LECTURES ON FANO MIRRORS

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ABSTRACT. These is a hack of the notes for my first two (out of five) KIAS lectures on *mirror constructions for Fano varieties* at Anmyundo castle, Korea (June 7-11, 2011).

First lecture is introductory and contains some examples. Second lecture is about certain inequalities and their extremes.

Third lecture was about understanding of mirror symmetry, mirror construction for minuscule varieties, historical overview of the field and toric degeneration hypothesis. Fourth lecture was on mutations of potentials and incarnations of Markov's equations. In the end of the course we constructed mirrors for 105 families of Fano threefolds and 10 families of del Pezzo surfaces. Evening discussions (that have to be added in the appendix) discuss some motivations, examples, corollaries, hints for the exercises and some challenging open problems.

1. FIRST LECTURE: EXAMPLES AND BACKGROUND ON ALGEBRAIC GEOMETRY.

Mirror symmetry is an experimentally observed duality between algebraic and symplectic geometries [1] .

Constructions of mirrors for Fano manifolds are interesting on their own, however they are of utter importance in the following research program ([1], van Straten, ...): classify Fano manifolds by classifying their mirrors. Moreover, to succeed in this program we need constructions of mirror objects before we even know if the Fano manifold itself exists or even what is a *Fano manifold* or *mirror symmetry*.

First of all I'll demonstrate some constructions of this kind, (later in the school we'll see their meaning and how it works).

Example 1.1 (1d)**.** Draw a dot. Then draw a star and two arrows: from the star to the dot, and from the dot to the star. Write 1 near the star and x near the dot. Take sum over two edges of the quotients of the letter written in the end by the letter written in the beginning: $W = \frac{x}{1} + \frac{1}{x}$. Then W is a mirror for the projective line $\mathbb{P}^1$.

Example 1.2 (Potentials associated with quivers)**.** More generally: start from any quiver (oriented graph) without cycles, add a new vertex star and draw extra edges from star to all vertices without any incoming arrow, and extra arrow to star from all vertices without any outgoing arrow; assign 1 to the star and independent variables to the vertices and consider the sum over all edges of the quotients of respective variables. Some quivers produce mirrors to some Fano manifolds.

Example 1.3 (1d-bis)**.** Consider a map $F : \mathbb{C}^* \to \mathbb{C}^*$: $F(x) = \frac{1}{x}$. It is 2-periodic: $F^2 = Id$. Take sum of all iterations. $W = x + F(x) = x + \frac{1}{x}$. Then W is a mirror for the projective line $\mathbb{P}^1$.

One can also consider a 2-periodic map $x \to \frac{q}{x}$ to construct $W = x + \frac{q}{x}$.

Example 1.4 (2d)**.** Consider a rational map $F_{N,r}(x, y) = (y, \frac{P_{N,r}(y)}{x})$ where $P_{N,r}$ is one of the following:

(1) $P_{4;2} = 1$
(2) $P_{3;3} = \frac{1}{y}$
(3) $P_{6,1} = y$
(4) $P_{5,1} = 1 + y$

Then $F_{N,r}$ has finite order N and sum of its iterations is a mirror for del Pezzo surface $S_{N,r}$ of index r and degree $\int_{[S]} c_1(S)^2 = rN$.

Exercise 1.5. *Note that any rational endomorphism can be considered as endomorphism of field of rational functions $\mathbb{C}(x, y)$. You can do these two exercises in any order:*

(1) *Show that maps $F_{N,r}$ are birational, i.e. they have inverse rational map G: $FG = GF = Id$.*

[1] Faithful practioners of the field believe that every algebro-geometric structure has a symplectic mirror-dual, and every symplectic structure (supported on some algebro-geometric object) has mirror-dual algebro-geometric.

(2) *Check that $F_{N,r}$ has order N.*

Exercise 1.6. *Prove that F constructed from any P cannot have order 2, and if it has order 3 or 4 then it lies in one-dimensional deformation of $P_{3,3}$ (respectively of $P_{4;2}$).*

Example 1.7 (3d)**.** Consider a rational map $F_n(x,y,z) = (y, z, \frac{P_n(y,z)}{x})$ where P_n is one of the following:

(1) 1
(2) $\frac{1}{yz}$
(3) $1 + y + z$

Then F_n has finite order and sum of its iterations is a mirror for Fano threefold $(\mathbb{P}^1)^3$, $\mathbb{P}^3$, V_{16}.

Exercise 1.8. *Compute the orders.*

Example 1.9 (Projective spaces $\mathbb{P}^n$)**.**

(1) Draw A_n graph with edges oriented from left to right, add a star, draw an edge from the star to the left vertex and an edge from right vertex to the star. Write $x_1, \ldots, x_n$ near the vertices and 1 near the star. Consider sum over edges like in the first example: $W = \frac{x_1}{1} + \frac{x_2}{x_1} + \cdots + \frac{1}{x_n}$. This function $W : (\mathbb{C}^*)^n \to \mathbb{C}$ is a mirror to the projective space $\mathbb{P}^n$.
(2) Consider a map $F(y_1, \ldots, y_n) = (y_2, \ldots, y_n, \frac{1}{y_1 \cdots y_n})$. Take the sum of iterations: $W' = y_1 + \cdots + y_n + \frac{1}{y_1 \cdots y_n}$.

Exercise 1.10. *Find a change of coordinates on the torus that maps W to W'.*

1.1. **Background on varieties: curves, surfaces, Calabi–Yau, Fano...** Every variety has the tangent bundle T_X, its dual cotangent bundle $\Omega^1_X = T^*_X$, canonical line bundle $K_X = \omega_X = \det \Omega^1_X$ and its dual anticanonical line bundle $-K_X = \det T_X$.

Naturality of these bundles makes them more important than the arbitrary ones.

Classification of curves C by their genus g (or equivalently any of: topological Euler characteristic $\chi_{top}(C) = \deg T_C = 2 - 2g$ or canonical degree $\deg K_C = 2g - 2$, or arithmetic genus $\dim H^1(\mathcal{O}_C) = g$)

There is a trichotomy:

- $g = 0$: projective line $\mathbb{P}^1$, canonical degree $\deg K_C = -2$ is negative; curvature is positive.
- $g = 1$: elliptic curves E, canonical class is trivial; curvature is zero.
- $g > 1$: higher genera curves C, canonical class is ample; curvature is negative.

For each g there is a moduli space of dimension $3g - 3 + \dim \mathrm{Aut}(C)$. Note there are infinitely many $g > 0$.

Hodge decomposition of cohomology $H^n(X, \mathbb{Z}) \otimes \mathbb{C} = \sum_{p+q=n} H^{p,q}(X, \mathbb{C})$ where $H^{p,q}(X, \mathbb{C})$ is p-holomorphic and q-antiholomorphic part, or equivalently $H^p(X, \Omega^q_X)$.

Symmetries of Hodge diamond of Kaehler (and hence projective) varieties: $(\mathbb{Z}/2\mathbb{Z})^2$.

Hodge diamond for curves. Serre duality: $H^0(C, \omega_C) = H^1(C, \mathcal{O}_C)^*$

Note that there is an extra symmetry when $g = 1$. This is the *mirror symmetry*! Elliptic curves are mirror dual to elliptic curves.

Trichotomy for projective varieties with $\mathrm{Pic}(V) = \mathbb{Z}$:

- $-K_V$ is ample: Fano manifolds
- $K_V = 0$: (generalized) Calabi–Yau varieties
- K_V is ample: varieties of general type.

Note that there are more intermediate options for varieties with $\mathrm{Pic}\, V \neq \mathbb{Z}$ (e.g. product of a line with a curve of higher genus).

Example 1.11 (varieties with $K = 0$)**.**

(1) Abelian varieties, e.g. products of elliptic curves. These exist in every dimension $\geqslant 1$.
(2) Holomorphic symplectic varieties (also known as hyperkahler manifolds): $K3$ surfaces (e.g. Kummer surfaces), Hilbert schemes of n points on $K3$ surfaces, and their deformations. These exist in even dimensions $\geqslant 2$.
(3) Honest Calabi-Yau varieties ($\pi_1(V) = 0$ and $h^{2,0}(V) = 0$), e.g. quintic in $\mathbb{P}^4$. These exist in any dimension $\geqslant 3$.

Bogomolov's theorem says that up to unramified covers any variety with $K = 0$ is product of these three types.

Example 1.12. Quotients of $K3$ surfaces by involutions acting without fixed points are called *Enriques surfaces.*

Remark 1.13 (Mirror symmetry). Note that Hodge diamonds of abelian and holomorphic symplectic varieties have extra symmetry with respect to line with angle 45 degree. This is not generally true for honest Calabi–Yau threefolds, but it has been noticed that many of them come in pairs V and V' so that their Hodge diamonds are mirror-symmetric to each other.

Kodaira dimension of variety V defined as an integer number $\varkappa(V)$ such that for some $C > 1$ we have $C^{-1} < \frac{\dim H^0(V,\omega_V^n)}{n^{\varkappa(V)i}} < C$ for big n. In case $H^0(V,\omega_V^n) = 0$ for all n number $\varkappa(V)$ is defined to be $-\infty$. Kodaira dimension is either $-\infty$ or in interval from 0 to $\dim V$. This is birational invariant of variety.

Enriques's classification of complex projective surfaces:

- $\varkappa = 2$: general type, K_S is numerically effective ($K_S \cdot C \geqslant 0$ for any curve $C \subset S$).
- $\varkappa = 1$: some elliptic surfaces (e.g. product of elliptic curve with curve of genus $g > 1$).
- $\varkappa = 0$: $K3$, Enriques, abelian, bi-elliptic (7 families).
- $\varkappa = -\infty$: *rational surfaces* or ruled surfaces over curve of genus $g > 0$.

Note that on the right we list *minimal models*, one can also blow them up.

Remark 1.14. Some elliptic surfaces have Kodaira dimension less than 1. In this lectures of particular importance will be *rational elliptic surfaces*, which can be obtained by blowing up the base loci of a pencil of elliptic curves on the plane.

Definition 1.15 (Iskovskikh)**.** Smooth complex variety V is called *a Fano manifold* if its anticanonical line bundle $-K_V = \det T_V$ is ample, i.e. some its tensor power gives the embedding of V into a projective space.

Fano manifolds in dimension two are also called *del Pezzo surfaces*.

Maximal $r \in \mathbb{Z}$ such that $-K_V = rH$ for some ample divisor $H \in \mathrm{Pic}(V)$ is called (Fano) index of Fano manifold.

Varieties V with $r(V) \geqslant \dim V - 1$ are sometimes called *del Pezzo varieties*.

Exercise 1.16. *What could be Kodaira dimension of a Fano manifold?*

Theorem 1.17 (del Pezzo, Castelnuovo?)**.** *Any del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or blowup of $0 \leqslant n \leqslant 8$ points in generic position on the projective plane $\mathbb{P}^2$. Generic position means: no three points lie on a line, no 6 on a conic, and similarly for singular cubics.*

In particular all del Pezzo surfaces (over $\mathbb{C}$) are rational; note that in dimensions $\geqslant 3$ some Fano manifolds are irrational. [2]

After second Ionut's lecture on J-function...

I was asked what is mirror symmetry for Fano. The point of these lectures is that in order to succeed in the original program we need some constructions that can be used by people who don't even know what is a Fano manifold.

However after Ionut's lecture it is a pleasure to give a pre-definition. Let V be a Fano manifold of dimension D.

Definition 1.18. Consider *very small* J-function, that is small J_V restricted to anticanonical direction with $z = 1$. Define G-function $G_V = \int_{[V]} J_V \cup [pt]$ as the fundamental term of very small J-function. For Laurent polynomial $W(y_1, \ldots, y_D)$ define its G-function as $G_W(t) = \int_{|y_i|=1} e^{tW}\omega$, where $\omega = \frac{1}{(2\pi i)^D}\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_D}{y_D}$. Laurent polynomial W is said to be *pre-shard of the mirror for* Fano manifold V if $G_V = G_W$ (this is $A = B'$ type equality). [3]

Exercise 1.19. *Prove that yesterday's examples of mirrors for $\mathbb{P}^n$ are indeed pre-shards of the mirror for it.*

Material from night exercise session.

Recall

Theorem 1.20 (Kodaira's vanishing theorem)**.** *If $\mathcal{L}$ is an ample line bundle on V then $H^i(V, \mathcal{L} \otimes K_V) = 0$ for any $i > 0$.*

Exercise 1.21. *If V is a Fano manifold then $H^i(V, \mathcal{O}_V) = 0$ for $i > 0$*

Exercise 1.22. *There is no smooth projective variety V' with Hodge diamond mirror symmetric to diamond of Fano manifold V.*

[2] End of the first lecture. Then first Brett's lecture on exploded manifolds.

[3] W is called Lax operator L in the original work of Eguchi-Hori-Xiong, and condition $G_V = G_W$ is essentially equation 4.26 from hep-th/9605225. Przyjalkowski calls this kind of W as *very weak Landau–Ginzburg model*.

This means mirror of a Fano manifold cannot be a projective variety. Instead **mirror to Fano manifold is a pencil of varieties with** $K = 0$, or in fact even simpler object – just a function (Laurent polynomial). It is not the geometry of the total space, but rather variation of the levels that matters. However the pencil should be a **very special one**: its fibers are Calabi–Yau varieties mirror dual to anticanonical sections of our Fano.

Exponential short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

gives an exact sequence of cohomologies

$$H^1(V, \mathcal{O}) \to H^1(V, \mathcal{O}^*) \to H^2(V, \mathbb{Z}) \to H^2(V, \mathcal{O})$$

Since by Kodaira vanishing $H^1(V, \mathcal{O}) = H^2(V, \mathcal{O}) = 0$ the map $c_1 : \mathrm{Pic}(V) \to H^2(V, \mathbb{Z})$ is an isomorphism.

Remark 1.23. Also all Fano manifolds are simply-connected and group $H^2(V, \mathbb{Z})$ equals to $\mathbb{Z}^\rho$, where ρ is Picard number.

So, rational, algebraic, numerical and homological equivalences of divisors on Fano manifolds coincide.

Remark 1.24. Same is true for honest Calabi–Yau varieties. However for $K3$ surfaces the map c_1 is merely an inclusion and is never an isomorphism.

Exercise 1.25. *Draw Hodge diamonds of Fano curves, surfaces and threefolds.*

Recall that projective space $\mathbb{P}^n = \mathbb{A}^{n+1}/\mathbb{C}^*$ inherits $(n+1)$ homogeneous coordinates $(X_0 : X_1 : \cdots : X_n)$, and in the chart $X_0 \neq 0$ it is parametrized by n inhomogeneous coordinates $x_1 = \frac{X_1}{X_0}, \ldots, x_n = \frac{X_n}{X_0}$.

Consider n-form $\omega = \frac{1}{(2\pi i)^n} \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_n}{x_n}$. This is a holomorphic volume form on the torus $(\mathbb{C}^*)^n$. However it is a meromorphic volume form on the projective space. Similarly one may consider n-form $\omega' = dx_1 \wedge \cdots \wedge dx_n$; ω' is holomorphic on the affine space and also meromorphic on $\mathbb{P}^n$.

Exercise 1.26. *Find divisors* (ω) *and* (ω')*.*

Exercise 1.27 (Projective space)**.** *Compute the canonical line bundle of the projective space* $\mathbb{P}^n$ *and show it is a Fano manifold. Hint: either consider determinants of the (dual) Euler sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0$$

or use the last exercise.

Example 1.28. Consider a hypersurface $i : X \subset \mathbb{P}^n$ in the projective space $\mathbb{P}^n$ given by zeroes of homogeneous equation of degree d in $n+1$ variables. Use the short exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|Y \to \mathcal{O}(d) \to 0$$

to compute the anticanonical bundle of X.

Exercise 1.29. (1) *Show that three-dimensional quintic is an honest Calabi–Yau variety.*

(2) *Show that any smooth section of anticanonical bundle on Fano manifold* V *is a generalized Calabi–Yau variety, and if* $\dim V \geqslant 4$ *then it is honest.*

So smooth hypersurface of degree $d \leqslant n$ in $\mathbb{P}^n$ is a Fano manifold of index $(n+1-d)$, hypersurface of degree $d = n+1$ is Calabi–Yau variety and hypersurface of degree $d > n+1$ is a variety of general type.

Next theorem is of fundamental importance:

Theorem 1.30 (Kollar, Miyaoka, Mori)**.** *In any given dimension* D *there are only finitely many deformation classes of Fano manifolds.*

Remark 1.31. The main reason why this holds is because **Fano manifolds have lots of rational curves** on them: they are rationally connected, which means there is a rational curve passing through any two generic points. We would like to see Gromov–Witten theory as some qualitative analogue of Mori's theory (to be explained later).

Deformation is a flat projective morphism over connected (irreducible) base with fibers being our varieties.

Note that Fano manifolds could be also deformation equivalent to infinitely many different non-Fano manifolds.

Exercise 1.32. (1) *Show that* $\mathbb{P}^1 \times \mathbb{P}^1$ *is a deformation of* $F_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(2))$

(2) *Consier Hirzebruch surfaces* $F_a = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(a))$*. Show that* F_a *is deformation of* F_{a+2b} *for any* $b > 0$*, but surfaces with odd and even* a*'s are not deformations of each other. Hint: moreover, they are not diffeomorphic; consider intersection form in the second cohomology.*

(3) *Show that all Hirzebruch surfaces with even a (resp. with odd a) are symplectomorphic to each other.*

Remark 1.33. Minimals models of rational surfaces are: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and F_n for $n \geqslant 2$.

By Ehresman's theorem deformation-equivalent smooth projective varieties are diffeomorphic. Moreover Fano manifolds from the same deformation class with symplectic form choosen in anticanonical class are symplectomorphic (symplectic geometry is topological).

Since Fano manifold with symplectic form in anticanononical class is rigid as symplectic variety, its mirror should be a rigid object in algebro-geometric world.

Question 1.34. *How to single out mirrors for Fano D-folds from the space of all Laurent polynomials in D variables?*

In two even lectures we will discuss two experimentally observed properties: *extremality* (in second lecture) and *mutability* (clusterity/troicity) (in fourth lecture).

2. Second lecture: inequalities and their extremes.

Natural source of "rigid" objects are extremal ones. We think about *extremality* as a condition when some natural inequality is saturated (becomes equality). Our point here is metamathematical. Under the assumption that the inequality is *natural* [4]:

- extremal objects are distinguished objects of mathematics
- they have essentilly discrete, rigid, combinatorial, arithmetic origin

The remainder of the lecture is a couple of examples of this principle.

(1) Mason–Stothers inequality and Belyi functions (Grothendieck dessin d'enfant); abc inequality after Smirnov
(2) Szpiro's inequality and extremal (rational) elliptic surfaces (of Beauville, Hirzebruch, Miranda-Persson); arithmetic Szpiro
(3) Golyshev's inequality and Extremal Local Systems
(4) Bogomolov-Miyaoka-Yau and Ball Quotients (esp. FPP); Parshin's *arithmetic Bogomolov*
(5) Number of objects in a full exceptional collection and the Minifolds
(6) Bertini–del Pezzo inequality and classification of varieties of minimal degree
(7) Kobayashi–Ochiai's characterization of manifolds of high Fano index (then also del Pezzo manifolds, Iskovskikh-Mukai's manifolds, Calabi-Yau threefolds, etc)
(8) Last but not least: inequality between arithmetic and geometric means and the mirror symmetry

Example 2.1. There is a famous inequality between arithmetic mean and geometric mean: if $X_1, \dots, X_n$ are real positive numbers, then

$$\frac{X_1 + \cdots + X_n}{n} \geqslant (X_1 \cdot \cdots \cdot X_n)^{\frac{1}{n}}$$

and the equality holds only if $X_1 = \cdots = X_N$.

Put $x_i = \frac{X_i}{X_n}$ and consider the modified equality:

$$\frac{(x_1 + \cdots + x_{n-1} + 1)^n}{x_1 \dots x_{n-1}} = n^n \lambda$$

Then this is the (shifted rescaled) shard of the mirror for hypersurface of degree $n-1$ in $\mathbb{P}^{n-1}$. The original equality ($\lambda = 1$) corresponds to its singular (conifold) fiber.

Exercise 2.2. *Prove it.*

Exercise 2.3. *Play this game with some other inequality between means (arithmetic, geometric, harmonic, etc).*

Now let us be more serious and consider the equation

$$a + b = c \tag{2.4}$$

Is there any natural inequality associated with 2.4?

[4]In practice it is very hard to tell a priori if the inequality is natural or not. We will see it in the examples below.

Theorem 2.5 (Stothers (1981), Mason (1983)). *If $a, b, c \in \mathbb{C}[x]$ are coprime polynomials then number of distinct roots of abc is at least their greatest degree plus one.*

Proof. Rational function $f = \frac{a}{c}$ can be considered as a map $f : \mathbb{P}^1 \to \mathbb{P}^1$. Degree d of this map (number of preimages of the generic point) equals to maximal degree of a, b and c. For $y \in C$ we have $a(y) = 0 \iff f(y) = 0$, $b(y) = 0 \iff f(y) = 1$ and $c(y) = 0 \iff f(y) = \infty$. $f(\infty)$ could be 0 or 1 or ∞ or neither. So up to point $\infty \in \mathbb{P}^1$ all distinct roots of abc correspond to distinct points in $f^{-1}\{0, 1, \infty\}$. Now the theorem follows from the following

Lemma 2.6. *Let $f : C \to \mathbb{P}^1$ be a map of degree d from Riemann surface of genus g to $\mathbb{P}^1$. Let N denote number of distinct points in the preimage $f^{-1}\{0, 1, \infty\}$. Then*

$$N \geqslant d + 2 - 2g, \tag{2.7}$$

and the inequality is saturated $\iff$ map f is unramified outside $\{0, 1, \infty\}$.

Proof. Recall Hurwitz formula, for map of degree d from curve C to curve C' one has the *equality*

$$\deg K_C = \deg f \cdot \deg K_{C'} + \sum_{P \in C} e_P \tag{2.8}$$

where e_P is the local ramification index in point P. In our case

$$2g - 2 = d \cdot (-2) + \sum e_P$$

For any fiber number of points in it counted *with multiplicity* $(1 + e_P)$ equals to degree d. So d equals number of distinct points plus sum of all ramification indices in the fiber. Respectively $3d$ equals N plus sum of all ramifications in these 3 fibers. Combine it with Hurwitz formula to obtain

$$N = (d + 2 - 2g) + E_U \tag{2.9}$$

where E_U is the sum of all ramification indices over $U = \mathbb{P}^1 \backslash \{0, 1, \infty\}$. Thus inequality 2.7 is equivalent to

$$E_U \geqslant 0 \tag{2.10}$$

which is obvious, so the lemma and the theorem are proven. □ □

Exercise 2.11. *Prove "Fermat's last theorem for polynomials": if three polynomials of one variable $X, Y, Z \in \mathbb{C}[u]$ and integer $n \geqslant 3$ satisfy equation $X^n + Y^n = Z^n$ then all these polynomials are constants.*

Definition 2.12. Rational functions on curve C that saturate inequality 2.7 are called *Belyi functions*. Equivalently these are the maps ramified over at most three points.[5] They have this name after famous

Theorem 2.13 (Belyi (1979)). *Curve C is defined over a number field $\iff$ it supports some Belyi function.*

We will say Belyi function is *pure* if all ramification indices over 1 equals 2, and *very pure* if additionally all ramification indices over 0 equals 3.

Definition 2.14 (Grothendieck (1984)).
1. *Dessin d'enfant* is the tiling of orientable surface by black and white triangles.
2. *Pure dessin* is a graph embedded into orientable surface, such that its complement is union of contractible faces.
3. *Very pure dessin* is a trivalent pure dessin.

Exercise 2.15.
1. *Orientation on the surface induces a cyclic order on the sets of edges adjacent to every vertex. Given a graph with this collection of cyclic orders - reconstruct the surface.*
2. *Draw all very pure dessins on with 2 vertices*
3. *Draw all very pure dessins on sphere with 4 vertices.*
4. *Make a list of gonalities of their faces.*

Definition 2.16 (Grothendieck).
1. Cartographic group is a free group $\mathbb{Z} \star \mathbb{Z} = \pi_1(U)$,
2. pure cartographic group is $\mathbb{Z} \star (\mathbb{Z}/2\mathbb{Z}) = \pi_1(\mathbb{P}(1, 2) \backslash \{0, 1\})$,
3. very pure cartographic group is $(\mathbb{Z}/3\mathbb{Z}) \star (\mathbb{Z}/2\mathbb{Z}) = PSL(2, \mathbb{Z}) = \pi_1(\mathbb{P}(2, 3) \backslash \{1\})$.

Proposition 2.17. *There are bijections between the following classes of objects:*

[5] Up to $PGL(2)$ change of coordinate on the target $\mathbb{P}^1$.

(1) *Belyi functions of degree d,*
(2) *Dessins d'enfant with d white and d black triangles,*
(3) *Homomorphisms of cartographic group into S_d.*

Similarly (very) pure Belyi functions of degree d correspond to (trivalent) graphs with $\frac{d}{2}$ edges correspond to representations of (very) pure cartographic group into S_d.

Proof.
- From Belyi function to the graph: consider the preimage of the real line (or real interval $[0, 1]$.
- From Belyi function to representation: consider the *monodromy* $\rho : \pi_1(U) \to \mathrm{Aut}(f^{-1}t)$.

□

Exercise 2.18. *Finish the proof.*

Exercise 2.19. *Find Belyi functions corresponding to dessins from exercise 2.15.*

This is the proto-typical example of our philosophy: the extremal objects (that saturate some inequality) have rigid, combinatorial (and arithmetic) nature.

By the way, $W = x + \frac{1}{x}$ is a Belyi map, of course.

We think about *extremal Laurent polynomials* as a special higher-dimensional generalization of Belyi functions.

Exercise 2.20. *Recall Frey's trick and the relation between Szpiro's inequality and abc conjecture.*

The emblematic example is that of **rational elliptic surfaces** (RES).

Exercise 2.21.
(1) *Show that RES has* 8 *moduli, one less than tuple of* 12 *points on* $\mathbb{P}^1$.
(2) *Show that generic RES has* 12 *distinct singular fibers.*

Question 2.22 (Szpiro)**.** *What is the minimal number $C'(\pi)$ of singular fibers for non-isotrivial elliptic surface $\pi : S \to \mathbb{P}^1$?*

Theorem 2.23 (Beauville (1981))**.** $C'(\pi) \geqslant 3$

Proof. Consider local system $R^1\pi_*\mathbb{Z}$ and note that moduli space of curves is hyperbolic. □

This inequality is not strong enough. Let us formulate a stronger and more precise one.

Assume our surface has a section (Jacobian).

Weierstrass model for rational elliptic surface is given as

$$y^2 = x^3 + Ax + B \tag{2.24}$$

where A and B are homogeneous polynomials of two variables $(X : Y)$ of degrees 4 and 6 respectively.

Recall (see page2 of the Russian handout) Kodaira's classification of the singular fibers of minimal Neron-Kodaira's model of elliptic surface (no (-1)-curves in fibers). Singular fibers are classified either by their local monodromy T or by local valuations a, b, δ of A, B and discriminant $\Delta = 27B^2 + 4A^3$. Fibers of type I_n $(n \geqslant 1)$ are *semi-stable*, others are not. I_n and I_n^* has multiplicative type, others additive.

Theorem 2.25 (Beauville (1981-1982))**.** *If $\pi : S \to \mathbb{P}^1$ is non-trivial elliptic surface with all singular fibers being semi-stable. Then*
(1) $C'(\pi) \geqslant 4$
(2) *If $C'(\pi) = 4$ then $\pi : S \to \mathbb{P}^1$ is one of* 6 *explicitely written Shioda's modular elliptic surfaces (see page3 of the Russian handout).*

Note that 4 coincides with naive parameter count: $12 - 8 = 4$, if one thinks that collisions of two fibers happens in codimension one and they are independent.

Exercise 2.26 (Charts and shards)**.**
(1) *Find some charts isomorphic to $(\mathbb{C}^*)^2$ inside $S \backslash \pi^{-1}\infty$, i.e. rewrite these families as families of pencils of some Laurent polynomials. For the first example: divide by XYZ, put $Z = 1$, $X = x_1$, $Y = x_2$, $t \to \frac{-1}{t}$, the resulting family is $1 - tW = 0$ where $W = x_1 + \frac{x_2}{x_1} + \frac{1}{x_2}$.*
(2) *Show that Laurent polynomials you constructed in the previous example are pre-shards of the mirrors for del Pezzo surfaces of degrees* 9, 8, 6 *and* 5.

Fibers of additive type have local monodromy of finite order. Map from Kodaira-Neron model to Weierstrass model contracts rank r root system of (-2)-curves.

Definition 2.27 (The defects)**.** Consider the defect (local conductor) c defined in any of the following ways:

(1) $c_P = (\delta_P - r_P)$
(2) $c_P = 2 - \operatorname{rk} L^{T_P}$, where $L = R^1\pi_*\mathbb{Z}$ is a local system of rank 2 and L^{T_P} is the space invariant with respet to the local monodromy T_P.
(3) c equals 0 for nonsingular fibers, 1 for semi-stable fibers and 2 for non-semistable fibers.

Define global defect (conductor) C as the sum of defects of all singular fibers.

Exercise 2.28. *Prove the equivalence of first two definitions of local defects.*

We can generalize Beauville's theorem as follows

Theorem 2.29 (Miranda–Persson (1986))**.** *Let $\pi : S \to \mathbb{P}^1$ be any elliptic surface, then*
(1) $C(\pi) \geqslant 4$
(2) $C(\pi) = 4 \iff S$ *is either one of* 6 *Beaville's surfaces, or one of* 6 *explicit rational surfaces with* 3 *singular fibers, or isotrivial surface with* 2 *singular fibers.* *(See page 4 of the Russian handout)*

These surfaces are also called extremal rational elliptic surfaces*.*

Proof. Note that

$$\deg \Delta = \sum_P \delta_P = \sum \chi_{top}(S_P) = \chi_{top}(S) = 12\chi(S, \mathcal{O}_S) \tag{2.30}$$

Let $MW(S)$ be the Mordell–Weil group of sections $s : \mathbb{P}^1 \to S$ s.t. $\pi s = Id_{\mathbb{P}^1}$. $\operatorname{Pic}(S)$ is generated by sections and irreducible components of the fibers, and Shioda-Tate's formula measures this explicitly:

$$\rho(S) = 2 + \sum_P r_P + \operatorname{rk} MW(S) \tag{2.31}$$

Consider the difference between 2.30 and 2.31:

$$C(\pi : S \to \mathbb{P}^1) = 12\chi(S, \mathcal{O}_S) - \rho(S) + 2 + \operatorname{rk} MW(S) \tag{2.32}$$

Note that

$$\rho(S) \leqslant h^{1,1}(S) = 10\chi(S, \mathcal{O}_S) \tag{2.33}$$

Since $\chi(S, \mathcal{O}_S) = 1 + p_g(S)$

$$C(S) \geqslant 4 + 2p_g(S) \tag{2.34}$$

Moreover, the last inequality is saturated only if $\rho(S) = h^{1,1}(S)$ and $MW(S)$ is finite group. This proves the first statement and also shows that extremal elliptic surface is rational ($p_g(S) = h^{2,0}(S) = 0$).

For the second statement consider j-invariant as a map from the base to $\mathbb{P}^1$. This map is a Belyi function. In Beauville's cases this Belyi function is very pure and is one from exercise 2.19 corresponding to one of 6 trivalent graphs on sphere from exercise 2.15. For Miranda–Persson's cases - look for not very pure Belyi functions. □

Exercise 2.35. *Fill the gaps in the proof.*

Exercise 2.36 (Charts and shards 2)**.** *For Miranda–Persson's examples*
(1) *Find charts like in exercise 2.26,*
(2) *Show the respective Laurent polynomials are pre-shards of the mirrors for del Pezzo surfaces of degrees* 1*,* 2*,* 3 *and* 4*.*

Note that in inequality $C(\pi) \geqslant 4$ left hand side depends only on local system $L = R^1\pi_*\mathbb{Z}$ and right hand side equals $4 = 2 \cdot 2 = 2 \cdot \operatorname{rk} L$.

So this is a special case of *Golyshev's inequality*:

$$\sum_P (\operatorname{rk} L - \operatorname{rk} L^{T_P}) \geqslant 2 \operatorname{rk} L \tag{2.37}$$

and if the inequality is saturated we call local system L *extremal local system.*

As we have seen from exercises 2.26 and 2.36 extremal local systems seem to have something to do with mirror symmetry.

We speculate the following hypothesis: Let W be a ((pre-)shard of) the mirror for a Fano manifold. Then W is *extremal Laurent polynomial* i.e. local system $R^\bullet W_*$ tend to have non-trivial extremal subquotient.

Remark 2.38. There are exceptional pecularities in even dimension. Note, we haven't yet constructe the mirrors for two del Pezzo surfaces (of degrees 7 and 8). These surfaces are toric and one can construct their mirrors $W_8 = xy + x + y + \frac{1}{xy}$ and $W_7 = xy + x + y + \frac{1}{x} + \frac{1}{y}$ and check that they are not extremal.

Exercise 2.39. *Mordell-Weil group for W_7 and W_8 should have rank* 1. *Describe it explicitly.*

Now we are going to prove Golyshev's inequality.

Extremality is acyclicity:

Euler-Poincare. Let F be a constructible sheaf of $\mathbb{C}$–vector spaces on a complex analytic smooth projective curve C. Denote by U an open subset over which F is locally constant, j the open embedding. Let u be a point in U and let $C \setminus U = U_0 = \{u_i\}$. Denote by F_u the fiber of F over u. This turns F_u into a $\pi_1(U)$–module. One has the Euler–Poincare formula:

$$\chi(C, F) = \sum (-1)^r h^r(C, F) = (2 - 2g) \dim F_u - \sum (\dim F_{u_i} - \dim F_u).$$

e.g. Milne Etale cohomology, V.2.12.

Comments: recall that a sheaf F of finite dimensional $\mathbb{C}$-vector spaces on an analytic variety V is constructible if there exists a stratification of V by analytic subspaces such that the restriction of F on every stratum is locally constant. A locally constant sheaf on U is the same as a local system and is given by a rep of $\pi_1(U, u)$ as follows: consider the fiber F_u, then any loop defines the monodromy in $GL(F_u)$. Vice versa, a rep of $\pi_1(U, u)$ defines a locally constant sheaf as follows: denote by $\tilde{U}$ the universal cover of U and quotient $\tilde{U} \times F_u$ by the action of $\pi_1(U, u)$.

Thus, a constructible sheaf on a proper analytic curve C roughly 'consists of' the skyscraper constituents, the locally constant sheaf F_U on a sufficiently small dense open subset $j : U \longrightarrow C$, and the information on how F_U extends to $C \setminus U$.

Derived categories of constructible sheaves (i.e. cohomologically bounded cohomologically constructible complexes up to quasiisomorphisms) are respected by six operations. In particular, direct images of constructible sheaves are again constructible.

For a local system L on U, it is easy to describe a fiber of the sheaf j_*F over any point u_i: by definition, it is isomorphic to the invariants of F_u under the action of the loop around u_i (local monodromy).

Proposition: Let L be a non-constant irreducible [in our situation, typically self–dual] local system on $U \subset P^1$. Then the inequality $R(L) \geqslant 2 \operatorname{rk} L$ holds.

Proof. Let $F = j_*L$. Note that i) $H^0(F) = 0$, since $H^0(\mathbb{P}^1, F) = H^0(U, L) =$ invariants of $\pi_1(U)$ acting on L_u; ii) by e.g. Milne Etale cohomology V.2.2 c) $H^2(C, j_*L)$ is dual to $H^0_c(C, j_*L^\vee))$, and the latter space is isomorphic to the invariants of $\pi_1(U)$ acting on $L^\vee_u$, hence is zero as well.

Hence, in our situation, with $C = \mathbb{P}^1$ and $H^0(C, F) = 0$ and $H^2(C, F) = 0$, the Euler-Poincare formula becomes

$$-h^1(X, F) = 2 \dim L_u - \sum (\dim L_{u_i} - \dim L_u),$$

and $2 \operatorname{rk} L - R \leqslant 0$, q.e.d.

Thus, for non-constant irreducibles, $\boxed{\text{extremal} \iff H^1(X, F) = 0}$.

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