GENERIC VANISHING THEORY VIA MIXED HODGE MODULES

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Abstract. We extend the dimension and strong linearity results of generic vanishing theory to bundles of holomorphic forms and rank one local systems, and more generally to certain coherent sheaves of Hodge-theoretic origin associated to irregular varieties. Our main tools are Saito's mixed Hodge modules, the Fourier-Mukai transform for $\mathcal{D}$-modules on abelian varieties introduced by Laumon and Rothstein, and Simpson's harmonic theory for flat bundles. In the process, we discover two natural categories of perverse coherent sheaves on the Picard variety and on the moduli space for Higgs line bundles.

A. Introduction

1. Generic vanishing theory. The attempt to understand cohomology vanishing statements on irregular varieties in the absence of strong positivity has led to what is usually called *generic vanishing theory*. Perhaps the most famous result is the generic vanishing theorem of Green and Lazarsfeld [GL87], which in a weak form states that on a smooth complex projective variety X, the cohomology of a generic line bundle $L \in \operatorname{Pic}^0(X)$ vanishes in degrees less than $\dim a(X)$, where $a\colon X \to \operatorname{Alb}(X)$ denotes the Albanese mapping of X. This theorem and its variants have found a surprising number of applications, ranging from results about singularities of theta divisors [EL97] to recent work on the birational geometry of irregular varieties, including a proof of Ueno's conjecture [CH11].

One can consider the set of those line bundles for which the cohomology in a given degree does not vanish, and thanks to the work of many people, the structure of these sets is very well understood. This is more precisely the content of generic vanishing theory. Denoting, for any coherent sheaf $\mathcal{F}$ on X, by

$$V^i(\mathcal{F}) := \{L \in \operatorname{Pic}^0(X) \mid H^i(X, \mathcal{F} \otimes L) \neq 0\} \subseteq \operatorname{Pic}^0(X)$$

the i-th cohomological support locus of $\mathcal{F}$, its main statements are roughly the following:

Dimension (D): One has $\operatorname{codim} V^i(\omega_X) \geq i - \dim X + \dim a(X)$ for all i [GL87, GL91]. This implies the generic vanishing theorem via Serre duality.

Linearity (L): The irreducible components of each $V^i(\omega_X)$ are torsion translates of abelian subvarieties of $\operatorname{Pic}^0(X)$ [GL91, Ara92, Sim93].

Strong linearity (SL): If $p_2 : X \times \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(X)$ is the second projection, and P is a Poincaré line bundle on $X \times \operatorname{Pic}^0(X)$, then $\mathbf{R}p_{2*}P$ is locally around each point quasi-isomorphic to a linear complex [GL91]. A precise

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version of this result is known to imply (L), except for the torsion statement, and based on this also (D).

Analogous results have been considered for the cohomology of local systems, replacing $\mathrm{Pic}^0(X)$ by $\mathrm{Char}(X)$, the algebraic group parametrizing rank one local systems [Ara92, Sim92, Sim93]. New approaches and extensions for the theory on $\mathrm{Pic}^0(X)$ have been introduced more recently, for example in [CH02, Hac04, PP11]. On the other hand, important gaps have remained in our understanding of some of the most basic objects. For instance, while (L) is also known for the sheaf of holomorphic p-forms Ω_X^p with $p < n$, a good generic Nakano-type vanishing statement as in (D) has eluded previous efforts, despite several partial results [GL87, PP11]. The same applies to the case of local systems of rank one, where the perhaps the even more interesting property (SL) has been missing as well.

In this paper, we answer those remaining questions, and at the same time recover the previous results of generic vanishing theory mentioned above (with the exception of the statement about torsion points, which is of a different nature) by enlarging the scope of the study to the class of filtered $\mathcal{D}$-modules associated to mixed Hodge modules on abelian varieties. In fact, there is a version of the Fourier-Mukai transform for $\mathcal{D}$-modules, introduced by Laumon [Lau96] and Rothstein [Rot96]; it takes $\mathcal{D}$-modules on an abelian variety to complexes of coherent sheaves on $A^\sharp$, the moduli space of line bundles on A with integrable connection. Our main results can be summarized briefly as describing the *Fourier-Mukai transform of the trivial* $\mathcal{D}$*-module* $\mathscr{O}_X$ on an irregular variety X.

2. Why mixed Hodge modules? To motivate the introduction of mixed Hodge modules into the problem, let us briefly recall the very elegant proof of the generic vanishing theorem discovered by Hacon [Hac04]. It goes as follows.

Let $A = \mathrm{Alb}(X)$ denote the Albanese variety of an irregular smooth complex projective variety X, and $a\colon X \to A$ its Albanese mapping (for some choice of base point, which does not matter here). Let $\widehat{A} = \mathrm{Pic}^0(A)$ denote the dual abelian variety. Using a well-known theorem of Kollár on the splitting of the direct image $\mathbf{R}a_*\omega_X$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_A)$ and standard manipulations, it is enough to prove that

$$\operatorname{codim} V^\ell(R^i a_*\omega_X) \geq \ell \quad \text{for all } i = 0, 1, \dots, k := \dim X - \dim a(X).$$

In terms of the Fourier-Mukai transform $\mathbf{R}\Phi_P\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_A) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{\widehat{A}})$, this is equivalent, via base change arguments, to the statement that

$$\operatorname{codim} \operatorname{Supp} R^\ell\Phi_P(R^i a_*\omega_X) \geq \ell$$

in the same range. Now the sheaves $R^i a_*\omega_X$ still satisfy a Kodaira-type vanishing theorem, and together with the special geometry of abelian varieties this implies after some work that the Fourier-Mukai transform of $R^i a_*\omega_X$ is the dual of a coherent sheaf $\mathscr{F}_i$ on $\widehat{A}$, i.e.

$$\mathbf{R}\Phi_P(R^i a_*\omega_X) \simeq \mathbf{R}\mathcal{H}om(\mathscr{F}_i, \mathscr{O}_{\widehat{A}}).$$

The desired inequality for the codimension of the support becomes

$$\operatorname{codim} \operatorname{Supp} R^\ell\Phi_P(R^i a_*\omega_X) = \operatorname{codim} \operatorname{Supp} \mathscr{E}xt^\ell(\mathscr{F}_i, \mathscr{O}_{\widehat{A}}) \geq \ell,$$

which is now a consequence of a general theorem about regular local rings. This proves the dimension statement (D), and hence the generic vanishing theorem for topologically trivial line bundles.

One of the subjects of this paper is to use this general framework in order to prove a generic vanishing theorem for general objects of Hodge-theoretic origin, and in particular for the sheaves Ω_X^p with $p < \dim X$. The role of Kollár's theorem is played by the decomposition theorem [BBD82], or more precisely by its Hodge-theoretic version due to Morihiko Saito [Sai90]. This is one main reason why mixed Hodge modules form a natural setting here. Another is the existence of a very general Kodaira-type vanishing theorem for mixed Hodge modules, again due to Saito, which becomes particularly useful on abelian varieties. This vanishing theorem allows us to generalize the second half of the proof above to any coherent sheaf of Hodge-theoretic origin on an abelian variety. Finally, in order to extract the relevant information about the sheaves Ω_X^p, one needs a result by Laumon and Saito on the behavior of filtered $\mathcal{D}$-modules under direct images, which only works well in the case of $\mathcal{D}$-modules that underlie mixed Hodge modules.

3. The main results. Let us now give a summary of the results we obtain. There are essentially two parts: vanishing and dimension results, for which which Hodge modules are crucially needed, and linearity results, which apply to certain Hodge modules, but for which the general theory of $\mathcal{D}$-modules and the harmonic theory of flat line bundles suffice in the proofs. The theory of mixed Hodge modules is reviewed in §4 below.

The starting point is a general Kodaira-type vanishing theorem for the graded pieces of the de Rham complex of a mixed Hodge module, proved by Saito. On an abelian variety A, this can be improved to a vanishing theorem for coherent sheaves of the form $\mathrm{gr}_k^F \mathcal{M}$, where $(\mathcal{M}, F)$ is any filtered $\mathcal{D}$-module underlying a mixed Hodge module on A (see Lemma 8.1 below). We use this observation to produce natural classes of perverse coherent sheaves [AB10, Kas04] on the dual abelian variety $\widehat{A}$, and on the parameter space for Higgs line bundles $\widehat{A} \times H^0(A, \Omega_A^1)$.

We first show that every mixed Hodge module on A gives rise to a collection of perverse coherent sheaves on $\widehat{A}$ (with respect to the dual standard t-structure).

Theorem 3.1. *Let A be a complex abelian variety, and M a mixed Hodge module on A with underlying filtered $\mathcal{D}$-module $(\mathcal{M}, F)$. Then for each $k \in \mathbb{Z}$, the coherent sheaf $\mathrm{gr}_k^F \mathcal{M}$ is a GV-sheaf on A, i.e.*

$$\operatorname{codim} V^i(\mathrm{gr}_k^F \mathcal{M}) \geq i \quad \text{for all } i.$$

Consequently, its Fourier-Mukai transform $\mathbf{R}\Phi_P(\mathrm{gr}_k^F \mathcal{M})$ is a perverse coherent sheaf on $\widehat{A}$.

This uses Hacon's general strategy, as in §2 above, and the correspondence established in [Pop09, PP11] between objects satisfying generic vanishing (or GV-objects) and perverse coherent sheaves in the sense of Kashiwara [Kas04].

In order to obtain a generic Nakano-type vanishing statement similar to (D), or statements for cohomological support loci of rank one local systems, we apply Theorem 3.1 to the direct image of the trivial Hodge module on an irregular variety under the Albanese map. Here our main tools are the decomposition theorem for Hodge modules [Sai88], extending the well-known result of [BBD82], and a formula due to Laumon [Lau85] for the behavior of the associated graded objects under projective direct images (which is true for mixed Hodge modules).

Our main results in this direction are the following. Let X be a smooth complex projective variety of dimension n, with nonzero irregularity $g = h^1(X, \mathscr{O}_X)$. Let $a\colon X \to A = \mathrm{Alb}(X)$ be the Albanese map of X. Consider the *defect of semi-smallness* of the Albanese map $a\colon X \to A$, which is defined by the formula

$$\delta(a) = \max_{\ell \in \mathbb{N}} \bigl(2\ell - \dim X + \dim A_\ell\bigr),$$

where $A_\ell = \bigl\{\, y \in A \bigm| \dim f^{-1}(y) \geq \ell \,\bigr\}$ for $\ell \in \mathbb{N}$. By applying the decomposition theorem and our results about mixed Hodge modules on abelian varieties to the direct image of the trivial Hodge module $\mathbb{Q}_X^H[n]$, we obtain the following theorem.

Theorem 3.2. *Let X be a smooth complex projective variety of dimension n. Then*

$$\operatorname{codim} V^q(\Omega_X^p) \geq |p + q - n| - \delta(a).$$

for every $p, q \in \mathbb{N}$.

A slightly stronger (and optimal) result, involving a natural invariant $k(X) \leq \delta(a)$ coming from the direct sum decomposition of $a_*\mathbb{Q}_X^H[n]$, is given in Theorem 10.2. The statement above is the appropriate generalization of the original generic vanishing theorem [GL87], which dealt with the case $p = n$. Note that, unlike in [GL87], the codimension bound depends on the entire Albanese mapping, not just on the dimension of the generic fiber. In the language of [PP11], our theorem is equivalent to the fact that, for each $p \in \mathbb{Z}$, the bundle Ω_X^p is a $\mathrm{GV}_{p-n-\delta(a)}$*-sheaf* with respect to the Fourier-Mukai transform induced by the Poincaré bundle on $X \times \mathrm{Pic}^0(X)$. Since the condition $\delta(a) = 0$ is equivalent to the Albanese map being semi-small, in particular one obtains:

Corollary 3.3. *Suppose that the Albanese map of X is semi-small. Then*

$$\operatorname{codim} V^q(\Omega_X^p) \geq |p + q - n|,$$

for every $p, q \in \mathbb{N}$, and so X satisfies the generic Nakano vanishing theorem.

Unlike in the case of ω_X, it is not sufficient to assume that the Albanese map is generically finite over its image; this was already pointed out in [GL87]. Nevertheless, our method also recovers the stronger statement for ω_X [GL87] and its higher direct images [Hac04] (see the end of §10). This is due to the special properties of the first non-zero piece of the Hodge filtration on mixed Hodge modules, established by Saito.

This approach also leads to a dimension theorem of type (D) for rank one local systems. Let $\mathrm{Char}(X) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ be the algebraic group of characters of X, and for each i consider the cohomological support loci

$$\Sigma^k(X) = \bigl\{\, \rho \in \mathrm{Char}(X) \bigm| H^k(X, \mathbb{C}_\rho) \neq 0 \,\bigr\},$$

where $\mathbb{C}_\rho$ denotes the local system of rank one associated to a character ρ.

Theorem 3.4. *Let X be a smooth complex projective variety. Then for each $k \in \mathbb{N}$,*

$$\operatorname{codim}_{\mathrm{Char}(X)} \Sigma^k(X) \geq 2\bigl(|k - \dim X| - k(X)\bigr).$$

To deduce this from the arguments leading to Theorem 3.2, we need to appeal to the structure results and the relationship with the space of Higgs bundles, proved by Simpson [Sim92, Sim93] and Arapura [Ara92]; see §12.

Note. While editing this paper, we learned of the very interesting preprint [KW11] by T. Krämer and R. Weissauer, who prove vanishing theorems for perverse sheaves on abelian varieties. They also obtain a generic vanishing theorem for Ω_X^p and for rank one local systems, involving the same quantity $\delta(a)$ as in Theorem 3.2, but without precise codimension bounds for the cohomological support loci. Their methods are very different from ours.

Two additional theorems complete the picture, by describing in detail the Fourier-Mukai transform of the trivial $\mathcal{D}$-module $\mathscr{O}_X$; they include results of type (D), (L) and (SL) on the space of rank one local systems on X. Here it is important to consider two different kinds of Fourier-Mukai transforms, corresponding in Simpson's terminology [Sim93] to the Dolbeault realization (via Higgs bundles) and the de Rham realization (via line bundles with integrable connection) of $\operatorname{Char}(X)$.

Setting $V = H^0(A, \Omega_A^1)$, one can naturally extend the usual Fourier-Mukai functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_A) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{\widehat{A}})$ to a relative transform (see §9)

$$\mathbf{R}\Phi_P \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\bigl(\mathscr{O}_{A\times V}\bigr) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\bigl(\mathscr{O}_{\widehat{A}\times V}\bigr).$$

One of the two theorems describes how the complex of filtered $\mathcal{D}$-modules $a_*(\mathscr{O}_X, F)$ underlying $a_*\mathbb{Q}_X^H[n] \in \mathrm{D}^{\mathrm{b}}\operatorname{MHM}(A)$ behaves with respect to this transform. We first show in Proposition 9.1 below that the associated graded complex satisfies

$$\operatorname{gr}_\bullet^F a_*(\mathscr{O}_X, F) \simeq \mathbf{R}a_*\Bigl[\mathscr{O}_X \otimes S^{\bullet-g} \to \Omega_X^1 \otimes S^{\bullet-g+1} \to \dots \to \Omega_X^n \otimes S^{\bullet-g+n}\Bigr],$$

where $S = \operatorname{Sym} V^*$, and the complex in brackets is placed in degrees $-n, \dots, 0$, with differential induced by the evaluation morphism $\mathscr{O}_X \otimes V \to \Omega_X^1$. Since this is a complex of finitely generated graded modules over $\operatorname{Sym}\Theta_A$, it naturally corresponds to a complex of coherent sheaves on cotangent bundle $T^*A = A \times V$, namely

$$\mathscr{C} = \mathbf{R}(a \times \mathrm{id})_*\Bigl[p_1^*\mathscr{O}_X \to p_1^*\Omega_X^1 \to \dots \to p_1^*\Omega_X^n\Bigr].$$

Theorem 3.5. *Let $a\colon X \to A$ be the Albanese map of a smooth complex projective variety of dimension n, and let $p_1\colon X \times V \to X$ be the first projection.*

(i) In the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A\times V})$, we have a non-canonical splitting

$$\mathscr{C} \simeq \bigoplus_{i,j} \mathscr{C}_{i,j}[-i]$$

where each $\mathscr{C}_{i,j}$ is a Cohen-Macaulay sheaf of dimension $\dim A$.

(ii) The support of each $\mathbf{R}\Phi_P\mathscr{C}_{i,j}$ is a finite union of torsion translates of triple tori in $\widehat{A} \times V$, subject to the inequalities

$$\operatorname{codim} \operatorname{Supp} R^\ell\Phi_P\mathscr{C}_{i,j} \geq 2\ell \qquad \text{for all } \ell \in \mathbb{Z}.$$

(iii) The dual objects $\mathbf{R}\mathcal{H}om\bigl(\mathbf{R}\Phi_P\mathscr{C}_{i,j}, \mathscr{O}_{\widehat{A}\times V}\bigr)$ also satisfy (ii).

A more refined version of this can be found in Theorem 13.2 below, where we also interpret the result in terms of perverse coherent sheaves: the objects $\mathbf{R}\Phi_P\mathscr{C}_{i,j}$ belong to a self-dual subcategory of the category of perverse coherent sheaves, with respect to a certain t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A\times V})$ introduced in §6. Note that this t-structure is different from the "dual standard t-structure" that appears in the generic vanishing theory of topologically trivial line bundles [Pop09].

The second of the two theorems is best stated in terms of the generalized Fourier-Mukai transform for $\mathcal{D}$-modules on abelian varieties, introduced by Laumon [Lau96] and Rothstein [Rot96]. Their work gives an equivalence of categories

$$\mathbf{R}\Phi_{P^\sharp} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_A) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A^\sharp}),$$

where $A^\sharp$ is the moduli space of line bundles with integrable connection on A, with universal cover $W = H^1(A, \mathbb{C}) = H^1(X, \mathbb{C})$. By composing with the Albanese mapping, there is also an induced functor

$$\mathbf{R}\Phi_{P^\sharp} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A^\sharp}).$$

We review the construction, both algebraically and analytically, in §15 below. This Fourier-Mukai transform is the right context for a strong linearity result (SL) for the $\mathcal{D}$-module $\mathscr{O}_X$, extending the result for topologically trivial line bundles in [GL91].

Theorem 3.6. *Let X be a smooth complex projective variety. Then the generalized Fourier-Mukai transform $\mathbf{R}\Phi_{P^\sharp}\mathscr{O}_X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A^\sharp})$ is locally, in the analytic topology, quasi-isomorphic to a linear complex. More precisely, let (L, ∇) be any line bundle with integrable connection on X, and $\mathcal{L}$ the associated local system. Then the pullback of $\mathbf{R}\Phi_{P^\sharp}\mathscr{O}_X$ from a neighborhood of (L, ∇) to the covering space W is quasi-isomorphic to (restriction of) the complex*

$$H^0(X, \mathcal{L}) \otimes \mathscr{O}_W \to H^1(X, \mathcal{L}) \otimes \mathscr{O}_W \to \cdots \to H^{2n}(X, \mathcal{L}) \otimes \mathscr{O}_W,$$

with differential given by the formula

$$H^k(X, \mathcal{L}) \otimes \mathscr{O}_W \to H^{k+1}(X, \mathcal{L}) \otimes \mathscr{O}_W, \quad \alpha \otimes f \mapsto \sum_{j=1}^{2g} (\varepsilon_j \wedge \alpha) \otimes z_j f.$$

Here $\varepsilon_1, \ldots, \varepsilon_{2g}$ is any basis of $H^1(X, \mathbb{C})$, and $z_1, \ldots, z_{2g}$ are the corresponding holomorphic coordinates on the affine space W.

As discussed in §20, every direct summand of a linear complex (in the derived category) is again quasi-isomorphic to a linear complex. It follows that all direct summands of $\mathbf{R}\Phi_{P^\sharp}\mathscr{O}_X$ coming from the decomposition

$$a_* \mathbb{Q}_X^H[n] \simeq \bigoplus_i M_{i,j}[-i] \in \mathrm{D}^{\mathrm{b}} \operatorname{MHM}(A)$$

have the same linearity property (see Corollary 15.3). Note also that using base change for local systems and the description of tangent cones to cohomological support loci as in [Lib02], via arguments as in[GL91] §4 (which we will not repeat here), Theorem 3.6 gives another proof of the linearity to the cohomological support loci $\Sigma^k(X)$, i.e. the statement of type (L).

Our proof of Theorem 3.6 relies on the harmonic theory for flat line bundles developed by Simpson [Sim92]. As in [GL91], the idea is that after pulling the complex $\mathbf{R}\Phi_{P^\sharp}\mathscr{O}_X$ back to the universal covering space of $A^\sharp$, one can use harmonic forms to construct a linear complex that is quasi-isomorphic to the pullback in a neighborhood of a given point. Additional technical difficulties arise however from the fact that the wedge product of two harmonic forms is typically no longer harmonic (which was true in the special case of $(0, q)$-forms needed in [GL91], and let to a quasi-isomorphism naturally being present in the picture). The new insight in this part of the paper is that a quasi-isomorphism can still be constructed by more involved analytic methods.

The last part of the paper contains a few statements extending our results to more general classes of $\mathcal{D}$-modules on abelian varieties, not necessarily underlying mixed Hodge modules. The proofs are of a different nature, and will be presented elsewhere. We propose a few natural open problems as well.

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B. Preliminaries

4. Mixed Hodge modules. In this section, we recall a few aspects of Morihiko Saito's theory of mixed Hodge modules [Sai90] which, as explained in the introduction, offers a convenient setting for the results of this paper. A very good survey can also be found in [Sai94]. We only discuss the case of a complex algebraic variety X, say of dimension n.

Saito defines an abelian category $\mathrm{MHM}(X)$ of *graded-polarizable mixed Hodge modules* on X. A mixed Hodge module is, roughly speaking, a variation of mixed Hodge structure with singularities. Over a one-point space it is the same thing as a graded-polarizable mixed Hodge structure with coefficients in $\mathbb{Q}$.

As in classical Hodge theory, there are pure and mixed objects. We shall denote by $\mathrm{HM}_\ell(X)$ the subcategory of polarizable (pure) Hodge modules of weight ℓ. On a one-point space, such an object is just a polarizable Hodge structure of weight ℓ. More generally, any polarizable variation of Hodge structure of weight k on a Zariski-open subset of some irreducible subvariety $Z \subseteq X$ uniquely extends to a Hodge module of weight $\dim Z + k$ on X. The existence of polarizations makes the category $\mathrm{HM}_\ell(X)$ semi-simple: each object admits a decomposition by support, and simple objects with support equal to an irreducible subvariety $Z \subseteq X$ are obtained from polarizable variations of Hodge structure on Zariski-open subsets of Z.

When $M \in \mathrm{HM}_\ell(X)$ is pure of weight ℓ, and we are in the presence of a projective morphism $f\colon X \to Y$, Saito shows that the direct image $f_* M \in \mathrm{D^b}\,\mathrm{MHM}(Y)$ splits non-canonically into the sum of its cohomology objects $H^i f_* M \in \mathrm{HM}_{\ell+i}(Y)$. The resulting isomorphism

$$f_* M \simeq \bigoplus_i H^i f_* M[-i]$$

is the analogue for Hodge modules of the decomposition theorem of [BBD82]. The proof of this result with methods from algebraic analysis and $\mathcal{D}$-modules is one of the main achievements of Saito's theory.

The precise definition of a mixed Hodge module is very involved; it uses regular holonomic $\mathcal{D}$-modules, perverse sheaves, and the theory of nearby and vanishing cycles. The familiar equivalence between local systems and flat vector bundles is replaced by the Riemann-Hilbert correspondence between perverse sheaves and regular holonomic $\mathcal{D}$-modules. Let $\mathcal{M}$ be a $\mathcal{D}$-module on X; in this paper, this always means a left module over the sheaf of algebraic differential operators $\mathcal{D}_X$. Recall that the de Rham complex of $\mathcal{M}$ is the $\mathbb{C}$-linear complex

$$\mathrm{DR}_X(\mathcal{M}) = \Big[\mathcal{M} \to \Omega_X^1 \otimes \mathcal{M} \to \cdots \to \Omega_X^n \otimes \mathcal{M}\Big],$$

placed in degrees $-n, \dots, 0$. When $\mathcal{M}$ is holonomic, $\mathrm{DR}_X(\mathcal{M})$ is constructible and satisfies the axioms for a perverse sheaf with coefficients in $\mathbb{C}$. According to the Riemann-Hilbert correspondence of Kashiwara and Mebkhout, the category of regular holonomic $\mathcal{D}$-modules is equivalent to the category of perverse sheaves $\operatorname{Perv}(\mathbb{C}_X)$ through the functor $\mathcal{M} \mapsto \operatorname{DR}_X(\mathcal{M})$. Internally, a *mixed Hodge module* M on a smooth complex algebraic variety X has three components:

(1) A regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, together with a good filtration $F_\bullet \mathcal{M}$ by $\mathscr{O}_X$-coherent subsheaves such that $\operatorname{gr}_\bullet^F \mathcal{M}$ is coherent over $\operatorname{gr}_\bullet^F \mathcal{D}_X$. This filtration plays the role of a Hodge filtration on $\mathcal{M}$.
(2) A perverse sheaf $K \in \operatorname{Perv}(\mathbb{Q}_X)$, together with an isomorphism
$$\alpha \colon \operatorname{DR}_X(\mathcal{M}) \to K \otimes_{\mathbb{Q}} \mathbb{C}.$$
This isomorphism plays the role of a $\mathbb{Q}$-structure on the $\mathcal{D}$-module $\mathcal{M}$. (When M is a mixed Hodge structure, K is the underlying $\mathbb{Q}$-vector space; when M corresponds to a polarizable variation of Hodge structure H on a Zariski-open subset of $Z \subseteq X$, K is the intersection complex $\operatorname{IC}_Z(H)$ of the underlying local system.)
(3) A finite increasing *weight filtration* $W_\bullet M$ of M by objects of the same kind, compatible with α, such that the graded quotients $\operatorname{gr}_\ell^W M = W_\ell M / W_{\ell-1} M$ belong to $\operatorname{HM}_\ell(X)$.

These components are subject to several conditions, which are defined by induction on the dimension of the support of $\mathcal{M}$. On a one-point space, a mixed Hodge module is a graded-polarizable mixed Hodge structure; in general, Saito's conditions require that the nearby and vanishing cycles of $\mathcal{M}$ with respect to any locally defined holomorphic function are again mixed Hodge modules (now on a variety of dimension $n-1$); the existence of polarizations; etc. An important fact is that these local conditions are preserved after taking direct images, and that they are sufficient to obtain global results such as the decomposition theorem.

The most basic Hodge module on a smooth variety X is the trivial Hodge module, denoted by $\mathbb{Q}_X^H[n]$. Its underlying $\mathcal{D}$-module is the structure sheaf $\mathscr{O}_X$, with the trivial filtration $F_0 \mathscr{O}_X = \mathscr{O}_X$; its underlying perverse sheaf is $\mathbb{Q}_X[n]$, the constant local system placed in degree $-n$. The comparison isomorphism is nothing but the well-known fact that the usual holomorphic de Rham complex
$$\mathscr{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \cdots \to \Omega_X^n$$
is a resolution of the constant local system $\mathbb{C}_X$. In general, all admissible variations of mixed Hodge structures are mixed Hodge modules, and any mixed Hodge module is generically one such. The usual cohomology groups of X can be obtained as the cohomology of the complex $p_* p^* \mathbb{Q}^H \in \mathrm{D}^{\mathrm{b}} \operatorname{MHM}(pt)$, where $p \colon X \to pt$ denotes the map to a one-point space, and $\mathbb{Q}^H$ is the Hodge structure of weight zero on $\mathbb{Q}$. There are similar expressions for cohomology with compact supports, intersection cohomology, and so forth, leading to a uniform description of the mixed Hodge structures on all of these groups.

5. Three theorems about mixed Hodge modules. In this section, we recall from the literature three useful theorems about the associated graded object $\operatorname{gr}_\bullet^F \mathcal{M}$, for a filtered $\mathcal{D}$-module $(\mathcal{M}, F)$ underlying a mixed Hodge module.

During the discussion, X will be a smooth complex projective variety of dimension n, and $M \in \operatorname{MHM}(X)$ a mixed Hodge module on X. As usual, we denote the underlying filtered $\mathcal{D}$-module by $(\mathcal{M}, F)$. Recall that the associated graded of the sheaf of differential operators $\mathcal{D}_X$, with respect to the filtration by order of differential operators, is isomorphic to $\mathcal{A}_X^\bullet = \operatorname{Sym}^\bullet \Theta_X$, the symmetric algebra of the tangent sheaf of X. Since $\operatorname{gr}_\bullet^F \mathcal{M}$ is finitely generated over this sheaf of algebras, it defines a coherent sheaf $\mathscr{C}(\mathcal{M}, F)$ on the cotangent bundle T^*X. The support of this sheaf is the characteristic variety of the $\mathcal{D}$-module $\mathcal{M}$, and therefore of pure dimension n because $\mathcal{M}$ is holonomic.

The first of the three theorems is Saito's generalization of the Kodaira vanishing theorem. Before we state it, observe that the filtration $F_\bullet \mathcal{M}$ is compatible with the $\mathcal{D}$-module structure on $\mathcal{M}$, and therefore induces a filtration on the de Rham complex of $\mathcal{M}$ by the formula

$$F_k \operatorname{DR}_X(\mathcal{M}) = \Big[F_k\mathcal{M} \to \Omega_X^1 \otimes F_{k+1}\mathcal{M} \to \cdots \to \Omega_X^n \otimes F_{k+n}\mathcal{M} \Big]. \tag{5.1}$$

The associated graded complex for the filtration in (5.1) is

$$\operatorname{gr}_k^F \operatorname{DR}_X(\mathcal{M}) = \Big[\operatorname{gr}_k^F \mathcal{M} \to \Omega_X^1 \otimes \operatorname{gr}_{k+1}^F \mathcal{M} \to \cdots \to \Omega_X^n \otimes \operatorname{gr}_{k+n}^F \mathcal{M} \Big],$$

which is now a complex of coherent sheaves of $\mathscr{O}_X$-modules in degrees $-n, \dots, 0$. This complex satisfies the following Kodaira-type vanishing theorem.

Theorem 5.2 (Saito)**.** *Let $(\mathcal{M}, F)$ be the filtered $\mathcal{D}$-module underlying a mixed Hodge module on a smooth projective variety X, and let L be any ample line bundle.*

(1) One has $\mathbf{H}^i\big(X, \operatorname{gr}_k^F \operatorname{DR}_X(\mathcal{M}) \otimes L\big) = 0$ for all $i > 0$.
(2) One has $\mathbf{H}^i\big(X, \operatorname{gr}_k^F \operatorname{DR}_X(\mathcal{M}) \otimes L^{-1}\big) = 0$ for all $i < 0$.

Proof. The proof works by reducing the assertion to a vanishing theorem for perverse sheaves on affine varieties, with the help of Saito's formalism. More details can be found in [Sai90, Proposition 2.33]. □

Example 5.3. For the trivial Hodge module $M = \mathbb{Q}_X^H[n]$ on X, we have $\mathcal{M} = \mathscr{O}_X$, and therefore $\operatorname{gr}_{-n}^F \operatorname{DR}_X(\mathscr{O}_X) = \omega_X$. This shows that Theorem 5.2 generalizes the Kodaira vanishing theorem. Since $\operatorname{gr}_{-p}^F \operatorname{DR}_X(\mathscr{O}_X) = \Omega_X^p[n-p]$, it also generalizes the Nakano vanishing theorem.

The second theorem gives more information about the coherent sheaf $\mathscr{C}(\mathcal{M}, F)$ on the cotangent bundle of X. Before stating it, we recall the definition of the *Verdier dual* $M' = \mathbf{D}_X M$ of a mixed Hodge module. If $\operatorname{rat} M$ is the perverse sheaf underlying M, then $\operatorname{rat} M'$ is simply the usual topological Verdier dual. On the level of $\mathcal{D}$-modules, note that since $\mathcal{M}$ is left $\mathcal{D}$-module, the dual complex

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, \mathcal{D}_X[n]\big)$$

is naturally a complex of right $\mathcal{D}$-modules; since $\mathcal{M}$ is holonomic, it is quasi-isomorphic to a single right $\mathcal{D}$-module. If $\mathcal{M}'$ denotes the left $\mathcal{D}$-module underlying the Verdier dual M', then that right $\mathcal{D}$-module is $\omega_X \otimes_{\mathscr{O}_X} \mathcal{M}'$. Thus we have

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, \mathcal{D}_X[n]\big) \simeq \omega_X \otimes_{\mathscr{O}_X} \mathcal{M}',$$

where the right $\mathcal{D}$-module structure on $\omega_X \otimes \mathcal{M}'$ is given by the rule $\xi \cdot (\omega \otimes m') = (\xi\omega) \otimes m' - \omega \otimes (\xi m')$ for $\xi \in \Theta_X$. The Hodge filtration on $\mathcal{M}$ induces a filtration on the dual complex and hence on $\mathcal{M}'$. The following result shows that this induced filtration is well-behaved, in a way that makes duality and passage to the associated graded compatible with each other.

Theorem 5.4 (Saito). *Let M be a mixed Hodge module on a smooth complex algebraic variety X of dimension n, and let M' denote its Verdier dual. Then*

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}_X^\bullet}\big(\mathrm{gr}_\bullet^F \mathcal{M}, \mathcal{A}_X^\bullet[n]\big) \simeq \omega_X \otimes_{\mathscr{O}_X} \mathrm{gr}_{\bullet+2n}^F \mathcal{M}',$$

where sections of $\mathrm{Sym}^k \Theta_X$ act with an extra factor of $(-1)^k$ on the right-hand side. If we consider both sides as coherent sheaves on T^*X, we obtain

$$\mathbf{R}\mathcal{H}om\big(\mathscr{C}(\mathcal{M}, F), \mathscr{O}_{T^*X}[n]\big) \simeq p^*\omega_X \otimes (-1)_{T^*X}^* \mathscr{C}(\mathcal{M}', F),$$

where $p\colon T^*X \to X$ is the projection. In particular, $\mathscr{C}(\mathcal{M}, F)$ is always a Cohen-Macaulay sheaf of dimension n.

Proof. That the coherent sheaf $\mathscr{C}(\mathcal{M}, F)$ on T^*X is Cohen-Macaulay is proved in [Sai88, Lemme 5.1.13]. For an explanation of how this fact implies the formula for the dual of the graded module $\mathrm{gr}_\bullet^F \mathcal{M}$, see [Sch11, §3.1]. □

The last of the three theorems gives a formula for the associated graded object of f_*M, where $f\colon X \to Y$ is a projective morphism between two smooth complex algebraic varieties. The formula itself first appears in a paper by Laumon [Lau85, Construction 2.3.2], but it is not true in the generality claimed there (that is to say, for arbitrary filtered $\mathcal{D}$-modules).

Before stating the precise result, we give an informal version. The following diagram of morphisms, induced by f, will be used throughout:

$$\begin{tikzcd} T^*X & T^*Y \times_Y X \arrow[l, "df"'] \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ & T^*Y \arrow[r, "p"] & Y \end{tikzcd}$$

Let $M \in \mathrm{MHM}(X)$ be a mixed Hodge module, let $(\mathcal{M}, F)$ be its underlying filtered $\mathcal{D}$-module, and write $\mathscr{C}(\mathcal{M}, F)$ for the associated coherent sheaf on T^*X. We shall denote the complex of filtered holonomic $\mathcal{D}$-modules underlying the direct image $f_*M \in \mathrm{D}^{\mathrm{b}}\,\mathrm{MHM}(Y)$ by the symbol $f_*(\mathcal{M}, F)$. Then *Laumon's formula* claims that

$$\mathscr{C}\big(f_*(\mathcal{M}, F)\big) \simeq \mathbf{R}p_{1*}\big(\mathbf{L}df^*\mathscr{C}(\mathcal{M}, F) \otimes p_2^*\omega_{X/Y}\big)$$

as objects in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{T^*Y})$. Unfortunately, the passage to coherent sheaves on the cotangent bundle loses the information about the grading; it is therefore better to work directly with the graded modules. As above, we write $p\colon T^*X \to X$ for the projection, and denote the graded sheaf of $\mathscr{O}_X$-algebras $p_*\mathscr{O}_{T^*X} = \mathrm{Sym}\,\Theta_X$ by the symbol $\mathcal{A}_X$.

Theorem 5.5 (Laumon). *Let $f\colon X \to Y$ be a projective morphism between smooth complex algebraic varieties, and let $M \in \mathrm{MHM}(X)$. Then with notation as above,*

$$\mathrm{gr}_\bullet^F \mathbf{R}f_*(\mathcal{M}, F) \simeq \mathbf{R}f_*\Big(\omega_{X/Y} \otimes_{\mathscr{O}_X} \mathrm{gr}_{\bullet+\dim X - \dim Y}^F \mathcal{M} \otimes_{\mathcal{A}_X} f^*\mathcal{A}_Y\Big).$$

Proof. This can be easily proved using Saito's formalism of induced $\mathcal{D}$-modules; both the factor of $\omega_{X/Y}$ and the shift in the grading come from the transformation between left and right $\mathcal{D}$-modules that is involved. To illustrate what is going on, we shall outline a proof based on factoring f through its graph. By this device, it suffices to verify the formula in two cases: (1) for a regular closed embedding $f\colon X \hookrightarrow Y$; (2) for a projection $f\colon Y \times Z \to Y$ with Z smooth and projective.

We first consider the case where $f\colon X \hookrightarrow Y$ is a regular closed embedding of codimension r. Here $\omega_{X/Y} \simeq \det N_{X|Y}$. Working locally, we may assume without loss of generality that $Y = X \times \mathbb{C}^r$. Let $t_1, \dots, t_r$ denote the coordinates on $\mathbb{C}^r$, and let $\partial_1, \dots, \partial_r$ be the corresponding vector fields. Then

$$f^* \mathcal{A}_Y = \mathcal{A}_X[\partial_1, \dots, \partial_r],$$

and so the formula we need to prove is that

$$\operatorname{gr}_\bullet^F f_*(\mathcal{M}, F) \simeq f_*\Big(\operatorname{gr}_{\bullet-r}^F \mathcal{M} \otimes_{\mathcal{A}_X} \mathcal{A}_X[\partial_1, \dots, \partial_r]\Big).$$

By [Sai88, p. 850], we have $f_*(\mathcal{M}, F) \simeq \mathcal{M}[\partial_1, \dots, \partial_r]$, with filtration given by

$$F_p\, f_*(\mathcal{M}, F) \simeq \sum_{k+|\nu| \le p-r} f_*\big(F_k \mathcal{M}\big) \otimes \partial^\nu.$$

Consequently, we get

$$\operatorname{gr}_p^F f_*(\mathcal{M}, F) \simeq \bigoplus_{\nu \in \mathbb{N}^r} f_*\big(\operatorname{gr}_{p-r-|\nu|}^F \mathcal{M}\big) \otimes \partial^\nu,$$

which is the desired formula. Globally, the factor of $\det N_{X|Y}$ is needed to make the above isomorphism coordinate independent.

Next, consider the case where $X = Y \times Z$, with Z smooth and projective of dimension r, and $f = p_1$. Then $\omega_{X/Y} \simeq p_2^* \omega_Z$. In this case, we have

$$f_*(\mathcal{M}, F) = \mathbf{R} p_{1*} \operatorname{DR}_{Y \times Z/Y}(\mathcal{M}),$$

where $\operatorname{DR}_{Y \times Z/Y}(\mathcal{M})$ is the relative de Rham complex

$$\Big[\mathcal{M} \to \Omega^1_{Y \times Z/Y} \otimes \mathcal{M} \to \cdots \to \Omega^r_{Y \times Z/Y} \otimes \mathcal{M}\Big],$$

supported in degrees $-r, \dots, 0$. As in (5.1), the Hodge filtration on the $\mathcal{D}$-module $\mathcal{M}$ induces a filtration on the relative de Rham complex by the formula

$$F_k \operatorname{DR}_{Y \times Z/Y}(\mathcal{M}) = \Big[F_k \mathcal{M} \to \Omega^1_{Y \times Z/Y} \otimes F_{k+1} \mathcal{M} \to \cdots \to \Omega^r_{Y \times Z/Y} \otimes F_{k+r} \mathcal{M}\Big].$$

Now the key point is that since Z is smooth and projective, the induced filtration on the direct image complex is strict by [Sai90, Theorem 2.14]. It follows that

$$\operatorname{gr}_\bullet^F f_*(\mathcal{M}, F) \simeq \mathbf{R} p_{1*}\Big[\operatorname{gr}_\bullet^F \mathcal{M} \to p_2^* \Omega^1_Z \otimes \operatorname{gr}_{\bullet+1}^F \mathcal{M} \to \cdots \to p_2^* \Omega^r_Z \otimes \operatorname{gr}_{\bullet+r}^F \mathcal{M}\Big].$$

On the other hand, a graded locally free resolution of ω_Z as a graded $\mathcal{A}_Z$-module is given by the complex

$$\Big[\mathcal{A}_Z^{\bullet-r} \to \Omega^1_Z \otimes \mathcal{A}_Z^{\bullet-r+1} \to \cdots \to \Omega^r_Z \otimes \mathcal{A}_Z^\bullet\Big],$$

again in degrees $-r, \dots, 0$. Therefore $\omega_{X/Y} \otimes_{\mathscr{O}_X} f^* \mathcal{A}_Y$ is naturally resolved, as a graded $\mathcal{A}_X$-module, by the complex

$$\Big[\mathcal{A}_X^{\bullet-r} \to p_2^* \Omega^1_Z \otimes \mathcal{A}_X^{\bullet-r+1} \to \cdots \to p_2^* \Omega^r_Z \otimes \mathcal{A}_X^\bullet\Big],$$

and so $\operatorname{gr}_{\bullet+r}^F \mathcal{M} \otimes_{\mathcal{A}_X} f^* \mathcal{A}_Y \otimes_{\mathscr{O}_X} \omega_{X/Y}$ is represented by the complex

$$\Big[\operatorname{gr}_\bullet^F \mathcal{M} \to p_2^* \Omega^1_Z \otimes \operatorname{gr}_{\bullet+1}^F \mathcal{M} \to \cdots \to p_2^* \Omega^r_Z \otimes \operatorname{gr}_{\bullet+r}^F \mathcal{M}\Big],$$

from which the desired formula follows immediately. $\square$

6. Perverse coherent sheaves. Let X be a smooth complex algebraic variety. In this section, we record some information about perverse t-structures on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, emphasizing two that are of interest to us here. We follow the notation introduced by Kashiwara [Kas04]. For a (possibly non-closed) point x of the scheme X, we write $\kappa(x)$ for the residue field at the point, $i_x\colon \operatorname{Spec}\kappa(x) \hookrightarrow X$ for the inclusion, and $\operatorname{codim}(x) = \dim \mathscr{O}_{X,x}$ for the codimension of the closed subvariety $\overline{\{x\}}$.

A *supporting function* on X is a function $p\colon X \to \mathbb{Z}$ from the topological space of the scheme X to the integers, with the property that $p(y) \geq p(x)$ whenever $y \in \overline{\{x\}}$. Given such a supporting function, one defines two families of subcategories

$$ {}^p\mathrm{D}^{\leq k}_{\mathrm{coh}}(\mathscr{O}_X) = \left\{ M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \;\middle|\; \mathbf{L}i_x^* M \in \mathrm{D}^{\leq k+p(x)}_{\mathrm{coh}}\bigl(\kappa(x)\bigr) \text{ for all } x \in X \right\} $$

and

$$ {}^p\mathrm{D}^{\geq k}_{\mathrm{coh}}(\mathscr{O}_X) = \left\{ M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \;\middle|\; \mathbf{R}i_x^! M \in \mathrm{D}^{\geq k+p(x)}_{\mathrm{coh}}\bigl(\kappa(x)\bigr) \text{ for all } x \in X \right\}. $$

The following fundamental result is proved in [Kas04, Theorem 5.9] and, based on an idea of Deligne, in [AB10, Theorem 3.10].

Theorem 6.1. *The above subcategories define a bounded t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ iff the supporting function has the property that $p(y) - p(x) \leq \operatorname{codim}(y) - \operatorname{codim}(x)$ for every pair of points $x, y \in X$ with $y \in \overline{\{x\}}$.*

For example, $p = 0$ corresponds to the standard t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. An equivalent way of putting the condition in Theorem 6.1 is that the dual function $\hat{p}(x) = \operatorname{codim}(x) - p(x)$ should again be a supporting function. If that is the case, one has the identities

$$ \begin{aligned} {}^{\hat{p}}\mathrm{D}^{\leq k}_{\mathrm{coh}}(\mathscr{O}_X) &= \mathbf{R}\mathcal{H}om\Bigl({}^p\mathrm{D}^{\geq -k}_{\mathrm{coh}}(\mathscr{O}_X), \mathscr{O}_X\Bigr) \\ {}^{\hat{p}}\mathrm{D}^{\geq k}_{\mathrm{coh}}(\mathscr{O}_X) &= \mathbf{R}\mathcal{H}om\Bigl({}^p\mathrm{D}^{\leq -k}_{\mathrm{coh}}(\mathscr{O}_X), \mathscr{O}_X\Bigr), \end{aligned} $$

which means that the duality functor $\mathbf{R}\mathcal{H}om(-, \mathscr{O}_X)$ exchanges the two perverse t-structures defined by p and $\hat{p}$. The heart of the t-structure defined by p is denoted

$$ {}^p\mathrm{Coh}(\mathscr{O}_X) = {}^p\mathrm{D}^{\leq 0}_{\mathrm{coh}}(\mathscr{O}_X) \cap {}^p\mathrm{D}^{\geq 0}_{\mathrm{coh}}(\mathscr{O}_X), $$

and is called the abelian category of *p-perverse coherent sheaves.*

We are interested in two special cases of Kashiwara's result. One is that the set of objects $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ with $\operatorname{codim} \operatorname{Supp} H^i(E) \geq i$ for all i forms part of a t-structure (the "dual standard t-structure" in Kashiwara's terminology); the other is that the same is true for the set of objects with $\operatorname{codim} \operatorname{Supp} H^i(E) \geq 2i$ for all i. This can be formalized in the following way.

We first define a supporting function $c\colon X \to \mathbb{Z}$ by the formula $c(x) = \operatorname{codim}(x)$. Since the dual function is $\hat{c} = 0$, it is clear that c defines a t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, namely the dual of the standard t-structure. The following result by Kashiwara [Kas04, Proposition 4.3] gives an alternative description of ${}^c\mathrm{Coh}(\mathscr{O}_X)$.

Lemma 6.2. *The following three conditions on $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ are equivalent:*

1. *(1) E belongs to ${}^c\mathrm{Coh}(\mathscr{O}_X)$.*
2. *(2) The dual object $\mathbf{R}\mathcal{H}om(E, \mathscr{O}_X)$ is a coherent sheaf.*
3. *(3) E is quasi-isomorphic to a bounded complex of flat $\mathscr{O}_X$-modules in non-negative degrees, and satisfies $\operatorname{codim} \operatorname{Supp} H^i(E) \geq i$ for every $i \geq 0$.*

We now define a second function $m\colon X \to \mathbb{Z}$ by the formula

$$m(x) = \left\lfloor \tfrac{1}{2} \operatorname{codim}(x) \right\rfloor .$$

It is easily verified that both m and the dual function

$$\hat{m}(x) = \left\lceil \tfrac{1}{2} \operatorname{codim}(x) \right\rceil$$

are supporting functions. As a consequence of Theorem 6.1, m defines a bounded t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$; objects of the heart ${}^m\mathrm{Coh}(\mathscr{O}_X)$ will be called m-perverse coherent sheaves. (We use this letter because m and $\hat{m}$ are as close as one can get to "middle perversity". Since the equality $m = \hat{m}$ can never be satisfied, there is of course no actual middle perversity for coherent sheaves.)

The next lemma follows easily from [Kas04, Lemma 5.5].

Lemma 6.3. *The perverse t-structures defined by m and $\hat{m}$ satisfy*

$$\begin{aligned} {}^m\mathrm{D}^{\leq k}_{\mathrm{coh}}(\mathscr{O}_X) &= \left\{ E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \;\middle|\; \operatorname{codim} \operatorname{Supp} H^i(E) \geq 2(i-k) \textit{ for all } i \in \mathbb{Z} \right\} \\ {}^{\hat{m}}\mathrm{D}^{\leq k}_{\mathrm{coh}}(\mathscr{O}_X) &= \left\{ E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \;\middle|\; \operatorname{codim} \operatorname{Supp} H^i(E) \geq 2(i-k) - 1 \textit{ for all } i \in \mathbb{Z} \right\}. \end{aligned}$$

By duality, this also describes the subcategories with $\geq k$.

Consequently, an object $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ is an m-perverse coherent sheaf iff

$$\operatorname{codim} \operatorname{Supp} H^i(E) \geq 2i \quad \text{and} \quad \operatorname{codim} \operatorname{Supp} R^i\mathcal{H}om(E, \mathscr{O}_X) \geq 2i - 1$$

for every $i \in \mathbb{Z}$. This shows one more time that the category of m-perverse coherent sheaves is not preserved by the duality functor $\mathbf{R}\mathcal{H}om(-, \mathscr{O}_X)$.

Lemma 6.4. *If $E \in {}^m\mathrm{Coh}(\mathscr{O}_X)$, then $E \in \mathrm{D}^{\geq 0}_{\mathrm{coh}}(\mathscr{O}_X)$.*

Proof. This is obvious from the fact that $m(x) \geq 0$. □

7. Integral functors and GV-objects. Let X and Y be smooth projective complex varieties of dimensions n and g respectively, and let $P \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X \times Y})$ be an object inducing integral functors

$$\mathbf{R}\Phi_P\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y),\ \mathbf{R}\Phi(\cdot) := \mathbf{R}p_{Y*}(p_X^*(-) \overset{\mathbf{L}}{\otimes} P)$$

and

$$\mathbf{R}\Psi_P\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X),\ \mathbf{R}\Psi(\cdot) := \mathbf{R}p_{X*}(p_Y^*(-) \overset{\mathbf{L}}{\otimes} P).$$

Let $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. There is a useful cohomological criterion for checking whether the Fourier transform $\mathbf{R}\Phi_P E$ is a perverse coherent sheaf on Y with respect to the t-structure defined by c from the previous subsection. Note first that by base change, for any sufficiently positive ample line bundle L on Y, the transform $\mathbf{R}\Psi_P(L^{-1})$ is supported in degree g, and $R^g\Psi_P(L^{-1})$ is a locally free sheaf on A. The following is contained in [Pop09, Theorem 3.8 and 4.1], and is partly based on the previous [Hac04, Theorem 1.2] and [PP11, Theorem A].

Theorem 7.1. *The following are equivalent:*

(1) $\mathbf{R}\Phi_P E$ *belongs to* ${}^c\mathrm{D}^{\leq 0}_{\mathrm{coh}}(\mathscr{O}_Y)$.

(2) For any sufficiently ample line bundle L on Y, and every $i > 0$,

$$H^i\big(A, E \otimes R^g\Psi_P(L^{-1})\big) = 0.$$

Moreover, if E is a sheaf—or more generally a geometric GV*-object, in the language of* [Pop09, Definition 3.7]*—satisfying the two equivalent conditions above, then $\mathbf{R}\Phi_P E$ is a perverse sheaf in ${}^c\mathrm{Coh}(\mathscr{O}_Y)$.*

Definition 7.2. An object E satisfying the equivalent conditions in the Theorem is called a GV-*object* (with respect to P); if it is moreover a sheaf, then it is called a GV-*sheaf.*

C. Constructing perverse coherent sheaves

8. Mixed Hodge modules on abelian varieties. Let A be a complex abelian variety of dimension g, and let $M \in \mathrm{MHM}(A)$ be a mixed Hodge module on A. As usual, we denote the underlying filtered holonomic $\mathcal{D}$-module by $(\mathcal{M}, F)$. From Theorem 5.2, we know that the associated graded pieces of the de Rham complex $\mathrm{DR}(\mathcal{M}, F)$ satisfy an analogue of Kodaira vanishing. A key observation is that on abelian varieties, the same vanishing theorem holds for the individual coherent sheaves $\mathrm{gr}_k^F \mathcal{M}$, due to the fact that the cotangent bundle of A is trivial. As we shall see, this implies that each $\mathrm{gr}_k^F \mathcal{M}$ is a GV-sheaf on A, and therefore transforms to a perverse coherent sheaf on $\widehat{A}$ (with respect to the dual standard t-structure).

Lemma 8.1. *Let $(\mathcal{M}, F)$ be the filtered $\mathcal{D}$-module underlying a mixed Hodge module on A, and let L be an ample line bundle. Then for each $k \in \mathbb{Z}$, we have*

$$H^i\big(A, \mathrm{gr}_k^F \mathcal{M} \otimes L\big) = 0$$

for every $i > 0$.

Proof. Consider for each $k \in \mathbb{Z}$ the complex of coherent sheaves

$$\mathrm{gr}_k^F \mathrm{DR}_A(\mathcal{M}) = \Big[\mathrm{gr}_k^F \mathcal{M} \to \Omega_A^1 \otimes \mathrm{gr}_{k+1}^F \mathcal{M} \to \cdots \to \Omega_A^g \otimes \mathrm{gr}_{k+g}^F \mathcal{M}\Big],$$

supported in degrees $-g, \dots, 0$. According to Theorem 5.2, this complex has the property that, for $i > 0$,

$$\mathbf{H}^i\big(A, \mathrm{gr}_k^F \mathrm{DR}_A(\mathcal{M}) \otimes L\big) = 0.$$

Using the fact that $\Omega_A^1 \simeq \mathscr{O}_A^{\oplus g}$, one can deduce the asserted vanishing theorem for the individual sheaves $\mathrm{gr}_k^F \mathcal{M}$ by induction on k. Indeed, since $\mathrm{gr}_k^F \mathcal{M} = 0$ for $k \ll 0$, inductively one has for each k a distinguished triangle

$$E_k \to \mathrm{gr}_k^F \mathrm{DR}_A(\mathcal{M}) \to \mathrm{gr}_{k+g}^F \mathcal{M} \to E_k[1],$$

with E_k an object satisfying $\mathbf{H}^i(A, E_k \otimes L) = 0$. □

We now obtain the first theorem of the introduction, by combining Lemma 8.1 with Theorem 7.1 and a trick invented by Mukai. As mentioned above, the method is the same as in Hacon's proof of the generic vanishing theorem [Hac04, PP11].

Proof of Theorem 3.1. Let L be an ample line bundle on $\widehat{A}$. By Theorem 7.1, it suffices to show that

$$H^i\big(A, \mathrm{gr}_k^F \mathcal{M} \otimes R^g\Psi_P(L^{-1})\big) = 0$$

for $i > 0$. Let $\varphi_L \colon \widehat{A} \to A$ be the isogeny induced by L. Then, by virtue of φ_L being étale,

$$H^i\big(A, \mathrm{gr}_k^F \mathcal{M} \otimes R^g\Psi_P(L^{-1})\big) \hookrightarrow H^i\big(\widehat{A}, \varphi_L^* \mathrm{gr}_k^F \mathcal{M} \otimes \varphi_L^* R^g\Psi_P(L^{-1})\big)$$

is injective, and so we are reduced to proving that the group on the right vanishes whenever $i > 0$.

Let $N = \varphi_L^* M$ be the pullback of the mixed Hodge module M to $\widehat{A}$. If $(\mathcal{N}, F)$ denotes the underlying filtered holonomic $\mathcal{D}$-module, then $F_k\mathcal{N} = \varphi_L^* F_k \mathcal{M}$ because φ_L is étale. On the other hand, by [Muk81] 3.11

$$\varphi_L^* R^g \Psi_P(L^{-1}) \simeq H^0(\widehat{A}, L) \otimes L.$$

We therefore get

$$H^i\big(\widehat{A}, \varphi_L^* \operatorname{gr}_k^F \mathcal{M} \otimes \varphi_L^* R^g \Psi_P(L^{-1})\big) \simeq H^0(\widehat{A}, L) \otimes H^i\big(\widehat{A}, \operatorname{gr}_k^F \mathcal{N} \otimes L\big),$$

which vanishes for $i > 0$ by Lemma 8.1. □

Going back to the de Rham complex, it is worth recording the complete information one can obtain about $\operatorname{gr}_k^F \mathrm{DR}_A(\mathcal{M})$ from Saito's theorem, since this produces further natural examples of GV-objects on A. One one hand, just as in the proof of Theorem 3.1, we see that each $\operatorname{gr}_k^F \mathrm{DR}_A(\mathcal{M})$ is a GV-object. On the other hand, since $\operatorname{gr}_k^F \mathrm{DR}_A(\mathcal{M})$ is supported in non-positive degrees, its Fourier-Mukai transform could a priori have cohomology in negative degrees. The following proposition shows that this is not the case.

Proposition 8.2. *If $(\mathcal{M}, F)$ underlies a mixed Hodge module on A, then*

$$\mathbf{R}\Phi_P\big(\operatorname{gr}_k^F \mathrm{DR}_A(\mathcal{M})\big) \in \mathrm{D}^{\geq 0}_{\mathrm{coh}}(\widehat{A}).$$

Proof. By a standard application of Serre vanishing (see [PP11, Lemma 2.5]), it suffices to show that for any sufficiently positive ample line bundle L on $\widehat{A}$,

$$\mathbf{H}^i\big(A, \operatorname{gr}_k^F \mathrm{DR}(\mathcal{M}, F) \otimes R^0\Psi_P(L)\big) = 0$$

for $i < 0$. Assuming that L is symmetric, $R^0\Psi_P(L)$ is easily seen to be the dual of the locally free sheaf $R^g\Psi_P(L^{-1})$, and so arguing as in the proof of Theorem 3.1, we are reduced to proving that

$$\mathbf{H}^i\big(\widehat{A}, \varphi_L^* \operatorname{gr}_k^F \mathrm{DR}(\mathcal{M}, F) \otimes L^{-1}\big) = 0$$

whenever $i < 0$. But since $\varphi_L^* \operatorname{gr}_k^F \mathrm{DR}(\mathcal{M}, F) \simeq \operatorname{gr}_k^F \mathrm{DR}(\mathcal{N}, F)$, this is an immediate consequence of part (2) of Saito's vanishing Theorem 5.2. □

Corollary 8.3. *If $(\mathcal{M}, F)$ underlies a mixed Hodge module on A, then*

$$\operatorname{gr}_k^F \mathrm{DR}_A(\mathcal{M}) \text{ and } \mathbf{R}\mathcal{H}om\big(\operatorname{gr}_k^F \mathrm{DR}_A(\mathcal{M}), \mathscr{O}_A[g]\big)$$

are GV*-objects on A. Therefore the graded pieces of the de Rham complexes associated to such $\mathcal{D}$-modules form a class of* GV*-objects which is closed under Grothendieck duality.*

Proof. Note that, by definition, the GV-objects on A are precisely those $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_A)$ for which $\mathbf{R}\Phi_P(E) \in {}^c\mathrm{D}^{\leq 0}_{\mathrm{coh}}(\mathscr{O}_{\widehat{A}})$, and that the Fourier-Mukai and duality functors satisfy the exchange formula

$$(\mathbf{R}\Phi_P E)^\vee \simeq \mathbf{R}\Phi_{P^{-1}}(E^\vee)[g]$$

for any object E in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_A)$ (see e.g. [PP11] Lemma 2.2). Observing that P and P^{-1} differ only by multiplication by -1, the statement follows from the Proposition above and the equivalence in [Kas04, Proposition 4.3], which says that $\mathrm{D}^{\geq 0}_{\mathrm{coh}}(\mathscr{O}_{\widehat{A}})$ is the category obtained by applying the functor $\mathbf{R}\mathcal{H}om(-, \mathscr{O}_{\widehat{A}})$ to ${}^c\mathrm{D}^{\leq 0}_{\mathrm{coh}}(\mathscr{O}_{\widehat{A}})$. □

D. Applications to generic vanishing

9. The decomposition theorem for the Albanese map. Let X be a smooth complex projective variety of dimension n, let $A = \mathrm{Alb}(X)$ be its Albanese variety, and let $a\colon X \to A$ be the Albanese map (for some choice of base point). As before, we set $g = \dim A$.

To understand the Hodge theory of the Albanese map, we consider the direct image $f_* \mathbb{Q}_X^H[n]$ in $\mathrm{D^b}\,\mathrm{MHM}(A)$. Here $\mathbb{Q}_X^H[n]$ is the trivial Hodge module on X; its underlying perverse sheaf is $\mathbb{Q}_X[n]$, and its underlying filtered holonomic $\mathcal{D}$-module is $(\mathscr{O}_X, F)$, where $\mathrm{gr}_k^F \mathscr{O}_X = 0$ for $k \neq 0$. Note that $\mathbb{Q}_X^H[n] \in \mathrm{HM}_n(X)$ is pure of weight n, because $\mathbb{Q}_X$ is a variation of Hodge structure of weight 0. According to the decomposition theorem [Sai88, Théorème 5.3.1 and Corollaire 5.4.8], we have a (non-canonical) decomposition

$$f_* \mathbb{Q}_X^H[n] \simeq \bigoplus_{i \in \mathbb{Z}} M_i[-i],$$

where each $M_i = H^i f_* \mathbb{Q}_X^H[n]$ is a pure Hodge module on A of weight $n+i$. Each M_i can be further decomposed (canonically) into a finite sum of simple Hodge modules

$$M_i = \bigoplus_j M_{i,j},$$

where $M_{i,j}$ has strict support equal to some irreducible subvariety $Z_{i,j} \subseteq A$; the perverse sheaf underlying $M_{i,j}$ is the intersection complex of a local system on a Zariski-open subset of $Z_{i,j}$. Note that since f is projective, we have the Lefschetz isomorphism [Sai90, Théorème 1]

$$M_{-i} \simeq M_i(i),$$

induced by i-fold cup product with the first Chern class of an ample line bundle. The Tate twist, necessary to change the weight of M_i from $n+i$ to $n-i$, requires some explanation. If $(\mathcal{M}_i, F)$ denotes the filtered $\mathcal{D}$-module underlying M_i, then the filtered $\mathcal{D}$-module underlying $M_i(i)$ is $(\mathcal{M}_i, F_{\bullet - i})$; thus the above isomorphism means that $F_k \mathcal{M}_{-i} \simeq F_{k-i} \mathcal{M}_i$.

To relate this decomposition with some concrete information about the Albanese map, we use Laumon's formula (Theorem 5.5) to compute the associated graded of the complex of filtered $\mathcal{D}$-modules $a_*(\mathscr{O}_X, F)$ underlying $a_* \mathbb{Q}_X^H[n] \in \mathrm{D^b}\,\mathrm{MHM}(A)$. To simplify the notation, we let

$$V = H^0(X, \Omega_X^1) = H^0(A, \Omega_A^1) \quad \text{and} \quad S = \mathrm{Sym}(V^*).$$

Proposition 9.1. *With notation as above, we have*

$$\mathrm{gr}_\bullet^F a_*(\mathscr{O}_X, F) \simeq \mathbf{R} a_* \Big[\mathscr{O}_X \otimes S^{\bullet - g} \to \Omega_X^1 \otimes S^{\bullet - g + 1} \to \cdots \to \Omega_X^n \otimes S^{\bullet - g + n} \Big],$$

with differential induced by the evaluation morphism $\mathscr{O}_X \otimes H^0(X, \Omega_X^1) \to \Omega_X^1$.

Proof. The cotangent bundle of A is isomorphic to the product $A \times V$, and so using the notation from §5, we have $\mathcal{A}_A = \mathscr{O}_A \otimes S$ as well as $a^* \mathcal{A}_A = \mathscr{O}_X \otimes S$. Consequently, $\omega_X \otimes a^* \mathcal{A}_A$ can be resolved by the complex

$$\Big[\mathcal{A}_X^{\bullet - n} \to \Omega_X^1 \otimes \mathcal{A}_X^{\bullet - n + 1} \to \cdots \to \Omega_X^n \otimes \mathcal{A}_X^{\bullet} \Big] \otimes S$$

placed as usual in degrees $-n, \dots, 0$. Applying Theorem 5.5, we find that

$$\operatorname{gr}_{\bullet}^F a_*(\mathscr{O}_X, F) \simeq \mathbf{R}a_*\Big[\mathscr{O}_X \otimes S^{\bullet - g} \to \Omega_X^1 \otimes S^{\bullet - g + 1} \to \dots \to \Omega_X^n \otimes S^{\bullet - g + n}\Big],$$

with Koszul-type differential induced by the morphism $\mathscr{O}_X \to \Omega_X^1 \otimes V^*$, which in turn is induced by the evaluation morphism $\mathscr{O}_X \otimes V \to \Omega_X^1$. $\square$

On the other hand, we know from the decomposition theorem that

$$a_*\mathbb{Q}_X^H[n] \simeq \bigoplus_{i,j} M_{i,j}[-i].$$

It follows that $\operatorname{gr}_{\bullet}^F a_*(\mathscr{O}_X, F)$ splits, as a complex of graded modules over $\mathcal{A}_A = \mathscr{O}_A \otimes S$, into a direct sum of modules of the form $\operatorname{gr}_{\bullet}^F \mathcal{M}_{i,j}$. Putting everything together, we obtain a key isomorphism which relates generic vanishing, zero sets of holomorphic one-forms on X, and the topology of the Albanese mapping.

Corollary 9.2. *With notation as above, we have*

$$\mathbf{R}a_*\Big[\mathscr{O}_X \otimes S^{\bullet - g} \to \Omega_X^1 \otimes S^{\bullet - g + 1} \to \dots \to \Omega_X^n \otimes S^{\bullet - g + n}\Big] \simeq \bigoplus_{i,j} \operatorname{gr}_{\bullet}^F \mathcal{M}_{i,j}[-i]$$

in the bounded derived category of graded $\mathscr{O}_A \otimes S$-modules.

Since the $M_{i,j}$ are Hodge modules on an abelian variety, it follows from Theorem 3.1 that each $\operatorname{gr}_k^F \mathcal{M}_{i,j}$ is a GV-sheaf on A. The isomorphism in Corollary 9.2 shows that whether or not the entire complex $\operatorname{gr}_k^F a_*(\mathscr{O}_X, F)$ satisfies a generic vanishing theorem is determined by the presence of nonzero $M_{i,j}$ with $i > 0$. We shall see in the next section how this leads to a generic vanishing theorem of Nakano-type.

To conclude this section, we recall a well-known condition on the fibers of the Albanese map that guarantees vanishing of the M_i in a certain range.

Proposition 9.3. *For $\ell \in \mathbb{N}$, let $A_\ell = \big\{ y \in A \bigm| \dim a^{-1}(y) \geq \ell \big\}$, and define the* defect of semi-smallness *of the map $a \colon X \to A$ to be*

$$\delta(a) = \max_{\ell \in \mathbb{N}} \big(2\ell - \dim X + \dim A_\ell\big).$$

Then we have $H^i a_* \mathbb{Q}_X^H[n] = 0$ whenever $|i| > \delta(a)$.

To illustrate the meaning of this numerical condition, suppose that X is of maximal Albanese dimension, so that $\dim X = \dim a(X)$. In this case, $2\ell - \dim X + \dim A_\ell = 2\ell - \operatorname{codim}\big(A_\ell, a(X)\big)$. We obtain for instance that a is *semi-small* iff $\delta(a) = 0$, in which case $M_i = 0$ for $i \neq 0$.

Proof. Since $M_{-i} \simeq M_i(i)$, it suffices to prove that if $M_i \neq 0$, then $i \leq \delta(a)$. Suppose that M is one of the simple summands of the Hodge module M_i, which means that $M[-i]$ is a direct factor of $a_*\mathbb{Q}_X^H[n]$. The strict support of M is an irreducible subvariety $Z \subseteq A$, and there is a variation of Hodge structure of weight $n + i - \dim Z$ on a dense open subset of Z whose intermediate extension is M. Let $z \in Z$ be a general point, and let $i \colon \{z\} \hookrightarrow A$ be the inclusion. Then $H^{-\dim Z} i^* M$ is the corresponding Hodge structure, and so we get $H^{-\dim Z} i^* M \neq 0$. This implies that we must have

$$H^{-\dim Z + i} i^* a_* \mathbb{Q}_X^H[n] \neq 0$$

as well. Now let $Y = a^{-1}(z)$. By the formula for proper base change [Sai90, (4.4.3)], we have $i^* a_* \mathbb{Q}_X^H[n] \simeq a_* \mathbb{Q}_Y^H[n]$, and so we find that

$$H^{-\dim Z + i} i^* a_* \mathbb{Q}_X^H[n] \simeq H^{n - \dim Z + i} a_* \mathbb{Q}_Y^H = H^{n - \dim Z + i}(Y)$$

is the usual mixed Hodge structure on the cohomology of the projective variety Y. By the above, this mixed Hodge structure is nonzero; it follows for dimension reasons that $n - \dim Z + i \leq 2 \dim Y$. Consequently, $i \leq \delta(a)$ as claimed. $\square$

10. Generic vanishing on the Picard variety. We now address the generic vanishing theorem of Nakano type, Theorem 3.2 of the introduction, and related questions. Ideally, such a theorem would say that

$$V^q(\Omega_X^p) = \Big\{ L \in \operatorname{Pic}^0(X) \;\Big|\; H^q\big(X, \Omega_X^p \otimes L\big) \neq 0 \Big\}$$

has codimension at least $|p + q - n|$ in $\widehat{A}$. Unfortunately, such a good statement is not true in general (see [GL87, §3] for an example). To simplify our discussion of what the correct statement is, we make the following definition.

Definition 10.1. Let X be a smooth projective variety. We say that X satisfies the *generic Nakano vanishing theorem with index k* if

$$\operatorname{codim} V^q(\Omega_X^p) \geq |p + q - n| - k$$

for every $p, q \in \mathbb{N}$.

Note that the absolute value is consistent with Serre duality, which implies that

$$V^q(\Omega_X^p) \simeq V^{n-q}(\Omega_X^{n-p}).$$

We can use the analysis in §9 to obtain a precise formula for the index k. In particular, Theorem 3.2 is a consequence of Proposition 9.3 and the following result.

Theorem 10.2. *Let X be a smooth complex projective variety of dimension n, with Albanese mapping $a \colon X \to A$. Define*

$$k(X) := \max\big\{ k \in \mathbb{Z} \;\big|\; H^k a_* \mathbb{Q}_X^H[n] \neq 0 \big\}.$$

Then X satisfies the generic Nakano vanishing theorem with index $k(X)$, but not with index $k(X) - 1$.

Proof. It follows from the equivalence established in [Pop09, §3] that generic Nakano vanishing with index k is equivalent to the statement that

$$\mathbf{R}\Phi_P\Big(\mathbf{R}a_* \Omega_X^p[n-p]\Big) \in {}^c\mathrm{D}_{\mathrm{coh}}^{\leq k}(\mathscr{O}_{\widehat{A}}). \tag{10.3}$$

We shall prove that this formula holds with $k = k(X)$ by descending induction on $p \geq 0$, starting from the trivial case $p = n+1$. To simplify the bookkeeping, we set

$$\mathscr{C}_\bullet = \mathbf{R}a_*\Big[\mathscr{O}_X \otimes S^{\bullet - g} \to \Omega_X^1 \otimes S^{\bullet - g + 1} \to \cdots \to \Omega_X^n \otimes S^{\bullet - g + n}\Big].$$

In particular, recalling that $(\mathcal{M}_i, F)$ is the filtered $\mathcal{D}$-module underlying the Hodge module $M_i = H^i a_* \mathbb{Q}_X^H[n]$, we get from Corollary 9.2 that

$$\mathscr{C}_{g-p} = \mathbf{R}a_*\Big[\Omega_X^p \to \Omega_X^{p+1} \otimes S^1 \to \cdots \to \Omega_X^n \otimes S^{n-p}\Big] \simeq \bigoplus_i \operatorname{gr}_{g-p}^F \mathcal{M}_i[-i].$$

From Theorem 3.1, we know that $\mathbf{R}\Phi_P(\mathrm{gr}^F_{g-p}\mathcal{M}_i) \in {}^c\mathrm{Coh}(\mathscr{O}_{\widehat{A}})$, and so we conclude that $\mathbf{R}\Phi_P\mathscr{C}_{g-p} \in {}^c\mathrm{D}^{\leq k}_{\mathrm{coh}}(\mathscr{O}_{\widehat{A}})$ for $0 \leq p \leq n$. It is clear from the definition of $\mathscr{C}_{g-p}$ that there is a distinguished triangle

$$\mathscr{C}'_{g-p} \to \mathscr{C}_{g-p} \to \mathbf{R}a_*\Omega^p_X[n-p] \to \mathscr{C}'_{g-p}[1],$$

in which $\mathscr{C}'_{g-p}$ is an iterated extension of $\mathbf{R}a_*\Omega^r_X[n-r]$ for $p+1 \leq r \leq n$. From the inductive hypothesis, we obtain that $\mathbf{R}\Phi_P\mathscr{C}'_{g-p} \in {}^c\mathrm{D}^{\leq k}_{\mathrm{coh}}(\mathscr{O}_{\widehat{A}})$. Now apply the functor $\mathbf{R}\Phi_P$ to the distinguished triangle to conclude that (10.3) continues to hold for the given value of p.

This argument can be reversed to show that $k(X)$ is the optimal value for the index. Indeed, suppose that X satisfies the generic Nakano vanishing theorem with some index k. Since each complex $\mathscr{C}_j$ is an iterated extension of $\mathbf{R}a_*\Omega^p_X[n-p]$, the above shows that $\mathbf{R}\Phi_P\mathscr{C}_j \in {}^c\mathrm{D}^{\leq k}_{\mathrm{coh}}(\mathscr{O}_{\widehat{A}})$, and hence that $\mathbf{R}\Phi_P(\mathrm{gr}^F_p\mathcal{M}_i) = 0$ for every $p \in \mathbb{Z}$ and $i > k$. Because the Fourier-Mukai transform is an equivalence of categories, we conclude that $\mathcal{M}_i = 0$ for $i > k$, which means that $k(X) \leq k$. □

Example 10.4. Our result explains the original counterexample from [GL87, §3]. The example consisted in blowing up an abelian variety A of dimension four along a smooth curve C of genus at least two; if X denotes the resulting variety, then $a\colon X \to A$ is the Albanese mapping, and a short computation shows that

$$H^2(X, \Omega^3_X \otimes a^*L) \simeq H^0(C, \omega_C \otimes L) \neq 0$$

for every $L \in \mathrm{Pic}^0(A)$. This example makes it clear that the index in the generic Nakano vanishing theorem is not equal to the dimension of the generic fiber of the Albanese mapping. On the other hand, it is not hard to convince oneself that

$${}^pH^k a_*\mathbb{Q}_X[4] \simeq \begin{cases} \mathbb{Q}_A[4] & \text{for } k = 0, \\ \mathbb{Q}_C[1] & \text{for } k = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $k(X) = 1$ is the correct value for the index in this case.

Theorem 3.2 and its proof improve the previously known generic vanishing results for Ω^p_X with $p < n$, and recover those for ω_X (or its higher direct images):

First, it was proved in [PP11, Theorem 5.11] that

$$\operatorname{codim} V^q(\Omega^p_X) \geq q + p - n - \mu(a),$$

with $\mu(a) = \max\{k, m-1\}$, k being the minimal dimension and m the maximal dimension of a fiber of the Albanese map. It is a routine check that $\delta(a) \leq \mu(a)$.

Secondly, in the case of the lowest nonzero piece of the filtration on $\mathbf{R}a_*(\mathscr{O}_X, F)$ the situation is better than what comes out of Theorem 10.2. This allows one to recover the original generic vanishing theorem for ω_X of [GL87], as well as its extension to higher direct images $R^j a_*\omega_X$ given in [Hac04]. Indeed, in the proof above note that

$$\mathscr{C}_{g-n} = \mathbf{R}a_*\omega_X \simeq \bigoplus_i \mathrm{gr}^F_{g-n}\mathcal{M}_i[-i]. \tag{10.5}$$

This shows that $R^i a_*\omega_X \simeq \mathrm{gr}^F_{g-n}\mathcal{M}_i$. Since these sheaves are torsion-free by virtue of Kollár's theorem, it follows that $\mathrm{gr}^F_{g-n}\mathcal{M}_i = 0$ unless $0 \leq i \leq \dim X - \dim a(X)$. Thus one recovers the original generic vanishing theorem of Green and

Lazarsfeld. A similar argument works replacing ω_X by higher direct images $R^i f_* \omega_Y$, where $f\colon Y \to X$ is a projective morphism with Y smooth and X projective and generically finite over A.

Note. It is worth noting that since for an arbitrary projective morphism $f\colon X \to Y$ a decomposition analogous to (10.5) continues to hold, one recovers the main result of [Kol86, Theorem 3.1], namely the splitting of $\mathbf{R} f_* \omega_X$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$. This is of course not a new observation: it is precisely Saito's approach to Kollár's theorem and its generalizations.

E. Cohomological support loci for local systems

11. Supports of transforms. Let A be an abelian variety of dimension g, and set as before $V = H^0(A, \Omega_A^1)$ and $S = \operatorname{Sym} V^*$. Let M be a mixed Hodge module on A, with underlying filtered $\mathcal{D}$-module $(\mathcal{M}, F)$. Then $\operatorname{gr}_{\bullet}^F \mathcal{M}$ is a finitely-generated graded module over $\operatorname{Sym} \Theta_A \simeq \mathscr{O}_A \otimes S$, and we denote the associated coherent sheaf on $T^*A = A \times V$ by $\mathscr{C}(\mathcal{M}, F)$. We may then define the *total Fourier-Mukai transform* of $\operatorname{gr}_{\bullet}^F \mathcal{M}$ to be

$$\mathbf{R}\Phi_P\big(\operatorname{gr}_{\bullet}^F \mathcal{M}\big) = \bigoplus_{k \in \mathbb{Z}} \mathbf{R}\Phi_P(\operatorname{gr}_k^F \mathcal{M}),$$

which belongs to the bounded derived category of graded modules over $\mathscr{O}_{\widehat{A}} \otimes S$. The geometric interpretation is as follows: the object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\big(\mathscr{O}_{\widehat{A} \times V}\big)$ corresponding to the total Fourier-Mukai transform of $\operatorname{gr}_{\bullet}^F \mathcal{M}$ is

$$\mathbf{R}\Phi_P \mathscr{C}(\mathcal{M}, F) = \mathbf{R} p_{23*}\Big(p_{13}^* \mathscr{C}(\mathcal{M}, F) \otimes p_{12}^* P\Big)$$

where the notation is as in the diagram

$$\begin{tikzcd}
A \times V & A \times \widehat{A} \times V \arrow[l, "p_{13}"'] \arrow[r, "p_{23}"] \arrow[d, "p_{12}"'] & \widehat{A} \times V \\
& A \times \widehat{A}. &
\end{tikzcd}$$

Let now X be a smooth projective variety of dimension n, let $a\colon X \to A$ its Albanese map, and consider again the decomposition

$$a_* \mathbb{Q}_X^H[n] \simeq \bigoplus_{i,j} M_{i,j}[-i] \in \mathrm{D}^{\mathrm{b}} \operatorname{MHM}(A).$$

Denote by $\mathscr{C}_{i,j} = \mathscr{C}(\mathcal{M}_{i,j}, F)$ the coherent sheaf on $T^*A = A \times V$ determined by the Hodge module $M_{i,j}$. The supports in $\widehat{A} \times V$ of the total Fourier-Mukai transforms $\mathbf{R}\Phi_P \mathscr{C}_{i,j}$ are of a very special kind; this follows by using a result of Arapura [Ara92]. To state the result, we recall the following term coined by Simpson [Sim93, p. 365].

Definition 11.1. A *triple torus* is any subvariety of $\widehat{A} \times H^0(A, \Omega_A^1)$ of the form

$$\operatorname{im}\Big(\varphi^*\colon \hat{B} \times H^0(B, \Omega_B^1) \hookrightarrow \widehat{A} \times H^0(A, \Omega_A^1)\Big),$$

for a surjective morphism $\varphi\colon A \to B$ to another abelian variety B. A subvariety is called a *torsion translate of a triple torus* if it is a translate of a triple torus by a point of finite order in $A \times H^0(A, \Omega_A^1)$.

Proposition 11.2. *With notation as above, every irreducible component of the support of* $\mathbf{R}\Phi_P\mathscr{C}_{i,j}$ *is a torsion translate of a triple torus in* $\widehat{A}\times V$.

Proof. It is convenient to prove a more general statement. For simplicity, denote by $\mathscr{C}\in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\big(\mathscr{O}_{A\times V}\big)$ the object corresponding to $\operatorname{gr}^F_\bullet a_*(\mathscr{O}_X,F)$. Recall from Proposition 9.1 and Corollary 9.2 that we have

$$\mathscr{C}\simeq \mathbf{R}(a\times \mathrm{id})_*\Big[p_1^*\mathscr{O}_X\to p_1^*\Omega_X^1\to\cdots\to p_1^*\Omega_X^n\Big]\simeq\bigoplus_{i,j}\mathscr{C}_{i,j}[-i], \tag{11.3}$$

where $p_1\colon X\times V\to X$, and the complex in brackets is placed in degrees $-n,\dots,0$, and has differential induced by the evaluation map $\mathscr{O}_X\otimes V\to\Omega_X^1$.

Now define, for $m\in\mathbb{N}$, the following closed subsets of $\widehat{A}\times V$:

$$\begin{aligned}Z^k(m)&=\Big\{(L,\omega)\in\widehat{A}\times V\;\Big|\;\dim\mathbf{H}^k\big(A,L\otimes\mathscr{C}\big|_{A\times\{\omega\}}\big)\geq m\Big\}\\&=\Big\{(L,\omega)\in\widehat{A}\times V\;\Big|\;\dim\mathbf{H}^k\big(X,L\otimes K(\Omega_X^1,\omega)\big)\geq m\Big\},\end{aligned}$$

where we denote the Koszul complex associated to a single one-form $\omega\in V$ by

$$K(\Omega_X^1,\omega)=\Big[\mathscr{O}_X\xrightarrow{\wedge\omega}\Omega_X^1\xrightarrow{\wedge\omega}\cdots\xrightarrow{\wedge\omega}\Omega_X^n\Big].$$

It follows from [Ara92, Corollary on p. 312] and [Sim93] that each irreducible component of $Z^k(m)$ is a torsion translate of a triple torus. To relate this information to the support of the total Fourier-Mukai transforms $\mathbf{R}\Phi_P\mathscr{C}_{i,j}$, we also introduce

$$Z^k_{i,j}(m)=\Big\{(L,\omega)\in\widehat{A}\times V\;\Big|\;\dim\mathbf{H}^k\big(L\otimes\mathscr{C}_{i,j}\big|_{A\times\{\omega\}}\big)\geq m\Big\}.$$

The decomposition in (11.3) implies that

$$Z^k(m)=\bigcup_\mu\bigcap_{i,j}Z^{k-i}_{i,j}\big(\mu(i,j)\big),$$

where the union is taken over the set of functions $\mu\colon\mathbb{Z}^2\to\mathbb{N}$ with the property that $\sum_{i,j}\mu(i,j)=m$. As in [Ara92, p. 312], this formula implies that each irreducible component of $Z^k_{i,j}(m)$ is also a torsion translate of a triple torus.

Finally, we observe that each irreducible component of $\operatorname{Supp}\big(\mathbf{R}\Phi_P\mathscr{C}_{i,j}\big)$ must also be an irreducible component of one the sets $Z^k_{i,j}:=Z^k_{i,j}(1)$, which concludes the proof. More precisely, we have

$$\operatorname{Supp}\big(\mathbf{R}\Phi_P\mathscr{C}_{i,j}\big)=\bigcup_k Z^k_{i,j}.$$

Indeed, the base change theorem shows that $\operatorname{Supp}\big(R^k\Phi_P\mathscr{C}_{i,j}\big)\subset Z^k_{i,j}$ for all k, with equality if $k\gg 0$. Now assume that $(L,\omega)\in Z^k_{i,j}$ is a general point of a component which is not contained in $\operatorname{Supp}\big(R^k\Phi_P\mathscr{C}_{i,j}\big)$. We claim that then $(L,\omega)\in Z^{k+1}_{i,j}$, which concludes the proof by descending induction. If this were not the case, then again by the base change theorem, the natural map

$$R^k\Phi_P\mathscr{C}_{i,j}\otimes\mathbb{C}(L,\omega)\to\mathbf{H}^k\big(A,L\otimes\mathscr{C}_{i,j}\big|_{A\times\{\omega\}}\big)$$

would be surjective, which would contradict $(L,\omega)\notin\operatorname{Supp}\big(R^k\Phi_P\mathscr{C}_{i,j}\big)$. $\square$

12. Generic vanishing for rank one local systems. We continue to use the notation from above. Let $\mathrm{Char}(X) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ be the algebraic group of characters of X. We are interested in bounding the codimension of the cohomological support loci

$$\Sigma^i(X) = \left\{ \rho \in \mathrm{Char}(X) \;\middle|\; H^i(X, \mathbb{C}_\rho) \neq 0 \right\},$$

where $\mathbb{C}_\rho$ denotes the local system of rank one associated to a character ρ. The structure of these loci has been studied by Arapura [Ara92] and Simpson [Sim93], who showed that they are finite unions of torsion translates of algebraic subtori of $\mathrm{Char}(X)$. This is established via an interpretation in terms of Higgs line bundles. (We have already used the more precise result for Higgs bundles from [Ara92] during the proof of Proposition 11.2.)

We shall assume for simplicity that $H^2(X, \mathbb{Z})$ is torsion-free (though the argument goes through in general, the only difference being more complicated notation). In this case, the space of Higgs line bundles on X consisting of pairs (L, θ) of holomorphic line bundles with zero first Chern class and holomorphic one-forms can be identified as the complex algebraic group

$$\mathrm{Higgs}(X) \simeq \widehat{A} \times V.$$

One can define an isomorphism of real (but not complex) algebraic Lie groups

$$\mathrm{Char}(X) \to \mathrm{Higgs}(X), \quad \rho \to (L_\rho, \theta_\rho),$$

where θ_ρ is the $(1,0)$-part of $\log\|\rho\|$ interpreted as a cohomology class via the isomorphism $H^1(X, \mathbb{R}) \simeq \mathrm{Hom}(\pi_1(X), \mathbb{R})$.

Note. L_ρ is not the holomorphic line bundle $\mathbb{C}_\rho \otimes_{\mathbb{C}} \mathscr{O}_X$, except when the character ρ is unitary; this point is stated incorrectly in [Ara92, p.312]. Please refer to [Sim93, p. 364] or page 31 below for the correct description of L_ρ.

By means of this identification, the local system cohomology of $\mathbb{C}_\rho$ can be computed in terms of the Dolbeault cohomology of the Higgs bundle (L_ρ, θ_ρ).

Lemma 12.1 ([Sim92, Lemma 2.2])**.** *There is an isomorphism*

$$\begin{aligned} H^k(X, \mathbb{C}_\rho) &\simeq \mathbf{H}^{k-n}\big(L_\rho \otimes K(\Omega_X^1, \theta_\rho)\big) \\ &= \mathbf{H}^{k-n}\Big(X, \Big[L_\rho \xrightarrow{\wedge\theta_\rho} L_\rho \otimes \Omega_X^1 \xrightarrow{\wedge\theta_\rho} \cdots \xrightarrow{\wedge\theta_\rho} L_\rho \otimes \Omega_X^n\Big]\Big), \end{aligned}$$

where the Koszul-type complex in brackets is again placed in degrees $-n, \dots, 0$.

We can now obtain the generic vanishing theorem for local systems of rank one, stated in the introduction.

Proof of Theorem 3.4. Note first that, since Verdier duality gives an isomorphism

$$H^k(X, \mathbb{C}_\rho) \simeq H^{2n-k}(X, \mathbb{C}_{-\rho}),$$

we only need to prove the asserted inequality for $k \geq n$. Furthermore, Lemma 12.1 shows that it is enough to prove, for $k \geq 0$, the analogous inequality

$$\mathrm{codim}_{\mathrm{Higgs}(X)} Z^k(1) \geq 2\big(k - k(X)\big),$$

for the subsets $Z^k(1) \subseteq \mathrm{Higgs}(X)$ that were introduced during the proof of Proposition 11.2. Recall from there that

$$Z^k(1) = \bigcup_{i,j} Z^{k-i}_{i,j}(1).$$

Finally, we saw in the proof of Theorem 13.2 that

$$\operatorname{codim} \operatorname{Supp} R^\ell \Phi_P \mathscr{C}_{i,j} \geq 2\ell$$

for all ℓ. By the same base change argument as before, this is equivalent to $\operatorname{codim} Z^\ell_{i,j}(1) \geq 2\ell$ for all ℓ. Since $\mathscr{C}_{i,j}$ is zero unless $i \leq k(X)$, we conclude that

$$\operatorname{codim} Z^k(1) \geq \min_i \operatorname{codim} Z^{k-i}_{i,j}(1) \geq 2\bigl(k - k(X)\bigr),$$

which is the desired inequality. $\square$

13. Duality and perversity for total transforms. In this section, we take a closer look at the behavior of the object $\mathscr{C}(\mathcal{M}, F)$ under the total Fourier-Mukai transform. We begin with a result that shows how the total Fourier-Mukai transform interacts with Verdier duality for mixed Hodge modules.

Lemma 13.1. *Let M be a mixed Hodge module on A, let M' be its Verdier dual, and let $\mathscr{C}(\mathcal{M}, F)$ and $\mathscr{C}(\mathcal{M}', F)$ be the associated coherent sheaves on $A \times V$. Then*

$$\mathbf{R}\mathcal{H}om\bigl(\mathbf{R}\Phi_P \mathscr{C}(\mathcal{M}, F), \mathscr{O}_{\widehat{A}\times V}\bigr) \simeq \iota^*\bigl(\mathbf{R}\Phi_P \mathscr{C}(\mathcal{M}', F)\bigr),$$

where $\iota = (-1_{\widehat{A}}) \times (-1_V)$.

Proof. Recall that the Grothendieck dual on a smooth algebraic variety X is given by $\mathbf{D}_X(-) = \mathbf{R}\mathcal{H}om\bigl(-, \omega_X[\dim X]\bigr)$. To simplify the notation, set $\mathscr{C} = \mathscr{C}(\mathcal{M}, F)$ and $\mathscr{C}' = \mathscr{C}(\mathcal{M}', F)$. Then

$$\begin{aligned}
\mathbf{D}_{\widehat{A}\times V}\bigl(\mathbf{R}\Phi_P \mathscr{C}(\mathcal{M}, F)\bigr) &= \mathbf{D}_{\widehat{A}\times V}\Bigl(\mathbf{R}p_{23*}\bigl(p_{13}^*\mathscr{C} \otimes p_{12}^* P\bigr)\Bigr) \\
&\simeq \mathbf{R}p_{23*}\Bigl(p_{13}^*\bigl(\mathbf{D}_{A\times V}\mathscr{C}\bigr)[g] \otimes p_{12}^* P^{-1}\Bigr) \\
&\simeq \mathbf{R}p_{23*}\Bigl(\bigl(\mathrm{id}\times\mathrm{id}\times(-1_V)\bigr)^*(p_{13}^*\mathscr{C}') \otimes \bigl(\mathrm{id}\times(-1_{\widehat{A}})\times\mathrm{id}\bigr)^*(p_{12}^*P)\Bigr)[2g] \\
&\simeq \bigl((-1_{\widehat{A}})\times(-1_V)\bigr)^*\bigl(\mathbf{R}\Phi_P\mathscr{C}'\bigr)[2g] = \iota^*\bigl(\mathbf{R}\Phi_P \mathscr{C}(\mathcal{M}', F)\bigr)[2g].
\end{aligned}$$

For the first isomorphism we use Grothendieck duality, while for the second we use Theorem 5.4. Since $\dim \widehat{A}\times V = 2g$, this implies the result. $\square$

Now let $a\colon X \to A$ be the Albanese map of a smooth projective variety of dimension n. Consider the coherent sheaves $\mathscr{C}_{i,j} = \mathscr{C}(\mathcal{M}_{i,j}, F)$ that arise from the decomposition theorem applied to $a_*\mathbb{Q}^H_X[n] \in \mathrm{D^b}\,\mathrm{MHM}(A)$; see Corollary 9.2 for the notation. Many of the results of the previous sections can be stated very succinctly in the following way, using the second t-structure introduced in §6.

Theorem 13.2. *With notation as above, each $\mathbf{R}\Phi_P\mathscr{C}_{i,j}$ is a m-perverse coherent sheaf on $\widehat{A}\times V$. More precisely, the support of the object $\mathbf{R}\Phi_P\mathscr{C}_{i,j}$ is a finite union of torsion translates of triple tori, subject to the inequality*

$$\operatorname{codim} \operatorname{Supp} R^k \Phi_P \mathscr{C}_{i,j} \geq 2k$$

for every $k \in \mathbb{Z}$. Moreover, the dual objects

$$\mathbf{R}\mathcal{H}om\bigl(\mathbf{R}\Phi_P\mathscr{C}_{i,j}, \mathscr{O}_{\widehat{A}\times V}\bigr)$$

have the same properties.

Proof. Let $q\colon \widehat{A} \times V \to \widehat{A}$ be the first projection. Since we are dealing with sheaves of graded modules, the support of the quasi-coherent sheaf

$$q_*\big(R^\ell \Phi_P \mathscr{C}_{i,j}\big) = \bigoplus_{k \in \mathbb{Z}} R^\ell \Phi_P\big(\mathrm{gr}_k^F \mathcal{M}_{i,j}\big)$$

is the image of $\operatorname{Supp} R^\ell \Phi_P \mathscr{C}_{i,j}$ under the map q. Thanks to Theorem 3.1, each $\mathbf{R}\Phi_P\big(\mathrm{gr}_k^F \mathcal{M}_{i,j}\big)$ is a c-perverse coherent sheaf on $\widehat{A}$, so each irreducible component of the image has codimension at least ℓ. On the other hand, by Proposition 11.2, the support of $R^\ell \Phi_P \mathscr{C}_{i,j}$ is a finite union of translates of triple tori. Since a triple torus is always of the form $\hat{B} \times H^0(B, \Omega_B^1)$, we obtain $\operatorname{codim} \hat{B} \geq \ell$, and therefore

$$\operatorname{codim} \operatorname{Supp} R^\ell \Phi_P \mathscr{C}_{i,j} \geq 2\ell.$$

This implies that $\mathbf{R}\Phi_P \mathscr{C}_{i,j}$ belongs to ${}^m\mathrm{D}_{\mathrm{coh}}^{\leq 0}\big(\mathscr{O}_{\widehat{A} \times V}\big)$. Since Verdier duality for mixed Hodge modules commutes with direct images by proper maps, we have

$$\mathbf{D}_A a_* \mathbb{Q}_X^H[n] \simeq a_* \mathbf{D}_X \mathbb{Q}_X^H[n] \simeq a_* \mathbb{Q}_X^H(n)[n],$$

which shows that each module $\mathbf{D}_A(M_{i,j})$ is again one of the direct factors in the decomposition of $a_* \mathbb{Q}_X^H[n]$. By Lemma 13.1, the dual complex is thus again of the same type, hence lies in ${}^m\mathrm{D}_{\mathrm{coh}}^{\leq 0}\big(\mathscr{O}_{\widehat{A} \times V}\big)$ as well. It is then clear from the description of the t-structure in §6 that both objects actually belong to the heart ${}^m\mathrm{Coh}(\mathscr{O}_{\widehat{A} \times V})$, hence are m-perverse coherent sheaves. $\square$

Note. More precisely, each $\mathbf{R}\Phi_P(\mathscr{C}_{i,j})$ belongs to the subcategory of ${}^m\mathrm{Coh}(\mathscr{O}_{\widehat{A} \times V})$ consisting of objects whose support is a finite union of translates of triple tori. It is not hard to see that this subcategory—unlike the category of m-perverse coherent sheaves itself—is closed under the duality functor $\mathbf{R}\mathcal{H}om\big(-, \mathscr{O}_{\widehat{A} \times V}\big)$, because each triple torus has even dimension.

14. Perverse coherent sheaves on the space of holomorphic one-forms. In this section we observe that, in analogy with the method described in [Pop09], our method also produces natural perverse coherent sheaves on the affine space $V = H^0(A, \Omega_A^1)$, where A is an abelian variety of dimension g. We shall use the following projection maps:

$$A \xleftarrow{p} A \times V \xrightarrow{q} V \qquad\qquad \widehat{A} \xleftarrow{p} \widehat{A} \times V \xrightarrow{q} V$$

Lemma 14.1. *Let L be an ample line bundle on the abelian variety $\widehat{A}$. For any object $E \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}\big(\mathscr{O}_{A \times V}\big)$, we have*

$$\mathbf{R}\mathcal{H}om\big(\mathbf{R}q_*(p^*L \otimes \mathbf{R}\Phi_P E), \mathscr{O}_V\big) \simeq \mathbf{R}q_*\big(E' \otimes p^*(-1_A)^* R^g \Psi_P(L^{-1})\big),$$

where we set

$$E' = (\mathrm{id} \times (-1_V))^* \mathbf{R}\mathcal{H}om\big(E, \mathscr{O}_{A \times V}[g]\big).$$

Proof. This is an exercise in interchanging Grothendieck duality with pushforward by proper maps and pullback by smooth maps. $\square$

We can rephrase Lemma 8.1, which was basically Saito's version of the Kodaira vanishing theorem for mixed Hodge modules on abelian varieties, as follows.

Lemma 14.2. *Let M be a mixed Hodge module on an abelian variety A, with underlying filtered $\mathcal{D}$-module $(\mathcal{M}, F)$, and let $\mathscr{C}(\mathcal{M}, F)$ be the coherent sheaf on $A \times V$ associated to $\mathrm{gr}_\bullet^F \mathcal{M}$. Then for every ample line bundle L on A, one has*

$$\mathbf{R}q_*\big(p^*L \otimes \mathscr{C}(\mathcal{M}, F)\big) \in \mathrm{Coh}(\mathscr{O}_V).$$

Proof. V being affine, it suffices to prove that the hypercohomology of the complex is concentrated in degree 0. But this hypercohomology is equal to

$$\mathbf{H}^i\big(A \times V, L \otimes p_*\mathscr{C}(\mathcal{M}, F)\big) \simeq \bigoplus_{k \in \mathbb{Z}} H^i\big(A, L \otimes \mathrm{gr}_k^F \mathcal{M}\big),$$

which vanishes for $i > 0$ because of Lemma 8.1. □

We can now obtain perverse coherent sheaves on the affine space V by pushing forward along the projection $q\colon A \times V \to V$.

Proposition 14.3. *Let M be a mixed Hodge module on A, with underlying filtered $\mathcal{D}$-module $(\mathcal{M}, F)$, and let $\mathscr{C}(\mathcal{M}, F)$ be the coherent sheaf on $A \times V$ associated to $\mathrm{gr}_\bullet^F \mathcal{M}$. Then for every ample line bundle L on $\widehat{A}$, one has*

$$\mathbf{R}q_*\Big(p^*L \otimes \mathbf{R}\Phi_P \mathscr{C}(\mathcal{M}, F)\Big) \in {}^c\mathrm{Coh}(\mathscr{O}_V).$$

Proof. By Lemma 6.2, it suffices to prove that

$$\mathbf{R}\mathcal{H}om\Big(\mathbf{R}q_*\big(p^*L \otimes \mathbf{R}\Phi_P \mathscr{C}(\mathcal{M}, F)\big), \mathscr{O}_V\Big) \in \mathrm{Coh}(\mathscr{O}_V).$$

By Lemma 14.1 and Theorem 5.4, this object is isomorphic to

$$\mathbf{R}q_*\Big(\mathscr{C}(\mathcal{M}', F) \otimes p^* R^g \Psi_P(L^{-1})\Big), \tag{14.4}$$

where $\mathscr{C}(\mathcal{M}', F)$ is associated to the Verdier dual $M' = \mathbf{D}_A M$. Now we apply the usual covering trick. Let $\varphi_L\colon \widehat{A} \to A$ be the isogeny defined by L. Then the object in (14.4) will belong to $\mathrm{Coh}(\mathscr{O}_V)$ provided the same is true for

$$\mathbf{R}q_*\Big(\varphi_L^* \mathscr{C}(\mathcal{M}', F) \otimes p^*L\Big) \otimes H^0\big(\widehat{A}, L\big). \tag{14.5}$$

Since $\varphi_L^* \mathscr{C}(\mathcal{M}', F)$ comes from the mixed Hodge module $\varphi_L^* M'$, this is a consequence of Lemma 14.2. □

F. Strong linearity results

15. Fourier-Mukai for D-modules. A stronger version of the results on cohomological support loci in [GL87] was given in [GL91] (see also various extensions in [CH02]). This is the statement (SL) in the introduction; roughly speaking, it states that the standard Fourier-Mukai transform $\mathbf{R}\Phi_P \mathscr{O}_X$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{A})$ is represented by a linear complex in a neighborhood of any point in $\widehat{A}$. Here we extend this to the setting of the trivial $\mathcal{D}$-module $\mathscr{O}_X$, and consequently to all the Hodge modules $M_{i,j}$ appearing in the previous sections. In order to do this, we shall make use of the Fourier-Mukai transform for $\mathcal{D}$-modules on abelian varieties, introduced by Laumon [Lau96] and Rothstein [Rot96].

We start by setting up some notation. Let A be a complex abelian variety of dimension g, and let $A^\sharp$ be the moduli space of algebraic line bundles with flat connection on A. Note that $A^\sharp$ naturally has the structure of a quasi-projective algebraic variety: on $\widehat{A}$, there is a canonical vector bundle extension

$$0 \to \widehat{A} \times H^0(A, \Omega_A^1) \to E_{\widehat{A}} \to \widehat{A} \times \mathbb{C} \to 0,$$

and $A^\sharp$ is isomorphic to the preimage of $\widehat{A} \times \{1\}$ inside $E_{\widehat{A}}$. The projection

$$\pi \colon A^\sharp \to \widehat{A}, \qquad (L, \nabla) \mapsto L$$

is thus a torsor for the trivial bundle $\widehat{A} \times H^0(A, \Omega_A^1)$; this corresponds to the fact that $\nabla + \omega$ is again a flat connection for any $\omega \in H^0(A, \Omega_A^1)$. Note that $A^\sharp$ is a group under tensor product, and that the trivial line bundle $(\mathscr{O}_A, d)$ plays the role of the zero element.

Recall now that Laumon [Lau96] and Rothstein [Rot96] have extended the Fourier-Mukai transform to $\mathcal{D}$-modules. Their generalized Fourier-Mukai transform takes bounded complexes of coherent algebraic $\mathcal{D}$-modules on A to bounded complexes of algebraic coherent sheaves on $A^\sharp$; we briefly describe it following the presentation in [Lau96, §3], which is more convenient for our purpose. On the product $A \times A^\sharp$, the pullback $P^\sharp$ of the Poincaré bundle P is endowed with a universal flat connection $\nabla^\sharp \colon P^\sharp \to \Omega^1_{A \times A^\sharp/A^\sharp} \otimes P^\sharp$, relative to $A^\sharp$. Given any algebraic left $\mathcal{D}$-module $\mathcal{M}$ on A, interpreted as a quasi-coherent sheaf with integrable connection, we consider $p_1^*\mathcal{M} \otimes P^\sharp$ on $A \times A^\sharp$, endowed with the natural tensor product integrable connection ∇ relative to $A^\sharp$. We then define

$$\mathbf{R}\Phi_{P^\sharp}(\mathcal{M}) := \mathbf{R}p_{2*}\mathrm{DR}(p_1^*\mathcal{M} \otimes P^\sharp, \nabla), \tag{15.1}$$

where $\mathrm{DR}(p_1^*\mathcal{M} \otimes P^\sharp, \nabla)$ is the usual (relative) de Rham complex

$$\Big[p_1^*\mathcal{M} \otimes P^\sharp \xrightarrow{\nabla} p_1^*\mathcal{M} \otimes P^\sharp \otimes \Omega^1_{A \times A^\sharp/A^\sharp} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} p_1^*\mathcal{M} \otimes P^\sharp \otimes \Omega^g_{A \times A^\sharp/A^\sharp} \Big]$$

placed in degrees $-g, \dots, 0$. As all of the entries in this complex are relative to $A^\sharp$, it follows that $\mathbf{R}\Phi_{P^\sharp}(\mathcal{M})$ is represented by a complex of algebraic quasi-coherent sheaves on $A^\sharp$. Restricted to coherent $\mathcal{D}$-modules, this is shown to induce an equivalence of categories

$$\mathbf{R}\Phi_{P^\sharp} \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_A) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A^\sharp}).$$

The same result is obtained by a different method in [Rot96, Theorem 6.2].

Note. Since $A^\sharp$ is not compact, it is essential to consider algebraic coherent sheaves on $A^\sharp$ in the above equivalence. On A, this problem does not arise, because the category of coherent analytic $\mathcal{D}$-modules on a smooth projective variety is equivalent to the category of coherent algebraic $\mathcal{D}$-modules by a version of the GAGA theorem.

Now let X be a smooth projective variety with Albanese map $a \colon X \to A$. By first pushing forward to A, or equivalently by working with the pullback of $(P^\sharp, \nabla^\sharp)$ to $X \times A^\sharp$, one can similarly define

$$\mathbf{R}\Phi_{P^\sharp} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A^\sharp}).$$

In this and the following subsections, our goal is to prove the following linearity property for the Fourier-Mukai transform of the trivial $\mathcal{D}$-module $\mathscr{O}_X$ (see Definition 20.1 below for the definition of a linear complex over a local ring).

Theorem 15.2. *Let X be a smooth projective variety of dimension n, and let $\mathbf{R}\Phi_{P^\sharp}(\mathscr{O}_X) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A^\sharp})$ be the Fourier-Mukai transform of the trivial $\mathcal{D}$-module on X. Let $(L, \nabla) \in A^\sharp$ be an any point, and denote by $R = \mathscr{O}^{an}_{A^\sharp,(L,\nabla)}$ the local ring in the analytic topology. Then the stalk $\mathbf{R}\Phi_{P^\sharp}(\mathscr{O}_X) \otimes_{\mathscr{O}_X} R$ is quasi-isomorphic to a linear complex over R.*

In combination with Proposition 20.2 below, we obtain that every direct summand of $\mathbf{R}\Phi_{P^\sharp}(\mathscr{O}_X)$ is also isomorphic to a linear complex in an analytic neighborhood of any given point on $A^\sharp$.

Corollary 15.3. *Suppose that a Hodge module M occurs as a direct factor of some $H^i a_* \mathbb{Q}^H_X[n]$. Let $(\mathcal{M}, F)$ be the filtered holonomic $\mathcal{D}$-module underlying M, and let $\mathbf{R}\Phi_{P^\sharp}(\mathcal{M}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A^\sharp})$ be its Fourier-Mukai transform. Then the stalk $\mathbf{R}\Phi_{P^\sharp}(\mathcal{M}) \otimes_{\mathscr{O}_X} R$ is quasi-isomorphic to a linear complex over R.*

The proof of Theorem 15.2 takes up the next four sections. We will in fact prove the more precise version given in Theorem 3.6 in the introduction, by producing the natural analogue of the *derivative complex* of [GL91]. As in that paper, the idea is to pull E back to a complex on the universal covering space $H^1(X, \mathbb{C}) = H^1(A, \mathbb{C})$, and to use harmonic forms to construct a linear complex that is isomorphic to the pullback in a neighborhood of the preimage of the given point. We will encounter however an important additional difficulty, due to the behavior of general harmonic forms under wedge product.

16. Analytic description. We begin by giving an analytic description of the space $A^\sharp$, and of the Fourier-Mukai transform for $\mathcal{D}$-modules. For the remainder of the discussion, we shall identify $H^1(A, \mathbb{C})$ with the space of translation-invariant complex one-forms on the abelian variety A. As complex manifolds, we then have

$$A^\sharp \simeq H^1(A, \mathbb{C})/H^1(A, \mathbb{Z}) \qquad \text{and} \qquad \widehat{A} \simeq H^1(A, \mathscr{O}_A)/H^1(A, \mathbb{Z}),$$

and this is compatible with the exact sequence

$$0 \to H^0(A, \Omega^1_A) \to H^1(A, \mathbb{C}) \to H^1(A, \mathscr{O}_A) \to 0.$$

In particular, the universal covering space of $A^\sharp$ is isomorphic to $H^1(A, \mathbb{C})$. Given a translation-invariant one-form $\tau \in H^1(A, \mathbb{C})$, let $\tau^{1,0}$ be its holomorphic and $\tau^{0,1}$ its anti-holomorphic part. Then the image of τ in $A^\sharp$ corresponds to the trivial bundle $A \times \mathbb{C}$, endowed with the holomorphic structure given by the operator $\bar{\partial} + \tau^{0,1}$ and the integrable connection given by $\partial + \tau^{1,0}$. We can summarize this by saying that τ corresponds to the trivial smooth vector bundle $A \times \mathbb{C}$, endowed with the smooth integrable connection $d_\tau = d + \tau$.

The Albanese mapping $a \colon X \to A$ induces an isomorphism $H^1(A, \mathbb{C}) = H^1(X, \mathbb{C})$, under which translation-invariant one-forms on A correspond to harmonic one-forms on X for any choice of Kähler metric. To shorten the notation, we set

$$W := H^1(A, \mathbb{C}) = H^1(X, \mathbb{C}).$$

Then a harmonic form $\tau \in W$ corresponds to the trivial smooth line bundle $X \times \mathbb{C}$, endowed with the smooth integrable connection $d_\tau = d + \tau$.

We can similarly interpret the pullback of the Poincaré bundle to the complex manifold $X \times W$. Let $\mathscr{C}^k(X \times W/W)$ be the sheaf of smooth complex-valued relative k-forms on $X \times W$ that are in the kernel of $\bar{\partial}_W$, meaning holomorphic in the direction of W, and denote by

$$C^k(X \times W/W) := p_{2*}\mathscr{C}^k(X \times W/W)$$

its pushforward to W. Let $T \in C^1(X \times W/W)$ denote the tautological relative one-form on $X \times W$: the restriction of T to the slice $X \times \{\tau\}$ is equal to τ. Then the pullback of $(P^\sharp, \nabla^\sharp)$ is isomorphic to the trivial smooth bundle $X \times W \times \mathbb{C}$, with complex structure given by the operator $\bar{\partial}_X + \bar{\partial}_W + T^{0,1}$, and with relative integrable connection given by the operator $\partial_X + T^{1,0}$.

This leads to the following analytic description of the Fourier-Mukai transform (similar to [GL91, Proposition 2.3]). Let

$$D\colon C^k(X \times W/W) \to C^{k+1}(X \times W/W)$$

be the differential operator defined by the rule $D(\alpha) = d_X\alpha + T \wedge \alpha$. Using that $d_X T = 0$, it is easy to see that $D \circ D = 0$.

Lemma 16.1. *The complex of $\mathscr{O}_W$-modules $\big(C^\bullet(X \times W/W), D\big)$ is quasi-isomorphic to the pullback $\pi^*\mathbf{R}\Phi_{P^\sharp}(\mathscr{O}_X)$, where $\pi : W \to A^\sharp$ is the universal cover.*

Proof. By the definition of the Fourier transform, $\mathbf{R}\Phi_{P^\sharp}(\mathscr{O}_X)$ is the derived push-forward, via the projection $p_2\colon X \times A^\sharp \to A^\sharp$, of the complex

$$\mathrm{DR}(P^\sharp, \nabla^\sharp) = \left[P^\sharp \xrightarrow{\nabla^\sharp} \Omega^1_{X \times A^\sharp/A^\sharp} \otimes P^\sharp \xrightarrow{\nabla^\sharp} \cdots \xrightarrow{\nabla^\sharp} \Omega^g_{X \times A^\sharp/A^\sharp} \otimes P^\sharp\right],$$

where $(P^\sharp, \nabla^\sharp)$ denotes the pullback of the Poincaré bundle to $X \times A^\sharp$. Since $\pi\colon W \to A^\sharp$ is a covering map, we thus obtain

$$\pi^*\mathbf{R}\Phi_{P^\sharp}(\mathscr{O}_X) \simeq \mathbf{R}p_{2*}\left((\mathrm{id} \times \pi)^* \,\mathrm{DR}(P^\sharp, \nabla^\sharp)\right).$$

As noted above, the pullback of $(P^\sharp, \nabla^\sharp)$ to $X \times W$ is isomorphic to the trivial smooth bundle $X \times W \times \mathbb{C}$, with complex structure given by $\bar{\partial}_X + \bar{\partial}_W + T^{0,1}$, and relative integrable connection given by $\partial_X + T^{1,0}$. By a version of the Poincaré lemma, we therefore have

$$(\mathrm{id} \times \pi)^* \,\mathrm{DR}(P^\sharp, \nabla^\sharp) \simeq \big(\mathscr{C}^\bullet(X \times W/W), D\big).$$

To obtain the desired result, it suffices then to note that

$$R^i p_{2*}\mathscr{C}^k(X \times W/W) = 0, \text{ for all } k \text{ and all } i > 0.$$

This follows from a standard partition of unity argument as in [GH78, p. 42]. □

To prove Theorem 15.2, it now suffices to show that the stalk of the complex $\big(C^\bullet(X \times W/W), D\big)$ at any given point $\tau \in W$ is quasi-isomorphic to a linear complex. Choose a basis $e_1, \dots, e_{2g} \in H^1(X, \mathbb{C})$ for the space of harmonic one-forms on X, and let $z_1, \dots, z_{2g}$ be the corresponding system of holomorphic coordinates on W, centered at the point τ. In these coordinates, we have

$$T = \tau + \sum_{j=1}^{2g} z_j e_j \in C^1(X \times W/W).$$

Let $R = \mathscr{O}^{an}_{W,\tau}$ be the analytic local ring at the point τ, with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = \mathbb{C}$. To simplify the notation, we put

$$C^k := C^k(X \times W/W) \otimes_{\mathscr{O}_W} R.$$

Then the R-module C^k consists of all convergent power series of the form

$$\alpha = \sum_I \alpha_I \otimes z^I := \sum_{I \in \mathbb{N}^{2g}} \alpha_I \otimes z_1^{I(1)} \cdots z_{2g}^{I(2g)}$$

where $\alpha_I \in A^k(X)$ are smooth complex-valued k-forms on X, and the summation is over all multi-indices $I \in \mathbb{N}^{2g}$. To describe the induced differential in the complex, we define the auxiliary operators $d_\tau = d + \tau$ and

$$e \colon C^k \to C^{k+1}, \qquad e(\alpha) = \sum_{I,j} e_j \wedge \alpha_I \otimes z_j z^I.$$

Then each differential $D \colon C^k \to C^{k+1}$ is given by the formula

$$D\alpha = \sum_I (d\alpha_I + \tau \wedge \alpha_I) \otimes z^I + \sum_{I,j} e_j \wedge \alpha_I \otimes z_j z^I = (d_\tau + e)\alpha. \tag{16.2}$$

Note that each R-module in the complex has infinite rank; moreover, in the formula for the differential D, the first of the two terms is not linear in $z_1, \dots, z_{2g}$. Our goal is then to build a linear complex quasi-isomorphic to

$$C^0 \to C^1 \to \cdots \to C^{2n-1} \to C^{2n},$$

by using harmonic forms with coefficients in the flat line bundle corresponding to τ. The space of d_τ-harmonic forms has the advantage of being finite-dimensional; in addition, any such form $\alpha \in A^k(X)$ satisfies $d_\tau \alpha = 0$, and hence

$$D\alpha = \sum_{j=1}^{2g} e_j \wedge \alpha \otimes z_j.$$

This shows that the differential is linear when restricted to the free R-module generated by the d_τ-harmonic forms. The only problem is that we do not obtain a subcomplex of $(C^\bullet, D)$ in this way, because the wedge product $e_j \wedge \alpha$ is in general no longer harmonic. This difficulty can be overcome by constructing a more careful embedding of the space of d_τ-harmonic k-forms into C^k, as we now explain.

17. Harmonic theory for flat line bundles. In this section, we summarize the theory of harmonic forms with coefficients in a flat line bundle, developed by Simpson [Sim92]. Let $A^k(X)$ be the space of smooth complex-valued k-forms on X. Choose a Kähler metric on X, with Kähler form $\omega \in A^2(X)$, and denote by

$$L \colon A^k(X) \to A^{k+2}(X), \qquad L(\alpha) = \omega \wedge \alpha$$

the associated Lefschetz operator. The metric gives rise to the $*$-operator

$$* \colon A^k(X) \to A^{2n-k}(X),$$

where $n = \dim X$, and the formula

$$(\alpha, \beta)_X = \int_X \alpha \wedge * \overline{\beta}$$

defines a Hermitian inner product on the space $A^k(X)$. With respect to these inner products, the adjoint of $L \colon A^k(X) \to A^{k+2}(X)$ is the operator $\Lambda \colon A^k(X) \to A^{k-2}(X)$. Likewise, the adjoint of the exterior derivative $d \colon A^k(X) \to A^{k+1}(X)$ is the operator $d^* \colon A^k(X) \to A^{k-1}(X)$, described more explicitly as $d^*\alpha = - * d * \alpha$. We use the notation $d = \partial + \bar\partial$ for the decomposition of d by type; thus ∂ maps (p,q)-forms to $(p+1,q)$-forms, and $\bar\partial$ maps (p,q)-forms to $(p,q+1)$-forms.

Now fix a holomorphic line bundle L with flat connection ∇. Let $A^k(X, L)$ denote the space of smooth k-forms with coefficients in L. The theorem of Corlette [Sim92, Theorem 1] shows that there is (up to rescaling) a unique metric on the underlying smooth bundle that makes (L, ∇) into a harmonic bundle. Together with the Kähler metric on X, it defines Hermitian inner products on the spaces $A^k(X, L)$. In the case at hand, the harmonic metric may be described concretely as follows. According to our previous discussion, there is a harmonic one-form $\tau \in A^1(X)$, such that L is isomorphic to the trivial smooth bundle $X \times \mathbb{C}$, with complex structure given by $\bar{\partial} + \tau^{0,1}$ and flat connection given by $\partial + \tau^{1,0}$. The harmonic metric on L is then simply the standard metric on the bundle $X \times \mathbb{C}$. Consequently, we have

$$A^k(X, L) = A^k(X),$$

and both the $*$-operator and the inner product induced by the harmonic metric agree with the standard ones defined above.

As before, we let $d_\tau = d + \tau$ be the operator encoding the complex structure and flat connection on (L, ∇); concretely,

$$d_\tau \colon A^k(X) \to A^{k+1}(X), \qquad \alpha \mapsto d\alpha + \tau \wedge \alpha.$$

Let $d_\tau^* \colon A^k(X) \to A^{k-1}(X)$ be the adjoint of d_τ with respect to the inner products, and let $\Delta_\tau = d_\tau d_\tau^* + d_\tau^* d_\tau$ be the Laplace operator; it is an elliptic operator of second order. If we denote by

$$\mathcal{H}_\tau^k = \ker\bigl(\Delta_\tau \colon A^k(X) \to A^k(X)\bigr)$$

the space of d_τ-harmonic k-forms, then $\mathcal{H}_\tau^k$ is finite-dimensional, and Hodge theory gives us a decomposition

$$A^k(X) = \mathcal{H}_\tau^k(X) \oplus \Delta_\tau A^k(X), \tag{17.1}$$

orthogonal with respect to the inner product on $A^k(X)$. Let $H_\tau \colon A^k(X) \to \mathcal{H}_\tau^k$ be the orthogonal projection to the space of harmonic forms. It is not hard to see that any $\alpha \in A^k(X)$ can be uniquely written in the form

$$\alpha = H_\tau \alpha + \Delta_\tau G_\tau \alpha, \tag{17.2}$$

where $G_\tau \alpha \in \Delta_\tau A^k(X)$ is the so-called Green's operator. The uniqueness of the decomposition implies that $d_\tau G_\tau = G_\tau d_\tau$ and $d_\tau^* G_\tau = G_\tau d_\tau^*$.

Following Simpson, we have a decomposition $d_\tau = \partial_\tau + \bar{\partial}_\tau$, where

$$\partial_\tau = \partial + \frac{\tau^{1,0} - \overline{\tau^{0,1}}}{2} + \frac{\tau^{0,1} + \overline{\tau^{1,0}}}{2},$$
$$\bar{\partial}_\tau = \bar{\partial} + \frac{\tau^{0,1} - \overline{\tau^{1,0}}}{2} + \frac{\tau^{1,0} + \overline{\tau^{0,1}}}{2}.$$

The justification for defining these two peculiar operators is that they satisfy the usual Kähler identities (which fail for the naive choice $\partial + \tau^{1,0}$ and $\bar{\partial} + \tau^{0,1}$).

Theorem 17.3 (Simpson)**.** *The following are true:*

1. *We have $\partial_\tau \partial_\tau = \bar{\partial}_\tau \bar{\partial}_\tau = \partial_\tau \bar{\partial}_\tau + \bar{\partial}_\tau \partial_\tau = 0$.*
2. *Let ∂_τ^* and $\bar{\partial}_\tau^*$ denote the adjoints of ∂_τ and $\bar{\partial}_\tau$, respectively. Then the first-order Kähler identities*
$$\partial_\tau^* = i[\Lambda, \bar{\partial}_\tau], \qquad \bar{\partial}_\tau^* = -i[\Lambda, \partial_\tau], \qquad d_\tau^* = i[\Lambda, \bar{\partial}_\tau - \partial_\tau].$$
are satisfied.

(3) *We have* $\bar{\partial}_\tau \partial_\tau^* + \partial_\tau^* \bar{\partial}_\tau = \partial_\tau \bar{\partial}_\tau^* + \bar{\partial}_\tau^* \partial_\tau = 0$.
(4) *The Laplace operator satisfies*

$$\Delta_\tau = 2(\partial_\tau \partial_\tau^* + \partial_\tau^* \partial_\tau) = 2(\bar{\partial}_\tau \bar{\partial}_\tau^* + \bar{\partial}_\tau^* \bar{\partial}_\tau),$$

and consequently, d_τ*-harmonic forms are both* ∂_τ*-closed and* ∂_τ^**-closed.*
(5) *We have* $H\partial_\tau = H\bar{\partial}_\tau = H\partial_\tau^* = H\bar{\partial}_\tau^* = 0$.
(6) *The Green's operator* G_τ *commutes with* ∂_τ, $\bar{\partial}_\tau$, ∂_τ^*, *and* $\bar{\partial}_\tau^*$.

Proof. It is easy to see from the definition that (1) holds. The first-order Kähler identities in (2) are proved in [Sim92, p. 14]; in this simple case, they can also be verified by hand by a calculation on $\mathbb{C}^n$ with the Euclidean metric. From this, (3) and (4) follow as in the case of the usual Kähler identities [Sim92, p. 22]. Finally, (5) is a consequence of (4), and (6) follows from the previous results by the uniqueness of the decomposition in (17.2). ☐

Note. The Higgs bundle associated to (L, ∇) is the smooth vector bundle $X \times \mathbb{C}$, with complex structure defined by the operator

$$\bar{\partial} + \frac{\tau^{0,1} - \overline{\tau^{1,0}}}{2},$$

and with Higgs field

$$\theta = \frac{\tau^{1,0} + \overline{\tau^{0,1}}}{2}.$$

Note that θ is holomorphic on account of $\bar{\partial}_\tau \bar{\partial}_\tau = 0$. The complex structure on the original flat line bundle is defined by the operator $\bar{\partial} + \tau^{0,1}$, which means that the two line bundles are different unless $\tau^{0,1} = -\overline{\tau^{1,0}}$.

Harmonic theory can be used to solve equations involving ∂_τ (or any of the other operators), as follows. Suppose that we are given an equation of the form $\partial_\tau \alpha = \beta$. A necessary and sufficient condition for the existence of a solution α is that $\partial_\tau \beta = 0$ and $H_\tau \beta = 0$. If this is the case, then among all possible solutions, there is a unique one that is ∂_τ^*-exact, namely $2\partial_\tau^* G_\tau \beta$. In fact, this is the solution of minimal norm. Note that we can always define $\alpha = 2\partial_\tau^* G_\tau \beta$; but since

$$\partial_\tau \bigl(2\partial_\tau^* G_\tau \beta\bigr) = \beta - H_\tau \beta - 2\partial_\tau^* G_\tau(\partial_\tau \beta),$$

we only obtain a solution to the original equation when $H_\tau \beta = 0$ and $\partial_\tau \beta = 0$. This idea will appear again in the construction below.

18. Sobolev spaces and norm estimates. At some point of the construction below, we will need to prove the convergence of certain power series. This requires estimates for the norms of the two operators $\bar{\partial}_\tau$ and G_τ introduced above, which hold in suitable Sobolev spaces. Since this is standard material in the theory of partial differential equations, we only give the briefest possible summary; all the results that we use can be found, for example, in [Wel08, Chapter IV].

From the Kähler metric on X, we get an L^2-norm on the space $A^k(X)$ of smooth k-forms, by the formula

$$\|\alpha\|_0^2 = (\alpha, \alpha)_X = \int_X \alpha \wedge *\bar{\alpha}.$$

It is equivalent to the usual L^2-norm, defined using partitions of unity. There is also a whole family of higher Sobolev norms: for $\alpha \in A^k(X)$, the m-th order

Sobolev norm $\|\alpha\|_m$ controls the L^2-norms of all derivatives of α of order at most m. The Sobolev space $W_m^k(X)$ is the completion of $A^k(X)$ with respect to the norm $\|-\|_m$; it is a Hilbert space. Elements of $W_m^k(X)$ may be viewed as k-forms α with measurable coefficients, all of whose weak derivatives of order at most m are square-integrable. Here is the first result from analysis that we need.

Theorem 18.1 (Sobolev lemma)**.** *If $\alpha \in W_m^k(X)$ for every $m \in \mathbb{N}$, then α agrees almost everywhere with a smooth k-form, and hence $\alpha \in A^k(X)$.*

The second result consists of a pair of norm inequalities, one for the differential operator ∂_τ^*, the other for the Green's operator G_τ.

Theorem 18.2. *Let the notation be as above.*

(1) There is a constant $C > 0$, depending on $m \geq 1$, such that

$$\|\partial_\tau^* \alpha\|_{m-1} \leq C \cdot \|\alpha\|_m$$

for every $\alpha \in W_m^k(X)$ with $m \geq 1$.

(2) There is another constant $C > 0$, depending on $m \geq 0$, such that

$$\|G_\tau \alpha\|_{m+2} \leq C \cdot \|\alpha\|_m$$

for every $\alpha \in W_m^k(X)$.

Proof. The inequality in (1) is easy to prove, using the fact that ∂_τ^* is a first-order operator and X is compact. On the other hand, (2) follows from the open mapping theorem. To summarize the argument in a few lines, (17.1) is actually derived from an orthogonal decomposition

$$W_m^k(X) = \mathcal{H}_\tau^k \oplus \Delta_\tau W_{m+2}^k(X)$$

of the Hilbert space $W_m^k(X)$. It implies that the bounded linear operator

$$\Delta_\tau \colon W_{m+2}^k(X) \cap (\mathcal{H}_\tau^k)^\perp \to W_m^k(X) \cap (\mathcal{H}_\tau^k)^\perp$$

is bijective; by the open mapping theorem, the inverse must be bounded as well. Since G_τ is equal to this inverse on $(\mathcal{H}_\tau^k)^\perp$, and zero on $\mathcal{H}_\tau^k$, we obtain the desired inequality. □

19. Construction of the linear complex. We now return to the proof of Theorem 15.2. Recall that, after pullback to the universal covering space W of $A^\sharp$, the stalk of the Fourier-Mukai transform of the $\mathcal{D}$-module $\mathscr{O}_X$ is represented by the complex of R-modules

$$C^0 \to C^1 \to \cdots \to C^{2n-1} \to C^{2n},$$

with differential

$$D(\alpha) = (d_\tau + e)\alpha = \sum_I d_\tau \alpha_I \otimes z^I + \sum_{I,j} e_j \wedge \alpha_I \otimes z_j z^I.$$

Our goal is to show that $(C^\bullet, D)$ is quasi-isomorphic to a linear complex over R.

We begin by constructing a suitable linear complex from the finite-dimensional spaces of d_τ-harmonic forms. Let $(\mathcal{H}_\tau^\bullet \otimes R, \delta)$ be the complex

$$\mathcal{H}_\tau^0 \otimes R \to \mathcal{H}_\tau^1 \otimes R \to \cdots \to \mathcal{H}_\tau^{2n-1} \otimes R \to \mathcal{H}_\tau^{2n} \otimes R,$$

with differential obtained by R-linear extension from

$$\delta\colon \mathcal{H}_\tau^k \to \mathcal{H}_\tau^{k+1} \otimes R, \qquad \delta(\alpha) = \sum_j H_\tau(e_j \wedge \alpha) \otimes z_j.$$

This is indeed a complex, but to show that $\delta \circ \delta = 0$ requires some computation.

Lemma 19.1. *We have $\delta \circ \delta = 0$.*

Proof. Before we begin the actual proof, let us make a useful observation: namely, that for every $\alpha \in A^k(X)$, one has

$$d_\tau(e_j \wedge \alpha) = d(e_j \wedge \alpha) + \tau \wedge e_j \wedge \alpha = -e_j \wedge d\alpha - e_j \wedge \tau \wedge \alpha = -e_j \wedge d_\tau \alpha,$$

due to the fact that e_j is a closed one-form. Now take $\alpha \in \mathcal{H}_\tau^k$. Then

$$H_\tau(e_j \wedge \alpha) = e_j \wedge \alpha - \Delta_\tau G_\tau(e_j \wedge \alpha) = e_j \wedge \alpha - d_\tau d_\tau^* G_\tau(e_j \wedge \alpha),$$

because $d_\tau(e_j \wedge \alpha) = 0$ by the above observation. Consequently,

$$e_k \wedge H_\tau(e_j \wedge \alpha) = e_k \wedge e_j \wedge \alpha + d_\tau\bigl(e_k \wedge d_\tau^* G_\tau(e_j \wedge \alpha)\bigr),$$

and since $H_\tau d_\tau = 0$, this allows us to conclude that

$$\delta(\delta\alpha) = \sum_{j,k} H_\tau\bigl(e_k \wedge H_\tau(e_j \wedge \alpha)\bigr) \otimes z_j z_k = \sum_{j,k} H_\tau(e_k \wedge e_j \wedge \alpha) \otimes z_j z_k = 0,$$

which means that $\delta \circ \delta = 0$. $\square$

Note that it is clear from the representation of cohomology via harmonic forms that the complex thus constructed is quasi-isomorphic to the stalk at a point mapping to (L, ∇) of the complex appearing in the statement of Theorem 3.6.

We shall now construct a sequence of maps $f^k\colon \mathcal{H}_\tau^k \to C^k$, in such a way that, after R-linear extension, we obtain a morphism of complexes $f\colon \mathcal{H}_\tau^\bullet \otimes R \to C^\bullet$. In order for the f^k to define a morphism of complexes, the identity

$$Df^k(\alpha) = f^{k+1}(\delta\alpha) \tag{19.2}$$

should be satisfied for every $\alpha \in \mathcal{H}_\tau^k$. As a first step, we shall find a *formal* solution to the problem, ignoring questions of convergence for the time being. Let $\hat{C}^k$ be the space of all formal power series

$$\alpha = \sum_I \alpha_I \otimes z^I := \sum_{I \in \mathbb{N}^{2g}} \alpha_I \otimes z_1^{I(1)} \cdots z_{2g}^{I(2g)}$$

with $\alpha_I \in A^k(X)$ smooth complex-valued k-forms. We extend the various operators from $A^k(X)$ to $\hat{C}^k$ by defining, for example,

$$d_\tau \alpha = \sum_I d_\tau \alpha_I \otimes z^I, \qquad e(\alpha) = \sum_{I,j} e_j \wedge \alpha_I \otimes z_j z^I, \qquad \text{etc.}$$

Note that $C^k \subseteq \hat{C}^k$ is precisely the subspace of those power series that converge in some neighborhood of $X \times \{z = 0\}$.

To make sure that $f^k(\alpha)$ induces the correct map on cohomology, we require that $H_\tau f^k(\alpha) = \alpha$. Following the general strategy for solving equations with the help of harmonic theory, we impose the additional conditions $\partial_\tau f^k(\alpha) = 0$ and $\bar{\partial}_\tau^* f^k(\alpha) = 0$. Under these assumptions, (19.2) reduces to

$$\bar{\partial}_\tau f^k(\alpha) + e f^k(\alpha) = f^{k+1}(\delta\alpha).$$

On account of $\bar\partial_\tau^* f^k(\alpha) = 0$ and the Kähler identities, we should then have

$$\begin{aligned} f^k(\alpha) &= H_\tau f^k(\alpha) + \Delta_\tau G_\tau f^k(\alpha) = \alpha + 2\big(\bar\partial_\tau \bar\partial_\tau^* + \bar\partial_\tau^* \bar\partial_\tau\big) G_\tau f^k(\alpha) \\ &= \alpha + 2\bar\partial_\tau^* G_\tau \big(\bar\partial_\tau f^k(\alpha)\big) = \alpha - 2\bar\partial_\tau^* G_\tau \big(e f^k(\alpha)\big). \end{aligned}$$

This suggests that we try to solve the equation $\big(\mathrm{id} + 2\bar\partial_\tau^* G_\tau e\big) f^k(\alpha) = \alpha$.

Lemma 19.3. *For any d_τ-harmonic form $\alpha \in \mathcal{H}_\tau^k$, the equation*

$$\big(\mathrm{id} + 2\bar\partial_\tau^* G_\tau e\big)\beta = \alpha \tag{19.4}$$

has a unique formal solution $\beta \in \hat{C}^k$. This solution has the property that $H_\tau \beta = \alpha$, as well as $\bar\partial_\tau^* \beta = 0$ and $\partial_\tau \beta = 0$.

Proof. To solve the equation formally, we write

$$\beta = \sum_{\ell=0}^{\infty} \sum_{|I|=\ell} \beta_I \otimes z^I = \sum_{\ell=0}^{\infty} \beta_\ell,$$

making β_ℓ homogeneous of degree ℓ in $z_1, \dots, z_{2g}$. Taking harmonic parts in (19.4), it is clear that we must have $\beta_0 = \alpha$ and $H_\tau \beta_\ell = 0$ for $\ell \geq 1$; for the rest, the equation dictates that

$$\beta_{\ell+1} = -\sum_j 2 z_j \bar\partial_\tau^* G_\tau \big(e_j \wedge \beta_\ell\big), \tag{19.5}$$

for every $\ell \geq 0$, which means that there is a unique formal solution β. It is apparent from the equation that $\bar\partial_\tau^* \beta = 0$, and so to prove the lemma, it remains to show that we have $\partial_\tau \beta = 0$. Since $\partial_\tau \beta_0 = 0$, we can proceed by induction on $\ell \geq 0$. Using the Kähler identity $\partial_\tau \bar\partial_\tau^* = -\bar\partial_\tau^* \partial_\tau$, and the fact that $\partial_\tau(e_j \wedge \beta_\ell) = -e_j \wedge \partial_\tau \beta_\ell$, we deduce from (19.5) that

$$\partial_\tau \beta_{\ell+1} = -\sum_j 2 z_j \partial_\tau \bar\partial_\tau^* G_\tau \big(e_j \wedge \beta_\ell\big) = -\sum_j 2 z_j \bar\partial_\tau^* G_\tau \big(e_j \wedge \partial_\tau \beta_\ell\big),$$

and so $\partial_\tau \beta_\ell = 0$ implies $\partial_\tau \beta_{\ell+1} = 0$, as required. □

The next step is to prove the convergence of the power series defining the solution to (19.4). For $\varepsilon > 0$, let

$$W_\varepsilon = \left\{ \tau + z \in W \;\middle|\; \sum_j |z_j| < \varepsilon \right\},$$

which is an open neighborhood of the point $\tau \in W$.

Lemma 19.6. *There is an $\varepsilon > 0$, such that for all $\alpha \in \mathcal{H}_\tau^k$, the formal power series*

$$\beta = (\mathrm{id} + 2\bar\partial_\tau^* G_\tau e)^{-1} \alpha \in \hat{C}^k$$

converges absolutely and uniformly on $X \times W_\varepsilon$ to an element of $C^k(X \times W_\varepsilon / W_\varepsilon)$.

Proof. If we apply the estimates from Theorem 18.2 to the relation in (19.5), we find that for every $m \geq 1$, there is a constant $C_m > 0$, such that

$$\|\beta_{\ell+1}\|_m \leq C_m \sum_j |z_j| \|\beta_\ell\|_{m-1} \leq C_m \varepsilon \cdot \|\beta_\ell\|_{m-1} \tag{19.7}$$

holds for all $\ell \geq 1$, provided that $z \in W_\varepsilon$. Given that $\beta_0 = \alpha$, we conclude that

$$\|\beta_\ell\|_0 \leq (C_1 \varepsilon)^\ell \|\alpha\|_0.$$

Now choose a positive real number $\varepsilon < 1/C_1$. We then obtain

$$\sum_{\ell=0}^{\infty} \|\beta_\ell\|_0 \leq \sum_{\ell=0}^{\infty} (C_1\varepsilon)^\ell \|\alpha\|_0 = \frac{\|\alpha\|_0}{1 - C_1\varepsilon},$$

from which it follows that β is absolutely and uniformly convergent in the L^2-norm as long as $z \in W_\varepsilon$. To prove that β is actually smooth, we return to the original form of (19.7). It implies that, for any $m \geq 1$,

$$\sum_{\ell=0}^{\infty} \|\beta_\ell\|_m \leq \|\alpha\|_m + C_m\varepsilon \cdot \sum_{\ell=0}^{\infty} \|\beta_\ell\|_{m-1}.$$

By induction on $m \geq 0$, one now easily shows that $\sum_\ell \|\beta_\ell\|_m$ converges absolutely and uniformly on $X \times W_\varepsilon$ for every $m \geq 0$; because of the Sobolev lemma, this means that β is smooth on $X \times W_\varepsilon$. Since we clearly have $\bar\partial_W \beta = 0$, it follows that $\beta \in C^k(X \times W_\varepsilon / W_\varepsilon)$. □

The preceding lemmas justify defining

$$f^k \colon \mathcal{H}^k_\tau \to C^k, \qquad f^k(\alpha) = (\mathrm{id} + 2\bar\partial^*_\tau G_\tau e)^{-1}\alpha.$$

It remains to show that we have found a solution to the original problem (19.2).

Lemma 19.8. *For every $\alpha \in \mathcal{H}^k_\tau$, we have $Df^k(\alpha) = f^{k+1}(\delta\alpha)$.*

Proof. Let $\beta = f^k(\alpha)$, so that $(\mathrm{id} + 2\bar\partial^*_\tau G_\tau e)\beta = \alpha$ and $\partial_\tau \beta = 0$. Noting that $\delta(\alpha) = H_\tau(e\alpha)$, we need to show that

$$(\mathrm{id} + 2\bar\partial^*_\tau G_\tau e)(\bar\partial_\tau + e)\beta = H_\tau(e\alpha).$$

Since $e \circ e = 0$ and $e(\bar\partial_\tau \beta) = -\bar\partial_\tau(e\beta)$, we compute that

$$(\mathrm{id} + 2\bar\partial^*_\tau G_\tau e)(\bar\partial_\tau + e)\beta = \bar\partial_\tau\beta + e\beta + 2\bar\partial^*_\tau G_\tau e \bar\partial_\tau \beta = \bar\partial_\tau\beta + e\beta - 2\bar\partial^*_\tau \bar\partial_\tau G_\tau e\beta.$$

We always have $e\beta = H_\tau(e\beta) + 2\bar\partial_\tau\bar\partial^*_\tau G_\tau e\beta + 2\bar\partial^*_\tau\bar\partial_\tau G_\tau e\beta$, and so we can simplify the above to

$$\begin{aligned}(\mathrm{id} + 2\bar\partial^*_\tau G_\tau e)(\bar\partial_\tau + e)\beta &= \bar\partial_\tau\beta + H_\tau(e\beta) + 2\bar\partial_\tau\bar\partial^*_\tau G_\tau e\beta \\ &= H_\tau(e\beta) + \bar\partial_\tau\big(\beta + 2\bar\partial^*_\tau G_\tau e\beta\big) = H_\tau(e\beta) + \bar\partial_\tau \alpha = H_\tau(e\beta).\end{aligned}$$

Thus it suffices to prove that $H_\tau(e\beta) = H_\tau(e\alpha)$. But this is straightforward: from $H_\tau(\beta) = \alpha$ and $\partial_\tau\beta = 0$, we get $\beta = \alpha + 2\partial_\tau\partial^*_\tau G_\tau\beta$, and therefore

$$e\beta = e\alpha - 2\partial_\tau\big(e\partial^*_\tau G_\tau\beta\big).$$

Since $H_\tau\partial_\tau = 0$, we obtain the desired identity $H_\tau(e\beta) = H_\tau(e\alpha)$. □

Note. The decomposition $\beta = \alpha + 2\partial_\tau\partial^*_\tau G_\tau\beta$ is the reason for imposing the additional condition $\partial_\tau f^k(\alpha) = 0$. Without this, it would be difficult to relate the d_τ-harmonic parts of $e\alpha$ and $e\beta$ in the final step of the proof.

If we extend R-linearly, we obtain maps of R-modules $f^k \colon \mathcal{H}^k_\tau \otimes R \to C^k$. Because (19.2) is satisfied, they define a morphism of complexes $f \colon \big(\mathcal{H}^\bullet_\tau \otimes R, \delta\big) \to \big(C^\bullet, D\big)$.

Lemma 19.9. *$f \colon \mathcal{H}^\bullet_\tau \otimes R \to C^\bullet$ is a quasi-isomorphism.*

Proof. We use the spectral sequence (20.4). The complex $\mathcal{H}_\tau^\bullet \otimes R$ is clearly linear, and so the associated spectral sequence

$$ {}^1E_1^{p,q} = \mathcal{H}_\tau^{p+q} \otimes \operatorname{Sym}^p(\mathfrak{m}/\mathfrak{m}^2) \Longrightarrow H^{p+q}\big(\mathcal{H}_\tau^\bullet \otimes R, \delta\big) $$

degenerates at E_2 by Lemma 20.5. On the other hand, the complex $C^\bullet$ also satisfies the conditions neede to define (20.4), giving us a second convergent spectral sequence with

$$ {}^2E_1^{p,q} = H^{p+q}(X, \ker \nabla) \otimes \operatorname{Sym}^p(\mathfrak{m}/\mathfrak{m}^2) \Longrightarrow H^{p+q}\big(C^\bullet, D\big), $$

where $\ker \nabla$ is the local system corresponding to (L, ∇). The morphism f induces a morphism between the two spectral sequences; at E_1, it restricts to isomorphisms ${}^1E_1^{p,q} \simeq {}^2E_1^{p,q}$, because $\mathcal{H}_\tau^k \simeq H^k(X, \ker \nabla)$. It follows that the second spectral sequence also degenerates at E_2; it is then not hard to see that f must be indeed a quasi-isomorphism. □

We have now shown that the complex $\big(C^\bullet, D\big)$ is isomorphic, in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$, to a linear complex. This completes the proof of Theorem 15.2.

20. Filtered complexes and linear complexes. This section contains the homological algebra used in the constructions and proofs of the previous sections. It reviews and expands some of the content of [LPS10, §1], the main improvement with respect to that paper being Proposition 20.2.

Definition 20.1. Let $(R, \mathfrak{m})$ be a regular local k-algebra of dimension n, with residue field $k = R/\mathfrak{m}$. A *linear complex* over R is a bounded complex $\big(K^\bullet, d\big)$ of finitely generated free R-modules with the following property: there is a system of parameters $t_1, \dots, t_n \in \mathfrak{m}$, such that every differential of the complex is a matrix of linear forms in $t_1, \dots, t_n$. The property of being linear is obviously not invariant under isomorphisms. To have a notion that works in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$, let us say that a complex is *quasi-linear* over R if it is quasi-isomorphic to a linear complex over R.

It is an interesting problem to try and find natural necessary and sufficient conditions for quasi-linearity in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$. For instance, if $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$, then is it true that E is quasi-linear over R iff $E \otimes_R \hat{R}$ is quasi-linear over the completion $\hat{R}$ with respect to the $\mathfrak{m}$-adic topology?

Here we only treat a special case that was needed in Corollary 15.3., namely the fact that any direct summand in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$ of a quasi-linear complex is again quasi-linea, plus a necessary condition for quasi-linearity. The basic tool that we shall use is the notion of a minimal complex. Recall that a bounded complex $(K^\bullet, d)$ of finitely generated free R-modules is called *minimal* if $d(K^i) \subseteq \mathfrak{m}K^{i+1}$ for every $i \in \mathbb{Z}$. This means that every differential of the complex is a matrix with entries in the maximal ideal $\mathfrak{m}$. A basic fact is that every object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$ is isomorphic to a minimal complex; moreover, two minimal complexes are isomorphic as objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$ if and only if they are isomorphic as complexes (see [Rob80] for more details). Linear complexes are clearly minimal; it follows that a minimal complex is quasi-linear iff it is isomorphic (as a complex) to a linear complex.

Proposition 20.2. *Let $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$ be quasi-linear. If E' is a direct summand of E in the derived category, then E' is also quasi-linear.*

Proof. Let $K^\bullet$ be a linear complex quasi-isomorphic to E, and $L^\bullet$ a minimal complex quasi-isomorphic to E'. Since E' is a direct summand of E in the derived category, there are morphisms of complexes

$$s\colon L^\bullet \to K^\bullet \qquad \text{and} \qquad p\colon K^\bullet \to L^\bullet$$

such that $p \circ s$ is homotopic to the identity morphism of $L^\bullet$. Because $L^\bullet$ is minimal, it follows that $p \circ s$ reduces to the identity modulo $\mathfrak{m}$, and is thus an isomorphism. After replacing p by $(p \circ s)^{-1} \circ p$, we may therefore assume without loss of generality that $p \circ s = \mathrm{id}$.

Now choose a system of parameters $t_1, \dots, t_n \in \mathfrak{m}$ with the property that each differential of the complex $K^\bullet$ is a matrix of linear forms in $t_1, \dots, t_n$. We are going to construct a new complex $L_0^\bullet$ that has the same property, and is isomorphic to $L^\bullet$. Set $L_0^i = L^i$, and define the differential $d_0\colon L_0^i \to L_0^{i+1}$ to be the linear part of $d\colon L^i \to L^{i+1}$. That is to say, let d_0 be the unique matrix of linear forms in $t_1, \dots, t_n$ with the property that $(d - d_0)(L^i) \subseteq \mathfrak{m}^2 L^{i+1}$ for every $i \in \mathbb{Z}$. It is easy to see that $d_0 \circ d_0 = 0$, and so $(L_0^\bullet, d_0)$ is a linear complex.

Likewise, we may define $s_0\colon L_0^i \to K^i$ as the constant part of $s\colon L^i \to K^i$; that is to say, as the unique matrix with entries in the field k such that $(s - s_0)(L^i) \subseteq \mathfrak{m} K^i$. Now consider the commutative diagram

$$\begin{tikzcd} L^i \arrow[r, "d"] \arrow[d, "s"'] & L^{i+1} \arrow[d, "s"'] \\ K^i \arrow[r, "d"] & K^{i+1}. \end{tikzcd}$$

By taking linear parts in the identity $d \circ s = s \circ d$, and using the fact that $K^\bullet$ is a linear complex, we find that $d \circ s_0 = s_0 \circ d_0$; consequently, $s_0\colon L_0^\bullet \to K^\bullet$ is a morphism of complexes. To conclude the proof, we consider the composition

$$p \circ s_0\colon L_0^\bullet \to L^\bullet.$$

By construction, $p \circ s_0$ reduces to the identity modulo $\mathfrak{m}$, and is therefore an isomorphism. This shows that $L^\bullet$ is indeed isomorphic to a linear complex, as claimed. $\square$

A necessary condition for quasi-linearity is the degeneration of a certain spectral sequence. To state this, we first recall some general facts. Let $\bigl(K^\bullet, F\bigr)$ be a filtered complex in an abelian category. We assume that the filtration is decreasing, meaning that $F^p K^n \supseteq F^{p+1} K^n$, and satisfies

$$\bigcup_{p \in \mathbb{Z}} F^p K^n = K^n \qquad \text{and} \qquad \bigcap_{p \in \mathbb{Z}} F^p K^n = \{0\}.$$

Moreover, the differentials should respect the filtration, in the sense that $d\bigl(F^p K^n\bigr) \subseteq F^p K^{n+1}$. Under these assumptions, the filtered complex gives rise to a spectral sequence (of cohomological type)

$$E_1^{p,q} = H^{p+q}\bigl(F^p K^\bullet / F^{p+1} K^\bullet\bigr) \Longrightarrow H^{p+q}\bigl(K^\bullet\bigr). \tag{20.3}$$

It converges by the standard convergence criterion [McC01, Theorem 3.2].

Going back now to the situation of a regular local k-algebra $(R, \mathfrak{m})$ as above, on any bounded complex $K^\bullet$ of R-modules with finitely generated cohomology, we can define the $\mathfrak{m}$-adic filtration by setting

$$F^p K^n = \mathfrak{m}^p K^n$$

for all $p \geq 0$. Noting that $\mathfrak{m}^p/\mathfrak{m}^{p+1} \simeq \mathrm{Sym}^p(\mathfrak{m}/\mathfrak{m}^2)$, we have

$$F^p K^n / F^{p+1} K^n \simeq \bigl(K^n \otimes_R k\bigr) \otimes_k \mathrm{Sym}^p(\mathfrak{m}/\mathfrak{m}^2).$$

Provided that each K^n has the property that

$$\bigcap_{p=1}^{\infty} \mathfrak{m}^p K^n = \{0\},$$

the filtration satisfies the conditions necessary to define (20.3), and we obtain a convergent spectral sequence

$$E_1^{p,q} = H^{p+q}\bigl(K^\bullet \otimes_R k\bigr) \otimes_k \mathrm{Sym}^p(\mathfrak{m}/\mathfrak{m}^2) \Longrightarrow H^{p+q}\bigl(K^\bullet\bigr). \tag{20.4}$$

It follows from the Artin-Rees theorem that the induced filtration on the limit is $\mathfrak{m}$-good; in particular, the completion of $H^n\bigl(K^\bullet\bigr)$ with respect to this filtration is isomorphic to $H^n\bigl(K^\bullet\bigr) \otimes_R \hat{R}$. In this sense, the spectral sequence describes the formal analytic stalk of the original complex.

Lemma 20.5 ([LPS10, Lemma 1.5])**.** *If $K^\bullet$ is a linear complex, then the spectral sequence*

$$E_1^{p,q} = H^{p+q}\bigl(K^\bullet \otimes_R k\bigr) \otimes_k \mathrm{Sym}^p(\mathfrak{m}/\mathfrak{m}^2) \Longrightarrow H^{p+q}\bigl(K^\bullet\bigr)$$

degenerates at the E_2-page.

Note that degeneration of the spectral sequence is not a sufficient condition for being quasi-linear: over $R = k[\![x,y]\!]$, the complex

$$R \xrightarrow{\binom{x^2}{y}} R^{\oplus 2},$$

is an example of a minimal complex that is not isomorphic to a linear complex, but where the spectral sequence nevertheless degenerates at E_2.

G. Generalizations and open problems

21. Extensions to D-modules on abelian varieties. In light of our results, it should be clear that all the theorems in generic vanishing theory are in reality statements about a certain class of filtered $\mathcal{D}$-modules on abelian varieties, namely those underlying mixed Hodge modules. Moreover, the dimension (D), linearity (L), and strong linearity (SL) results that we have discussed should be viewed as properties of the Fourier-Mukai transforms of such $\mathcal{D}$-modules.

A natural question is then whether there is a larger (and more easily described) class of $\mathcal{D}$-modules on abelian varieties for which the same results are true. It is known that when $(\mathcal{M}, F)$ underlies a mixed Hodge module M, the $\mathcal{D}$-module $\mathcal{M}$ is always regular and holonomic; when M is pure, $\mathcal{M}$ is in addition semi-simple. This suggests that generic vanishing theory might extend to $\mathcal{D}$-modules with those properties on abelian varieties.

Using the recent work of Sabbah [Sab05,Sab09] and Mochizuki [Moc07a,Moc07b] on the correspondence between semi-simple holonomic $\mathcal{D}$-modules and polarizable twistor $\mathcal{D}$-modules, such an extension is indeed possible. The results are as follows.

Theorem 21.1. *Let $\mathcal{M} \in \mathrm{D}_h^{\mathrm{b}}(\mathcal{D}_A)$ be a complex of $\mathcal{D}$-modules with bounded holonomic cohomology on a complex abelian variety A. Then for every $k, m \in \mathbb{Z}$, the cohomological support locus*

$$\Sigma_m^k(\mathcal{M}) := \left\{ (L, \nabla) \in A^\sharp \;\middle|\; \dim \mathbf{H}^k\Big(A, \mathrm{DR}_A\big(\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla)\big)\Big) \geq m \right\}$$

is a finite union of translates of triple tori in $A^\sharp$; the translates are by torsion points when $\mathcal{M}$ is of geometric origin. The cohomological support loci satisfy

$$\operatorname{codim} \Sigma_m^k(\mathcal{M}) \geq 2k \qquad \text{for every } k \in \mathbb{Z}$$

in the special case when $\mathcal{M}$ is a single holonomic $\mathcal{D}$-module.

The theorem implies the analogous result for cohomological support loci of constructible complexes and perverse sheaves, which are now subsets of $\mathrm{Char}(A)$; this is because of the Riemann-Hilbert correspondence. By the usual base change arguments, one derives the following properties of the Fourier-Mukai transform.

Corollary 21.2. *For $\mathcal{M} \in \mathrm{D}_h^{\mathrm{b}}(\mathcal{D}_A)$, the support of the Fourier-Mukai transform $\mathbf{R}\Phi_{P^\sharp}\mathcal{M} \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathscr{O}_{A^\sharp})$ is a finite union of translates of triple tori in $A^\sharp$; the translates are by torsion points when $\mathcal{M}$ is of geometric origin. When $\mathcal{M}$ is a single holonomic $\mathcal{D}$-module, the system of inequalities*

$$\operatorname{codim} \operatorname{Supp} R^\ell \Phi_{P^\sharp}\mathcal{M} \geq 2\ell \qquad \text{for every } \ell \in \mathbb{Z}$$

is satisfied, which implies that $\mathbf{R}\Phi_{P^\sharp}\mathcal{M}$ is an m-perverse coherent sheaf on $A^\sharp$.

For semi-simple holonomic $\mathcal{D}$-modules, there is also a result analogous to (SL).

Theorem 21.3. *If $\mathcal{M}$ is a semi-simple holonomic $\mathcal{D}$-module on A, then the Fourier-Mukai transform $\mathbf{R}\Phi_{P^\sharp}\mathcal{M}$ is locally, in the analytic topology, quasi-isomorphic to a linear complex constructed from the cohomology of twists of $\mathcal{M}$.*

The proofs will appear elsewhere. They are based on an extension of the Fourier-Mukai transform to twistor $\mathcal{D}$-modules, which produces complexes of coherent analytic sheaves on the twistor space $\mathrm{Tw}(A^\sharp)$ of the quaternionic manifold $A^\sharp$, and on the powerful results about twistor $\mathcal{D}$-modules by Mochizuki and Sabbah. This treatment unifies the results for the two spaces $\widehat{A} \times H^0(A, \Omega_A^1)$ and $A^\sharp$ that we used in this paper: they appear as two different fibers of $\mathrm{Tw}(A^\sharp) \to \mathbb{P}^1$. (This can also be used to deduce that the main result of [GL91] is implied by Theorem 3.6 by passage to graded pieces.)

22. Open problems. Given the discussion above, a very interesting problem in this context is the following:

Problem 22.1. Describe classes of $\mathcal{D}$-modules—such as holonomic, regular holonomic, or semi-simple holonomic $\mathcal{D}$-modules—on an abelian variety in terms of their Fourier-Mukai transforms. In other words, characterize the subcategories of $\mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathscr{O}_{A^\sharp})$ that correspond to the categories of such $\mathcal{D}$-modules under $\mathbf{R}\Phi_{P^\sharp}$.

The two results above give several necessary conditions, so the problem is really to find sufficient conditions. Such a description might also shed some light on the difficult question of which $\mathcal{D}$-modules are of geometric origin.

One may also wonder whether there are extensions of various results in this paper in the non-projective setting.

Problem 22.2. Does the analogue of Theorem 3.1 hold on arbitrary complex tori?

Note that the (SL) type result, Theorem 3.6, generalizes to compact Kähler manifolds, since the proof only uses harmonic theory. This raises the question whether our statements of type (D), here relying heavily on vanishing theorems for ample line bundles, extend to that context as well.

Problem 22.3. Are there analogues of the generic vanishing theorems 3.2 and 3.4 in the Kähler setting?

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