GROMOV-WITTEN THEORY OF ELLIPTIC ORBIFOLD $\mathbb{P}^1$ AND QUASI-MODULAR FORMS

TODOR MILANOV & YONGBIN RUAN

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1. Introduction

A major problem in geometry and physics is to compute the Gromov-Witten invariants of a given target manifold. In general, this is a complicated problem. However, in certain special situations, the computations lead to beautiful objects, such as modular forms. It is clearly an important problem to locate all these special examples where the modularity exists. The simplest example of this phenomenon is the genus-1, degree-d invariants $n_{1,d}$ of an elliptic curve E. It is well-known that their generating function can be expressed in terms of the Dedekind η-function

$$\exp\Big(-\sum_{d\geq 1} n_{1,d}q^d\Big) = q^{-1/24}\ \eta(q).$$

One can say much more. Let us introduce some notation. Let X be a projective manifold and $\overline{\mathcal{M}}_{g,k}(X,\beta)$ be the moduli space of genus g stable maps with k markings and fundamental class β. Let e_i be the evaluation map at the i-th marked point x_i and ψ_i be the first Chern class of the cotangent line bundle at x_i. Choose a basis ϕ_i of $H^*(X,\mathbb{Q})$ with $\phi_0 = 1$. The numerical GW invariants are defined by

$$\langle \tau_{l_1}(\phi_{i_1}),\dots,\tau_{l_k}(\phi_{i_k})\rangle^X_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,k}(X,\beta)]^{vir}} \prod_i (e_i^*\phi_i)\psi_1^{l_i}.$$

The above invariant is zero unless

$$\sum_i (\deg(\phi_i) + 2l_i) = 2(c_1(TX)(\beta) + (3-n)(g-1) + k).$$

The advantage of Calabi-Yau manifolds, such as the elliptic curve E, is that $c_1(TX) = 0$ and hence the dimension constraint is independent of β. For the elliptic curve E, the degree β can be identified with a non-negative integer d. Then, it is natural to define

$$\langle \tau_{l_1}(\phi_{i_1}),\dots,\tau_{l_k}(\phi_{i_k})\rangle^E_{g}(q) = \sum_{d\geq 0}\langle \tau_{l_1}(\phi_{i_1}),\dots,\tau_{l_k}(\phi_{i_k})\rangle^E_{g,d}q^d. \tag{1}$$

The genus-1 invariant $n_{1,d}$ from above corresponds to $\langle\ \rangle^E_1(q)$. By the dilaton and the divisor equations, the invariants with insertion $\tau_1(1), \tau_0(\phi_{-1})$ can be deduced from other invariants. Without loss of generality, we assume that $\tau_l(\phi_i) \neq \tau_1(1), \tau_0(\phi_{-1})$. Then, Okounkov-Pandharipande [27, 28, 29] showed that the invariant (1) converges to a quasi-modular form of $SL_2(\mathbb{Z})$ with the change of variable $q = e^{2\pi i\tau}$. Together with a result of Krawitz-Shen [22], we shall prove the modularity for another class of examples, the elliptic orbifold $\mathbb{P}^1$ with weights $(3,3,3), (2,4,4), (2,3,6)$. These orbifolds are the quotients of an elliptic curve E. Our methods however, are completely different from the methods of Okounkov–Pandharipande.

To state the theorem, choose a basis $\phi_i, i = -1, 0, \dots$ of H^*_{CR} such that ϕ_{-1} is the divisor class and $\phi_0 = 1$. In the above cases, $c_1(TX) = 0$ and we can define

$$\langle \tau_{l_1}(\phi_{i_1}), \dots, \tau_{l_k}(\phi_{i_k}) \rangle^X_g(q) \tag{2}$$

similarly. The main result of the current paper is the following modularity theorem.

Theorem 1.1. *Suppose that $\tau_l(\phi) \neq \tau_1(1), \tau_0(\phi_{-1})$ and X is one of the three elliptic orbifolds $\mathbb{P}^1$ from above. For any multi-indices l_j, i_j, the GW invariant* (2) *converges to a quasi-modular form of an appropriate weight for a finite index subgroup Γ of $SL_2(\mathbb{Z})$ under the change of variables $q = e^{2\pi i\tau/3}$, $e^{2\pi i\tau/4}$, $e^{2\pi i\tau/6}$, respectively (see section 6 for the subgroups Γ and the weights of the quasi-modular forms).*

We would like to remark that if we include insertions of the form $\tau_1(1)$ then a similar statement holds. In this case however, we need to perform a dilaton shift which amounts to taking linear combinations of the above invariants.

The modular invariance has been at the center of recent physical developments of Gromov-Witten theory by Klemm and his collaborators [2, 16]. Some of the key ideas such as anti-holomorphic completion were directly inspired by their work, for which the authors express their special thanks. There is a work of similar flavor by Coates-Iritani on modularity of GW invariants of local $\mathbb{P}^2$ [7]. We are informed that Paul Johnson has an independent approach to the results in this paper. We thank them for interesting discussions. When this paper is finished, we notice a related paper of Costello-Li where they constructed a B-model high genus theory of elliptic curve and obtained corresponding mirror symmetry [8]. Finally, Satake–Takahashi [37] established an isomorphism between the quantum cohomology of the above orbifold projective lines and the Milnor rings of the simple elliptic singularities, which is an important step in our main construction (although we do not make use of their results).

1.1. **Relation to the work of Krawitz-Shen.** There is a companion article by Krawitz-Shen [22]. Together, we completely solved all the problems regarding the GW theory and related topics for the above three classes of orbifolds. The idea is from the Landau-Ginzburg/Calabi-Yau correspondence. Since the general philosophy applies to many other examples, let us briefly outline it.

Recall that a polynomial W is called *quasi-homogeneous* if there are rational numbers q_i, called the *degrees* or the *charges* of x_i, such that

$$W(\lambda^{q_0}x_0, \lambda^{q_1}x_1, \dots, \lambda^{q_N}x_N) = \lambda W(x_0, x_1, \dots, x_N)$$

for all $\lambda \in \mathbb{C}^*$. The polynomial W is called *non-degenerate* if: (1) W defines a unique singularity at zero; (2) the choice of q_i is unique. A diagonal matrix

$\operatorname{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N)$ is called an *abelian* or *diagonal symmetry* of W if

$$W(\lambda_0 x_0, \lambda_1 x_1, \ldots, \lambda_N x_N) = W(x_0, x_1, \ldots, x_N).$$

The diagonal symmetries form a group G_{max} which is always nontrivial since it contains the element

$$J_W = \operatorname{diag}(e^{2\pi i q_0}, e^{2\pi i q_1}, \ldots, e^{2\pi i q_N}).$$

When W satisfies the *Calabi-Yau condition* $\sum_i q_i = 1$, $X_W = \{W = 0\}$ defines a Calabi-Yau hypersurface in the weighted projective space $\mathbb{P}^N(c_0, c_1, \ldots, c_N)$, where $q_i = c_i/d$ for a common denominator d. The element J_W acts trivially on X_W, while for any subgroup G such that $\langle J_W \rangle \subseteq G \subseteq G_{max}$, the group $\widetilde{G} = G/\langle J \rangle$ acts faithfully on X_W. The LG/CY correspondence predicts that the FJRW theory of (W, G), up to analytic continuation and the quantization of a symplectic transformation, is equivalent to the Gromov-Witten theory of $X_W/\widetilde{G}$ [30]. The case studied here are *mirror* of the three classes of simple elliptic singularities: $\widetilde{E}_N (N = 6, 7, 8)$. More precisely,

$$\begin{aligned}
\mathbb{P}^1(3,3,3) =& \{P_8^T := x_0^3 + x_1^3 + x_2^3 = 0\}/\widetilde{G}_{max}, \\
\mathbb{P}^1(2,4,4) =& \{X_9^T := x_0^2 x_1 + x_1^3 + x_0 x_2^2 = 0\}/\widetilde{G}_{max}, \\
\mathbb{P}^1(2,3,6) =& \{J_{10}^T := x_0^3 + x_1^3 + x_1 x_2^2 = 0\}/\widetilde{G}_{max}.
\end{aligned}$$

Chiodo–Ruan (see [5]), proposed a three-step approach to the LG/CY correspondence based on the B-model. Let us take simple elliptic singularities to simplify the notation. By Berglund-Hübsch-Krawitz, P_8^T, X_9^T, J_{10}^T with G_{max} is mirror to P_8, X_9, J_{10} with the trivial group. There is a B-model construction in the latter case in terms of Saito-Givental theory. More precisely, consider the miniversal deformation of the simple elliptic singularities in the so-called *marginal direction*: $P_8 + s x_0 x_1 x_2, X_9 + s x_0 x_1 x_2$, or $J_{10} + s x_0 x_1 x_2$ for all the nonsingular values of s. According to Saito, the above miniversal deformation space admits a generic semisimple Frobenius manifold structure. Givental has constructed a higher genus generating function $\mathcal{F}_g$ over semisimple points. We should mention that the original Saito-Givental theory is defined for a germ of singularities. On the other hand, we study a "global" version of the Saito-Givental theory, where the marginal parameter s is deformed from zero to infinity. In fact, the modularity arises only from this global point of view. To emphasis this key perspective, we often refer to it as a *global Saito-Givental theory*.

Chiodo–Ruan (see [5]) proposed that (i) FJRW theory of (W, G_{max}) is mirror to a global Saito-Givental theory at $s = 0$; (ii) GW theory of $X_W/\widetilde{G}_{max}$ is mirror to global Saito-Givental theory at $s = \infty$; (iii) global Saito-Givental theory at $s = 0$ is related to global Saito-Givental theory at $s = \infty$ by analytic continuation and quantization of a symplectic transformation. This article and that of Krawitz-Shen studied completely different aspects of this problem and can be treated as a single package. In particular, Krawitz and Shen proved (i), (ii) in [22] by a direct computation of the A-model for both GW theory and FJRW theory. In this article, we gave a detailed study of the B-model and affirm (iii). Therefore, by using our theorem (see Theorem 4.2), Krawitz-Shen deduced the LG/CY correspondence of all genera for the above examples.

Simple elliptic singularities are usually organized in one-parameter families, which according to Saito's interpretation (see [34]) can be viewed as a pull-back of some universal family parametrized by the modular curve. Let us point out that we are slightly abusing notation, because for X_9 and J_{10} we use respectively the normal forms $x_0^2x_2 + x_0x_1^3 + x_2^2$ and $x_0^3 + x_1^3x_2 + x_2^2$ instead of $x_0^4+x_1^4+x_2^2$ and $x_0^6+x_1^3+x_2^2$. The main motivation for our choice is to simplify the exposition. The LG/CY correspondence can be proved for other normal forms as well. We will deal with the remaining cases in a separate publication.

On the other hand, we proved much more than just (iii), namely the modularity of global Saito-Givental theory! This is not a consequence of the LG/CY correspondence and represents an entirely new direction in GW theory. However, to draw the consequence for the A-model such as GW theory, we use Krawitz-Shen's theorems at critical places. Namely, we use (i) to prove the extendibility of global Saito-Givental theory over the caustic and (ii) to connect our result to GW theory.

Finally, the appearance of modularity in the B-model comes from the global behavior of the primitive form used to define the Frobenius structure. In general, it is a difficult problem to compute the primitive forms. However, in the cases under consideration, a primitive form and the corresponding Frobenius structure are determined by a choice of symplectic basis of H_1 of the corresponding elliptic curve. This basis determines a point $\tau \in \mathbb{H}$ on the upper half-plane for each value of the parameter s. The domain of the parameter s can be identified with the quotient of $\mathbb{H}$ by the monodromy group Γ. Therefore, the global Saito-Givental function $\mathcal{F}_g$ should be viewed as a function of $\tau \in \mathbb{H}$. In this paper, we study the transformation of $\mathcal{F}_g(\tau)$ under $\tau \to \nu(\tau)$ for $\nu \in \Gamma$. The transformation of $\mathcal{F}_g(\tau)$ is given by the quantization of a certain symplectic transformation. This confirms (iii). We want to emphasise that $\mathcal{F}_g$ does *not* transform as a modular form. A crucial idea, motivated by physics, is to complete $\mathcal{F}_g(\tau)$ in a specific way to a *non-holomorphic* function $\mathcal{F}_g(\tau, \bar{\tau})$. The anti-holomorphic completion is the generalization of the corresponding construction of quasi-modular form. Then, we can show

Theorem 1.2. *The modified non-holomorphic function $\mathcal{F}_g(\tau, \bar{\tau})$ transforms as an almost holomorphic modular form (see the detailed statement in Section 4).*

This implies that the original $\mathcal{F}_g(\tau)$ is quasi-modular. Using the results of Krawitz-Shen, it implies Theorem 1.1.

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2. Global Frobenius structures

To simplify the notation, we shall focus on the simple elliptic singularities of the P_8-family:

$$f(\sigma, x) = x_0^3 + x_1^3 + x_2^3 + \sigma x_0 x_1 x_2, \quad x = (x_0, x_1, x_2).$$

σ takes values in the punctured complex line $\Sigma = \{\sigma^3 + 27 \neq 0\}$, so that the origin $x = 0$ is an isolated critical point. The remaining two cases can be analyzed in a similar way. The necessary modifications are explained in Section 6.

2.1. **Basic set-up.** Let us first recall the basic set-up of Saito's Frobenius manifold structure on the miniversal deformation of a singularity. We will use our example to illustrate the procedure.

2.1.1. *The space of miniversal deformations.* Recall (see [1]) the action of the group of germs of holomorphic changes of the coordinates $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ on the space of all germs at 0 of holomorphic functions. Given a holomorphic germ $f(x)$ with an isolated critical point at $x = 0$ we say that the family of functions $F(s, x)$ is a *miniversal deformation* of f if it is transversal to the orbit of f. One way to construct a miniversal deformation is to choose a $\mathbb{C}$-linear basis $\{\phi_i(x)\}$ in the Jacobi algebra $\mathcal{O}_{\mathbb{C}^3,0}/\langle \partial_{x_0} f, \partial_{x_1} f, \partial_{x_2} f\rangle$. Then the following family provides a miniversal deformation:

$$F(s, x) = f(x) + \sum_{i=1}^{\mu} s_i \phi_i(x), \quad s = (s_1, s_2, \ldots, s_\mu) \in \mathbb{C}^\mu,$$

where μ is the dimension of the Jacobi algebra, also known as the *Milnor number* or the *multiplicity* of the critical point.

In our setting the Milnor number is $\mu = 8$. It is convenient to use the index set $\{-1, 0, 1, \ldots, 6\}$ instead of $\{1, \ldots, \mu\}$. We choose the following set of monomials to construct a miniversal deformation: $\phi_{-1} = x_0 x_1 x_2$, $\phi_0 = 1$, and ϕ_i $i = 1, 2, \ldots, 6$ are given respectively by:

$$x_0, \quad x_1, \quad x_2, \quad x_0 x_1, \quad x_0 x_2, \quad x_1 x_2.$$

Note that s_{-1} is naturally identified with σ. Let us assign weight $1/3$ to each variable x_i so that $f(\sigma, x)$ and $\phi_i(x)$ become weighted-homogeneous of degree respectively 1 and $1 - d_i$, where

$$d_{-1} = 0,\ d_0 = 1,\ d_1 = d_2 = d_3 = 2/3, \quad \text{and} \quad d_4 = d_5 = d_6 = 1/3.$$

Note that assigning weight d_i to each s_i turns F into a weighted-homogeneous function of weight 1.

Put $\mathcal{S} = \Sigma \times \mathbb{C}^{\mu-1}$ and $X = \mathcal{S} \times \mathbb{C}^3$. Then we have the following maps:

$$\begin{array}{ccc} \mathcal{S}\times\mathbb{C}^3 & & \\ \varphi\big\downarrow & \searrow & \\ \mathcal{S}\times\mathbb{C} & \xrightarrow[p]{} & \mathcal{S} \end{array} \qquad \begin{array}{l} \varphi(s,x) = (s, F(s,x)), \\ \\ p(s,\lambda) = s. \end{array}$$

By definition the critical set C of F is the support of the sheaf

$$\mathcal{O}_C := \mathcal{O}_X / \langle \partial_{x_0} F, \partial_{x_1} F, \partial_{x_2} F \rangle.$$

The map $\partial/\partial s_i \mapsto \partial F/\partial s_i$ induces an isomorphism between the sheaf $\mathcal{T}_\mathcal{S}$ of holomorphic vector fields on $\mathcal{S}$ and $q_* \mathcal{O}_C$, where $q = p \circ \varphi$. In particular, each tangent space $T_s\mathcal{S}$ is equipped with an associative commutative multiplication $\bullet_s$ depending holomorphically on $s \in \mathcal{S}$. If in addition we have a volume form $\omega = g(s,x) d^3x$, where $d^3x = dx_0 dx_1 dx_2$ is the standard volume form; then $q_*\mathcal{O}_C$ (hence $\mathcal{T}_\mathcal{S}$ as well) is equipped with the *residue pairing*:

$$(\psi_1, \psi_2) = \frac{1}{(2\pi i)^3} \int_{\Gamma_\epsilon} \frac{\psi_1(s,y)\psi_2(s,y)}{F_{y_0} F_{y_1} F_{y_2}}\, \omega, \tag{3}$$

where $y = (y_0, y_1, y_2)$ are unimodular coordinates for the volume form, i.e., $\omega = d^3y$, and Γ_ϵ is a real 3-dimensional cycle supported on $|F_{x_0}| = |F_{x_1}| = |F_{x_2}| = \epsilon$.

Given a semi-infinite cycle

$$\mathcal{A} \in \varprojlim H_3(\mathbb{C}^3, (\mathbb{C}^3)_{-m}; \mathbb{C}) \cong \mathbb{C}^\mu, \tag{4}$$

where

$$(\mathbb{C}^3)_m = \{x \in \mathbb{C}^3 \mid \mathrm{Re}(F(s,x)/z) \le m\}, \tag{5}$$

put

$$J_\mathcal{A}(s,z) = (-2\pi z)^{-3/2}\, z d_\mathcal{S} \int_\mathcal{A} e^{F(s,x)/z} \omega, \tag{6}$$

where $d_\mathcal{S}$ is the de Rham differential on $\mathcal{S}$. The oscillatory integrals $J_\mathcal{A}$ are by definition sections of the cotangent sheaf $\mathcal{T}^*_\mathcal{S}$.

According to Saito's theory of primitive forms [33, 36], there exists a volume form ω such that the residue pairing is flat and the oscillatory integrals satisfy a system of differential equations, which in flat-homogeneous coordinates $t = (t_{-1}, t_0, \ldots, t_6)$ has the form

$$z\partial_i J_{\mathcal{A}}(t,z) = \partial_i \bullet_t J_{\mathcal{A}}(t,z), \tag{7}$$

where $\partial_i := \partial/\partial t_i$ $(-1 \le i \le 6)$ and the multiplication is defined by identifying vectors and covectors via the residue pairing. Due to homogeneity the integrals satisfy a differential equation with respect to the parameter $z \in \mathbb{C}^*$:

$$(z\partial_z + E)J_{\mathcal{A}}(t,z) = \theta\, J_{\mathcal{A}}(t,z), \tag{8}$$

where

$$E = \sum_{i=-1}^{6} d_i t_i \partial_i, \quad (d_i := \deg t_i = \deg s_i),$$

is the *Euler vector field* and θ is the so-called *Hodge grading operator* :

$$\theta : \mathcal{T}_S^* \to \mathcal{T}_S^*, \quad \theta(dt_i) = \Big(\frac{1}{2} - d_i\Big) dt_i.$$

The compatibility of the system (7)–(8) implies that the residue pairing, the multiplication, and the Euler vector field give rise to a *conformal Frobenius structure* of conformal dimension 1. We refer to B. Dubrovin [9] for the definition and more details on Frobenius structures.

For the simple elliptic singularities of type P_8 the primitive forms can be described as follows. Let $\pi(\sigma)$ be a solution to the differential equation

$$\frac{d^2u}{d\sigma^2} + \frac{3\sigma^2}{\sigma^3+27}\,\frac{du}{d\sigma} + \frac{\sigma}{\sigma^3+27}\, u = 0; \tag{9}$$

then the form $\omega = d^3x/\pi(\sigma)$ is primitive. For the reader's convenience we prove this statement in Appendix A.

2.2. **Global Frobenius structures.** Traditionally, one studies the germ at $s = 0$ of the Frobenius structure in singularity theory, because in general the primitive form is known to exist only locally (as a germ with respect to the deformation parameters s). For our purposes however, we would like to vary the Frobenius structure from $s_{-1} = 0$ to $s_{-1} = \infty$. This leads to the study of *global Frobenius structure.* In this subsection, we shall treat the construction of Saito's Frobenius manifold structure with this purpose in mind. Our first goal is to define primitive forms globally in the sense that they vary with s in a nice fashion.

2.2.1. *Periods of elliptic curves and global primitive forms.* Put $X_{s,\lambda}$ for the fiber $\varphi^{-1}(s,\lambda)$ and let $(\mathcal{S}\times\mathbb{C})'$ be the set of all (s,λ) that parametrize non-singular fibers $X_{s,\lambda}$. The complement of $(\mathcal{S}\times\mathbb{C})'$ is a analytic hypersurface, called *the discriminant*, and the union X' of all non-singular fibers $X_{s,\lambda}$ is a smooth fibration over $(\mathcal{S}\times\mathbb{C})'$, called *the Milnor fibration.* Following Looijenga [23] we compactify the fibers $X_{s,\lambda}$ by adding an elliptic curve. Namely, the map

$$\mathcal{S}\times\mathbb{C}^3\to\mathcal{S}\times\mathbb{C}P^3,\quad (s,x)\mapsto(s,[x_0,x_1,x_2,1])$$

is an embedding and we denote by $\overline{X}$ the Zariski closure of X in $\mathcal{S}\times\mathbb{C}P^3$. The map $\varphi: X\to\mathcal{S}\times\mathbb{C}$ naturally extends to a map $\overline{X}\to\mathcal{S}\times\mathbb{C}$. We denote by $\overline{X}_{s,\lambda}$ the corresponding fibers. It is easy to check that the intersection of $\overline{X}_{s,\lambda}$ with the hyperplane $\{z_3=0\}$ (here $[z_0,z_1,z_2,z_3]$ are homogeneous coordinates of $\mathbb{C}P^3$) is the elliptic curve (known also as the *elliptic curve at infinity*):

$$E_\sigma:\quad z_0^3+z_1^3+z_2^3+\sigma z_0z_1z_2=0,$$

where $\sigma=s_{-1}$. Moreover, the Gelfand–Leray form d^3x/dF gives rise to a holomorphic form on $X_{s,\lambda}$ that has a simple pole along E_σ, and therefore its Poincaré residue $\mathrm{Res}_{E_\sigma}[d^3x/dF]$ is a holomorphic 1-form on E_σ of degree 0, so it depends only on $\sigma=s_{-1}$ but not on $s_0,s_1,\dots,s_6$ (see [23]).

According to K. Saito (see [33]) the primitive forms for simple elliptic singularities are parametrized by the periods of E_σ

$$\pi_A(\sigma):=2\pi i\int_{A_\sigma}\mathrm{Res}_{E_\sigma}[d^3x/dF]\ ,\tag{10}$$

where $A\in H_1(E_{\sigma_0},\mathbb{C})$ is any non-zero 1-cycle and A_σ is a flat family of cycles uniquely determined by A for all σ in a small neighborhood of σ_0. In Appendix A we prove that the space of solutions to (9) coincides with the space of all periods $\pi_A(\sigma)$. Slightly abusing the notation, we often omit the index σ from A_σ and use A to denote the flat family of cycles induced by A.

For our purposes it is convenient to rewrite the integral (10) as a period of the Gelfand–Leray form. Namely, let $X_{s,\lambda}$ be any non-singular fiber of the Milnor fibration such that $s_{-1}=\sigma$. The boundary of any tubular neighborhood of E_σ in $\overline{X}_{s,\lambda}$ is a circle bundle over E_σ that induces via pullback an injective *tube map* $L: H_1(E_\sigma)\to H_2(X_{s,\lambda})$. Let $a=L(A)$; then we have

$$\pi_A(\sigma)=\pi_a(s):=\int_a\frac{d^3x}{dF}.\tag{11}$$

We refer to a as a *tube* or *toroidal* cycle. The space of all toroidal cycles coincides with the kernel of the intersection pairing on $H_2(X_{s,\lambda};\mathbb{C})$ (see [11, 23]).

A flat family of cycles A is a multi-valued object; therefore the induced global primitive form and global Frobenius structure are multi-valued as well. This leads to the key observation that, when discussing a global Frobenius structure, it is more natural to replace Σ by its universal cover. The latter is naturally identified with the upper half-plane $\mathbb{H}$. Namely, fix a reference point, say $\sigma_0 = 0$. The points in the universal cover $\tilde{\Sigma}$ of Σ are pairs consisting of a point $\sigma \in \Sigma$ and a homotopy class of paths $l(t)$ with $l(0) = \sigma_0, l(1) = \sigma$. We fix a symplectic basis $\{A', B'\}$ of $H_1(E_{\sigma_0}; \mathbb{Z})$ once and for all. The map $(\sigma, l(t)) \mapsto \tau' = \pi_{B'}/\pi_{A'}$, where the periods $\pi_{B'}$ and $\pi_{A'}$ are analytically continued along the path $l(t)$, defines an analytic isomorphism between the universal cover of $\tilde{\Sigma}$ and the upper half-plane $\mathbb{H}$. In other words, we have a Frobenius structure on $\mathbb{H} \times \mathbb{C}^{\mu-1}$ for any non-zero cycle

$$A = dA' + cB' \in H_1(E_{\sigma_0}; \mathbb{C}), \quad -d/c \notin \mathbb{H}. \tag{12}$$

2.2.2. *Flat coordinates.* The goal in this section is to construct a flat homogeneous coordinate system $t = (t_{-1}, t_0, \dots, t_6)$. The idea is to expand the oscillatory integrals into a power series near $z = \infty$. The flat coordinates will be identified with the leading coefficients in these expansions. The problem of analyzing the Gauss–Manin connection at $z = \infty$ was addressed by M. Noumi [25] while the construction of flat coordinates for simple and simple elliptic singularities can be found in [26]. The combination of these two articles implies the result that we need. However, our point of view is somewhat different from the one in [25]. For the reader's convenience as well as to avoid any misunderstanding we give a self-contained exposition.

Let $\alpha(\sigma, 1) \in H_2(X_{\sigma,1}; \mathbb{Z})$ be a flat family of cycles defined for σ near $\sigma_0 = 0$. Using the rescaling $x \mapsto \lambda^{1/3}x$ we obtain a cycle $\alpha(\sigma, \lambda) \in H_2(X_{\sigma,\lambda}; \mathbb{C})$. The cycle $\mathcal{A}$ formed by $\alpha(\sigma, z\lambda)$ as λ varies along the path

$$\lambda : [0, \infty) \to \mathbb{C}, \quad \lambda(t) = -t,$$

is a semi-infinite cycle of the type (4). The corresponding oscillatory integral takes the form

$$\int_{\mathcal{A}} e^{F/z}\omega = z \int_0^{-\infty + i0} e^{\lambda} \int_{\alpha(\sigma,\lambda)} e^{\sum_{j=0}^{6} s_j \phi_j(x) z^{-d_j}} \frac{\omega}{df}\, d\lambda.$$

Rescaling $x \mapsto \lambda^{1/3}x$ and expanding the integrand in powers of z we get

$$(-2\pi z)^{-3/2} \int_{\mathcal{A}} e^{F/z}\omega = z^{-1/2} \sum_{\delta} \Big(\int_{\alpha(\sigma,1)} c_\delta(s, x) \frac{\omega}{df} \Big) z^{-\delta}, \tag{13}$$

where the sum is over all non-negative elements of the lattice in $\mathbb{Q}$ spanned over $\mathbb{Z}$ by the degrees d_i and $c_\delta(s, x)$ is

$$\sum \widetilde{\Gamma}(k_0(1 - d_0) + k_1(1 - d_1) + \cdots) \frac{s_0^{k_0}}{k_0!} \frac{s_1^{k_1}}{k_1!} \cdots (\phi_0(x))^{k_0} (\phi_1(x))^{k_1} \cdots,$$

where the summation is over all integers $k_i \geq 0$ such that $k_0d_0 + k_1d_1 + \cdots = \delta$ (note that the sum is finite) and

$$\widetilde{\Gamma}(k) := (-2\pi)^{-3/2}\,(-1)^{k+1}\,\Gamma(k+1) = (2\pi)^{-3/2}e^{\pi i(k-1/2)}\int_0^\infty e^{-t}t^k dt.$$

Given a set of middle homology cycles α_i $(-1 \leq i \leq 6)$ we define the matrix Π whose entries are the following periods

$$\Pi_{\delta,i} = \int_{\alpha_i(\sigma,1)} c_\delta(s,x)\frac{\omega}{df}, \quad i = -1, 0, \ldots, 6,$$

where the index δ takes values in $\{0, 1, 1/3, 2/3\}$. The order in the latter set is such that it matches the rows in which the entries $\Pi_{\delta,i}$ should be placed.

Let $t = (t_{-1}, t_0, \ldots, t_6)$ be a flat-homogeneous coordinate system with degrees $\deg t_i = d_i$. It is convenient to introduce the following involution $'$ on the index set $\{-1, 0, 1, \ldots, 6\}$:

$$(-1)' = 0, \quad 0' = -1, \quad \text{and} \quad i' = 7 - i \quad \text{for} \quad 1 \leq i \leq 6. \tag{14}$$

Let us fix flat coordinates such that the residue pairing has the form $(\partial_i, \partial_j) = \delta_{ij'}$. Finally, we form the following matrix:

$$\begin{bmatrix} t_{-1} & 1 \\ t^2/2 & t_0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & t_4 & t_5 & t_6 \\ t_1 & t_2 & t_3 & 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

where $t^2 = \sum_{i=-1}^6 t_i t_{i'}$. By direct sum $M_1 \oplus M_2$ of two matrices M_1 and M_2 (not necessarily diagonal!) we mean a block-diagonal matrix with M_1 and M_2 on the diagonal.

Lemma 2.1. *There are cycles α_i such that:*

(a) *The period matrix Π coincides with (15).*
(b) *The cycles α_i $(-1 \leq i \leq 6)$ are eigenvectors of the classical monodromy operator with eigenvalues $e^{-2\pi\sqrt{-1}\,d_i}$.*
(c) *The cycle $\alpha_0 = -(-2\pi)^{3/2}L(A)$, where $L(A)$ is the tube cycle that parametrizes the Frobenius structure.*

Proof. Let $\mathcal{A}_i$ be the semi-infinite cycles constructed from α_i via rescaling. The oscillatory integrals $J_{\mathcal{A}_i}(s,z)$ satisfy the differential equations (7) and (8). The coordinates of $J_{\mathcal{A}_i}(s,z)$ with respect to the 1-forms $dt_{-1}, dt_0, \ldots, dt_6$ give rise to column vectors and we put $J(s,z)$ for the matrix formed by these columns. Using (8) we get that $J(s,z)$ has the following form:

$$(S_0 + S_1z^{-1} + S_2z^{-2} + \cdots)\, z^\theta,$$

while (7) implies that S_0 is a constant matrix independent of t and z. Changing the cycles α_i if necessary we can arrange that $S_0 = 1$.

(a) Let us compare the coefficients in front of $z^{-\delta}$ for $0 \leq \delta \leq 1$ in

$$J_{\mathcal{A}_i}(t,z) = S(t,z) z^{\theta} dt_i. \tag{16}$$

The RHS equals

$$z^{-d_i+1/2} S(t,z) dt_i = z^{1/2} \left(z^{-d_i} dt_i + \delta_{i,-1} z^{-1} S_1 dt_{-1} + \cdots \right),$$

where the dots stand for terms involving $z^{-\delta}$ with $\delta > 1$. We have $S_1 dt_{-1} = \sum t_i dt_{i'}$, because both co-vectors satisfy the differential equations $Lie_{\partial_i} v(t) = dt_{i'}$ and the initial condition $v(0) = 0$. Therefore, $S_1 dt_{-1} = dt^2/2$. Comparing the coefficients in front of $z^{-\delta}$ for $0 \leq \delta < 1$ in (16) we get (using also (13)) that $d_{\mathcal{S}} \Pi_{\delta,i}$ is either 0 if $d_i \neq \delta$, or dt_i if $d_i = \delta$. When $\delta = 1$ we have:

$$d_{\mathcal{S}} \Pi_{1,0} = dt_0 \quad \text{and} \quad d_{\mathcal{S}} \Pi_{1,-1} = dt^2/2.$$

In other words, up to some constant 4×8 matrix C the period matrix has the form that we want. In order to fix the constants we set $t_0 = \cdots = t_6 = 0$. Up to some non-zero constant factors the differential forms $c_\delta(s,x)\omega/df$, $\delta = 0, 1, 1/3, 2/3$, coincide repectively with

$$\frac{\omega}{df}, \quad (s_0 + \cdots)\frac{\omega}{df}, \quad \Big(\sum_{i=4}^{6} s_i \phi_i(x) + \cdots \Big) \frac{\omega}{df}, \quad \Big(\sum_{i=1}^{3} s_i \phi_i(x) + \cdots \Big) \frac{\omega}{df},$$

where the dots stand for at least quadratic polynomials in $s_0, s_1, \ldots, s_6$. All periods vanish when $t_0 = \cdots = t_6 = 0$, except for $\Pi_{0,-1}$ and $\Pi_{0,0}$ (note that $\Pi_{0,i} = 0$, $1 \leq i \leq 6$, follows from $\Pi_{1,i} = 0$). We need to prove only that $C_{0,-1} = C_{0,0} = 0$. We return to these identities once we establish (b) and (c).

(b) The Gelfand–Leray forms $\phi_i(x)\omega/df$ give rise to a basis of eigenvectors for the classical monodromy operator with eigenvalues $e^{2\pi\sqrt{-1}d_i}$. Since we already proved in (a) that $\Pi_{0,1} = \Pi_{1,1} = \Pi_{\frac{1}{3},1} = 0$, we get

$$\int_{\alpha_1} \phi_i \omega/df = 0 \quad \text{for} \quad i = -1, 0, 4, 5, 6.$$

In other words, α_1 belongs to the dual space of the space of middle cohomology classes spanned by $\phi_i(x)\omega/df$, $1 \leq i \leq 3$. The latter is the eigenspace with eigenvalue $e^{2\pi\sqrt{-1}d_1}$; hence α_1 is an eigenvector with eigenvalue $e^{-2\pi\sqrt{-1}d_1}$. The remaining cases are analyzed in a similar way.

(c) Let us substitute $t_1 = \cdots = t_6 = 0$ in the 2×2 block of Π formed by the intersection of the rows $d = 0, 1$ and the columns $i = -1, 0$. We get the following table of identities:

$$-(-2\pi)^{-3/2}\int_{\alpha_{-1}(\sigma,1)}\frac{\omega}{df}=t_{-1}+C_{0,-1}\qquad -(-2\pi)^{-3/2}\int_{\alpha_0(\sigma,1)}\frac{\omega}{df}=1+C_{0,0}$$

$$-(-2\pi)^{-3/2}s_0\int_{\alpha_{-1}(\sigma,1)}\frac{\omega}{df}=t_0t_{-1}\qquad -(-2\pi)^{-3/2}s_0\int_{\alpha_0(\sigma,1)}\frac{\omega}{df}=t_0.$$

Put $\alpha_0=m\,a+n\alpha_{-1}$, where $a=L(A)$. Then the $(0,0)$-identity (keep in mind also the $(0,-1)$-identity) turns into

$$-(-2\pi)^{-3/2}m+n(t_{-1}+C_{0,-1})=1+C_{0,0},$$

It follows that $n=0$. The $(1,0)$-identity gives $-(-2\pi)^{-3/2}ms_0=t_0$, while the remaining two identities give $s_0(t_{-1}+C_{0,-1})=t_0t_{-1}=-(-2\pi)^{-3/2}ms_0t_{-1}$. From here we get $m=-(-2\pi)^{3/2}$ and $C_{0,-1}=0$, which imply also that $C_{0,0}=0$. □

2.3. **Modular transformations of the Frobenius structure.** Every closed loop C in Σ based at σ_0 induces a monodromy transormation ν of both $H_2(X_{\sigma_0};\mathbb{C})$ and $H^2(X_{\sigma_0};\mathbb{C})$. We refer to ν as a *modular transformation*, while the set of all modular transformations forms a group which we call the *modular group* of the family of singularities at hand. Let us fix a basis of cycles α_i satisfying the conditions in Lemma 2.1.

2.3.1. *Modular transformations.* The middle cohomology groups $H^2(X_{s,\lambda};\mathbb{C})$ form a vector bundle equipped with a flat Gauss–Manin conncetion. Given a holomorphic form $\phi(s,x)d^3x$ the integrals $\int\phi(s,x)d^3x/dF$ and $\int d^{-1}(\phi(s,x)d^3x)$ determine naturally sections of the middle cohomology bundle. Here d is the de Rham differential with respect to $x\in\mathbb{C}^3$ and $d^{-1}\omega$ means any 2-form η such that $d\eta=\omega$. We have the following formulas for the covariant derivatives of such sections (see [1]):

$$\nabla_{\partial/\partial s_i}\int d^{-1}(\phi(s,x)d^3x)=-\int\frac{\partial F}{\partial s_i}\,\phi(s,x)\,\frac{d^3x}{dF}+\int d^{-1}\mathrm{Lie}_{\partial/\partial s_i}(\phi(s,x)d^3x)$$

and

$$\nabla_{\partial/\partial\lambda}\int d^{-1}(\phi(s,x)d^3x)=\int\phi(s,x)\frac{d^3x}{dF}.$$

The second formula implies the following identity:

$$\partial_\lambda\int_\alpha\phi(s,x)\partial_{x_i}F\frac{d^3x}{dF}=\int_\alpha\partial_{x_i}\phi(s,x)\frac{d^3x}{dF},\tag{17}$$

where α is some middle homology cycle. Indeed, the integrand on the LHS equals $\phi(x)d^3x/dx_i$ while the one on the RHS is $d(\phi(x)d^3x/dx_i)/dF$.

Lemma 2.2. *Let ν be a modular transformation; then the matrix of ν with respect to the basis $\{\alpha_i\}_{i=-1}^{6}$ has the following block-diagonal form:*

$$g \oplus \mathrm{Diag}(e^{2\pi i d_1 k}, \dots, e^{2\pi i d_6 k}), \tag{18}$$

for some $(g,k) \in \mathrm{SL}(2;\mathbb{C}) \times \mathbb{Z}$.

Proof. Let us compute the monodromy of the following sections $\int \phi_i(x) d^3x/df$, $-1 \le i \le 6$. The space spanned by the sections with $i=-1$ and $i=0$ is dual to the space of toroidal cycles, i.e., it is isomorphic to the homology group $H_1(E_\sigma;\mathbb{C})$. Note that

$$\partial_\sigma \int_\alpha x_0 \frac{d^3x}{df} = -\partial_\lambda \int_\alpha x_0^2 x_1 x_2 \frac{d^3x}{df}.$$

On the other hand

$$x_0^2 x_1 x_2 = \frac{9}{\sigma^3+27} x_1 x_2 f_{x_0} + \frac{\sigma^2}{\sigma^3+27} x_0 x_1 f_{x_1} - \frac{3\sigma}{\sigma^3+27} x_1^2 f_{x_2}$$

Using formula (17) we get (recall that $\phi_1(x) = x_0$)

$$\partial_\sigma \int_\alpha \phi_1(x) \frac{d^3x}{df} = -\frac{\sigma^2}{\sigma^3+27} \int_\alpha \phi_1(x) \frac{d^3x}{df}.$$

Solving this differential equation for σ we get

$$\int \phi_1(x) \frac{d^3x}{df} = (\sigma^3+27)^{-1/3} A_1,$$

where $A_1 \in H^2(X_{\sigma_0,1};\mathbb{C})$ is a flat section of the middle cohomology bundle. Under analytic continuation along a simple loop around $(-27)^{1/3}$ the RHS gains a factor of ϵ^{-1}, where $\epsilon = e^{2\pi i/3}$. Therefore, A_1 is an eigenvector of the corresponding monodromy transformation with eigenvalue ϵ. Similarly, one proves that

$$\int \phi_i(x) \frac{d^3x}{df} = (\sigma^3+27)^{-1+d_i} A_i, \quad 1 \le i \le 6,$$

where A_i $(1 \le i \le 6)$ are eigenvectors with eigenvalues $e^{-2\pi i d_i}$.

Finally, using Lemma 2.1 we get

$$\int_{\alpha_i} \phi_j(x) \frac{\omega}{df} = 0$$

in the following two cases: (1) $i = 1,2,3$ and $j = -1,0,4,5,6$; (2) $i = 4,5,6$ and $j = -1,0,1,2,3$. This means that α_i and A_i belong to eigenspaces that are dual to each other. The lemma follows. $\square$

2.3.2. *Modular transformations of the flat coordinates.* According to Lemma 2.1 there is a cycle $B = bA' + aB'$, linearly independent from A, s.t.,

$$t_{-1} := \frac{\pi_B}{\pi_A} = \frac{a\tau' + b}{c\tau' + d} \tag{19}$$

is a flat coordinate and the residue pairing of the vector fields 1 and $\partial/\partial t_{-1}$ is 1. More precisely, in the notation of Lemma 2.1, we have

$$\alpha_{-1} = -(-2\pi)^{3/2}L(B), \quad \alpha_0 = -(-2\pi)^{3/2}L(A).$$

The intersection pairing on the elliptic curve at infinity, up to a sign, is the same as the Seifert form of the corresponding toroidal cycles. Using Theorem 10.28(i) from [18], we see that up to a sign the intersection number $A \circ B$ must be $\sqrt{-1}$.

Remark 2.3. *The basis $\{A, B\}$ is not symplectic and t_{-1} is not a modulus of the elliptic curve at infinity.*

The analytic continuation along a path C transforms t_{-1} into

$$g(t_{-1}) := \frac{n_{11}t_{-1} + n_{21}}{n_{12}t_{-1} + n_{22}}, \tag{20}$$

where (n_{ij}) is the matrix (from $\mathrm{SL}_2(\mathbb{C})$) that describes the parallel transport g of $\{\alpha_{-1}, \alpha_0\}$ along C, i.e.,

$$g(\alpha_{-1}) = n_{11}\alpha_{-1} + n_{21}\alpha_0, \quad \text{and} \quad g(\alpha_0) = n_{12}\alpha_{-1} + n_{22}\alpha_0. \tag{21}$$

For a given matrix $g = (n_{ij}) \in \mathrm{SL}_2(\mathbb{C})$ we adopt the number theorist's notation

$$j(g, t_{-1}) := n_{12}t_{-1} + n_{22}.$$

Lemma 2.4. *The analytic continuation along the loop C induces a coordinate change $t \mapsto \nu(t)$ of the following form:*

$$\nu(t)_{-1} = g(t_{-1}), \quad \nu(t)_0 = t_0 + \frac{n_{12}}{2j(g, t_{-1})} \sum_{i=1}^{6} t_i t_{i'}, \quad \nu(t)_i = \frac{e^{2\pi i d_i k}}{j(g, t_{-1})}\, t_i \ (1 \le i \le 6),$$

where k is some integer.

Proof. Note that the cycles α_{-1} and α_0 are transformed respectively into $n_{11}\alpha_{-1} + n_{21}\alpha_0$ and $n_{12}\alpha_{-1} + n_{22}\alpha_0$. The period $\int_{\alpha_0} d^3x/df$ is transformed into $(n_{12}t_{-1} + n_{22}) \int_{\alpha_0} d^3x/df$, which implies that the primitive form ω transforms into $\omega/j(\nu, t_{-1})$. According to Lemma 2.1, we have $t_0 = \int_{\alpha_0} c_1(s, x)\omega/df$. Hence t_0 is transformed into

$$\Big(n_{22}t_0 + n_{12} \int_{\alpha_{-1}} c_1(s, x)\omega/df\Big) j(\nu, t_{-1})^{-1}.$$

According to Lemma 2.1 the above integral is $\Pi_{1,-1} = t_{-1}t_0 + \frac{1}{2}\sum_{j=1}^{6} t_j t_{j'}$. This proves the transformation law for t_0. The remaining ones are proved in a similar way by using Lemmas 2.1 and 2.2. □

2.4. **Changing the Frobenius structures.** Let A_i, $i = 1, 2$ be two cycles of the form (12). We pick a cycle B_i for A_i as explained above and let g_i be the matrix such that $t^i_{-1} = g_i(\tau')$. Since the intersection numbers $A_i \circ B_i (i = 1, 2)$ are equal, the matrix

$$g := g_2 g_1^{-1} =: \begin{bmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{bmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

Let $k \in \mathbb{Z}$ be an arbitrary integer.

Lemma 2.5. *The map $t \mapsto \widetilde{t}$, defined by*

$$\widetilde{t}_{-1} = g(t_{-1}), \quad \widetilde{t}_0 = t_0 + \frac{n_{12}}{2j(g, t_{-1})} \sum_i t_i t_{i'}, \quad \widetilde{t}_i = \frac{e^{2\pi i d_i k}}{j(g, t_{-1})} t_i,$$

respects the residue pairings corresponding to A_1 and A_2.

The proof is straightforward and it is omitted. In particular, this lemma allows us to identify the flat vector fields arising from two different families of flat cycles A_1 and A_2. Note however, that the corresponding Frobenius multiplications are identical if and only if the cycle A_2 is obtained from A_1 by means of parallel transport along a closed loop. The reason for this is that every analytic isomorphism $t^1_{-1} = g_1(\tau') \mapsto t^2_{-1} = g_2(\tau')$ has the form $g_2 g g_1^{-1}$, where g is an automorphism of $\mathbb{H}$, i.e., $g \in \mathrm{SL}_2(\mathbb{R})$. The structure constants of the Frobenius multiplications are functions of σ; therefore if we think of σ as a function on the upper half-plane, then it should be g-invariant. But the automorphisms of $\mathbb{H}$ that preserve σ are precisely the elements of the modular group of the P_8-singularity, i.e., the group of deck transformations of the universal cover $\mathbb{H} \to \Sigma$.

3. Givental's higher genus potential

In this section, we introduce Givental's higher genus potential $\mathcal{F}_{g,formal}$ which will be our object of study.

3.1. **Symplectic vector space.** Let H be the space of flat vector fields on $\mathcal{S}$ equipped with the residue pairing $(\, , \,)$. Following Givental we introduce the vector space $\mathcal{H} = H((z))$ of formal Laurent series in z^{-1} equipped with the symplectic structure

$$\Omega(f(z), g(z)) = \mathrm{res}_{z=0}(f(-z), g(z))dz.$$

Using the polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = H[[z^{-1}]]z^{-1}$ we identify $\mathcal{H}$ with the cotangent bundle $T^*\mathcal{H}_+$. Let us fix flat coordinates

$$t = (t_{-1}, t_0, \dots, t_6), \quad (\partial_i, \partial_j) = \delta_{i,j'},$$

where $\partial_i = \partial/\partial t^i$ and $'$ is the involution (14).

Using the residue pairing we identify the tangent and the cotangent bundle $T\mathcal{S} \cong T^*\mathcal{S}$. Using the flat coordinates we trivialize the cotangent bundle $T^*\mathcal{S} \cong \mathcal{S} \times H$. In this way, H turns into the space of flat holomorphic differential 1-forms. We use the basis $\{dt_i\}_{i=-1}^6$ of H in order to represent the linear transformations of H by matrices of size μ.

3.1.1. *The stationary phase asymptotics.* Let $s \in \mathcal{S}$ be a semi-simple point, i.e., the critical values u_i $(1 \le i \le \mu)$ form locally near s a coordinate system. Then we have an isomorphism

$$\Psi : \mathbb{C}^\mu \to H \cong T_s\mathcal{S}, \quad e_i \mapsto \sqrt{\Delta_i}\,\partial/\partial u_i, \quad (\partial/\partial u_i, \partial/\partial u_j) = \delta_{ij}/\Delta_i,$$

that diagonalizes the Frobenius multiplication and the residue pairing:

$$e_i \bullet e_j = \sqrt{\Delta_i} e_i \delta_{i,j}, \quad (e_i, e_j) = \delta_{ij}.$$

The system of differential equations (7) and (8) admits a unique formal solution of the type

$$\Psi R(s,z) e^{U/z}, \quad R(s,z) = 1 + R_1(s) z + R_2(s) z^2 + \cdots$$

where U is a diagonal matrix with entries $u_1, \dots, u_\mu$ on the diagonal and $R_k(s) \in \mathrm{End}(\mathbb{C}^\mu)$. Alternatively this formal solution coincides with the stationary phase asymptotics of the following integrals. Let $\mathcal{B}_i$ be the semi-infinite cycle of the type (4) consisting of all points $x \in \mathbb{C}^3$ such that the gradient trajectories of $-\mathrm{Re}(F/z)$ flow into the critical value u_i. Then

$$(-2\pi z)^{-3/2} \; z d_S \int_{\mathcal{B}_i} e^{F(s,x)/z} \omega \;\sim\; e^{u_i/z} \Psi R(s,z) e_i \quad \text{as} \quad z \to 0.$$

We refer to [1, 15] for more details and proofs.

3.2. **The total ancestor potential.** Let us fix the Darboux coordinate system on $\mathcal{H}$ given by the linear functions q_k^i, $p_{k,i}$ defined as follows:

$$\mathbf{f}(z) = \sum_{k=0}^{\infty} \sum_{i=-1}^{6} (q_k^i\, \partial_i\, z^k + p_{k,i}\, dt_i\, (-z)^{-k-1}) \quad \in \quad \mathcal{H},$$

where dt_i is identified via the residue pairing with $\partial_{i'}$.

It is known (and it is easy to prove) that R is a symplectic transformation, i.e., ${}^{\mathrm{T}}R(-z)R(z) = I_\mu$. Note that R has the form $e^{A(z)}$, where $A(z)$ is an infinitesimal symplectic transformation. On the other hand, a linear transformation $A(z)$ is infinitesimal symplectic if and only if the map $\mathbf{f} \in \mathcal{H} \mapsto A\mathbf{f} \in \mathcal{H}$ defines a Hamiltonian vector field with Hamiltonian given by the quadratic function $h_A(\mathbf{f}) = \frac{1}{2}\Omega(A\mathbf{f},\mathbf{f})$. By definition, the quantization of e^A is given by the differential operator $e^{\widehat{h}_A}$, where the quadratic Hamiltonians are quantized according to the following rules:

$$(p_{k,i}p_{l,j})\widehat{\ } = \hbar\frac{\partial^2}{\partial q_k^i\partial q_l^j},\quad (p_{k,i}q_l^j)\widehat{\ } = (q_l^j p_{k,i})\widehat{\ } = q_l^j\frac{\partial}{\partial q_k^i},\quad (q_k^i q_l^j)\widehat{\ } = q_k^i q_l^j/\hbar.$$

Note that the quantization defines a projective representation of the Poisson Lie algebra of quadratic Hamiltonians:

$$[\widehat{F},\widehat{G}] = \{F,G\}\widehat{\ } + C(F,G),$$

where F and G are quadratic Hamiltonians and the values of the cocycle C on a pair of Darboux monomials is non-zero only in the following cases:

$$C(p_{k,i}p_{l,j}, q_k^i q_l^j) = \begin{cases} 1 & \text{if } (k,i)\neq(l,j), \\ 2 & \text{if } (k,i)=(l,j). \end{cases} \tag{22}$$

The action of the operator $\widehat{R}$ on an element $F(\mathbf{q}) \in \mathbb{C}_\hbar[[q_0, q_1+1, q_2, \dots]]$, whenever it makes sense, is given by the following formula:

$$\widehat{R}\,F(\mathbf{q}) = \Big(e^{\frac{\hbar}{2}V\partial^2}F(\mathbf{q})\Big)\Big|_{\mathbf{q}\mapsto R^{-1}\mathbf{q}}, \tag{23}$$

where $V\partial^2$ is the quadratic differential operator $\sum_{k,l}(\partial^a, V_{kl}\partial^b)\partial_{q_k^a}\partial_{q_l^b}$, whose coefficients V_{kl} are given by

$$\sum_{k,l=0}^{\infty} V_{kl}(-z)^k(-w)^l = \frac{{}^{\mathrm{T}}R(z)R(w) - 1}{z+w}. \tag{24}$$

By definition, the Kontsevich–Witten tau-function is the following generating series:

$$\mathcal{D}_{\rm pt}(\hbar; q(z)) = \exp\Big(\sum_{g,n}\frac{1}{n!}\hbar^{g-1}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^{n}(q(\psi_i)+\psi_i)\Big), \tag{25}$$

where $q(z) = \sum_k q_k z^k$, $(q_0, q_1, \dots)$ are formal variables, ψ_i $(1 \le i \le n)$ are the first Chern classes of the cotangent line bundles on $\overline{\mathcal{M}}_{g,n}$. The function is interpreted as a formal series in $q_0, q_1+1, q_2, \dots$ whose coefficients are Laurent series in $\hbar$..

Let $s \in \mathcal{S}$ be *a semi-simple* point, i.e., the critical values $u^i(s)$ $(1 \le i \le \mu)$ of $F(s,x)$ form a coordinate system. Let $t = (t_{-1}, t_0, \dots, t_6)$ be the flat coordinates of s. Motivated by the Gromov–Witten theory of symplectic manifolds Givental introduced the notion of the *total ancestor potential* of a semi-simple

Frobenius structure. In particular the definition makes sense in singularity theory as well. Namely, the total ancestor potential is by definition the following formal function on $H[z]$:

$$\mathcal{A}_t(\hbar;\mathbf{q}) := \widehat{\Psi}\,\widehat{R}\,e^{\widehat{U/z}}\prod_{i=1}^{\mu}\mathcal{D}_{\rm pt}(\hbar\,\Delta_i;{}^i\mathbf{q}(z)\sqrt{\Delta_i}) \tag{26}$$

where

$$\mathbf{q}(z)=\sum_{k=0}^{\infty}\sum_{a=-1}^{6}q_k^a z^k\partial_a,\quad {}^i\mathbf{q}(z)=\sum_{k=0}^{\infty}{}^iq_kz^k.$$

The quantization $\widehat{\Psi}$ is interpreted as the change of variables

$$\sum_{i=1}^{\mu}{}^i\mathbf{q}(z)e_i=\Psi^{-1}\mathbf{q}(z)\quad\text{i.e.}\quad {}^iq_k\sqrt{\Delta_i}=\sum_{a=-1}^{6}(\partial_a u^i)\,q_k^a. \tag{27}$$

The correctness of definition (26) is not quite obvious. The problem is that the substitution $\mathbf{q}\mapsto R^{-1}\mathbf{q}$, which, written in more detail, reads

$$q_0\mapsto q_0,\quad q_1\mapsto\overline{R}_1q_0+q_1,\quad q_2\mapsto\overline{R}_2q_0+\overline{R}_1q_1+q_2,\quad\dots,$$

where

$$R^{-1}=1+\overline{R}_1z+\overline{R}_2z^2+\cdots,$$

is not a well-defined operation on the space of formal series. This complication however, is offset by a certain property of the Kontsevich–Witten tau function. By definition, an *asymptotic* function is an element of the Fock space $\mathbb{C}_\hbar[[q_0,q_1,\dots]]$ of the form

$$\mathcal{A}=\exp\Big(\sum_{g=0}^{\infty}\overline{\mathcal{F}}^{(g)}(\mathbf{q})\hbar^{g-1}\Big).$$

It is called *tame* if the following $(3g-3+r)$-jet constraints are satisfied:

$$\left.\frac{\partial^r\mathcal{F}^{(g)}}{\partial q_{k_1}^{i_1}\cdots\partial q_{k_r}^{i_r}}\right|_{\mathbf{q}=0}=0\quad\text{if } k_1+\cdots+k_r\geq 3g-3+r.$$

The Kontsevich–Witten tau function, up to the shift $q_1\mapsto q_1+\mathbf{1}$, is tame for dimensional reasons: $\dim\mathcal{M}_{g,r}=3g-3+r$. It follows that the action of $\widehat{R}$ is well-defined. Moreover, according to Givental [13], $\widehat{R}$ preserves the class of tame asymptotic functions. In other words, the total ancestor potential is a tame asymptotic function in the Fock space $\mathbb{C}_\hbar[[q_0,q_1+\mathbf{1},q_2,\dots]]$.

Let us point out the following homogeneity property of the Kontsevich–Witten tau-function:

$$\mathcal{D}_{\rm pt}(c^2\hbar;cQ(z))=c^{-1/24}\,\mathcal{D}_{\rm pt}(\hbar;Q(z))\quad\text{for all } c\in\mathbb{C}.$$

This formula can be used to rewrite the definition of the ancestor potential (26) in a different form, which looks simpler. However, we prefer to work with formula (26), otherwise $\mathcal{A}_t$ will be a formal series in a different Fock space.

3.2.1. *The total ancestor potential at non-semi-simple points.* Equation (8) can be rewritten as

$$\nabla_t J = 0, \quad \text{where} \quad \nabla_t := d - z^{-1}\theta + z^{-2} E \bullet_t \ .$$

One may think of ∇_t as an *isomonodromic* family of connection operators on $\mathbb{C} \setminus \{0\}$ parametrized by $t \in \mathcal{S}$. Let $S(t,z)$ be gauge transformations of the form

$$1 + S_1(t) z^{-1} + S_2(t) z^{-2} + \cdots ,$$

conjugating ∇_t and $\nabla_0 = d - z^{-1}\theta$:

$$\nabla_t = S(t,z)\, \nabla_0\, S(t,z)^{-1}.$$

The series $S(t,z)$ is also a symplectic transformation so it can be quantized in the same way as R. The quantized symplectic transformation $\widehat{S}$ acts as follows:

$$\widehat{S}^{-1}\, F(q) = e^{W(\mathbf{q},\mathbf{q})/2\hbar} F([S\mathbf{q}]_+), \tag{28}$$

where $W(\mathbf{q},\mathbf{q})$ is the quadratic form $\sum_{k,l}(W_{kl} q_l, q_k)$ whose coefficients are defined by

$$\sum_{k,l\geq 0} W_{kl} z^{-k} w^{-l} = \frac{{}^{\mathrm{T}}S(z) S(w) - 1}{z^{-1} + w^{-1}}. \tag{29}$$

The + sign in (28) means truncation of all negative powers of z, i.e., in $F(\mathbf{q})$ we have to substitute:

$$q_k \mapsto q_k + S_1 q_{k+1} + S_2 q_{k+2} + \cdots , \quad k = 0, 1, 2, \ldots \ .$$

This operation is well-defined on the space of formal series. Note however that $S_1 \partial_0 = t$ (see the proof of Lemma 2.1), where $t = \sum t_a \partial_a \in H$ are the flat coordinates of the point $s \in \mathcal{S}$. Therefore, we have an isomorphism

$$\widehat{S}^{-1} : \mathbb{C}_\hbar[[q_0, q_1 + \mathbf{1}, q_2, \ldots]] \to \mathbb{C}_\hbar[[q_0 - t, q_1 + \mathbf{1}, q_2, \ldots]].$$

Following Givental, we define the so-called *total descendant potential*:

$$\mathcal{D}(\hbar; \mathbf{q}) = e^{F^{(1)}(t)}\, \widehat{S(t,z)}^{-1} \mathcal{A}_t(\hbar; \mathbf{q}), \tag{30}$$

where

$$F^{(1)}(t) := \frac{1}{2} \sum_{i=1}^{\mu} \int R_1^{ii} du^i + \frac{1}{48} \sum_{i=1}^{\mu} \ln(\Delta_i), \tag{31}$$

is called the *genus*-1 potential. Since $S(t, z)$ and $\Psi R e^{U/z}$ satisfy the same differential equations with respect to t, one can check that the definition (30) is independent of t, i.e., $\partial_i \mathcal{D} = 0$ for all i. By setting $t = q_0$, we get that the total descendant potential is a formal series in $q_1 + \mathbf{1}, q_2, q_3, \ldots,$ whose coefficients are analytic multi-valued functions on $\mathcal{S}$ with poles along the *caustic* $\mathcal{K}$. Here, multi-valued means that they are single-valued on the universal cover of $\mathcal{S}$, while the caustic is the subset of $\mathcal{S}$ of all non-semi-simple points.

Since the calibration S is defined for all $s \in \mathcal{S}$, we can use equation (30) to define $\mathcal{A}_t$ for all t as well. Note however that in the setting of an arbitrary semi-simple Frobenius structure $\mathcal{A}_t$ might not be a power series in q_0. In the setting of singularity theory, Givental (see [15]) conjectured that

Conjecture 3.1. *The coefficients of the total descendant potential* $\mathcal{D}(\hbar; \mathbf{q})$ *extend holomorphically through the caustic* $\mathcal{K}$.

In particular, if this is true then the total ancestor potential $\mathcal{A}_t$ is a power series in $q_0, q_1 + \mathbf{1}, q_2, \ldots$ whose coefficients are holomorphic functions in t.

3.3. **Extending $\mathcal{A}_t$ over the non-semi-simple locus.** The primitive form is multi-valued on Σ, but it is analytic on the universal cover $\widetilde{\Sigma} \cong \mathbb{H}$ of Σ (see Subsection 2.4). Therefore, the Frobenius structure, which a priori is defined only for $s \in \mathcal{S}$ such that s_{-1} is near σ_0, induces a holomorphic Frobenius structure on $\mathbb{H} \times \mathbb{C}^{\mu-1}$. Let $\widetilde{\mathcal{K}}$ be the lift of the caustic to the universal cover, i.e., the set of all $t \in \mathbb{H} \times \mathbb{C}^{\mu-1}$ such that the critical values $\{u^i(t)\}_{i=-1}^{6}$ fail to form a coordinate system. Then the total ancestor potential is a formal series whose coefficients are holomorphic on $\mathbb{H} \times \mathbb{C}^{\mu-1} \setminus \widetilde{\mathcal{K}}$.

Lemma 3.2. *Assume that the coefficients of the total ancestor potential are holomorphic in a neighborhood of some point* $\tau_0 \times 0 \in \mathbb{H} \times \mathbb{C}^{\mu-1}$. *Then they extend holomorphically across the caustic* $\widetilde{\mathcal{K}}$.

Proof. Let $a(t)$ be one of the coefficients of $\mathcal{A}_t$. Since the operators R_k and the Hessians Δ_i have only finite order poles along $\widetilde{\mathcal{K}}$ the same is true for $a(t)$. In other words the set $\widetilde{\mathcal{K}}_a$ of all points $t \in \mathbb{H} \times \mathbb{C}^{\mu-1}$ such that $a(t)$ is not holomorphic is a analytic subset. Let us assume that $\widetilde{\mathcal{K}}_a$ is non-empty. Due to the Hartogs extension theorem the codimension of $\widetilde{\mathcal{K}}_a$ is at least 1 and hence it is exactly 1. According to the assumption of the lemma $\mathbb{H} \times 0$ is not contained in $\widetilde{\mathcal{K}}_a$. It follows that $\widetilde{\mathcal{K}}_a$ intersects $\mathbb{H} \times 0$ in a discrete subset $\{\tau_i \times 0\}$. Moreover, due to homogeneity $\widetilde{\mathcal{K}}_a$ is invariant with respect to the rescaling action (with approriate weights) of $\mathbb{C}^*$ on $\mathbb{H} \times \mathbb{C}^{\mu-1}$. Therefore every irreducible component of $\widetilde{\mathcal{K}}_a$ intersects $\mathbb{H} \times 0$, because the coordinates on $\mathbb{C}^{\mu-1}$ have positive weights, so every $\mathbb{C}^*$-orbit intersects $\mathbb{H} \times 0$. It follows that $\widetilde{\mathcal{K}}_a$ is a disjoint union of irreducible components of the type $\{\tau_i\} \times \mathbb{C}^{\mu-1}$. In particular, the caustic $\widetilde{\mathcal{K}}$

has irreducible components of this type as well. But this is not true: it is easy to see that $\{\tau_i\} \times \mathbb{C}^{\mu-1}$ has a semi-simple point for every τ_i. □

According to M. Krawitz and Y. Shen [22], if we choose the cycle A in such a way that $\pi_A(0) = 1$ and $\pi'_A(0) = 0$ then the total descendant potential coincides with the generating function for the FJRW invariants of the singularity $f(0, x) = x_0^3 + x_1^3 + x_2^3$. Then the assumptions of the lemma are satisfied with $\sigma_0 = 0$. Now if we choose a different primitive form, then it is easy to see that the total ancestor potential changes by a formula similar to the one in Theorem 4.2. In particular, the new potential depends holomorphically on t as well.

3.4. **The poles at the cusps.** To close the section, we prove that

Lemma 3.3. *The coefficients of the total ancestor potential have at most finite order poles at the cusps.*

Proof. Recall the notation from Section 3.1.1. The coefficients R_k are determined recursively by the following relations:

$$(d + \Psi^{-1} d\Psi \wedge) R_k = [dU, R_{k+1}],$$

which determines the off-diagonal entries of R_{k+1}^{ij} in terms of the entries of R_k, and

$$R_{k+1}^{ii} = \frac{1}{k+1} \sum_{j \neq i} R_1^{ij}\, R_{k+1}^{ji}\, (u_i - u_j).$$

These formulas are derived from the fact that the asymptotic operator $\Psi R e^{U/z}$ is a solution to the system of differential equations (7) and (8) (see [15]). In order to prove that the coefficients of the ancestor potential have finite order poles at the cusps, it is enough to prove that the asymptotic operator has finite order poles at $\sigma = \infty$ and $\sigma^3 + 27 = 0$.

Given a point $t = (t_{-1}, t_0, \dots, t_{\mu-2})$ we put $'t = (t_0, \dots, t_{\mu-2})$ and view the critical values as functions in $(\sigma, 't) \in \Sigma \times \mathbb{C}^{\mu-1}$. We need to prove that (for $'t$ fixed) $u_i(\sigma, 't)$ has a finite order pole at the punctures of the Riemann sphere Σ. Let us show how the argument works for one of the finite punctures σ_0, i.e., σ_0 is such that $\sigma_0^3 + 27 = 0$. For the puncture at $\sigma = \infty$ the argument is similar.

It is well known that the critical values are eigenvalues of the multiplication by the Euler vector field, i.e., they are the zeroes of an algebraic equation

$$u^\mu + \sum_{k=1}^{\mu} a_k(\sigma,\ 't) u^{\mu-k} := \det(u\, I_\mu - E \bullet_t) = 0.$$

It is easy to see that there is an integer m and a constant C such that

$$|a_k(\sigma,\ 't)| < C\, (\sigma^3 + 27)^{-m}$$

for all $(\sigma,\ 't)$ in a fixed neighborhood of $(\sigma_0, 0)$. Since we have

$$1 + \sum_{k=1}^{\mu} a_k(\sigma,\ 't) u_i(\sigma,\ 't)^{-k} = 0$$

at least for one k we must have

$$|a_k(\sigma,\ 't) u_i(\sigma,\ 't)^{-k}| \geq 1/\mu.$$

From here one gets easily that

$$|u_i(\sigma,\ 't)\ (\sigma^3 + 27)^{m/k}| \leq C\mu. \quad \square$$

4. Transformations of the ancestor potentials

Let us fix a flat coordinate system

$$t = (t_{-1}, t_0, \ldots, t_6), \quad (\partial_i, \partial_j) = \delta_{ij'},$$

corresponding to an arbitrary primitive form. It is convenient to denote the remaining coordinates by $'t = (t_0, t_1, \ldots, t_6)$. Slightly abusing the notation we will sometimes identify t_{-1} with the point $(t_{-1}, 0)$.

Recall that analytic continuation along some closed loop in Σ transforms the flat coordinates $t \mapsto \nu(t)$ according to the formulas in Lemma 2.4. In this section we would like to calculate how the total ancestor potential $\mathcal{A}_{t_{-1}}(\hbar; \mathbf{q})$ transforms under analytic continuation. We will assume that Conjecture 3.1 holds, so that we can view the total ancestor potential as a formal series in $q_0, q_1 + \mathbf{1}, q_2, \ldots$. This assumption is not really necessary in order to prove the transformation law of $\mathcal{A}_{t_{-1}}$, but it is necessary later on in order to prove that the coefficients of $\mathcal{A}_{t_{-1}}$ are quasi-modular forms.

4.1. **Modular transformations.** We start by determining how the operator $\Psi R e^{U/z}$ changes under analytic continuation along some closed loop in Σ. Let ν be the corresponding modular transformation of the middle homology group $H_2(X_{\sigma_0,1}; \mathbb{Z})$. Note that if we fix a Morse coordinate system near each critical point ξ_i, then analytic continuation will simply permute the basis. Hence, the monodromy transformation of the stationary phase asymptotics of the oscillatory integrals $\int_{\mathcal{B}_i} e^{F/z}\omega$ is represented by some permutation matrix P. Finally, given $\nu = (g, k) \in \mathrm{SL}_2(\mathbb{C}) \times \mathbb{Z}$, put

$$M_\nu = \begin{bmatrix} j(g,t_{-1})^{-1} & * & * & * \\ 0 & j(g,t_{-1}) & 0 & 0 \\ 0 & * & \epsilon^{2k} I_3 & 0 \\ 0 & * & 0 & \epsilon^k I_3 \end{bmatrix} \tag{32}$$

where the $*$ entries in the first row and the second column are respectively

$$M_{-1,j} = -e^{2\pi i d_j k}\, n_{12} j(g, t_{-1})^{-1} t_j, \quad 1 \leq j \leq 6$$

and

$$M_{-1,0} = -n_{12}z - \frac{n_{12}^2}{2j(g,t_{-1})}\sum_{i=1}^{6} t_i t_{i'}, \quad M_{i,0} = n_{12}t_{i'} \quad 1 \le i \le 6,$$

$\epsilon = e^{2\pi i/3}$.

Let us point out that the transposition of a given matrix A with respect to the residue pairing has the following form:

$$({}^{\mathrm{T}}A)_{ij} = A_{j'i'}, \quad -1 \le i, j \le 6. \tag{33}$$

Lemma 4.1. *Analytic continuation changes the operator* $\Psi R e^{U/z}$ *into*

$${}^{\mathrm{T}}M_\nu\,(\Psi R e^{U/z})\,P,$$

where $\nu = (g,k)$ *is the corresponding modular transformation.*

Proof. By definition

$$(\Psi R e^{U/z} e_i, \partial_j) = (-2\pi z)^{-3/2}\; z\partial_j\, I_{\mathcal{B}_i}[e^{F/z}\omega], \tag{34}$$

where $I_{\mathcal{B}_i}[e^{F/z}\omega]$ denotes the stationary phase asymptotics. Under analytic continuation the primitive form becomes $\omega/j(g,t_{-1})$, while the asymptotics $I_{\mathcal{B}_i}$ changes into $I_{P(\mathcal{B}_j)}$. It remains only to determine the monodromy of the flat vector fields

$$\partial_j = \sum_a (\partial_j s_a)\partial/\partial s_a \quad \mapsto \quad \sum_a m_{aj}\,\partial_a.$$

Recall that under analytic continuation the flat coordinates t are transformed into $\nu(t)$ (see Lemma 2.4). Put $s = (s_{-1}, s_0, \ldots, s_6)$. Then we have $s(\nu(t)) = s(t)$. Using the chain rule

$$\frac{Ds}{Dt}(\nu(t))\frac{D\nu}{Dt}(t) = \frac{Ds}{Dt}(t)$$

we get that the matrix with entries m_{aj} coincides with the Jacobian matrix $\left(\frac{D\nu}{Dt}\right)^{-1}$. The latter is straightforward to compute:

$$\begin{bmatrix} j(g,t_{-1})^2 & 0 & 0 & 0 \\ -\frac{n_{12}^2}{2}\sum_{i=1}^{6} t_i t_{i'} & 1 & -\epsilon^k n_{12} m' & -\epsilon^{2k} n_{12} m'' \\ j(g,t_{-1}) n_{12}{}^{\mathrm{T}}m'' & 0 & \epsilon^k j(g,t_{-1}) I_3 & 0 \\ j(g,t_{-1}) n_{12}{}^{\mathrm{T}}m' & 0 & 0 & \epsilon^{2k} j(g,t_{-1}) I_3 \end{bmatrix}, \tag{35}$$

where

$$m' = [t_6, t_5, t_4], \quad m'' = [t_3, t_2, t_1]$$

and ${}^{\mathrm{T}}m'$ and ${}^{\mathrm{T}}m''$ are the columns with entries respectively (from top to bottom) t_4, t_5, t_6 and t_1, t_2, t_3 Therefore, analytic continuation transforms the RHS of formula (34) into

$$(-2\pi z)^{-3/2} \sum_{a=-1}^{6} m_{aj}\, z\partial_a \Big(I_{P(\mathcal{B}_i)}[e^{F/z}\omega]/j(g,t_{-1}) \Big). \tag{36}$$

On the other hand the derivative in (36) is

$$\delta_{a,-1} \frac{(-n_{12}z)}{j(g,t_{-1})^2}\, z\partial_0\, I_{P(\mathcal{B}_i)}[e^{F/z}\omega] + \frac{1}{j(g,t_{-1})}\, z\partial_a\, I_{P(\mathcal{B}_i)}[e^{F/z}\omega].$$

It is convenient to introduce the linear operator $P : \mathbb{C}^\mu \to \mathbb{C}^\mu$ whose action on the standard basis $\{e_i\}$ corresponds to the permutation of the Morse coordinate systems $\mathcal{B}_i \mapsto P(\mathcal{B}_i)$. Then formula (36) takes the form

$$\delta_{-1,j}(-n_{12}z)(\Psi R e^{U/z} P(e_i), \partial_0) + \sum_{a=-1}^{6} j(g,t_{-1})^{-1}(\Psi R e^{U/z} P(e_i), m_{aj}\partial_a).$$

Note that

$$\delta_{a,0}\delta_{-1,j}(-n_{12}z) + m_{aj} j(g,t_{-1})^{-1}$$

is the (a',j')-entry of the matrix M (see (32)). To finish the proof, it remains only to use that $\partial_a = dt_{a'}$, $\partial_j = dt_{j'}$. □

Define

$$J(\nu,t_{-1}) = \begin{bmatrix} 1 & 0 \\ 0 & j(g,t_{-1})^2 \end{bmatrix} \oplus j(g,t_{-1})\, \epsilon^{2k}\, I_3 \oplus j(g,t_{-1})\, \epsilon^{k}\, I_3 \tag{37}$$

and

$$X_{\nu,t_{-1}}(z) = \begin{bmatrix} 1 & -n_{12}z/j(g,t_{-1}) \\ 0 & 1 \end{bmatrix} \oplus I_6, \tag{38}$$

for $\nu = (g,k) \in \mathrm{SL}_2(\mathbb{C}) \times \mathbb{Z}$.

Theorem 4.2. *Analytic continuation transforms*

$$\mathcal{A}_{t_{-1}}(\hbar;\mathbf{q}) \mapsto (\widehat{X}_{\nu,t_{-1}}\mathcal{A}_{t_{-1}})(\hbar j(\nu,t_{-1})^2; J(\nu,t_{-1})\mathbf{q}),$$

where we first apply the operator $\widehat{X}_{\nu,t_{-1}}$ and then we rescale $\hbar$ and $\mathbf{q}$.

Proof. The idea is to derive the transformation law for the ancestor potential $\mathcal{A}_t$ at some semi-simple point $t = (t_{-1}, {}'t)$ and then pass to the limit ${}'t \to 0$.

According to Lemma 4.1 the operator $\Psi R e^{U/z}$ is transformed into

$${}^{\mathrm{T}}M \Psi R e^{U/z} P.$$

We may assume that $P = 1$ because P is a permutation matrix, so its quantization $\widehat{P}$ will leave the product of Kontsevich–Witten tau functions invariant. Put $M = M_0 + M_1 z$. Then we have

$$ {}^{\mathrm{T}}M\Psi R e^{U/z} = \widetilde{\Psi}\widetilde{R}e^{U/z}, \quad \text{where} \quad \widetilde{\Psi} = M_0^{-1}\Psi, \quad \widetilde{R} = \Psi^{-1}M_0{}^{\mathrm{T}}M\Psi R. $$

The quantization is in general only a projective representation. However, the quantization of the operators $\Psi^{-1}M_0{}^{\mathrm{T}}M\Psi$ and R involves quantizing only p^2 and $p\,q$-terms. Since the cocycle (22) on such terms vanishes we get

$$ (\widetilde{R})\widehat{\ } = (\Psi^{-1}M_0{}^{\mathrm{T}}M\Psi)\widehat{\ }\,\widehat{R}. $$

The operators M_0 and Ψ are independent of z and their quantizations by definition are just changes of variables. Hence

$$ (\widetilde{\Psi}\widetilde{R})\widehat{\ } = \widehat{M_0}^{-1}(M_0{}^{\mathrm{T}}M)\widehat{\ }(\Psi R)\widehat{\ }. $$

By definition Δ_i^{-1} is $(\partial_{u_i}, \partial_{u_i})$, which gains a factor of $j(\nu, t_{-1})^{-2}$ under analytic continuation. The ancestor potential (26) is transformed into

$$ \widehat{M_0}^{-1}(M_0{}^{\mathrm{T}}M)\widehat{\ }\Big(\mathcal{A}_t(j(\nu,t_{-1})^2\hbar; j(\nu,t_{-1})\mathbf{q})\Big). \tag{39} $$

Let us take the limit $'t \to 0$. Using formula (32), we see that

$$ M_0^{-1} \to j(\nu, t_{-1})\ J(\nu, t_{-1})^{-1}, \quad M_0{}^{\mathrm{T}}M \to X_{\nu,t_{-1}}. $$

It remains only to notice that the rescaling

$$ (\hbar, \mathbf{q}) \mapsto (j(\nu,t_{-1})^2\hbar, j(\nu,t_{-1})\mathbf{q}) $$

commutes with the action of any quantized operator. □

4.2. **Non-modular transformations.** Assume now that we have two Frobenius structures corresponding to some cycles A_1 and A_2. According to Lemma 2.5 the relation between the flat coordinates for A_1 and A_2 can be described by a pair $\nu = (g, k) \in \mathrm{SL}_2(\mathbb{C}) \times \mathbb{Z}$.

Lemma 4.3. *Let $(\Psi R e^{U/z})_{A_i}$ be the asymptotic operator corresponding to the cycle $A_i (i = 1, 2)$; then $(\Psi R e^{U/z})_{A_2} = {}^{\mathrm{T}}M_\nu(\Psi R e^{U/z})_{A_1}$.*

The proof of this Lemma is similar to the proof of Lemma 4.1. Moreover, using the same argument as in the proof of Theorem 4.2, we get

Theorem 4.4. *Let $\mathcal{A}_{A_i,t_{-1}}(\hbar;\mathbf{q})$ be the total ancestor potential of the Frobenius structure corresponding to $A_i (i = 1, 2)$; then*

$$ \mathcal{A}_{A_2,\nu(t)_{-1}}(\hbar;\mathbf{q}) = (\widehat{X}_{\nu,t_{-1}}\mathcal{A}_{A_1,t_{-1}})(\hbar j(\nu,t_{-1})^2; J(\nu,t_{-1})\mathbf{q}), $$

where we first apply the operator $\widehat{X}_{\nu,t_{-1}}$ and then we rescale $\hbar$ and $\mathbf{q}$.

Let us emphasize the difference between Theorem 4.2 and Theorem 4.4. The former compares the values of the ancestor potential at *two different points* in $\mathbb{H}\times\mathbb{C}^{\mu-1}$. The latter compares the ancestor potentials of *two different* Frobenius structures at the *same point* in $\mathbb{H}\times\mathbb{C}^{\mu-1}$.

4.3. **The genus-1 potential.** We finish this section by describing the modular transformations of the genus-1 potential (31). The potential is a homogeneous function of degree 0 and therefore it depends only on the moduli $\tau = t_{-1}$ of the Frobenius structure. In fact, it is the derivative $\partial F^{(1)} := \partial F^{(1)}/\partial t_{-1}$ that transforms more naturally.

Proposition 4.5. *Let $\nu = (g,k) \in \mathrm{SL}_2(\mathbb{C})\times\mathbb{Z}$ be a modular transformation; then*

$$\partial F^{(1)}(\nu(t)) = j(g,t_{-1})^2\partial F^{(1)}(t) + \Big(\frac{\mu}{24}-\frac{1}{2}\Big)\, n_{12}\, j(g,t_{-1}).$$

Proof. Recall the notation in the proof of Theorem 4.2. Note that

$$\widetilde{R}_1 = R_1 + \Psi^{-1}M_0({}^{\mathrm{T}}M_1)\Psi$$

and

$$(\partial_{-1}\log\Delta_i)(\nu(t)) = j(g,t_{-1})^2(\partial_{-1}\log\Delta_i)(t) + 2n_{12}j(g,t_{-1}).$$

Since

$$\partial F^{(1)}(\nu(t)) = \partial_{-1}\Big(F^{(1)}(\nu(t))\Big)\; j(g,t_{-1})^2,$$

in order to prove the proposition, we need to verify that

$$\mathrm{tr}\Big(\Psi^{-1}M_0({}^{\mathrm{T}}M_1)\Psi(\partial_{-1}U)\Big) = -n_{12}/j(g,t_{-1}).$$

On the other hand, since by definition

$$\Psi\, dU\,\Psi^{-1} = A,\quad A = \sum_{i=-1}^{\mu-2}(\partial_i \bullet_t)\; dt_i,$$

the LHS of the above identity is precisely $\mathrm{tr}\Big(M_0({}^{\mathrm{T}}M_1)(\partial_{-1}\bullet_t)\Big)$. Since $F^{(1)}$ depends only on the moduli t_{-1}, we may assume that $'t = 0$. In this case however, the quantum multiplication operator $\partial_{-1}\bullet_{t_{-1}}$ and $M_0({}^{\mathrm{T}}M_1)$ are given by the following matrices:

$$\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\oplus(0\cdot I_6),\quad \text{and}\quad \begin{bmatrix}0 & -n_{12}/j(g,t_{-1})\\ 1 & 0\end{bmatrix}\oplus(0\cdot I_6).$$

The proposition follows. □

Corollary 4.6. *Let $\nu = (g,k) \in \mathrm{SL}_2(\mathbb{C}) \times \mathbb{Z}$ be a modular transformation; then*

$$F^{(1)}(\nu(t)) = F^{(1)}(t) + \Big(\frac{\mu}{24} - \frac{1}{2}\Big) \log j(g, t_{-1}).$$

Assume that we have two Frobenius structures corresponding to some cycles $A_i (i = 1, 2)$. Let $F^{(1)}_{A_i}$ be the genus-1 potentials and $\nu = (g,k) \in \mathrm{SL}_2(\mathbb{C}) \times \mathbb{Z}$ be the transformation identifying the two flat structures; then

Corollary 4.7. *The following formula holds:*

$$F^{(1)}_{A_2}(t) = F^{(1)}_{A_1}(t) + \Big(\frac{\mu}{24} - \frac{1}{2}\Big) \log j(g, t_{-1}).$$

The genus-1 potential of a simple elliptic singularity was computed by I. Strachan [40]. In the P_8 case (when $\mu = 8$) the answer is the following:

$$F^{(1)}_A = -\frac{1}{24} \log\Big((27 + \sigma^3)\pi_A^4\Big). \tag{40}$$

The computation in [40] is carried out for a specific choice of the cycle A. However, using Corollary 4.7, we get that the above formula is valid for all possible choices of A.

5. Anti-holomorphic completion

The transformation of the ancestor potential under the modular group from the previous section is slightly complicated. In particular, the potential is not modular. A magic trick to restore the modularity is to complete it to an anti-holomorphic function. We call it an *anti-holomorphic completion*. It is motivated by physics (see [2]) and it has its origin in the so-called *holomorphic anomaly equations*. Mathematically, it can be thought as generalizing the construction of quasi-modular forms.

5.1. Quasi-modular forms.

A function $f : \mathbb{H} \to \mathbb{C}$ is called a *holomorphic quasi-modular form of weight m* with respect to some finite-index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ if there are functions f_i, $1 \le i \le N$, holomorphic on $\mathbb{H}$, such that

(1) The functions $f_0 := f$ and f_i are holomorphic at the cusps of Γ.
(2)

$$f(\tau, \overline{\tau}) = f_0(\tau) + f_1(\tau)(\tau - \overline{\tau})^{-1} + \cdots + f_N(\tau)(\tau - \overline{\tau})^{-N}.$$

is modular, i.e.,

$$f(g\tau, g\overline{\tau}) = j(g,\tau)^m f(\tau, \overline{\tau}), \quad \text{for all } g \in \Gamma,$$

$f(\tau,\overline{\tau})$ is called the *anti-holomorphic completion* of $f(\tau)$.

For more details we refer to [19]. The key example of a quasi-modular form is the Eisenstein series

$$G_2 = -\frac{1}{24} + \sum_{n=1}^{\infty} \Big(\sum_{d|n} d\Big) q^n, \quad q = e^{2\pi i\tau}.$$

It is known that G_2 satisfies the following identity:

$$G_2(g(\tau)) = j(g,\tau)^2 G_2(\tau) - \frac{1}{4\pi i} n_{12} j(g,\tau), \quad g \in \mathrm{SL}_2(\mathbb{Z}),$$

where the matrix g and its action on τ are the same as in (20). The map $\tau \mapsto g(\tau)$ induces the following transformation:

$$-\frac{1}{\tau-\overline{\tau}} \mapsto -j(g,\tau)^2 \frac{1}{\tau-\overline{\tau}} + n_{12} j(g,\tau). \tag{41}$$

It follows that $G_2(\tau) - \frac{1}{4\pi i}(\tau-\overline{\tau})^{-1}$ transforms as a modular form of weight 2. It is not hard to prove that every quasi-modular form can be written uniquely as a polynomial in G_2 whose coefficients are modular forms on Γ.

For our purposes, we have to relax condition (1) in the definition of a quasi-modular forms. Namely, we will be assuming that the forms have finite order poles at the cusps.

5.2. **The anti-holomorphic completion of $\mathcal{A}_{t_{-1}}(\hbar;\mathbf{q})$.** We continue to denote by $t \mapsto \nu(t)$ the transformation of the flat coordinates corresponding to analytic continuation along a closed loop in Σ. Recall also that t_{-1} is identified with a point τ' on the upper-half plane via some fractional linear transformation g. Slightly abusing notation we define complex conjugation:

$$t_{-1} = g(\tau') := \frac{a\tau'+b}{c\tau'+d} \quad \mapsto \quad \overline{t}_{-1} = \frac{a\overline{\tau'}+b}{c\overline{\tau'}+d},$$

i.e., the complex conjugation of t_{-1} is the one induced from the upper half-plane. Note that the transformation law (41) still holds. Put

$$\widetilde{X}_{t_{-1},\overline{t}_{-1}}(z) = \begin{bmatrix} 1 & -z(t_{-1}-\overline{t}_{-1})^{-1} \\ 0 & 1 \end{bmatrix} \oplus I_6$$

and define

$$\mathcal{A}_{t_{-1},\overline{t}_{-1}}(\hbar;\mathbf{q}) = (\widetilde{X}_{t_{-1},\overline{t}_{-1}})^{\widehat{}}\, \mathcal{A}_{t_{-1}}(\hbar;\mathbf{q}) \, . \tag{42}$$

As a consequence of Theorem 4.2 we get the following corollary..

Corollary 5.1. *Analytic continuation transforms the anti-holomorphic completion* (42) *as follows:*

$$\mathcal{A}_{t_{-1},\overline{t}_{-1}}(\hbar;\mathbf{q}) \mapsto \mathcal{A}_{t_{-1},\overline{t}_{-1}}(j(\nu,t_{-1})^2\hbar; J(\nu,t)\mathbf{q}).$$

Proof. Using the transformation rule (41) we get that analytic continuation transforms

$$\widetilde{X}_{t_{-1},\bar{t}_{-1}}(z) \mapsto \widetilde{X}_{t_{-1},\bar{t}_{-1}}(j(\nu,t_{-1})^2 z)\, X^{-1}_{\nu,t_{-1}}(j(\nu,t_{-1})^2 z).$$

Since the quantization is a representation when restricted to the space of upper-triangular symplectic transformations, the quantization of the RHS of the above equation is just a composition of the quantizations of the two operators. It remains only to use Theorem 4.2 and the fact that under the rescaling

$$(\hbar,\mathbf{q}) \mapsto (j(\nu,t_{-1})^2\hbar, J(\nu,t_{-1})\mathbf{q})$$

the operators change as follows

$$(\widetilde{X}_{t_{-1},\bar{t}_{-1}}(z)\, X^{-1}_{\nu,t_{-1}}(z))\widehat{\ } \mapsto (\widetilde{X}_{t_{-1},\bar{t}_{-1}}(j(\nu,t_{-1})^2 z) X^{-1}_{\nu,t_{-1}}(j(\nu,t_{-1})^2 z))\widehat{\ }. \quad \square$$

5.3. **Quasi-modularity of the ancestor potential.** It is convenient to introduce the following multi-index convention. Given $I = (i_0, i_1, \dots)$, with only finitely many $i_k := (i_{k,-1}, i_{k,0}, \dots, i_{k,6}) \in \mathbb{Z}^\mu$ different from 0, we define the monomial

$$q_0^{i_0}(q_1+\mathbf{1})^{i_1} q_2^{i_2}\cdots,$$

where the raising of a vector variable to a vector power means raising each component of the variable by the corresponding component of the power and then taking their product. The anti-holomorphic ancestor potential (42) has the form

$$\mathcal{A}_{t_{-1},\bar{t}_{-1}}(\hbar;\mathbf{q}) = \exp\sum_g \hbar^{g-1}\mathcal{F}^{(g)}_{t_{-1},\bar{t}_{-1}}(\mathbf{q}),$$

where the genus-g potential $\mathcal{F}^{(g)}_{t_{-1},\bar{t}_{-1}}$ is a formal series of the following type:

$$\sum_I a_I^{(g)}(t_{-1},\bar{t}_{-1})\; q_0^{i_0}(q_1+\mathbf{1})^{i_1} q_2^{i_2}\cdots . \tag{43}$$

Similarly, we let $a_I^{(g)}(t_{-1})$ be the coefficients of the ancestor potential $\mathcal{A}_{t_{-1}}(\hbar;\mathbf{q})$. Finally, for each multi-index I we introduce the following two integers:

$$d(I) := \sum_k (i_{k,-1}d_{-1} + \cdots + i_{k,6}d_6),$$

and

$$m(I) := \sum_k (2i_{k,-1} + i_{k,1} + \cdots + i_{k,6}).$$

Theorem 5.2. *The coefficient $a_I^{(g)}(t_{-1})$ is non-zero only if $d(I)$ is an integer. Moreover, each non-zero coefficient is a quasi-modular form of weight $2g-2+m(I)$.*

Proof. Analytic continuation transforms the series (43) into

$$\sum_I a_I^{(g)}(\nu(t_{-1},\bar{t}_{-1}))\; q_0^{i_0}(q_1+\mathbf{1})^{i_1}q_2^{i_2}\cdots\ .$$

On the other hand the substitution

$$\mathbf{q}\mapsto J(\nu,t)\mathbf{q}$$

transforms the series (43) into

$$\sum_I j(\nu,t_{-1})^{m(I)}e^{-2\pi i d(I)}a_I^{(g)}(t_{-1},\bar{t}_{-1})\; q_0^{i_0}(q_1+\mathbf{1})^{i_1}q_2^{i_2}\cdots\ .$$

Recalling Corollary 5.1, we get the following formula:

$$\widetilde{a}_I^{(g)}(\nu(t_{-1})) = j(\nu,t_{-1})^{2g-2+m(I)}\ e^{-2\pi i d(I)k}\ a_I^{(g)}(t_{-1},\bar{t}_{-1}). \tag{44}$$

Let us prove that $d(I)$ is an integer. We claim that rescaling the asymptotics $\Psi R e^{U/z}$ via

$$t_i\mapsto e^{2\pi\sqrt{-1}\,d_i}t_i,\quad -1\le i\le 6, \tag{45}$$

transforms $\Psi R e^{U/z}$ according to the formula in Lemma 4.1, where $P=I_\mu$ and $\nu=(I_2,1)$. Assuming this claim, we get that under the rescaling (45) the descendant potential transforms according to Theorem 4.2. This means that formula (44) is still valid, and since $g=I_2$ we get that $e^{-2\pi i d(I)}=1$, i.e., $d(I)$ must be an integer.

The claim follows easily from the homogeneity property of the oscillatory integrals. Namely, from

$$(z\partial_z+E)\,J_{\mathcal{B}_i}(t,z)=\theta\,J_{\mathcal{B}_i}(t,z),$$

we get

$$J_{\mathcal{B}_i}(e^ct,e^cz)=e^{c\,\theta}\,J_{\mathcal{B}_i}(t,z),$$

where the scalar e^c acts on the i-th coordinate of t with weight d_i. Let $c\to 2\pi i$ and note that the limit of the LHS, up to sign, coincides with rescaling $J_{\mathcal{B}_i}(t,z)$ via (45). On the RHS the operator $e^{2\pi i\theta}$ coincides with the matrix $(-{}^{\mathrm{T}}M_\nu)$, where $\nu=(I_2,1)$ (and $'t=0$).

6. Relating to Gromov-Witten theory

We have finished the proof of quasi-modularity of global Saito-Givental theory for P_8. The remaining two cases of simple elliptic singularities are

$$X_9:\quad f(x,\sigma)=x_0^2x_2+x_0x_1^3+x_2^2+\sigma x_0x_1x_2\ ,\quad \sigma^3+27\neq 0,$$

and

$$J_{10}:\quad f(x,\sigma)=x_0^3x_2+x_1^3+x_2^2+\sigma x_0x_1x_2\ ,\quad \sigma^3+27\neq 0.$$

The proof of quasi-modularity for X_9, J_{10} is identical and we leave it for the readers to fill in the details. On the other hand, global Saito-Givental theory is considered to be a B-model theory. To draw consequences for A-model theory, such as the GW theory of an elliptic orbifold $\mathbb{P}^1$, we have to identify the A-model theory as a certain limit of the B-model. For our purposes, there are two important limits, the Gepner limit $\sigma = 0$ and the large complex structure limit $\sigma = \infty$. The Gepner limit corresponds to FJRW theory, while the large complex structure limit corresponds to GW theory. For simple elliptic singularities, the appropriate flat coordinates at the Gepner limit $\sigma = 0$ have been worked out already by Noumi-Yamada [26]. In our set-up, they correspond to the choice of a cycle A such that $\pi_A(0) = 1, \pi'_A(0) = 0$. We define

$$\mathcal{A}_{Gepner,t}(\hbar; \mathbf{q}) := e^{F_A^{(1)}(t)+\frac{1}{24}t_{-1}}\ \mathcal{A}_{A,t}(\hbar; \mathbf{q}),$$

where $F_A^{(1)}$ and $\mathcal{A}_{A,t}$ are the genus-1 potential and the total ancestor potential of the Frobenius structure corresponding to the cycles A. The following theorem is the LG-to-LG all genera mirror theorem of Krawitz-Shen [22].

Theorem 6.1. *For P_8, X_9, J_{10}, $e^{-t_{-1}/24}\mathcal{A}_{Gepner,t}$ coincides with the ancestor potential function of FJRW invariants, up to a linear identification of the flat coordinates.*

Since the FJRW ancestor potential function extends over the caustic, Lemma 4.3 implies

Corollary 6.2. *The conjecture 3.2 holds for P_8, X_9, J_{10}, i.e., global Saito-Givental theory extends over the caustic.*

The above corollary allows us to define the ancestor potential function at $t_i = 0$ for $i \geq 0$, which is crucial for our discussion of modularity.

6.1. **The divisor equation in singularity theory.** Before we start to discuss the large complex structure limit and the relation to Gromov-Witten theory, we discuss the divisor equation in the B-model. The latter is an important tool in the computation of Gromov-Witten theory and it is necessary for the LG-to-CY mirror theorem of Krawitz–Shen [22].

Let us denote by P the flat vector field $\partial/\partial t_{-1}$. Let $t = (t_{-1}, t_0, \ldots, t_6)$ be a generic semi-simple point. We will prove below that the correlators of the ancestor potential $\mathcal{A}_t(\hbar; \mathbf{q})$ are invariant with respect to the transformation $t_{-1} \to t_{-1} + 2\pi i$ and that they expand in a Fourier series in $q = e^{t_{-1}}$ near $q = 0$ (see Proposition 6.5).

On the other hand, the ancestor potential of the singularity satisfies the differential equation

$$\partial_{-1}\ \mathcal{A}_t = ((P \bullet_t /z)\widehat{\ } - \partial_{-1}F^{(1)}(t))\mathcal{A}_t.$$

This formula follows from the fact that the quantization operator $\Psi Re^{U/z}$ satisfies the quantum differential equations.

According to Corollary 4.5, $\partial_{-1}F^{(1)}(t)$ is a quasi-modular form (of weight 2). In particular, it admits a Fourier expansion near $q = 0$. Moreover, a straightforward computation shows that the constant term of this expansion is $-1/24$. For P_8, one has to use formula (40) and the Fourier expansions of σ and π_A from Section 6.2. In the other two cases the computation is again straightforward, thanks to the results of I. Strachan (see [40]). In other words,

$$\mathcal{A}_{LCS,t}(\hbar;\mathbf{q}) := e^{F^{(1)}(t)+\frac{1}{24}t_{-1}}\ \mathcal{A}_t(\hbar;\mathbf{q})$$

can be expanded into a Fourier series near $q = 0$. Since the ancestor extends through the caustic, we can take the limit of $\mathcal{A}_{LCS,t}$ as $'t = (t_0, t_1, \ldots, t_{\mu-2}) \to 0$. The resulting function, or more precisely its Fourier expansion, will be denoted by $\mathcal{D}_{LCS,q}(\hbar;\mathbf{q})$. It satisfies the following differential equation:

$$q\partial_q\ \mathcal{D}_{LCS,q}(\hbar;\mathbf{q}) = \Big((P\bullet/z)\widehat{\ }+\frac{1}{24}\Big)\mathcal{D}_{LCS,q}(\hbar;\mathbf{q}),$$

where $P\bullet$ is the Frobenius multiplication by P at $t = (t_{-1}, 0, \ldots, 0)$. Note that

$$P\bullet\phi_i(x) = \delta_{i,0}P \quad \text{for all } i.$$

Therefore, the differential equation from above coincides with the divisor equation in the GW theory of the corresponding orbifold projective line with Novikov variable q and divisor class P. More precisely, the differential equation gives the following relation between the *correlators* of $\mathcal{D}_{LCS,q}(\hbar;\mathbf{q})$:

$$\langle P, \phi_{a_1}\psi_1^k, \ldots, \phi_{k_n}\psi^{k_n}\rangle_{g,n+1,d}$$

equals

$$d\langle \phi_{a_1}\psi_1^k, \ldots, \phi_{k_n}\psi^{k_n}\rangle_{g,n,d} + \sum_{i=1}^{n}\langle \ldots, P\bullet\phi_i\psi_i^{k_i-1}, \ldots\rangle_{g,n,d},$$

for all (g, n, d), s.t., $d \neq 0$ or $2g-2+n > 0$. Here

$$\langle \phi_{a_1}\psi_1^k, \ldots, \phi_{k_n}\psi^{k_n}\rangle_{g,n,d}$$

is by definition the coefficient in front of $\hbar^{g-1}q_{k_1}^{a_1}\cdots q_{k_n}^{a_n}q^d$ in the generating function $\log(\mathcal{D}_{LCS,q})$.

In the rest of this section, we focus on the coordinates at the large complex structure limit $\sigma = \infty$ and the identification with GW theory. There is a large body of literature in the Calabi-Yau case. The identification is referred as *a mirror map*. In our case it goes as follows. On the A-model side, the A-model moduli space has a coordinate t_{-1} corresponding to a complexified Kähler class. GW theory involves power series in $q = e^{t_{-1}}$. The B-model moduli are parameterized by σ^3 (for P_8 and J_{10}) or σ^2 for X_9. For our purposes, it is convenient to work with σ directly. The mirror map is a map $\sigma \to t_{-1}(\sigma)$ defined locally around $\sigma = \infty$. By fixing a symplectic basis $\{A, B\}$ we can identify the Kähler class t_{-1} with the modulus of the complex structure $\tau = \pi_B/\pi_A$. We can treat the mirror map as a map $\tau \to \tau(\sigma)$. The latter can be described explicitly via the Picard–Fuchs equations satisfied by the periods π_A and π_B. We begin with the P_8-case.

6.2. **Large complex structure limit of the family P_8.** To begin with, let us construct a Frobenius manifold isomorphism between the Milnor ring of P_8 and the quantum cohomology of $\mathbb{P}^1(3,3,3)$.

The orbifold cohomology admits the following natural basis: $\Delta_0 := 1$, $\Delta_{-1} := P$ is the hyperplane class, and Δ_i and $\Delta_{i'}$, $i' = 7 - i$, are the cohomology classes 1 supported on the *twisted sectors* of the i-th orbifold point ($i = 1, 2, 3$) of *age* 1/3 and 2/3 respectively (see [4] for some background on orbifold GW theory). The only non-zero Poincaré pairings between these cohomology classes are

$$(\Delta_{-1}, \Delta_0) = 1, \quad (\Delta_i, \Delta_{i'}) = 1/3, \ i = 1, 2, \ldots, 6.$$

According to Krawitz–Shen [22] the quantum cohomology and the higher-genus theory are uniquely determined from the divisor equation and the correlators:

$$\langle \Delta_1, \Delta_2, \Delta_3 \rangle_{0,3,1} = 1, \quad \langle \Delta_i, \Delta_i, \Delta_i \rangle_{0,3,0} = 1/3.$$

To set up the B-model coordinates, we first have to choose a symplectic basis $\{A, B\}$. The corresponding periods π_A and π_B are solutions to the differential equation (9), which has a regular singular point at $\sigma = \infty$. We choose the cycles A, B in such a way that

$$\pi_A(\sigma) = -(-1)^{1/2}\,\sigma^{-1}\ {}_2F_1(1/3, 2/3; 1; -27/\sigma^3).. \tag{46}$$

and

$$\pi_B(\sigma) = -\frac{3}{2\pi i}\pi_A(\sigma)\log(-\sigma) + \frac{3}{2\pi i}\ (-3i\sigma^{-1})\sum_{k=1}^{\infty} b_k(-\sigma/3)^{-3k}\ . \tag{47}$$

The coefficients b_k can be determined uniquely from the recursion relation

$$-9k^2 b_k + (9k^2 - 9k + 2)b_{k-1} + (2k-1)a_{k-1} - 2k a_k = 0,$$

where a_k are the coefficients of the hypergeometric series ${}_2F_1(1/3, 2/3; 1; y)$, i.e.,

$$a_0 = 1, \quad a_k = \frac{(1/3)_k (2/3)_k}{(k!)^2},\ k \geq 1, \quad \text{where} \quad (b)_k = b(b+1)\cdots(b+k-1).$$

These formulas suggest that the correct parameter of the B-model moduli is $(-\sigma/3)^{-3}$, but this is not important for us. Exponentiating, we obtain an identity of the following form:

$$e^{2\pi i\tau/3} = -\sigma^{-1}\Big(1 + \sum_{k=1}^{\infty} c_k\, (-\sigma)^{-3k}\Big),$$

where the coefficients c_k can be written explicitly in terms of a_k and b_k. By inverting the above series we can obtain the Fourier expansion of $-\sigma^{-1}$ in terms of $q := e^{2\pi i\tau/3}$. Note that

$$\mathrm{res}_{x=0}\, \frac{x_0x_1x_2}{f_{x_0}f_{x_1}f_{x_2}}\, d^3x = \frac{1}{\sigma^3+27}$$

and that the monomials $(\sigma^3+27)^{\deg\phi_i}\phi_i(x)(-1 \le i \le 6)$ provide a basis in which the residue pairing with respect to the standard volume form is constant. It follows that the identifications

$$\begin{aligned} 1 &= 1 \\ P &= (\sigma^3+27)\pi_A^2(\sigma)\phi_{-1}(x) \\ 27^{\deg\Delta_i - \frac{1}{3}}\Delta_i &= (-1)^{\deg\phi_i-\frac{1}{2}}(\sigma^3+27)^{\deg\phi_i}\phi_i(x)\,\pi_A(\sigma), \quad 1 \le i \le 6, \end{aligned}$$

provide an isomorphism between the Poincaré and the residue pairings. Moreover, after a straightforward computation, we find the leading terms of the Fourier series of the following correlators:

$$\langle\Delta_1,\Delta_2,\Delta_3\rangle_{0,3} = -i\pi_A(\sigma) = q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + \cdots,$$

$$\langle\Delta_1,\Delta_1,\Delta_1\rangle_{0,3} = -i(-\sigma/3)\pi_A(\sigma) = \frac{1}{3} + 2q^3 + 2q^9 + 2q^{12} + \cdots.$$

Remark 6.3. *The Fourier expansion of $-i\pi_A(\sigma)$ coincides with Saito's eta product (see* [31]*)*

$$\eta_{E_6^{(1,1)}}(3\tau) := \eta(9\tau)^3\eta(3\tau)^{-1}.$$

Proposition 6.4. *The cycles A and B are integral up to a scalar factor and τ is a modulus of the elliptic curve at infinity.*

Proof. The j-invariant of the elliptic curve at infinity is

$$j(\sigma) = -\frac{\sigma^3(-216+\sigma^3)^3}{(27+\sigma^3)^3}.$$

According to Kodaira [20], there exists a symplectic basis $\{A', B'\}$ of $H_1(E_\sigma;\mathbb{Z})$ whose monodromy around $\sigma = \infty$ is the same as the monodromy of $\{\pi_A, \pi_B\}$. This implies that

$$A = c\,A', \quad B = c\,B' + d\,A'$$

for some constants c and d. On the other hand the Fourier expansion of the above j-invariant is

$$\frac{1}{q^3}+744+196884q^3+21493760q^6+\cdots, \quad q=e^{2\pi i\tau/3}.$$

Comapring with the well known Fourier expansion of the j-invariant, we get that $\tau=\tau'$ and hence the constant $d=0$. □

Note that under the identification between the quantum cohomology and the Milnor ring from above, the Kähler parameter (i.e., the coordinate along the hyperplane class P) becomes t_{-1}. The next Proposition guarantees that the divisor equations in singularity theory and in Gromov–Witten theory are the same.

Proposition 6.5. *The Kähler parameter is related to τ via the following mirror map: $t_{-1}=2\pi i\tau/3$.*

Proof. We want to compute the constant $(1,\partial/\partial\tau)_A$ (the index A means residue pairing with respect to d^3x/π_A). To begin with note that

$$\frac{\partial\tau}{\partial\sigma}\,(1,\partial/\partial\tau)_A=(1,\partial/\partial\sigma)_A=(1,x_0x_1x_2)_A=\frac{1}{(27+\sigma^3)\pi_A^2}. \tag{48}$$

On the other hand

$$\frac{\partial\tau}{\partial\sigma}=\frac{\pi_B'\pi_A-\pi_B\pi_A'}{\pi_A^2}.$$

The numerator is the Wronskian of the solutions π_B and π_A of the differential equation (9) and hence it equals

$$\mathrm{const}\cdot(27+\sigma^3)^{-1} \quad \sim \quad \mathrm{const}\cdot\sigma^{-3},$$

where we took the expansion near $\sigma=\infty$ and kept only the leading term. On the other hand using the expansions of π_A and π_B at $\sigma=\infty$ (see formulas (46) and (47)) one can check that the leading order term of the numerator is: $3\sigma^{-3}/2\pi i$. Therefore, the above constant is $3/2\pi i$. Now, from equation (48), we get $(1,\partial/\partial\tau)_A=2\pi i/3$. On the other hand, since $1=(1,P)=(1,\partial/\partial t_{-1})$, we must have $t_{-1}=2\pi i\tau/3$. □

All necessary conditions for the reconstruction theorem of Krawitz–Shen are satisfied. Therefore, we have the following theorem:

Theorem 6.6. *Under the identification of $q=e^{2\pi i\tau/3}$ with the Novikov variable, $\mathcal{D}_{LCS,q}$ is equal to the descendant potential function of the elliptic orbifold $\mathbb{P}^1$ with weights $(3,3,3)$.*

It is well known that the modular group, i.e., the monodromy group of the local system $H_1(E_\sigma;\mathbb{Z}),\sigma\in\Sigma$, is $\Gamma(3)$ – the principal congruence subgroup of level 3. Let us denote by $a_I^{(g)}(\tau)$ the coefficients of the ancestor potential $\mathcal{A}_{t_{-1}}$ of the singularity. An immediate consequence is

Corollary 6.7. *The following statements hold:*

(1) $a_I^{(g)}(\tau)$ *has no pole at the cusp* $\tau = i\,\infty$.
(2) $\mathcal{A}_{Gepner}$ *is related to* $\mathcal{A}_{LCS}$ *by the composition of analytic continuation and the quantization of a symplectic transformation (see Theorem 4.4).*
(2) *The coefficients of the total descendant potential for the GW theory of* $\mathbb{P}^1(3,3,3)$ *are quasi-modular forms on* $\Gamma(3)$.

Remark 6.8. *From the B-model alone, it is difficult to see whether* $a_I^{(g)}(\tau)$ *does not have a pole at the cusp, i.e., at* $q = 0$. *The situation is similar to the extendibility of Givental's function to the caustic. Again we draw the conclusion from the mirror A-model side by using Krawitz-Shen's GW-to-LG all genera mirror theorem.*

Remark 6.9. *One may wonder if analytic continuation alone will relate* $\mathcal{A}_{Gepner}$ *to* $\mathcal{A}_{LCS}$. *The answer is generally no. We are choosing different symplectic bases at* $\sigma = 0, \sigma = \infty$. *One basis may not be analytic continuation of the other. For example, we can often choose an integral basis at* $\sigma = \infty$. *But the basis at* $\sigma = 0$ *is not integral in general.*

Remark 6.10. *It is convenient to use the language of symplectic bases to describe the ideas. Technically, it is easier to work with Picard-Fuchs equations. Fortunately, the two approaches are equivalent. However, it is generally a difficult question to identify a solution of the Picard-Fuchs equation with the period of an explicit cycle.*

6.3. **Large complex structure limit of the family** X_9**.** The primitve forms are given by $d^3x/\pi_A(\sigma)$, where $\pi_A(\sigma)$ is a solution to the same differenrial equation as in the P_8-case.

6.3.1. *The Gauss–Manin connection in the marginal direction.* In order to identify the quantum cohomology with the Milnor ring, we need to find the monomials in the Milnor ring for which the residue pairing assumes a constant form. We fix the following basis in the Milnor ring: $\phi_{-1} = x_0x_1x_2$, $\phi_0 = 1$, and ϕ_i for $i = 1, 2, \ldots, 7$ are given respectively by

$$x_0, \quad x_1, \quad x_2, \quad x_1^2, \quad x_0x_1, \quad x_0x_2, \quad x_1x_2.$$

Put $\Phi_i(\sigma)$ for the section $\int \phi_i d^3x/df$ of the middle cohomology bundle. A straightforward computation, similar to the one in Section 2.3, gives the following differential equations:

$$\partial_\sigma \Phi_1 = -\frac{\sigma^2}{4(27+\sigma^3)}\,\Phi_1 - \frac{9}{2(27+\sigma^3)}\,\Phi_2,$$
$$\partial_\sigma \Phi_2 = \frac{3\sigma}{4(27+\sigma^3)}\,\Phi_1 - \frac{\sigma^2}{2(27+\sigma^3)}\,\Phi_2,$$

$$\begin{aligned}\partial_\sigma \Phi_3 &= -\frac{\sigma^2}{27+\sigma^3}\,\Phi_3 - \frac{9}{2(27+\sigma^3)}\,\Phi_5\\ \partial_\sigma \Phi_4 &= -\frac{9}{27+\sigma^3}\,\Phi_3 + \frac{3\sigma}{2(27+\sigma^3)}\,\Phi_5,\\ \partial_\sigma \Phi_5 &= \frac{3\sigma}{27+\sigma^3}\,\Phi_3 - \frac{\sigma^2}{2(27+\sigma^3)}\,\Phi_5,\end{aligned}$$

and

$$\begin{aligned}\partial_\sigma \Phi_6 &= -\frac{7\sigma^2}{4(27+\sigma^3)}\,\Phi_6 - \frac{9}{2(27+\sigma^3)}\,\Phi_7\\ \partial_\sigma \Phi_7 &= \frac{21\sigma}{4(27+\sigma^3)}\,\Phi_6 - \frac{\sigma^2}{2(27+\sigma^3)}\,\Phi_7.\end{aligned}$$

From here we get the following solutions:

$$\begin{aligned}\Phi_1(\sigma) &= \sigma^{-1/4}\Phi_{1,1}(\sigma)A_1 + \sigma^{-5/2}\Phi_{1,2}(\sigma)A_2,\\ \Phi_2(\sigma) &= -\sigma^{-5/4}\Phi_{2,1}(\sigma)A_1 + \frac{1}{2}\sigma^{-1/2}\Phi_{2,2}(\sigma)A_2,\end{aligned}$$

$$\begin{aligned}\Phi_3(\sigma) &= \sigma^{-1}\Phi_{3,1}(\sigma)A_3 + \sigma^{-5/2}\Phi_{3,2}(\sigma)A_5,\\ \Phi_4(\sigma) &= A_4 - \frac{1}{3}\sigma\Phi_3(\sigma),\\ \Phi_5(\sigma) &= -2\sigma^{-2}\Phi_{5,1}(\sigma)A_3 + \frac{1}{3}\sigma^{-1/2}\Phi_{5,2}(\sigma),\end{aligned}$$

$$\begin{aligned}\Phi_6(\sigma) &= \sigma^{-7/4}\Phi_{6,1}(\sigma)A_6 + \sigma^{-5/2}\Phi_{6,2}(\sigma)A_7\\ \Phi_7(\sigma) &= -\frac{7}{3}\sigma^{-11/4}\Phi_{7,1}(\sigma)A_6 + \frac{1}{6}\sigma^{-1/2}\Phi_{7,2}(\sigma)A_7,\end{aligned}$$

where

$$\begin{aligned}\Phi_{1,1}(\sigma) &= {}_2F_1(1/12, 5/12; 1/4; -27/\sigma^3),\\ \Phi_{1,2}(\sigma) &= {}_2F_1(5/6, 7/6; 7/4; -27/\sigma^3),\end{aligned}$$

$$\begin{aligned}\Phi_{2,1}(\sigma) &= {}_2F_1(5/12, 13/12; 5/4; -27/\sigma^3),\\ \Phi_{2,2}(\sigma) &= {}_2F_1(1/6, 5/6; 3/4; -27/\sigma^3),\end{aligned}$$

$$\begin{aligned}\Phi_{3,1}(\sigma) &= {}_2F_1(1/3, 2/3; 1/2; -27/\sigma^3),\\ \Phi_{3,2}(\sigma) &= {}_2F_1(5/6, 7/6; 3/2; -27/\sigma^3),\end{aligned}$$

$$\begin{aligned}\Phi_{5,1}(\sigma) &= {}_2F_1(2/3, 4/3; 3/2; -27/\sigma^3),\\ \Phi_{5,2}(\sigma) &= {}_2F_1(1/6, 5/6; 1/2; -27/\sigma^3),\end{aligned}$$

$$\begin{aligned}\Phi_{6,1}(\sigma) &= {}_2F_1(7/12, 11/12; 3/4; -27/\sigma^3),\\ \Phi_{6,2}(\sigma) &= {}_2F_1(5/6, 7/6; 5/4; -27/\sigma^3),\end{aligned}$$

$$\begin{aligned}\Phi_{7,1}(\sigma) &= {}_2F_1(11/12, 19/12; 7/4; -27/\sigma^3),\\ \Phi_{7,2}(\sigma) &= {}_2F_1(1/6, 5/6; 1/4; -27/\sigma^3),\end{aligned}$$

and A_i are flat sections of the middle cohomology bundle. Solving for A_i in terms of Φ_i we get certain polynomials in the Milnor ring which, according to our general construction of flat coordinates, should be part of a basis in which the residue pairing (with respect to the standard form d^3x) is constant.

6.3.2. *The orbifold quantum cohomology.* The orbifold cohomology of $\mathbb{P}^1(4,4,2)$ has the following natural basis: 1 is the unit, P is the hyperplane class, and the remaining cohomology classes are supported on the twisted sectors. Namely, $\Delta_{i1}, \Delta_{i2}, \Delta_{i3}$, $i = 1, 2$ are the units (Δ_{ik} has degree $k/4$) in the cohomology of the twisted sectors of the i-th $\mathbb{Z}/4\mathbb{Z}$-orbifold point, and Δ_{31} is the unit in the cohomology of the twisted sector of the $\mathbb{Z}/2\mathbb{Z}$-orbifold point. The only non-zero pairings are

$$(1, P) = 1, \quad (\Delta_{i1}, \Delta_{i3}) = 1/4, \quad (\Delta_{i2}, \Delta_{i2}) = 1/4, \quad (\Delta_{31}, \Delta_{31}) = 1/2,$$

where $i = 1, 2$. Also, the following correlators are easily computed because the corresponding moduli spaces are points.

$$\langle \Delta_{i1}, \Delta_{i1}, \Delta_{i2} \rangle_{0,3,0} = 1/4, \quad i = 1, 2,$$

$$\langle \Delta_{11}, \Delta_{21}, \Delta_{31} \rangle_{0,3,1} = 1.$$

According to Krawitz–Shen [22], the quantum cohomology and the higher genus theory are uniquely determined from the above relations and the divisor equation.

We specify the cycles $\{A, B\}$ by choosing the corresponding periods π_A and π_B. The period π_A is the same as in the P_8-case (see (46)), while π_B is 3 times larger:

$$\pi_B(\sigma) = -\frac{9}{2\pi i}\pi_A(\sigma)\log(-\sigma) + \frac{9}{2\pi i}\,(-3i\sigma^{-1})\sum_{k=1}^{\infty} b_k(-\sigma/3)^{-3k}\ .$$

Proposition 6.11. *Up to a scalar, the cycles A and B are integral and τ is a modulus of the elliptic curve at infinity.*

Proof. The argument is similar to the one in Proposition 6.4: we need to check that the j-invariant has the correct Fourier expansion in terms of $e^{2\pi i\tau}$. The j-invariant of the elliptic curve at infinity is

$$j(\sigma) = -\frac{(24\sigma + \sigma^4)^3}{(27 + \sigma^3)}.$$

Substituting in this formula the Fourier series of $-\sigma^{-1}$, we get

$$e^{-2\pi i\tau}+744+196884e^{2\pi i\tau}+\cdots. \quad \square$$

Since

$$\mathrm{res}_{x=0}\,\frac{x_0x_1x_2}{f_{x_0}f_{x_1}f_{x_2}}\,d^3x=\frac{9}{4(27+\sigma^3)},$$

it is easy to check that if we identify

$$1=1,\quad P=\frac{4}{9}(27+\sigma^3)\,x_0x_1x_2\,\pi_A^2,$$

and

$$\begin{aligned}
\Delta_{11} &= 2e^{\pi\sqrt{-1}/4}(27+\sigma^3)^{1/4}\Big(\frac{1}{2}\sigma^{-1/2}\Phi_{2,2}(\sigma)\,x_0-\sigma^{-5/2}\Phi_{1,2}(\sigma)\,x_1\Big)\,\pi_A\\
\Delta_{12} &= -(27+\sigma^3)^{1/2}\Big(2\sigma^{-2}\,\Phi_{5,1}(\sigma)\,x_2+\sigma^{-1}\,\Phi_{3,1}(\sigma)\,x_1x_2\Big)\pi_A,\\
\Delta_{13} &= -2e^{-\pi\sqrt{-1}/4}(27+\sigma^3)^{3/4}\Big(\frac{1}{6}\sigma^{-1/2}\Phi_{7,2}(\sigma)\,x_0x_2-\sigma^{-5/2}\Phi_{6,2}(\sigma)\,x_1x_2\Big)\pi_A\\
\Delta_{21} &= (27+\sigma^3)^{1/4}\Big(\sigma^{-1/4}\Phi_{1,1}(\sigma)\,x_1+\sigma^{-5/4}\Phi_{2,1}(\sigma)\,x_0\Big)\pi_A\\
\Delta_{22} &= e^{\pi\sqrt{-1}/2}\Big(x_1^2+\frac{1}{3}\sigma x_2+(27+\sigma^3)^{1/2}\Big(\frac{1}{3}\sigma^{-1/2}\Phi_{5,2}(\sigma)\,x_2-\sigma^{-5/2}\Phi_{3,2}(\sigma)\,x_0x_1\Big)\Big)\pi_A\\
\Delta_{23} &= (27+\sigma^3)^{3/4}\Big(\sigma^{-7/4}\Phi_{6,1}(\sigma)\,x_1x_2+\frac{7}{3}\sigma^{-11/4}\Phi_{7,1}\,x_0x_2\Big)\pi_A\\
\Delta_{31} &= e^{\pi\sqrt{-1}/2}\Big(-x_1^2-\frac{1}{3}\sigma x_2+2(27+\sigma^3)^{1/2}\Big(\frac{1}{3}\sigma^{-1/2}\Phi_{5,2}(\sigma)\,x_2-\sigma^{-5/2}\Phi_{3,2}(\sigma)\,x_0x_1\Big)\Big)\pi_A
\end{aligned}$$

then the residue and the Poincaré pairings coincide. Moreover, put $q=e^{2\pi i\tau/4}$; then we have the following formulas for the correlators:

$$\langle\Delta_{11},\Delta_{21},\Delta_{31}\rangle_{0,3}=q+2q^5+q^9+2q^{13}+2q^{17}+3q^{25}+O(q^{26}),$$

$$\langle\Delta_{11},\Delta_{11},\Delta_{12}\rangle_{0,3}=\frac{1}{4}+q^4+q^8+q^{16}+2q^{20}+O(q^{26}),$$

and

$$\langle\Delta_{11},\Delta_{11},\Delta_{22}\rangle_{0,3}=q^2+2q^{10}+q^{18}+O(q^{26}).$$

Proposition 6.12. *The Kähler parameter is related to τ via the following mirror map: $t_{-1}=2\pi i\tau/4$.*

The proof is along the same lines as Proposition 6.5 and it is left as an exercise. Using again the results of Krawitz-Shen we have the following GW-to-LG all genera mirror theorem.

Theorem 6.13. *Let $q = e^{2\pi i\tau/4}$. Under the above isomorphism between the quantum cohomology and the Milnor ring, $\mathcal{D}_{LCS,q}$ is equal to the total descendant potential function of the elliptic orbifold $\mathbb{P}^1$ with weights $(4,4,2)$.*

Theorem 5.2 and 6.13 yield

Corollary 6.14. *The Gromov-Witten total descendant potential function of the elliptic orbifold $\mathbb{P}^1$ with weights $(4,4,2)$ is quasi-modular for $q = e^{2\pi i\tau/4}$ and for some finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

6.4. **Large complex structure limit of the family J_{10}.** The primitive forms are given by $d^3x/\pi_A(\sigma)$, where π_A is a solution to the same differential equation as in the P_8-case.

6.4.1. *The Gauss–Manin connection in the marginal direction.* We fix the following basis in the Milnor ring: $\phi_{-1} = x_0x_1x_2$, $\phi_0 = 1$, and ϕ_i for $i = 1, 2, \ldots, 8$ are given by the monomials

$$x_0, \quad x_0^2, \quad x_1, \quad x_0^3, \quad x_0x_1, \quad x_0^4, \quad x_1^2, \quad x_0^5.$$

Put $\Phi_i(\sigma)$ for the section $\int \phi_i(x)d^3x/df$ of the middle cohomology bundle. They satisfy the following system of differential equations:

$$\partial_\sigma \Phi_1 = -\frac{\sigma^2}{2(27+\sigma^3)}\,\Phi_1, \quad \partial_\sigma \Phi_8 = -\frac{3}{2(27+\sigma^3)}\Big(\frac{24(18+\sigma^3)}{24\sigma+\sigma^4} + \sigma^2\Big)\,\Phi_8,$$

$$\partial_\sigma \Phi_2 = -\frac{9}{27+\sigma^3}\,\Phi_3, \quad \partial_\sigma \Phi_3 = -\frac{\sigma^2}{27+\sigma^3}\,\Phi_3,$$

$$\partial_\sigma \Phi_4 = \frac{\sigma^2}{2(27+\sigma^3)}\,\Phi_4 - \frac{18}{27+\sigma^3}\,\Phi_5, \quad \partial_\sigma \Phi_5 = -\frac{2\sigma^2}{27+\sigma^3}\,\Phi_5 - \frac{3\sigma}{2(27+\sigma^3)}\,\Phi_4,$$

$$\partial_\sigma \Phi_6 = \frac{1}{\sigma}\,\Phi_6 - \frac{162}{\sigma^2(27+\sigma^3)}\,\Phi_7, \quad \partial_\sigma \Phi_7 = \frac{27-\sigma^3}{\sigma(27+\sigma^3)}\,\Phi_7.$$

The solutions have the form

$$\Phi_1(\sigma) = (27+\sigma^3)^{-1/6}A_1, \quad \Phi_8(\sigma) = \frac{24\sigma+\sigma^4}{(27+\sigma^3)^{5/6}}A_8$$

$$\Phi_2(\sigma) = (-\sigma/3)(27+\sigma^3)^{-1/3}A_3 + A_2, \quad \Phi_3(\sigma) = (27+\sigma^3)^{-1/3}A_3,$$

$$\Phi_4(\sigma) = \Phi_{41}(\sigma)\,A_4 + \Phi_{42}(\sigma)\,A_5, \quad \Phi_5(\sigma) = \Phi_{51}(\sigma)\,A_4 + \Phi_{52}(\sigma)\,A_5,$$

$$\Phi_6(\sigma) = \frac{18+\sigma^3}{3(27+\sigma^3)^{2/3}}A_7 + \sigma A_6, \quad \Phi_7(\sigma) = \sigma(27+\sigma^3)^{-2/3}A_7,$$

where A_i are flat sections of the middle cohomology bundle and

$$\begin{aligned}
\Phi_{41} &= 3(-\sigma/3)^{1/2} \;\; {}_2F_1(-1/6,1/6;-1/2;-27/\sigma^3),\\
\Phi_{42} &= \tfrac{4}{9}(-\sigma/3)^{-4} \;\; {}_2F_1(\;4/3,5/3;\;\;5/2;-27/\sigma^3),\\
\Phi_{51} &= (-\sigma/3)^{-1/2} \;\; {}_2F_1(\;1/6,5/6;\;\;1/2;-27/\sigma^3),\\
\Phi_{52} &= (-\sigma/3)^{-2} \;\; {}_2F_1(\;2/3,4/3;\;\;3/2;-27/\sigma^3)..
\end{aligned}$$

From here we can determine the elements in the Milnor ring that correspond to the flat sections A_i. They should correspond to orbifold cohomology classes of $\mathbb{P}^1(6,3,2)$.

6.4.2. *The quantum cohomology.* A natural basis in the orbifold cohomology is: the unit 1, the hyperplane class P, and the units of the twisted sectors $\Delta_{1i}(1 \le i \le 5)$, $\Delta_{2j}(j=1,2)$, and Δ_{31}. The Poincaré pairing in this basis is non-zero only in the following cases:

$$(\Delta_{1,i},\Delta_{1,j}) = \delta_{i+j,6}/6, \quad (\Delta_{2,1},\Delta_{2,2}) = 1/3, \quad (\Delta_{31},\Delta_{3,1}) = 1/2.$$

According to Krawitz–Shen [22], the quantum cohomology and the higher-genus theory are uniquely determined by the divisor equation and the following correlators:

$$\begin{aligned}
&\langle \Delta_{11},\Delta_{11},\Delta_{14}\rangle_{0,3,0} = 1/6, \quad \langle \Delta_{11},\Delta_{12},\Delta_{13}\rangle_{0,3,0} = 1/6\\
&\langle \Delta_{21},\Delta_{21},\Delta_{21}\rangle_{0,3,0} = 1/3, \quad \langle \Delta_{11},\Delta_{21},\Delta_{31}\rangle_{0,3,1} = 1.
\end{aligned}$$

We specify the cycles $\{A,B\}$ by choosing the corresponding periods π_A and π_B. The period π_A is the same as in the P_8-case (see (46)), while π_B is 3 times larger:

$$\pi_B(\sigma) = -\frac{9}{2\pi i}\pi_A(\sigma)\log(-\sigma) + \frac{9}{2\pi i}\,(-3i\sigma^{-1})\sum_{k=1}^{\infty} b_k(-\sigma/3)^{-3k}\;.$$

From here we can express $-\sigma^{-1}$ as a Fourier series in $e^{2\pi i\tau/9}$, where $\tau := \pi_B/\pi_A$.

Proposition 6.15. *Up to a scalar, the cycles A and B are integral and τ is a modulus of the elliptic curve at infinity.*

Proof. The argument is similar to the one in Proposition 6.4: we need to check that the j-invariant has the correct Fourier expansion in terms of $e^{2\pi i\tau}$. The j-invariant of the elliptic curve at infinity is

$$j(\sigma) = -\frac{(24\sigma+\sigma^4)^3}{(27+\sigma^3)}.$$

Substituting in this formula the Fourier series of $-\sigma^{-1}$, we get

$$e^{-2\pi i\tau} + 744 + 196884e^{2\pi i\tau} + \cdots . \quad \square$$

In order to match the quantum cohomology and the Milnor ring we make the following identifications ($\eta = e^{2\pi i/6}$):

$$\begin{aligned}
(-1)^{\frac{1}{2}-\deg \Delta_{11}}\Delta_{11} &= (27+\sigma^3)^{1/6} x_0 \pi_A,\\
(-1)^{\frac{1}{2}-\deg \Delta_{12}}\Delta_{12} &= \Big(\eta x_0^2 + \frac{1}{3}\Big(\eta\sigma + (27+\sigma^3)^{1/3}\Big)x_1\Big)\pi_A,\\
(-1)^{\frac{1}{2}-\deg \Delta_{13}}\Delta_{13} &= \frac{1}{9}(27+\sigma^3)^{1/2}\Big(\Phi_{52}(\sigma)\, x_0^3 - \Phi_{42}(\sigma)\, x_0x_1\Big)\pi_A,\\
(-1)^{\frac{1}{2}-\deg \Delta_{14}}\Delta_{14} &= \Big(\frac{1}{3\sigma^2}\Big(\eta^2(18+\sigma^3) - \sigma(27+\sigma^3)^{2/3}\Big)x_1^2 - \eta^2 x_0^4/\sigma\Big)\pi_A,\\
(-1)^{\frac{1}{2}-\deg \Delta_{15}}\Delta_{15} &= \frac{(27+\sigma^3)^{5/6}}{24\sigma+\sigma^4} x_0^5\pi_A,\\
(-1)^{\frac{1}{2}-\deg \Delta_{21}}\Delta_{21} &= \Big(\frac{1}{3}\Big(-\eta\sigma + 2(27+\sigma^3)^{1/3}\Big)x_1 - \eta x_0^2\Big)\pi_A,\\
(-1)^{\frac{1}{2}-\deg \Delta_{22}}\Delta_{22} &= \Big(-\frac{1}{3\sigma^2}\Big(\eta^2(18+\sigma^3) + 2\sigma(27+\sigma^3)^{2/3}\Big)x_1^2 + \eta^2 x_0^4/\sigma\Big)\pi_A,\\
(-1)^{\frac{1}{2}-\deg \Delta_{31}}\Delta_{31} &= -\frac{1}{2\sqrt{3}}(27+\sigma^3)^{1/2}\Big(\Phi_{51}(\sigma)\, x_0^3 - \Phi_{41}(\sigma)\, x_0x_1\Big)\pi_A,
\end{aligned}$$

Since

$$\mathrm{res}_{x=0}\, \frac{x_0x_1x_2}{f_{x_0}f_{x_1}f_{x_2}}\, d^3x = \frac{3}{2(27+\sigma^3)}$$

the identifications for the other two classes should be

$$1 = 1, \quad P = \frac{2}{3}\,(27+\sigma^3)x_0x_1x_2\,\pi_A^2.$$

It is easy to check that the Poincaré and the residue pairings agree. We also have the analogue of Proposition 6.5.

Proposition 6.16. *The Kähler parameter is related to the modulus τ via the following mirror map: $t_{-1} = 2\pi i\tau/6$.*

The proof is again along the same lines and it is omitted. In order to recall the main result of Krawitz–Shen we just need to check that the Fourier expansions of the correlators in powers of $q := e^{2\pi i\tau/6}$ have the correct leading terms. For the first correlator we have

$$\langle \Delta_{11}, \Delta_{11}, \Delta_{14}\rangle_{0,3} = \frac{1}{18}\Big(\sigma + 2\eta^2(27+\sigma^3)^{1/3}\Big)\,(-1)^{1/2}\,\pi_A$$

and the Fourier expansion is the following:

$$\langle \Delta_{11}, \Delta_{11}, \Delta_{14}\rangle_{0,3} = \frac{1}{6} + q^6 + q^{18} + q^{24} + O(q^{30}).$$

The remaining correlators can be computed similarly. The computations are straightforward but quite cumbersome. Here is what we got (with the help of computer software):

$$\begin{aligned}\langle\Delta_{11},\Delta_{12},\Delta_{13}\rangle_{0,3} &= \frac{1}{6}+q^{12}+O(q^{31})\\ \langle\Delta_{21},\Delta_{21},\Delta_{21}\rangle_{0,3} &= \frac{1}{3}+2q^{6}+2q^{18}+2q^{24}+O(q^{30})\\ \langle\Delta_{11},\Delta_{21},\Delta_{31}\rangle_{0,3} &= q+2q^{7}+2q^{13}+2q^{19}+q^{25}+O(q^{31}).\end{aligned}$$

The main result of Krawitz-Shen in the case of the J_{10} singularity can be formulated in this way:

Theorem 6.17. *Let $q=e^{2\pi i\tau/6}$. Under the above identification of the quantum cohomology and the Milnor ring, $\mathcal{D}_{LCS,q}$ is equal to the descendant potential function of the elliptic orbifold $\mathbb{P}^1$ with weights $(6,3,2)$.*

Theorems 5.2 and 6.17 imply

Corollary 6.18. *The Gromov–Witten total descendant potential function of the elliptic orbifold $\mathbb{P}^1$ with weights $(6,3,2)$ is quasi-modular for $q=e^{2\pi i\tau/6}$ and a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

Appendix A. Primitive forms for simple elliptic singularities

A.1. **Oscillatory integrals.** If V is a vector space, then we denote by $V((z))$ (resp. $V[[z]]$) the space of formal Laurent (resp. power) series in z with coefficients in V. The space $\mathcal{H}_F$ of oscillatory integrals is defined formally as the third cohomology of the twisted de Rham complex:

$$q_*\Omega^{\bullet}_{X/S}((z)),\quad d=zd_{X/S}+dF\wedge .$$

Given a differential form $\omega\in q_*\Omega^3_{X/S}$, we denote by $\int e^{F/z}\omega$ its projection on $\mathcal{H}_F$. The sheaf $\mathcal{H}_F$ is equipped with a Gauss-Manin connection:

$$\nabla^{\mathrm{G.M}}_{\partial/\partial t^a}\int e^{F/z}g(t,x,z)d^3x=\int e^{F/z}\Big(z^{-1}\frac{\partial F}{\partial t^a}g+\frac{\partial g}{\partial t^a}\Big)d^3x$$

and

$$\nabla^{\mathrm{G.M}}_{\partial/\partial z}\int e^{F/z}g(t,x,z)d^3x=\int e^{F/z}\Big(-z^{-2}Fg+\frac{\partial g}{\partial z}\Big)d^3x,$$

where $d^3x=dx_0dx_1dx_2$ is the standard volume form.

We say that an element $\omega\in\mathcal{H}_F$ is homogeneous of degree r if

$$(z\nabla_{\partial/\partial z}+\nabla_E)\omega=r\omega,$$

where E is the Euler vector field. Let us denote by $\mathcal{H}_F^{(0)}$ the subspace of $\mathcal{H}_F$ consisting of power series in z. According to K. Saito there exists a sequence of bilinear pairings:

$$K_F^{(k)} : \mathcal{H}_F^{(0)} \times \mathcal{H}_F^{(0)} \to \mathcal{O}_{\mathcal{S}}, \quad k \geq 0,$$

satisfying the following properties:

(K1) The pairings are symmetric for k even and skew-symmetric for k odd.

(K2) The pairings are compatible with the Gauss-Manin connection:

$$\xi\, K_F^{(k)}(\omega_1, \omega_2) = K_F^{(k)}(\nabla_\xi \omega_1, \omega_2) + K_F^{(k)}(\omega_1, \nabla_\xi \omega_2)$$

for all $\xi \in \mathcal{T}_S$ and all $\omega_1, \omega_2 \in \mathcal{H}_F^{(0)}$..

(K3) We have $K_F^{(k)}(z\omega_1, \omega_2) = K_F^{(k-1)}(\omega_1, \omega_2)$.

(K4) If ω_1 and ω_2 are homogeneous of degrees r_1 and r_2 then $K_F^{(k)}(\omega_1, \omega_2)$ is homogeneous of degree $r_1 + r_2 - k - 3$ (the number 3 here corresponds to the fact that ω_1 and ω_2 are 3-forms).

(K5) If $\omega_i \in \mathcal{H}_F$ are represented by differential forms $g_i(t,x)d^3x$ independent of z then $K_F^{(0)}(\omega_1, \omega_2)$ coincides with the residue pairing

$$\Big(\frac{1}{2\pi i}\Big)^3 \int_{\Gamma_\epsilon} \frac{g_1(t,x)\; g_2(t,x)}{F_{x_0} F_{x_1} F_{x_2}}\; d^3x\ ,$$

where the integration cycle Γ_ϵ is supported on $|\frac{\partial F}{\partial x_0}| = |\frac{\partial F}{\partial x_1}| = |\frac{\partial F}{\partial x_2}| = \epsilon$.

A.2. **The primitive forms.** Let $g(s,x)d^3x$ be a volume form (i.e., $g(s,x) \neq 0$ for all $(s,x) \in \mathcal{S} \times \mathbb{C}^3$) and $\omega \in \mathcal{H}_F^{(0)}$ be the corresponding oscillatory integral. The period mapping

$$\partial/\partial s^a \mapsto z\nabla_{\partial/\partial s_a} \int e^{F/z} g(s,x) d^3x, \quad 1 \leq a \leq N, \tag{49}$$

induces an isomorphism between $\mathcal{T}_{\mathcal{S}}[[z]]$ and $\mathcal{H}_F^{(0)}$. The volume form is called *primitive* if it is homogeneous and it satisfies the following properties:

(P1) For all vector fields $\partial/\partial s^i$, $\partial/\partial s^j$ and all $k \geq 1$ we have

$$K_F^{(k)}\Big(z\nabla_{\partial/\partial s^i}\omega, z\nabla_{\partial/\partial s^j}\omega\Big) = 0$$

(P2) For all vector fields $\partial/\partial s^i$, $\partial/\partial s^j$, $\partial/\partial s^l$ and all $k \geq 2$ we have

$$K_F^{(k)}\Big(z\nabla_{\partial/\partial s^i} z\nabla_{\partial/\partial s^j}\omega, z\nabla_{\partial/\partial s^l}\omega\Big) = 0.$$

(P3) For all vector fields $\partial/\partial s^i$, $\partial/\partial s^j$ and for all $k \geq 2$ we have

$$K_F^{(k)}\Big(- z^2\nabla_{\partial/\partial z}\; z\nabla_{\partial/\partial s^i}\omega, z\nabla_{\partial/\partial s^j}\omega\Big) = 0.$$

Since $g(s,x)d^3x$ is a homogeneous volume form the function $g(s,x)$ must be homogeneous of degree 0, i.e., $g(s,x)$ depends only on the degree-0 variable $s_{-1}=\sigma$. Condition (P1) holds for any degree-0 function g, due to the homogeneity (K4) and the skew-symmetry (K1) of K_F.

We are going to prove that all primitive forms have the form $d^3x/\pi_a(\sigma)$, where a is a monodromy invariant cycle and π_a is the Gelfand-Leray period (11).

All identities involving holomorphic forms should be understood in the space $\mathcal{H}_F$ of oscillatory integrals, i.e., we work modulo $(zd_{X/S}+dF\wedge)$-exact forms. Note that we have the following identity:

$$\frac{\partial F}{\partial s^i}\frac{\partial F}{\partial s^j}d^3x = C_{ij}^k\frac{\partial F}{\partial s^k}d^3x + zB_{ij}^k\frac{\partial F}{\partial s^k}d^3x + z^2A_{ij}^k\frac{\partial F}{\partial s^k}d^3x, \tag{50}$$

where we adopted Einstein's convention for summation over repeating lower and upper indices. The coefficients are homogeneous functions on S of degree, respectively:

$$\begin{aligned} \deg C_{ij}^k &= \deg s_k - \deg s_i - \deg s_j + 1,\\ \deg B_{ij}^k &= \deg s_k - \deg s_i - \deg s_j,\\ \deg A_{ij}^k &= \deg s_k - \deg s_i - \deg s_j - 1. \end{aligned}$$

In particular, $A_{ij}^k\neq 0$ only for $i=j=-1$ and $k=0$. Let $\omega=g(\sigma)d^3x\in\mathcal{H}_F$ be a primitive form. A straightforward differentiation gives that $z\nabla_{\partial/\partial s^i}z\nabla_{\partial/\partial s^j}\omega$ is a sum of three terms:

$$C_{ij}^k(z\nabla_{\partial/\partial s^k}\omega),$$

$$z\Big(-C_{ij}^k\frac{1}{g}\frac{\partial g}{\partial s^k}z\nabla_{\partial/\partial s^0}\omega + \frac{1}{g}\frac{\partial g}{\partial\sigma}\Big(\delta_{i,-1}\,z\nabla_{\partial/\partial s^j}\omega + \delta_{j,-1}\,z\nabla_{\partial/\partial s^i}\omega\Big) + B_{ij}^k\,z\,\nabla_{\partial/\partial s^k}\omega\Big),$$

and

$$z^2\delta_{i,-1}\delta_{j,-1}\Big(A_{ij}^0(\sigma) - \frac{2}{g^2}\Big(\frac{\partial g}{\partial\sigma}\Big)^2 - B_{ij}^{-1}\frac{1}{g}\frac{\partial g}{\partial\sigma} + \frac{1}{g}\frac{\partial^2 g}{\partial\sigma^2}\Big)\,z\nabla_{\partial/\partial s^0}\omega$$

In order for ω to be primitive we have to arrange that

$$K_F^{(2)}(z\nabla_{\partial/\partial s^i}z\nabla_{\partial/\partial s^j}\omega, z\nabla_{\partial/\partial s^l}\omega)=0 \quad \text{for all } -1\leq i,j,l\leq 6.$$

We already know that $K_F^{(k)}(z\nabla_{\partial/\partial s^i}\omega, z\nabla_{\partial/\partial s^l})=0$ for all i and l, and all $k\geq 1$. Therefore, using property (K3) of the higher residue pairing, we get that it is enough to prove that the last of the above 3 terms is 0. In other words, g must be a solution to a second order differential equation. Put $u=1/g$; then the differential equation becomes:

$$\frac{\partial^2 u}{\partial \sigma^2} = B^{-1}_{-1,-1}(\sigma)\,\frac{\partial u}{\partial \sigma} + A^0_{-1,-1}(\sigma)\,u.$$

Comparing with equation (50) we see that the solutions of this differential equation can be constructed via the oscillatory integrals $\int e^{f/z} d^3x$ which are obtained from $\int e^{F/z} d^3x$ by specializing the parameters $s_0 = s_1 = \cdots = s_6 = 0$. Note that this substitution is necessary in order for $C^k_{-1,-1}$ to become 0 so that the relation (50) matches the above differential equation.

Alternatively, solutions of the differential equation can be constructed via the Laplace transform of the oscillatory integrals. Namely, the Gelfand-Leray periods π_α where α is any flat middle homology cycles. Note however, that the Gelfand-Leray periods vanish whenever α is an eigenvector of the classical monodromy with eigenvalue different from 1. Therefore, we may assume that α is an invariant cycle with respect to the classical monodromy, i.e., it is a tube cycle.

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IPMU, Todai Institute for Advanced Studies (TODIAS), University of Tokyo, Kashiwa, Chiba 277-8583, Japan
E-mail address: todor.milanov@ipmu.jp

Department of Mathematics, University of Michigan, Ann Arbor, MI
E-mail address: ruan@umich.edu