsilon\Psi_m = 0. \quad (\text{B.12})$$

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