The tensor structure on the representation category of the $\mathcal{W}_p$ triplet algebra

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December, 2011

Abstract

We study the braided monoidal structure that the fusion product induces on the abelian category $\mathcal{W}_p$-mod, the category of representations of the triplet W-algebra $\mathcal{W}_p$. The $\mathcal{W}_p$-algebras are a family of vertex operator algebras that form the simplest known examples of symmetry algebras of logarithmic conformal field theories. We formalise the methods for computing fusion products that are widely used in the physics literature and illustrate a systematic approach to calculating fusion products in non-semi-simple representation categories. We combine these methods with the general theory of braided monoidal categories to prove that $\mathcal{W}_p$-mod is a rigid braided monoidal category and that therefore the fusion product is bi-exact. The rigidity of $\mathcal{W}_p$-mod allows us to provide explicit formulae for the fusion product on the set of all simple and all projective $\mathcal{W}_p$-modules.

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1 Introduction

The theory of vertex operator algebras is an algebraic approach to describing the chiral symmetry algebras of conformal field theories, at least when the number of irreducible representations of the symmetry algebra is finite [1, 2, 3, 4]. Over the last few years a class of conformal field theories, called logarithmic conformal field theories, has been the subject of a lot of research. Logarithmic conformal field theories appear in the description of critical points of a number of interesting physical systems. Examples are polymers, spin chains, percolation and sand-pile models [5, 6, 7, 8, 9, 10]. Logarithmic conformal field theories generalise the conformal field theories most commonly considered, by allowing the singularities encountered in correlation functions, when two field insertion approach each other, to be logarithmic divergencies rather than just poles [11]. Two necessary consequences of the logarithmic divergencies are that L_0 the generator of scale transformations is no longer diagonalisable and that the representation theory of the symmetry algebra is non-semi-simple. The non-semi-simplicity in particular has made it quite challenging to find rigorous mathematical classifications of the representations of symmetry algebras associated to logarithmic conformal field theories.

Arguably the two best understood families of vertex operator algebras associated to logarithmic conformal field theories are the $\mathcal{W}_p$- and the $\mathcal{W}_{p_+,p_-}$-series, where $p \geq 2$ and $p_\pm \geq 2$, with p_+, p_- coprime respectively. The $\mathcal{W}_p$-series is by now quite well understood [12, 13, 14, 15, 16, 17, 18, 19, 20]. In particular the representation category $\mathcal{W}_p$-mod was completely classified in [21] as a $\mathbb{C}$-linear abelian category. On the other hand, the representation theory of $\mathcal{W}_{p_+,p_-}$-series is not so well understood yet. There are well supported conjectures for lists of all irreducible and all projective representations, but there is still a lot of work to be done [16, 22, 23, 24, 25, 26].

The purpose of this paper is to analyse the monoidal structure induced on $\mathcal{W}_p$-mod, the representation category of the $\mathcal{W}_p$ triplet algebra, by the fusion product of $\mathcal{W}_p$-representations, by making heavy use of all that is known of $\mathcal{W}_p$-mod as an abelian category. The main results can be summarised as follows:

- Section 2.3 in which a systematic description is given on how to define and compute fusion products in a non-semi-simple setting.
- Theorem 39 which states that $\mathcal{W}_p$-mod has the structure of a rigid braided monoidal category.
- Theorem 43 which gives explicit formulae for the fusion products of all

simple and all projective $\mathcal{W}_p$-modules.

The formulae in theorem 43 were first conjectured in [17].

As a final comment we would like to note that the $\mathcal{W}_p$-series is closely related to quantum groups at roots of unity [15, 16]. Indeed it was shown in [21] that the representation categories of $\mathcal{W}_p$ and its corresponding quantum group are equivalent as abelian categories. It was shown in [27] that the standard quantum group tensor product cannot coincide with the $\mathcal{W}_p$ fusion product, because it is not braided.

The paper is organised as follows. In section 2 we introduce our notation for vertex operator algebras, give a short definition of monoidal categories and explain how to define and compute the fusion product in the representation category V-mod of an arbitrary c_2-cofinite vertex operator algebra V. In section 3 we introduce the $\mathcal{W}_p$ triplet algebra and its representation category $\mathcal{W}_p$-mod. Sections 2 and 3 are introductory and serve to familiarise the reader with fusion products, the $\mathcal{W}_p$-algebra, representations of the $\mathcal{W}_p$-algebra and to introduce our notation.

Section 4 contains a detailed analysis of two simple $\mathcal{W}_p$-modules which we call X_1^- and X_2^+. We calculate their fusion products with all simple $\mathcal{W}_p$-modules and prove that they are rigid objects in $\mathcal{W}_p$-mod. This section relies heavily on the notions discussed in sections 2.2 and 2.3. In section 5 we prove this paper's two main theorems 39 and 43 by exploiting the rigidity of X_1^- and X_2^+ to compute the fusion product of X_1^- and X_2^+ with all projective modules. This allows to prove the rigidity of $\mathcal{W}_p$-mod and to compute the fusion product on the set of all simple and all projective modules as well as the induced product on the Grothendieck group $K(\mathcal{W}_p)$.

Acknowledgements

This work was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. The first author is supported by Kakenhi-grant number 22540010, while the second author is supported by the SNSF scholarship for prospective researchers and thanks the University of Tokyo, IPMU for its hospitality. The second author would also like to thank Matthias Gaberdiel and Ingo Runkel for helpful discussions and whose conjectured fusion rules inspired this paper.

2 The definition of fusion tensor products

2.1 Vertex operator algebras and current algebras

In this section we will briefly summarise our definitions and notation for vertex operator algebras. For a more detailed discussion see [28, 29].

Definition 1. *A tuple (V, Ω, T, Y) – consisting of a vector space V, two distinguished non-trivial elements Ω, T and a map Y – is called a vertex operator algebra (VOA for short), if it satisfies the following conditions.*

1. *The vectors pace V, is a complex non-negative integer graded vector space*

$$V = \bigoplus_{n=0}^{\infty} V[n] \,, \tag{2.1}$$

such that $V[0] = \mathbb{C}\Omega$, $\dim V[h] < \infty \ \forall h \geq 0$ and $T \in V[2]$.

2. *The map Y is a $\mathbb{C}$-linear map*

$$\begin{aligned} Y : V &\to End_{\mathbb{C}}[[z, z^{-1}]] \\ A &\mapsto Y(A; z) = A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-h} \,, \end{aligned} \tag{2.2}$$

for $A \in V[h_A]$, such that

$$Y(A; z)\Omega - A \ \in V[[z]]z \tag{2.3}$$

and

$$Y(\Omega; z) = \mathrm{id}_V \,. \tag{2.4}$$

3. *If we set*

$$Y(T; z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \,, \tag{2.5}$$

then the modes L_n satisfy the commutation relations of the Virasoro algebra with fixed central charge $c = c_V$

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c_V}{12}(m^3 - m)\delta_{m+n,0} \,. \tag{2.6}$$

4. *The zero mode of the Virasoro algebra L_0 acts semi-simply on V and*

$$V[h] = \{A \in V | L_0 A = hA\} \,. \tag{2.7}$$

5. *For any element $A \in V$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}z} Y(A; z) = Y(L_{-1}A; z) \,. \tag{2.8}$$

6. *For any elements $A, B \in V$, $Y(A; z)$ and $Y(B; z)$ are local with respect to each other*[1] *and satisfy the operator product expansion*

$$Y(A; z)Y(B; w) = \sum_{n \in \mathbb{Z}} Y(A_n B; w)(z - w)^{-n-h} \,, \tag{2.9}$$

where we have assumed that $A \in V[h]$ is a homogeneous element.

When there is no chance of confusion we will refer to a VOA just by its graded vector space V.

Remark 2. *By the above definition it follows that for $A \in V[h]$ the Virasoro generators L_0 and L_{-1} satisfy*

$$\begin{aligned} [L_{-1}, A_n] &= -(n + h - 1)A_{n-1} \\ [L_0, A_n] &= -nA_n \,. \end{aligned} \tag{2.10}$$

Next we introduce a finiteness condition due to Zhu [3].

Definition 3. *A VOA V is said to be c_2-cofinite if*

$$\dim_{\mathbb{C}} V/c_2(V) < \infty \,, \tag{2.11}$$

where $c_2(V)$ the subspace of V defined by

$$c_2(V) = \mathrm{span}\{A_n B |\ A \in V[h],\ B \in V,\ n \leq -(h + 1)\} \,. \tag{2.12}$$

In this paper will will only be considering c_2-cofinite VOAs. Among many other helpful properties c_2-cofiniteness guarantees that the V has only a finite number of irreducible representations.

The algebra of the modes of a VOA V can be understood by using the concepts of current the Lie algebra $\mathfrak{g}(V)$ and the current algebra $\mathcal{U}(V)$ of V.

[1] Locality is essentially equivalent to the the vertex operators $Y(A; z)$ and $Y(B; w)$ commuting. For a more precise definition we refer to [29].

The representation theory of a VOA V can be defined by left $\mathcal{U}(V)$-modules with some extra properties, which we will explain in sequel.

Let V be a VOA. Consider the spaces $V^{(1)} = \bigoplus_{h\geq 0} V[h]\otimes\mathbb{C}[[\xi,\xi^{-1}]](d\xi)^{1-h}$ and $V^{(0)} = \bigoplus_{h\geq 0} V[h]\otimes\mathbb{C}[[\xi,\xi^{-1}]](d\xi)^{-h}$ as well as the $\mathbb{C}$-linear map $\nabla : V^{(0)} \to V^{(1)}$ defined by

$$\nabla(v\otimes f(\xi)(\mathrm{d}\xi)^{-h}) = L_{-1}v\otimes f(\xi)(\mathrm{d}\xi)^{-h} + v\otimes \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}(\mathrm{d}\xi)^{1-h}. \tag{2.13}$$

Definition 4. *Define*

$$\mathfrak{g}(V) = V^{(1)}/\nabla V^{(0)}\,. \tag{2.14}$$

Then $\mathfrak{g}(V)$ has the structure of a Lie algebra given by

$$[v\otimes f(\mathrm{d}\xi)^{1-h_1}, w\otimes g(\mathrm{d}\xi)^{1-h_2}] = \sum_{m=0}^{h_1+h_2-1}\frac{1}{m!}v_{m-h_1+1}w\otimes\frac{d^m f}{\mathrm{d}\xi^m}g(\mathrm{d}\xi)^{m+2-h_1-h_2}. \tag{2.15}$$

For each element $A\in V[h]$ we denote

$$A_n = [A\otimes\xi^{n+h-1}(\mathrm{d}\xi)^{1-h}]\in\mathfrak{g}(V)\,. \tag{2.16}$$

and

$$L_n = [T\otimes\xi^{n-1}(\mathrm{d}\xi)^{1-n}]\in\mathfrak{g}(V)\,. \tag{2.17}$$

Then $\{L_n\}$ generate the Virasoro Lie algebra as a subalgebra in $\mathfrak{g}(V)$ and the Lie algebra $\mathfrak{g}(V)$ has a $\mathbb{Z}$-graded Lie algebra structure by

$$\mathfrak{g}(V)[d] = \{g\in\mathfrak{g}(V)\,|\,[L_0,g] = dg\}\ . \tag{2.18}$$

By definition A_n are the elements in $\mathfrak{g}(V)[-n]$.

We now define the current algebra $\mathcal{U}(V)$ of V. We first consider the universal enveloping algebra $U(\mathfrak{g}(V))$ of $\mathfrak{g}(V)$. Then $U(\mathfrak{g}(V))$ has the structure of a $\mathbb{Z}$-graded algebra by decomposition into L_0 eigenspaces

$$U(\mathfrak{g}(V))[d] = \{P\in U(\mathfrak{g}(V))\,|\,[L_0,P] = dP\}\ . \tag{2.19}$$

Consider the degreewise completion of $U(\mathfrak{g}(V))$

$$\overline{U(\mathfrak{g}(V))} = \sum_{d\in\mathbb{Z}}\overline{U(\mathfrak{g}(V))[d]} \tag{2.20}$$

and consider the degreewise closed two sided ideal

$$\overline{\mathcal{I}} = \sum_{d\in\mathbb{Z}} \overline{\mathcal{I}[d]} \tag{2.21}$$

of $\overline{U(\mathfrak{g}(V))}$ generated by the Borcherds relations which arise from the operator product expansion

$$Y(A;z)Y(B;w) = \sum_{n\in\mathbb{Z}} Y(A_nB;w)(z-w)^{-n-h}\,. \tag{2.22}$$

Definition 5. *The current algebra $\mathcal{U}(V)$ of V is the topological $\mathbb{Z}$-graded algebra*

$$\mathcal{U}(V) = \sum_d \mathcal{U}(V)[d] = \sum_d \overline{U(\mathfrak{g}(V))[d]}/\overline{\mathcal{I}[d]}\,. \tag{2.23}$$

The following proposition is very important in this paper, because it allows us to switch back and forth between calculations in the current Lie algebra and the current algebra.

Proposition 6. *The canonical $\mathbb{Z}$-graded Lie algebra map*

$$\mathfrak{g}(V) = \sum_d \mathfrak{g}(V)[d] \to \mathcal{U}(V) = \sum_d \mathcal{U}(V)[d] \tag{2.24}$$

has a dense image.

We define filtrations of $\mathcal{U}(V)$

$$\begin{aligned} \mathcal{F}_k(\mathcal{U}) &= \bigoplus_{d\geq k} \mathcal{U}(V)[d]\,, \\ \mathcal{F}^k(\mathcal{U}) &= \bigoplus_{d\leq k} \mathcal{U}(V)[d]\,, \end{aligned} \tag{2.25}$$

satisfying

$$\begin{aligned} \cdots \mathcal{F}_{k-1}(\mathcal{U}) \supset \mathcal{F}_k(\mathcal{U}) \supset \mathcal{F}_{k+1}(\mathcal{U}) \cdots\,, \\ \cdots \mathcal{F}^{k-1}(\mathcal{U}) \subset \mathcal{F}^k(\mathcal{U}) \subset \mathcal{F}^{k+1}(\mathcal{U}) \cdots\,. \end{aligned} \tag{2.26}$$

The category V-mod of representations of the VOA V is defined using left $\mathcal{U}(V)$-modules.

Definition 7. *A representation M of the VOA V, also called a V-module, is a left $\mathcal{U}(V)$-module containing a finite dimensional $\mathcal{F}_0(\mathcal{U})$ invariant subspace M_0 such that $\mathcal{U}(V)\cdot M_0 = M$. We denote the abelian category of V-modules by V-mod.*

For any V-module M we can define the $\mathbb{C}$-linear map

$$\begin{aligned} Y^M : V &\to \mathrm{End}_{\mathbb{C}}(M)[[z, z^{-1}]] \\ A \mapsto Y^M(A; z) &= \sum_{n\in\mathbb{Z}} \rho_M(A_n) z^{-n-h} \,, \end{aligned} \tag{2.27}$$

where ρ_M is a representation of the current Lie algebra $\mathfrak{g}(V)$ on M and we have assumed that $A \in V[h]$. Then we have the following formulae

$$\begin{aligned} Y^M(\Omega; z) &= \mathrm{id}_M \\ \frac{\mathrm{d}}{\mathrm{d}z} Y^M(A; z) &= Y^M(L_{-1}A; z) \end{aligned} \tag{2.28}$$

and for all $A \in V[h]$ and $B \in V$, the operators $Y^M(A; z)$ and $Y^M(B; w)$ are local with respect to each other and satisfy the operator product expansion

$$Y^M(A; z) Y^M(B; w) = \sum_{n\in\mathbb{Z}} Y^M(A_n B; w)(z - w)^{-n-h} \,. \tag{2.29}$$

Proposition 8. *1. The number of isomorphism classes of simple V-modules is finite.*

2. For any V-module M all Jordan-Hölder series

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = \{0\} \,, \tag{2.30}$$

such that M_i/M_{i+1} are simple V-modules, are finite.

3. Each V-module M decomposes into a direct sum of finite dimensional generalised L_0-eigenspaces $M[h]$

$$\begin{aligned} M &= \bigoplus_{h\in H} M[h] \,, \\ M[h] &= \{u \in M, (L_0 - h)^n u = 0, \text{ for some } n \geq 1\} \,, \end{aligned} \tag{2.31}$$

where H is the set of all weights, a discrete subset of $\mathbb{C}$ generated from the finite set H_0 of highest weights by adding all non-negative integers.

4. V-mod admits a contravariant endofunctor called the contragredient or contragredient dual.

Definition 9. *The current algebra $\mathcal{U}(V)$ admits the algebra anti-automorphism*

$$\sigma : \mathcal{U}(V)[d] \to \mathcal{U}(V)[-d] \,, \tag{2.32}$$

which for $A \in V[h]$ is defined by $\sigma(A_n) = (-1)^h A_{-n}$.

Definition 10. *For any V-module M the contragredient V-module M^* is defined by*

$$M^* = \sum_{h \in H} \mathrm{Hom}_{\mathbb{C}}(M[h], \mathbb{C}) \tag{2.33}$$

as a vector space. While the action of the current algebra is given by

$$\langle P\varphi, u \rangle = \langle \varphi, \sigma(P)u \rangle \,, \tag{2.34}$$

for $\varphi \in M^*, u \in M, P \in \mathcal{U}(V)$. The contragredient of the contragredient is again the original module $M = (M^*)^*$.

To better analyse V-mod we define finite dimensional $\mathbb{C}$-algebras $A_k(V)$, $k = 0, 1, \ldots$ in the following way. Consider the degreewise closed right $\mathcal{U}(V)$-ideal

$$\mathcal{I}_k = \overline{\mathcal{F}_{k+1}(\mathcal{U}) \cdot \mathcal{U}(V)} \subset \mathcal{U}(V) \tag{2.35}$$

and consider the two sided $\mathcal{F}_0(\mathcal{U})$ ideal

$$I_k = \mathcal{I}_k \cap \mathcal{F}_0(\mathcal{U}) \,. \tag{2.36}$$

Taking the quotient of $\mathcal{F}_0(\mathcal{U})$ by I_k we define a series of $\mathbb{C}$-algebras

$$A_k(V) = \frac{\mathcal{F}_0(\mathcal{U})}{I_k} \,. \tag{2.37}$$

Proposition 11. *For $k = 0, 1, 2, \ldots$ the $\mathbb{C}$-algebras $A_k(V)$ are all finite dimensional.*

We denote by $A_k(V)$-mod, the abelian category of finite dimensional left $A_k(V)$-modules. We define the covariant functor

$$\begin{aligned} A_k : V\text{-mod} &\to A_k(V)\text{-mod} \\ M &\to \mathcal{A}_k(M) = \frac{M}{I_k(M)} \,. \end{aligned} \tag{2.38}$$

Proposition 12.

1. *A V-module M is the zero module if and only if $\mathcal{A}_k(M)$ is the zero module.*
2. *If M is a simple V-module, then $\mathcal{A}_k(M)$ is simple and the set of isomorphism classes of simple V-modules are in one to one correspondence with the isomorphism classes of simple $A_k(V)$-modules.*

The $k = 0$ case is most important to our work at hand, we will refer to $A_0(V)$ as the zero mode algebra of the VOA V.

To define the notion of the fusion tensor product on V-mod, we prepare some additional concepts and definitions. As a first step we define left and right completions of the current algebra $\mathcal{U}(V)$. For each $k \in \mathbb{Z}$ define

$$\begin{aligned} \mathcal{F}^k(\mathcal{U}^L) &= \prod_{d \leq k} \mathcal{U}(V)[d] & \mathcal{F}_k(\mathcal{U}^R) &= \prod_{d \geq k} \mathcal{U}(V)[d] \\ \mathcal{U}^L &= \bigcup_k \mathcal{F}^k(\mathcal{U}^L) & \mathcal{U}^R &= \bigcup_k \mathcal{F}_k(\mathcal{U}^R)\,. \end{aligned} \tag{2.39}$$

Then $\mathcal{U}^L$ and $\mathcal{U}^R$ are topological $\mathbb{C}$-algebras with topologies defined by the filtrations $\mathcal{F}^k(\mathcal{U}^L)$ and $\mathcal{F}_k(\mathcal{U}^R)$ and the canonical inclusions

$$\begin{aligned} \mathcal{U}(V) &\to \mathcal{U}^L \\ \mathcal{U}(V) &\to \mathcal{U}^R \end{aligned} \tag{2.40}$$

have dense images.

For any object M of V-mod, we define its closure by

$$\overline{M} = \varprojlim_k M/\mathcal{F}_k(\mathcal{U})(M)\,. \tag{2.41}$$

Then there is a continuous action of $\mathcal{U}^R$ on $\overline{M}$. Note that M already has the structure of a $\mathcal{U}^L$-module. By the properties of projective limits $\overline{M}$ is equipped with a complete Hausdorff linear topology. Then the action of the current algebra $\mathcal{U}(V)$ is uniquely extended to an action of $\mathcal{U}^R$ on $\overline{M}$. The two spaces M and $\overline{M}$ share the same generalised L_0-eigenspaces $M[h] = \overline{M}[h]$

$$\begin{aligned} M[h] &= \{m \in M |\ (L_0 - h)^n m = 0,\ \text{for some } n \geq 1\} \\ \overline{M}[h] &= \{m \in \overline{M} |\ (L_0 - h)^n m = 0,\ \text{for some } n \geq 1\}\,, \end{aligned} \tag{2.42}$$

but unlike M, $\overline{M}$ also contains infinite sums of generalised L_0-eigenvectors

$$\overline{M} = \prod_{h \in H} M[h]\,. \tag{2.43}$$

The image of the canonical inclusion $M \to \overline{M}$ is dense.

For any V-module M there is a canonical surjective linear map

$$\mathcal{A}_{k+1}(M) \to \mathcal{A}_k(M) \tag{2.44}$$

and the projective limit

$$\varprojlim_k \mathcal{A}_k(M) \tag{2.45}$$

has a unique continuous $\mathcal{U}^R$ action. Indeed

$$\overline{M} = \varprojlim_k \mathcal{A}_k(M) \tag{2.46}$$

as a continuous $\mathcal{U}^R$-module. Also we have the canonical homomorphisms

$$\begin{aligned} \mathfrak{g}^L(V) &\to \mathcal{U}^L \\ \mathfrak{g}^R(V) &\to \mathcal{U}^R\,, \end{aligned} \tag{2.47}$$

which both have dense images.

For each $k \in \mathbb{Z}$ we define

$$\mathfrak{g}_k(V) = \sum_{d \geq k} \mathfrak{g}(V)[d]\,. \tag{2.48}$$

Then the canonical map

$$\mathfrak{g}_k(V) \to \mathcal{F}_k(\mathcal{U}) \to \mathcal{F}_k(\mathcal{U}^R) \tag{2.49}$$

has dense image. So we have the $\mathbb{C}$-linear isomorphism

$$\frac{M}{\mathfrak{g}_{k+1}(V)(M)} \to \frac{M}{I_k(M)} = \mathcal{A}_k(M)\,. \tag{2.50}$$

Therefore we have

$$\overline{M} = \varprojlim_k \frac{M}{\mathfrak{g}_{k+1}(V)(M)} \tag{2.51}$$

For later use we define for each V-module M the quotient

$$\frac{M}{c_1(M)} = \frac{M}{\mathrm{span}\{A_{-n}m \,|\, m \in M,\ A \in V[h],\ n \geq h > 0\}}\,. \tag{2.52}$$

Then $M/c_1(M)$ is a finite dimensional complex vector space and there exists a canonical surjective linear map

$$\frac{M}{c_1(M)} \to \mathcal{A}_0(M)\,. \tag{2.53}$$

Also for later use we introduce the following notation.

Definition 13. *Let M be a V-module, then we define L_0-graded subspaces M^0 and M^s, such that*

$$\begin{aligned} M &= M^0 \oplus \mathfrak{g}_1(V) \cdot M \,, \\ M &= M^s \oplus c_1(M) \,, \end{aligned} \tag{2.54}$$

respectively called the zero mode *and the* special subspace*, such that the canonical maps*

$$\begin{aligned} M^0 &\to \mathcal{A}_0(M) \,, \\ M^s &\to \frac{M}{c_1(M)} \,, \end{aligned} \tag{2.55}$$

are $\mathbb{C}$-linear isomorphisms.

Remark 14. *The subspaces M^0 and M^s are not uniquely defined. We will fix specific choices of subspaces in a later section.*

2.2 General properties of braided monoidal categories

In this section we introduce the concepts of monoidal categories, their rigidness and some general properties. We mainly follow the appendices of the seminal papers due to Kazhdan and Lusztig [30, Appendix A] as well as the standard reference for monoidal categories [31]. We will only be considering monoidal categories that are also $\mathbb{C}$-linear and abelian and we assume that the reader is familiar with basic notions of abelian categories such as exact sequences, projective modules, injective modules, etc.

Definition 15. *A monoidal category is a tuple $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ – just $(\mathcal{C}, \otimes, \mathbf{1})$ for short – where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the tensor product bifunctor, $\mathbf{1} \in \mathcal{C}$ is the tensor unit, $\alpha_{L,M,N} : L \otimes (M \otimes N) \xrightarrow{\sim} (L \otimes M) \otimes N$ is the associator, $\lambda_M : \mathbf{1} \otimes M \to M$ is the left unit isomorphism, and $\rho_M : M \otimes \mathbf{1} \to M$ is the right unit isomorphism. These data are subject to conditions, in particular α satisfies the pentagon axiom and λ, ρ, α obey the triangle axiom.*

Definition 16. *We say that an object M is weakly rigid if the contravariant functor*

$$F_M(-) = \mathrm{Hom}(- \otimes M, \mathbf{1}) \tag{2.56}$$

is representable, i.e. for all objects N there exits an object $M^\vee$ called the tensor dual[2] *such that*

$$\mathrm{Hom}(N \otimes M, 1) \cong \mathrm{Hom}(N, M^\vee)\,. \tag{2.57}$$

Therefore if M is weakly rigid there exits a morphism

$$e_M : M^\vee \otimes M \to \mathbf{1} \tag{2.58}$$

that is isomorphic to $\mathrm{id}_{M^\vee} \in \mathrm{Hom}(M^\vee, M^\vee)$ by the equivalence (2.57). A monoidal category is called weakly rigid if all its object are weakly rigid.

Definition 17. *An object M is said to be rigid of it is weakly rigid and there exits a morphism*

$$i_M : \mathbf{1} \to M \otimes M^\vee\,, \tag{2.59}$$

such that

$$\begin{aligned} \mathrm{id}_M &= \rho_M \circ (\mathrm{id}_M \otimes e_M) \circ \alpha^{-1}_{M,M^\vee,M} \circ (i_M \otimes \mathrm{id}_M) \circ \lambda^{-1}_M \\ \mathrm{id}_{M^\vee} &= \lambda_{M^\vee} \circ (e_M \otimes \mathrm{id}_{M^\vee}) \circ \alpha_{M,M^\vee,M} \circ (\mathrm{id}_{M^\vee} \otimes i_M) \circ \rho^{-1}_{M^\vee}\,. \end{aligned} \tag{2.60}$$

Definition 18. *A braiding b on a monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is a natural transformation between the functors $\otimes$ and $\otimes \circ P$, where $P : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the permutation $(M, N) \to (N, M)$. For M, N in $\mathcal{C}$, b defines a morphism $b_{M,N} \in \mathrm{Hom}(M \otimes N, N \otimes M)$ satisfying*

1. *All $b_{M,N}$ are isomorphisms,*
2. *For any L, M, N in $\mathcal{C}$ we have*

$$\begin{aligned} b_{L\otimes M,N} &= \alpha^{-1}_{N,L,M} \circ (b_{L,N} \circ \mathrm{id}_M) \circ \alpha_{L,N,M} \circ (\mathrm{id}_L \otimes b_{M,N}) \otimes \alpha^{-1}_{L,M,N} \\ b_{L,M\otimes N} &= \alpha_{M,N,L} \circ (\mathrm{id}_M \otimes b_{L,N}) \circ \alpha^{-1}_{M,L,N} \circ (b_{L,M} \circ \mathrm{id}_N) \otimes \alpha_{L,M,N}\,, \end{aligned} \tag{2.61}$$

3. *For all M in $\mathcal{C}$*

$$b_{M,\mathbf{1}} = b_{\mathbf{1},M} = \mathrm{id}_M\,. \tag{2.62}$$

A monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ satisfies a number of nice properties

[2]Strictly speaking $M^\vee$ is called the right dual. There is a similar notion of a left dual. However if the tensor category is braided then these notions are related. We will therefore only be considering right duals.

Proposition 19. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal category, then*

1. *For any rigid object M in $\mathcal{C}$ the functors*

$$M \otimes -, \ - \otimes M : \mathcal{C} \to \mathcal{C} \tag{2.63}$$

   are exact.

2. *For a rigid object M with dual $M^\vee$ and arbitrary objects N, L we have the isomorphism*

$$\mathrm{Hom}(N, M \dot{\otimes} L) \cong \mathrm{Hom}(M^\vee \otimes N, L) \tag{2.64}$$

3. *Let M in $\mathcal{C}$ be rigid with dual $M^\vee$. Then for any projective object P $M^\vee \otimes P$ is projective. For any injective object I, and $M \otimes I$ is injective in $\mathcal{C}$ if $M^\vee$ is also rigid.*

4. *Assume that*

   (a) *The abelian category $\mathcal{C}$ has enough projective and injective objects.*

   (b) *All projective objects are injective and all injective objects are projective.*

   (c) *All projective objects are rigid.*

   Then if

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{2.65}$$

   is an exact sequence in $\mathcal{C}$ such that two of L, M, N are rigid, then the third object is also rigid.

5. *If M, N in $\mathcal{C}$ are rigid objects, then $M \otimes N$ is also rigid and its dual $(M \otimes N)^\vee$ is given by*

$$(M \otimes N)^\vee = N^\vee \otimes M^\vee \,. \tag{2.66}$$

2.3 Fusion tensor products and their properties

Fusion plays a central role in analysing conformal field theories and is indeed the central theme of this paper. Fusion describes the short distance expansion of two fields on the level of the representations. The fusion product $M \dot{\otimes} N$ of two V-modules M and N is the smallest V-module in which all the fields appearing in the short distance expansions – of fields transforming in M with fields transforming in N – transform.

There is a wealth of literature on fusion tensor products in both mathematics and physics. In the case of rational conformal field theory the representation category is semi-simple and the theory of fusion tensor products is well established [32, 33]. If the conformal field theory is logarithmic, the representation category is not semi-simple. Fortunately there are computational methods in physics one can fall back on for defining fusion tensor products without assuming semi-simplicity [34, 35]. For VOAs arising from affine Lie algebras, there exists a mathematically rigorous definition of fusion tensor products due to Kazhdan and Lusztig [30] that does not rely on semi-simplicity. In this paper we will state our definition of fusion in the spirit of [30, 32, 33, 34, 35] in the case of c_2-cofinite VOAs in a mathematically rigorous way, but postpone the proof of its properties to [36]. We will be generalising the notions developed in [28].

We fix a c_2-cofinite VOA V and prepare some notation.

Definition 20. *We define the current Lie algebra on the Riemann sphere with punctures at* $0, 1, \infty$ *by*

$$\mathfrak{g}^{\mathbb{P}}(V) = \frac{\bigoplus_{n=0}^{\infty} V[n] \otimes \mathbb{C}[z, z^{-1}, (z-1)^{-1}] \mathrm{d}z^{1-h}}{\nabla(\bigoplus_{n=0}^{\infty} V[n] \otimes \mathbb{C}[z, z^{-1}, (z-1)^{-1}] \mathrm{d}z^{-h})}\,, \tag{2.67}$$

where ∇ *is defined as in* (2.13)

$$\nabla(A \otimes f(z)\mathrm{d}z^{-h}) = L_{-1}A \otimes f(z)\mathrm{d}z^{-h} + A \otimes \frac{\mathrm{d}f(z)}{\mathrm{d}z}\mathrm{d}z^{1-h}\,. \tag{2.68}$$

Then $\mathfrak{g}^{\mathbb{P}}(V)$ *has the structure of a Lie algebra given by*

$$\begin{aligned}[A \otimes f(z)\mathrm{d}z^{1-h_A}, B \otimes g(z)\mathrm{d}z^{1-h_A}] = \\ \sum_{m=0}^{\infty} \frac{1}{m!} A_{m-h_A+1}B \otimes \frac{\mathrm{d}^m f(z)}{\mathrm{d}z^m} g(z)\mathrm{d}z^{m+2-h_A-h_B}\,,\end{aligned} \tag{2.69}$$

for $A \in V[h_A]$ *and* $B \in V[h_B]$.

For $f(z) \in \mathbb{C}[z, z^{-1}, (z-1)^{-1}]$ we denote the Laurent expansions at 0, 1 and infinity by $f_0(\xi_0) \in \mathbb{C}((\xi_0))$, $f_1(\xi_1) \in \mathbb{C}((\xi_1))$ and $f_\infty(\xi_\infty) \in \mathbb{C}((\xi_\infty^{-1}))$ respectively. For example for

$$f(z) = \frac{1}{z-1} \tag{2.70}$$

the expansions and radii of convergence are given by

$$\begin{aligned} f_0(\xi_0) &= -\sum_{n\geq 0} \xi_0^n & \xi_0 &= z \quad 1 > |z| > 0 \\ f_1(\xi_1) &= \xi_1^{-1} & \xi_1 &= z-1 \quad 1 > |z-1| > 0 \\ f_\infty(\xi_\infty) &= \sum_{n\geq 0} \xi_\infty^{-(n+1)} & \xi_\infty &= z \quad |z| > 1\,. \end{aligned} \tag{2.71}$$

Then we can define Lie algebra homomorphisms

$$\begin{aligned} j_a^L &: \mathfrak{g}^{\mathbb{P}}(V) \to \mathfrak{g}^L \quad a = 0,1 \\ j_\infty^R &: \mathfrak{g}^{\mathbb{P}}(V) \to \mathfrak{g}^R\,, \end{aligned} \tag{2.72}$$

where $\mathfrak{g}^L$ and $\mathfrak{g}^R$ are the left and right completions of $\mathfrak{g}(V)$ defined in (2.47), by

$$\begin{aligned} j_a^L([A \otimes f(z)\mathrm{d}z^{1-h}]) &= [A \otimes f_a(\xi_a)\mathrm{d}\xi_a^{1-h}] \\ j_\infty^R([A \otimes f(z)\mathrm{d}z^{1-h}]) &= [A \otimes f_\infty(\xi_\infty)\mathrm{d}\xi_\infty^{1-h}]\,. \end{aligned} \tag{2.73}$$

The Lie algebra maps j_a^L and j_∞^R have dense images.

The analogue of $\mathfrak{g}_k(V)$ for the current Lie algebra on the Riemann sphere is given by

$$\mathfrak{g}_k^{\mathbb{P}}(V) = \mathrm{span}\{[A \otimes f(z)\mathrm{d}z^{1-h}] \in \mathfrak{g}^{\mathbb{P}}(V) |\ \mathrm{ord}_\infty(f(z)) \leq h-1-k\}\,, \tag{2.74}$$

where $\mathrm{ord}_\infty(f(z))$ is the order of the pole of $f(z)$ at infinity. The image of the map

$$\mathfrak{g}_k^{\mathbb{P}}(V) \xrightarrow{j_\infty^R} \mathfrak{g}_k^R \to \mathcal{F}_k(\mathcal{U}^R) \tag{2.75}$$

is dense and therefore the cannonical map

$$\frac{\mathfrak{g}^{\mathbb{P}}(V)}{\mathfrak{g}_k^{\mathbb{P}}(V)} \to \frac{\mathfrak{g}^R}{\mathfrak{g}_k^R} \tag{2.76}$$

is an isomorphism, hence

$$\varprojlim_k \frac{\mathfrak{g}^{\mathbb{P}}(V)}{\mathfrak{g}_k^{\mathbb{P}}(V)} \to \varprojlim_k \frac{\mathfrak{g}^R}{\mathfrak{g}_k^R} = \mathfrak{g}^R\,. \tag{2.77}$$

Now consider the map

$$j_{1,0} = j_1^L \otimes \mathbb{1} + \mathbb{1} \otimes j_0^L : \mathfrak{g}^{\mathbb{P}}(V) \to \mathfrak{g}^L \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{g}^L \tag{2.78}$$

Then for any two V-modules M, N, the vector space $M \otimes N$ is a left $\mathfrak{g}^{\mathbb{P}}(V)$-module by $j_{1,0}$.

Then we have the following. For each $k = 0, 1, 2, \ldots$

1. $\dim_{\mathbb{C}} M \otimes N / \mathfrak{g}_k^{\mathbb{P}}(V)(M \otimes N) < \infty$

2. The Lie algebra $\mathfrak{g}^R = \varprojlim_k \mathfrak{g}^{\mathbb{P}}(V)/\mathfrak{g}_k^{\mathbb{P}}(V)$ acts continuously on the projective limit

$$\overline{M \dot{\otimes} N} = \varprojlim_k \frac{M \otimes N}{\mathfrak{g}_k^{\mathbb{P}}(V)(M \otimes N)} \tag{2.79}$$

by $j_\infty^R : \mathfrak{g}^{\mathbb{P}}(V) \to \mathfrak{g}^R$.

Furthermore by $\mathfrak{g}^R \to \mathcal{U}^R$ the right completion $\mathcal{U}^R$ of the current algebra acts continuously on this space.

Define for each $h \in \mathbb{C}$,

$$M \dot{\otimes} N[h] = \{m \in \overline{M \dot{\otimes} N} |\ \exists n \geq 1 \text{ s.t. } (L_0 - h)^n m = 0\} . \tag{2.80}$$

Then the fusion product of M and N is given by

$$M \dot{\otimes} N = \bigoplus_{h \in \mathbb{C}} M \dot{\otimes} N[h] . \tag{2.81}$$

Proposition 21. *1. The space $M \dot{\otimes} N$ is a V-module.*

2. For each $k \geq 0$ we have the $\mathbb{C}$-linear isomorphisms

$$\mathcal{A}_k(M \dot{\otimes} N) = \frac{M \dot{\otimes} N}{I_k(M \dot{\otimes} N)} \cong \frac{M \otimes N}{\mathfrak{g}_{k+1}^{\mathbb{P}}(M \otimes N)} . \tag{2.82}$$

Most notably we have the $A_0(V)$-module isomorphism

$$\mathcal{A}_0(M \dot{\otimes} N) \cong \frac{M \otimes N}{\mathfrak{g}_1^{\mathbb{P}}(M \otimes N)} . \tag{2.83}$$

The proof of the following Theorem will be postponed to [36] a paper dedicated solely to giving a precise formulation of the fusion tensor product for c_2-cofinite VOAs on Riemann surfaces of arbitrary genus.

Theorem 22. *1. The triplet $(V\text{−mod}, \dot{\otimes}, V)$ is a braided monoidal category with unit object $V = \mathbf{1}$.*

2. For any V-module M, the contravariant functor from V-mod to $\mathbb{C}$-Vec the category of complex vector spaces

$$\begin{aligned} F_M : V\text{−mod} &\to \mathbb{C}\text{−Vec} \\ N &\mapsto F_M(N) = \mathrm{Hom}_{V-\mathrm{mod}}(N \dot{\otimes} M, V^*) , \end{aligned} \tag{2.84}$$

where V^* is the contragredient of V, is represented by M^* the contragredient of M, i.e.

$$F_M(N) \cong \mathrm{Hom}_{V-\mathrm{mod}}(N, M^*) . \tag{2.85}$$

The construction of the associators

$$\alpha_{L,M,N} : L\dot{\otimes}(M\dot{\otimes}N) \overset{\cong}{\to} (L\dot{\otimes}M)\dot{\otimes}N \tag{2.86}$$

and the braiding

$$b_{M,N} : M\dot{\otimes}N \overset{\cong}{\to} N\dot{\otimes}M \tag{2.87}$$

is highly non-trivial. To construct them one must determine a set of differential equations – called the KZ-equations – that characterise the N-point conformal blocks of the conformal field theory associated to V-mod. Some of the calculations required to prove theorem 22 have already appeared in [28]. The full proof will be given in [36].

Unfortunately the definition of $M\dot{\otimes}N$ is rather difficult to work with, because even though the image of canonical map

$$M \otimes N \to \overline{M\dot{\otimes}N} \tag{2.88}$$

is dense, it generally does not lie in $M\dot{\otimes}N$.

However for each $k \geq 0$ we can make use of the isomorphism

$$\mathcal{A}_k(M\dot{\otimes}N) \cong \frac{M \otimes N}{\mathfrak{g}^{\mathbb{P}}_{k+1}(V)(M \otimes N)} \,. \tag{2.89}$$

By analysing these quotients, we can study $M\dot{\otimes}N$ level for level. For any given element $m \otimes n \in M \otimes N$, we will denote the class it represents in $\mathcal{A}_k(M\dot{\otimes}N)$ by $[m \otimes n]$.

As we shall see in the sequel, for the purposes of this paper it will be sufficient to make statements about $\mathcal{A}_0(M\dot{\otimes}N)$ and in one case $\mathcal{A}_1(M\dot{\otimes}N)$. As a vector space $\mathfrak{g}^{\mathbb{P}}_1(V)$ is spanned by elements of the form

$$[v \otimes z^{-n+h-1}\mathrm{d}z^{1-h}], \quad v \in V[h] \tag{2.90}$$

for $n \geq 0$ and

$$[v \otimes (z-1)^{-m+h-1}\mathrm{d}z^{1-h}], \quad v \in V[h] \tag{2.91}$$

for $m \geq h$. From the expansions defined above it therefore follows that in $\mathcal{A}_0(M\dot{\otimes}N)$ for $1 \leq n \leq h-1$ we have the relations

$$\begin{aligned} &j_{1,0}([v \otimes z^{-n+h-1}\mathrm{d}z^{1-h}]) = \\ &\qquad \sum_{k=0}^{h-1-n} \binom{h-1-n}{k} v_{k-(h-1)} \otimes \mathbb{1} + \mathbb{1} \otimes v_{-n} = 0 \,, \end{aligned} \tag{2.92}$$

and for $m \geq h$

$$
\begin{aligned}
&j_{1,0}([v \otimes z^{-m+h-1}\mathrm{d}z^{1-h}]) = \\
&\qquad \sum_{k\geq 0} \binom{m-h+k}{m-h} (-1)^k v_{k-(h-1)} \otimes \mathbb{1} + \mathbb{1} \otimes v_{-m} = 0\,, \\
&j_{1,0}([v \otimes (z-1)^{-m+h-1}\mathrm{d}z^{1-h}]) = \\
&\qquad v_{-m} \otimes \mathbb{1} + \sum_{k\geq 0} \binom{m-h+k}{m-h} (-1)^{h-1-m} \mathbb{1} \otimes v_{k-(h-1)} = 0\,.
\end{aligned}
\tag{2.93}
$$

The action of the zero modes is given by

$$
\begin{aligned}
j_{1,0}(v_0) &= j_{1,0}([v \otimes z^{h-1}\mathrm{d}z^{1-h}]) \\
&= \sum_{k=0}^{h-1} \binom{h-1}{k} v_{k-(h-1)} \otimes \mathbb{1} + \mathbb{1} \otimes v_0\,.
\end{aligned}
\tag{2.94}
$$

For the generators of the Virasoro algebra this means

$$
\begin{aligned}
&L_{-1} \otimes \mathbb{1} \simeq -\mathbb{1} \otimes L_{-1} \\
&L_{-n} \otimes \mathbb{1} \simeq \sum_{j=0}^{\infty} \binom{n-2+j}{n-2} (-1)^n \mathbb{1} \otimes L_{j-1} \\
&\mathbb{1} \otimes L_{-n} \simeq -\sum_{j=0}^{\infty} \binom{n-2+j}{n-2} (-1)^j L_{j-1} \otimes \mathbb{1}\,,
\end{aligned}
\tag{2.95}
$$

for $n \geq 2$ and

$$
j_{1,0}(L_0) = L_{-1} \otimes \mathbb{1} + L_0 \otimes \mathbb{1} + \mathbb{1} \otimes L_0\,. \tag{2.96}
$$

To aid us in computing $\mathcal{A}_0(M\dot{\otimes}N)$, we make use of the special and zero mode subspaces in definition 13 to state the following proposition due to Nahm [34].

Proposition 23. *Let M and N be V-modules. Then the canonical $\mathbb{C}$-linear map*

$$
M^s \otimes N^0 \to \mathcal{A}_0(M\dot{\otimes}N) \to 0\,. \tag{2.97}
$$

is a surjective.

Proof. Let $m \otimes n \in M \otimes N$, $A \in V[h_A]$, $B \in V[h_B]$, $k \geq h_A$ and $\ell > 0$. We introduce two kinds manipulations by using the formulae in (2.93)

1. Moving modes to the right

$$[A_{-k}m \otimes n] = -\sum_{j\geq 0} \binom{k-h_A+j}{k-h_A} [m \otimes A_{j-(h_A-1)}n] \,. \tag{2.98}$$

Note that A_{-k} is replaced by modes with a mode number that is greater than $-k$, *i.e.* the grading is lowered.

2. Moving modes to the left

$$[m \otimes B_{-\ell}n] = -\sum_{j\geq 0} \binom{h_B-1-\ell}{j} [B_{j-(h_B-1)}m \otimes n] \,. \tag{2.99}$$

Note that $B_{-\ell}$ is replaced by modes with a mode number that is greater than or equal to $-\ell$, *i.e.* the grading is lowered or stays the same.

Since the image of the canonical map $\mathfrak{g}_k(V) \to \mathcal{F}_1(V)$ is dense and $\mathcal{F}_1(V)(M) = I_0(M)$, any element n in N is represented by $n = x \cdot n_0$ for some $x \in \mathfrak{g}_1(V)$ and $n_0 \in N^0$. Then by using formula (2.99), the class $[m \otimes n]$ can be represented by $m' \otimes n_0$ for some $m' \in M$. Consider $m \in c_1(M)$ and $n_0 \in N^0$, we can assume that m is homogeneous such that $m \in c_1(M)[h]$ for some h. By definition we can assume that m has the form

$$m = A_{-k}m_0 \tag{2.100}$$

for some $m_0 \in M^s$ and $A \in V[h_A],\ k \geq h_A$. Then by using formula (2.98)

$$[m \otimes n_0] = -\sum_{j\geq 0} \binom{k-h_A+j}{k-h_A} [m_0 \otimes A_{j-(h_A-1)}n_0] \,. \tag{2.101}$$

For each summand we use again the fact that the class $[m_0 \otimes A_{j-(h_A-1)}n_0]$ can be represented by an element $m_0' \otimes n_0' \in M \otimes N^0$. If we decompose m_0' into homogeneous summands, then the weights of the individual summands will all be less then the original weight h of m. Because the weights are bounded from below, a finite number of applications of the formulae (2.98) and (2.99) will yield a representative in $M^s \otimes N^0$ for any class $[m \otimes n]$. □

Before we end this section on the fusion product we consider the relation between the fusion product $\dot{\otimes}_V$ of a VOA V and the fusion product $\dot{\otimes}_{V'}$ of a subVOA $V' \subset V$.

Proposition 24. *Let V' be a c_2-cofinite subVOA of the VOA V. Let M and N be V-modules, then they are also V'-modules and there exits a surjective V'-module map*

$$M \dot{\otimes}_{V'} N \to M \dot{\otimes}_V N \,. \tag{2.102}$$

Proof. Since

$$\mathfrak{g}_k(V') \subset \mathfrak{g}_k(V) \tag{2.103}$$

there is a canonical surjection of $A_k(V')$-modules

$$\mathcal{A}_k(M\dot{\otimes}_{V'}N) = \frac{M \otimes N}{\mathfrak{g}^{\mathbb{P}}_{k+1}(V')} \to \frac{M \otimes N}{\mathfrak{g}^{\mathbb{P}}_{k+1}(V)} = \mathcal{A}_k(M\dot{\otimes}_V N)\,. \tag{2.104}$$

Note that for sufficiently large k, a given generalised L_0-eigenspace is stable, *i.e.*

$$\mathcal{A}_k(M\dot{\otimes}_V N)[h] = \mathcal{A}_{k+1}(M\dot{\otimes}_V N)[h] = (M\dot{\otimes}_V N)[h],\ k >> 0\,. \tag{2.105}$$

Therefore the surjection (2.104) implies a surjection

$$M\dot{\otimes}_{V'}N[h] \to M\dot{\otimes}_V N[h] \tag{2.106}$$

between generalised L_0-eigenspaces. This can be repeated for all values of h and therefore there exists a surjective V'-module map

$$M\dot{\otimes}_{V'}N \to M\dot{\otimes}_V N\,. \tag{2.107}$$

□

Proposition 25. *Let M be a V-module, then the covariant functors $M\dot{\otimes}-$ and $-\dot{\otimes}M$ are right exact.*

Proof. We prove the proposition for $M\dot{\otimes}-$. The proof for $-\dot{\otimes}M$ follows analogously.

Let $A, B, C \in V$-mod satisfy the exact sequence

$$0 \to A \to B \to C \to 0\,. \tag{2.108}$$

Then the sequence

$$M\dot{\otimes}A \to M\dot{\otimes}B \to M\dot{\otimes}C \to 0\,. \tag{2.109}$$

is exact if the restriction to the generalised L_0-eigenspaces

$$M\dot{\otimes}A[h] \to M\dot{\otimes}B[h] \to M\dot{\otimes}C[h] \to 0 \tag{2.110}$$

is exact. Because for sufficiently large k the generalised L_0-eigenspaces for fixed generalised eigenvalue h are stable under taking $\mathcal{A}_k$ quotients, we consider the sequence

$$\frac{M \otimes A}{\mathfrak{g}^{\mathbb{P}}_k(V)(M \otimes A)}[h] \to \frac{M \otimes B}{\mathfrak{g}^{\mathbb{P}}_k(V)(M \otimes B)}[h] \to \frac{M \otimes C}{\mathfrak{g}^{\mathbb{P}}_k(V)(M \otimes C)}[h] \to 0\,, \tag{2.111}$$

which is clearly exact. □

2.4 Algebra morphisms between products of modules

We have defined the fusion tensor product in V-mod. Now we introduce the concept of vertex operators in a conformal field theory on $\mathbb{P}$ associated to V-mod, by extending the notions in [32].

For a V-module M we have defined the topological completion

$$\overline{M} = \prod_{h \in H} M[h] \,. \tag{2.112}$$

For any two V-modules M, N we denote by $\mathrm{Hom}^c_{\mathbb{C}}(\overline{M}, \overline{N})$, the space of continuous $\mathbb{C}$-linear maps from $\overline{M}$ to $\overline{N}$. Then we have a $\mathbb{C}$-linear isomorphism

$$\mathrm{Hom}_{\mathcal{U}(V)}(M, N) \cong \mathrm{Hom}_{\mathcal{U}^R}(\overline{M}, \overline{N}). \tag{2.113}$$

Now for the V-modules L, M, N consider the complex vector spaces

$$\mathrm{Hom}_{\mathcal{U}(V)}(M \dot{\otimes} N, L) \cong \mathrm{Hom}_{\mathcal{U}^R}(\overline{M \dot{\otimes} N}, \overline{L}) \,. \tag{2.114}$$

We know that

$$\overline{M \dot{\otimes} N} = \varprojlim_k \frac{M \otimes N}{\mathfrak{g}^{\mathbb{P}}_{k+1}(M \otimes N)} \,. \tag{2.115}$$

So there exists a injective $\mathbb{C}$-linear map

$$\mathrm{Hom}_{\mathcal{U}(V)}(M \dot{\otimes} N, L) \to \mathrm{Hom}^c_{\mathbb{C}}(\overline{M \dot{\otimes} N}, \overline{L}) \to \mathrm{Hom}_{\mathbb{C}}(M \otimes N, \overline{L}) \,. \tag{2.116}$$

The following proposition characterises the image of this map.

Proposition 26. *For $\psi \in \mathrm{Hom}_{\mathbb{C}}(M \otimes N, \overline{L})$ the necessary and sufficient condition for ψ to lie in the image of the map from $\mathrm{Hom}_{\mathcal{U}()}(M \dot{\otimes} N, L)$ is that for all $f(z) \in \mathbb{C}[z, z^{-1}, (z-1)^{-1}]$, $A \in V[h]$, $m \in M$ and $n \in N$*

$$j^R_\infty([A \otimes f(z) \mathrm{d}z^{1-h}])(\psi(m)n) \,, \tag{2.117}$$

where we denote

$$\psi : m \otimes n \mapsto \psi(m)n \,. \tag{2.118}$$

Now consider the complex algebras $\mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}]$ and $\mathcal{R} = \mathbb{C}[y, y^{-1}]$. An element $f(z, y) \in \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}]$ can be thought of as a rational function on $\mathbb{P} \times \mathbb{P}$. We consider three domains.

1. $U_0 = \{(z, y) \in \mathbb{P} \times \mathbb{P} | |y| > |z| > 0\}$,

2. $U_y = \{(z,y) \in \mathbb{P} \times \mathbb{P} | |y| > |z-y| > 0\}$,

3. $U_\infty = \{(z,y) \in \mathbb{P} \times \mathbb{P} | |z| > |y| > 0\}$,

and define local coordinates ξ_a, y on U_a, $a = 0, y, \infty$ by defining $\xi_0 = z$, $\xi_y = z - y$ and $\xi_\infty = z$. Then we can define the algebra homomorphisms

$$j_a^L : \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}] \to \mathcal{R}((\xi_a)) \,, \quad a = 0, y \,, \tag{2.119}$$

and

$$j_\infty^R : \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}] \to \mathcal{R}((\xi_\infty^{-1})) \tag{2.120}$$

by expanding $f(z)$ on the open sets U_a by the local coordinates (ξ_a, y). We denote $j_a^L(f) = f_a(\xi_a; y)$ for $a = 0, y$ and $j_\infty^R(f) = f_\infty(\xi_\infty; y)$. Then we can define the current Lie algebra $\mathfrak{g}_{\mathcal{R}}^{\mathbb{P}}(V)$ over $\mathcal{R}$ by

$$\mathfrak{g}_{\mathcal{R}}^{\mathbb{P}}(V) = \frac{\bigoplus_{n=0}^\infty V[n] \otimes \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}] \mathrm{d}z^{1-h}}{\nabla(\bigoplus_{n=0}^\infty V[n] \otimes \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}] \mathrm{d}z^{-h})} \,, \tag{2.121}$$

where ∇ is defined as in (2.13)

$$\nabla(A \otimes f(z,y) \mathrm{d}z^{-h}) = L_{-1} A \otimes f(z,y) \mathrm{d}z^{-h} + A \otimes \frac{\mathrm{d}f(z,y)}{\mathrm{d}z} \mathrm{d}z^{1-h} \,. \tag{2.122}$$

Then $\mathfrak{g}_{\mathcal{R}}^{\mathbb{P}}(V)$ has the structure of a Lie Algebra given by

$$\begin{aligned} &[A \otimes f(z,y) \mathrm{d}z^{1-h_A}, B \otimes g(z,y) \mathrm{d}z^{1-h_A}] = \\ &\sum_{m=0}^\infty \frac{1}{m!} A_{m-h_A+1} B \otimes \frac{\mathrm{d}^m f(z,y)}{\mathrm{d}z^m} g(z,y) \mathrm{d}z^{m+2-h_A-h_B} \,, \end{aligned} \tag{2.123}$$

Therefore we can define Lie algebra homomorphisms over $\mathcal{R}$

$$\begin{aligned} j_a^L : \mathfrak{g}_{\mathcal{R}}^{\mathbb{P}}(V) \to \mathcal{R} \otimes \mathfrak{g}^L \,, \quad a = 0, y \,, \\ j_\infty^R : \mathfrak{g}_{\mathcal{R}}^{\mathbb{P}}(V) \to \mathcal{R} \otimes \mathfrak{g}^R \,, \end{aligned} \tag{2.124}$$

in the same way as in (2.73).

Definition 27. *For V-modules L, M, N a $\mathrm{Hom}_{\mathbb{C}}(M \otimes N, \overline{L})$-valued holomorphic function $\psi(y)$ on $\mathbb{C}^*$ – which may be multi-valued – is called a vertex operator of type $\binom{M}{L,N}$ if it satisfies the following two conditions.*

1. *For $f(z) \in \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}]$, $A \in V[h]$, $m \in M$ and $n \in N$ we have*

$$\begin{aligned} &j_\infty^R([A \otimes f(z,y) \mathrm{d}z^{1-h}])(\psi(m;y)n) = \\ &\psi(j_y^L([A \otimes f(z,y) \mathrm{d}z^{1-h}]m; y))n + \psi(m;y) j_0^L([A \otimes f(z,y) \mathrm{d}z^{1-h}])n \,. \end{aligned} \tag{2.125}$$

2. For $m \in M$ and $n \in N$

$$\frac{\mathrm{d}}{\mathrm{d}y}\psi(m;y)n = \psi(L_{-1}m;y)n\,. \tag{2.126}$$

We denote by $I^M_{L,N}$ the complex vector space of vertex operators of type $\binom{M}{L,N}$. By taking $y=1$ we can define linear maps

$$\begin{aligned} I^M_{L,N} &\to \mathrm{Hom}_{\mathbb{C}}(M\otimes N, \overline{L}) \\ \psi(-;y) &\mapsto \psi(-;1)\,. \end{aligned} \tag{2.127}$$

Then we have the following theorem.

Theorem 28. *The image of the map $\psi(-;y)\mapsto\psi(-;1)$ is contained in the image of the injection*

$$\mathrm{Hom}_{\mathcal{U}(V)}(M\dot{\otimes}N, L) \to \mathrm{Hom}_{\mathbb{C}}(M\otimes N, \overline{L})\,, \tag{2.128}$$

and the two images are equal.

By the above vertex operator one can define N-point conformal blocks and prove the validity of the associativity and braiding constraints. As we have mentioned before, this will be postponed to [36].

We will revisit these concepts for one special case when proving the rigidity of the $\mathcal{W}_p$-module X_2^+. We also make note of a slight abuse of notation we will be using. When considering an element $\psi\in\mathrm{Hom}_{\mathcal{U}(V)}(M\dot{\otimes}N, L)$ and $m\in M$, $n\in N$ we will identify $m\otimes n$ with $\psi(m)n$, when there is no chance of confusion.

3 The abelian category $\mathcal{W}_p$-mod

In this section we briefly review the structure of $\mathcal{W}_p$-mod as an abelian category following [21].

3.1 General properties of $\mathcal{W}_p$-mod

For $p\geq 2$ the VOA $\mathcal{W}_p$ is generated by the identity $\mathbb{1}$, the energy momentum tensor $T(z)$ and three weight $2p-1$ primary fields $W^\varepsilon(z)$, where $\varepsilon=\pm,0$ labels sl_2-charges. The central charge of the theory is given by

$$c_p = 1 - 6\frac{(p-1)^2}{p}\,. \tag{3.1}$$

It has been shown in [18] that $\mathcal{W}_p$ is c_2-cofinite.

As an abelian category $\mathcal{W}_p$-mod decomposes into a $\mathbb{C}$-linear sum of abelian subcategories

$$\mathcal{W}_p\text{-mod} = \bigoplus_{s=0}^{p} \mathcal{C}_s \,, \tag{3.2}$$

where for $1 \leq s \leq p$ the $\mathcal{C}_s$ are full abelian subcategories of $\mathcal{W}_p$-mod. For $s \neq s'$ and $M \in \mathcal{C}_s,\ M' \in \mathcal{C}_{s'}$the spaces $\mathrm{Ext}^i_{\mathcal{W}_p}(M, M') = 0$ for any $i \in \mathbb{Z}$, in particular $\mathrm{Hom}_{\mathcal{W}_p}(M, M') = \mathrm{Ext}^0_{\mathcal{W}_p}(M, M') = 0$. The two subcategories $\mathcal{C}_0$ and $\mathcal{C}_p$ are semi-simple and contain one simple object each

$$X_p^+ \in \mathrm{obj}(\mathcal{C}_p) \qquad\qquad X_p^- \in \mathrm{obj}(\mathcal{C}_0) \tag{3.3}$$

which are projective in $\mathcal{C}_p$ and $\mathcal{C}_0$ respectively as well as in $\mathcal{W}_p$-mod. We will therefore occasionally also denote these modules by $P_p^\varepsilon = X_p^\varepsilon$. For $1 \leq s \leq p-1$ the subcategories $\mathcal{C}_s$ are not semi-simple. They contain two simple objects each

$$X_s^+,\ X_{p-s}^- \in \mathrm{obj}(\mathcal{C}_s)\,. \tag{3.4}$$

We denote the projective covers of X_s^+ and X_{p-s}^- by P_s^+ and P_{p-s}^- respectively. They are characterised by the socle series[3] of length 3

$$\begin{aligned} X_s^+ &= S_0(P_s^+) \subset S_1(P_s^+) \subseteq S_2(P_s^+) = P_s^+ \\ X_{p-s}^- &= S_0(P_{p-s}^-) \subset S_1(P_{p-s}^-) \subseteq S_2(P_{p-s}^-) = P_{p-s}^-\,, \end{aligned} \tag{3.5}$$

such that

$$\begin{aligned} S_1(P_s^+)/S_0(P_s^+) &= 2X_{p-s}^- & S_2(P_s^+)/S_1(P_s^+) &= X_s^+ \\ S_1(P_{p-s}^-)/S_0(P_{p-s}^-) &= 2X_s^+ & S_2(P_{p-s}^-)/S_1(P_{p-s}^-) &= X_{p-s}^-\,. \end{aligned} \tag{3.6}$$

Both P_s^+ and P_{p-s}^- have two occurrences of X_s^+ and X_{p-s}^- as subquotients therefore they have identical characters.

The simple and the projective modules of $\mathcal{W}_p$-mod are all self-contragredient, *i.e.* $X_s^{\varepsilon *} = X_s^\varepsilon$ and $P_s^{\varepsilon *} = P_s^\varepsilon$ for $1 \leq s \leq p,\ \varepsilon = \pm$. In particular the vacuum representation, X_1^+ which is the tensor unit, is self contragredient. Therefore by proposition 19 $(\mathcal{W}_p\text{−mod}, \dot{\otimes}, X_1^+)$ is weakly rigid and for each $\mathcal{W}_p$-module M its weakly rigid dual $M^\vee$ coincides with its contragredient M^*.

For $1 \leq s \leq p-1$ the subcategories $\mathcal{C}_s$ also contain 6 families of indecomposable modules characterised by socle series of length 2. For $d \geq 1$ these are summarised in the table below.

[3] A socle series of a module M is a filtration of submodules $S_1(M) \subseteq \cdots S_n(M) = M$ such that $S_1(M)$ is the maximal semi-simple submodule of M and $S_i(M)/S_{i-1}(M)$ is the maximal semi-simple submodule of $S_{i+1}/S_{i-1}(M)$.

	$G^+_{s,d}$	$G^-_{p-s,d}$	$H^+_{s,d}$	$H^-_{p-s,d}$	$I^+_{s,d}(\lambda)$	$I^-_{p-s,d}(\lambda)$
S_1/S_0	$(d+1)X^+_s$	$(d+1)X^-_{p-s}$	dX^+_s	dX^-_{p-s}	dX^+_s	dX^-_{p-s}
S_2/S_1	dX^-_{p-s}	dX^+_s	$(d+1)X^-_{p-s}$	$(d+1)X^+_s$	dX^-_{p-s}	dX^+_s

Note that $I^+_{s,d}(\lambda)$ and $I^-_{p-s,d}(\lambda)$ are not uniquely characterised by their socle series alone. To each of the two series there corresponds a continuous family of inequivalent indecomposable modules parametrised by $\lambda \in \mathbb{P}$.

The simple modules $X^\pm_s$ can be decomposed into direct sums of simple Virasoro modules

$$X^+_s = \bigoplus_{m=1}^{\infty} (2m-1)\mathcal{L}_{h_{2m-1,s}} \tag{3.7}$$
$$X^-_s = \bigoplus_{m=1}^{\infty} 2m\mathcal{L}_{h_{2m,s}}\,,$$

where $\mathcal{L}_{h_{r,s}}$ is the highest weight irreducible Virasoro module of weight

$$h_{r,s} = \frac{1}{4p}\left((rp-s)^2 - (p-1)^2\right)\,, \tag{3.8}$$

therefore the weights of X^+_s and X^-_s, which we will denote by h^+_s and h^-_s, are $h_{1,s}$ and $h_{2,s}$ respectively.

The dimension of the highest weight spaces $X^+_s[h^+_s]$ is 1 and the dimension of the highest weight spaces $X^-_s[h^-_s]$ is 2. We fix non-zero vectors $u_s \in X^+_s[h^+_s]$ and we fix a basis v^+_s, v^-_s of $X^-_s[h^-_s]$ which satisfies the following conditions

$$W^+_0 v^+_s = 0, \qquad\qquad W^-_0 v^-_s = 0\,. \tag{3.9}$$

Then v^+_s and v^-_s are universally determined up to constants.

We have the following results.

1. The Virasoro submodules $\mathcal{U}(\mathcal{L})u_s$ are isomorphic to the irreducible Virasoro modules $\mathcal{L}_{h_{1,s}}$ and the Virasoro submodules $\mathcal{U}(\mathcal{L})v^\mu_s$ are isomorphic to the irreducible Virasoro modules $\mathcal{L}_{h_{2,s}}$.

2. The action of the first few modes of the W-fields on the highest weight vector u_s of $X^+_s,\ 1 \le s \le p$ is given by

$$W^\varepsilon_{-k} u_s = 0 \qquad \varepsilon = 0, \pm, \quad k < 2p - s\,. \tag{3.10}$$

3. The action of the first few modes of the W-fields on the highest weight vectors v^μ_s of $X^-_s,\ 1 \le s \le p,\ \mu = \pm$ is then given by

$$W^\varepsilon_{-k} v^\mu_s \begin{cases} = 0 & \varepsilon = \mu \\ \in \mathcal{U}(\mathcal{L})v^\mu_s & \varepsilon = 0 \\ \in \mathcal{U}(\mathcal{L})v^{-\mu}_s & \varepsilon = -\mu \end{cases} \qquad \varepsilon = \pm, 0 \quad k < 3p - s\,. \tag{3.11}$$

After defining $\mathcal{W}_p$-mod, we now turn to $A_0(\mathcal{W}_p)$-mod – the category of zero mode quotients of $\mathcal{W}_p$-modules. We define elements of $A_0(\mathcal{W}_p)$-mod, in the following way

$$\overline{X}_s^{\varepsilon} = \mathcal{A}_0(X_s^{\varepsilon})\,, \quad 1 \leq s \leq p,\ \varepsilon = \pm\,. \tag{3.12}$$

Then the $\overline{X}_s^{\varepsilon}$ are simple objects in $A_0(\mathcal{W}_p)$-mod and any simple object of $A_0(\mathcal{W}_p)$-mod is isomorphic to one of the objects $\overline{X}_s^{\varepsilon}$.

Just like $\mathcal{W}_p$-mod, $A_0(\mathcal{W}_p)$-mod also decomposes into a $\mathbb{C}$-linear direct sum of abelian subcategories

$$A_0(\mathcal{W}_p)\text{-mod} = \bigoplus_{s=1,\varepsilon=\pm}^{p} \overline{\mathcal{C}}_s^{\varepsilon}. \tag{3.13}$$

The subcategories $\overline{\mathcal{C}}_p^{+}$ and $\overline{\mathcal{C}}_s^{-}, 1 \leq s \leq p$ are semi-simple and each contain one simple module

$$\overline{X}_p^{+} \in \text{obj}(\overline{\mathcal{C}}_p^{+}) \qquad\qquad \overline{X}_s^{-} \in \text{obj}(\overline{\mathcal{C}}_s^{-})\,. \tag{3.14}$$

For $1 \leq s \leq p-1$ the subcategories $\overline{\mathcal{C}}_s^{+}$ are not semi-simple. In addition to one simple module

$$\overline{X}_s^{\varepsilon}, \in \text{obj}(\overline{\mathcal{C}}_s^{\varepsilon})\,, \tag{3.15}$$

they also contain and one reducible but indecomposable module $\widetilde{X}_s^{+}$ that is the projective cover of $\overline{X}_s^{+}$ and satisfies the exact sequence

$$0 \longrightarrow \overline{X}_s^{+} \longrightarrow \widetilde{X}_s^{+} \longrightarrow \overline{X}_s^{+} \longrightarrow 0\,. \tag{3.16}$$

The image of the indecomposable $\mathcal{W}_p$-modules in $A_0(\mathcal{W}_p)$-mod is

$$\begin{array}{lll}
\mathcal{A}_0(X_s^{\pm}) = \overline{X}_s^{\pm} & \mathcal{A}_0(I_{s,d}^{+}(\lambda)) = d\overline{X}_s^{+} & \mathcal{A}_0(G_{s,d}^{-}) = (d+1)\overline{X}_s^{-} \oplus d\overline{X}_{p-s}^{+} \\
\mathcal{A}_0(G_{s,d}^{+}) = (d+1)\overline{X}_s^{+} & \mathcal{A}_0(P_s^{+}) = \widetilde{X}_s^{+} & \mathcal{A}_0(H_{s,d}^{-}) = d\overline{X}_s^{-} \oplus (d+1)\overline{X}_{p-s}^{+} \\
\mathcal{A}_0(H_{s,d}^{+}) = d\overline{X}_s^{+} & \mathcal{A}_0(P_s^{-}) = \overline{X}_s^{-} \oplus 2\overline{X}_{p-s}^{+} & \mathcal{A}_0(I_{s,d}^{-}(\lambda)) = d\overline{X}_s^{-} \oplus d\overline{X}_{p-s}^{+}\,.
\end{array}$$

As one can see from the above table the indecomposable structure of $A_0(\mathcal{W}_p)$ is much simpler than that of $\mathcal{W}_p$-mod as only the images of P_s^{+} are non-semi-simple.

The detailed $\mathcal{W}_p$-module structure of X_1^{-} and X_2^{+} is crucial to calculations. We have the following results.

Proposition 29. *As complex vector spaces the $\mathcal{A}_0$ quotients of simple $\mathcal{W}_p$ modules satisfy*

1. *For* $1 \leq s \leq p$, $\dim \mathcal{A}_0(X_s^+) = 1$ *and the space is spanned by the equivalence class represented by* u_s *and therefore has conformal weight* h_s^+.

2. *For* $1 \leq s \leq p$, *we fix the zero mode subspace* $(X_s^+)^0$ *to be spanned by the highest weight vector* u_s.

3. *For* $1 \leq s \leq p$, $\dim \mathcal{A}_0(X_s^-) = 2$ *and the space is spanned by the equivalence classes represented by the two highest weight vectors* v_s^ε, $\varepsilon = \pm$ *and therefore has conformal weight* h_s^-.

4. *For* $1 \leq s \leq p$, *we fix the zero mode subspace* $(X_s^-)^0$ *to be spanned by the two highest weight vectors* v_s^ε, $\varepsilon = \pm$.

Proposition 30. *The the two copies of* $\mathcal{L}_{h_{2,1}}$ *in the simple module* X_1^- *each contain a null vector at level 2*

$$(L_{-1}^2 - pL_{-2})v_1^\mu = 0 \quad \mu = \pm \tag{3.17}$$

and a well defined choice for the special subspace $(X_1^-)^s$ *is given by*

$$(X_1^-)^s = \bigoplus_{j=0}^{1} \bigoplus_{\mu=\pm} \mathbb{C} L_{-1}^j v_1^\mu \,. \tag{3.18}$$

Proposition 31. *The Virasoro submodule* $\mathcal{L}_{h_{1,2}}$ *of the simple module* X_2^+ *contains a null vector at level 2*

$$(L_{-1}^2 - \tfrac{1}{p} L_{-2})u_2 = 0 \tag{3.19}$$

and a well defined choice for the special subspace $(X_2^+)^s$ *is given by*

$$(X_2^+)^s = \bigoplus_{j=0}^{1} \mathbb{C} L_{-1}^j u_2 \oplus \bigoplus_{\varepsilon=0,\pm} \mathbb{C} W_{-2p+2}^\varepsilon u_2 \,. \tag{3.20}$$

3.2 The free field realisation of $\mathcal{W}_p$

One can explicitly construct $\mathcal{W}_p$ as a subVOA of a free field VOA V_L on a lattice by the method of screening operators. The free field VOA is constructed by means of the Heisenberg algebra

$$\mathfrak{a} = \mathbb{C}\mathbb{1} \oplus \bigoplus_{n\in\mathbb{Z}} \mathbb{C} a_n \tag{3.21}$$

as well as an operator $\hat{a}$, satisfying the commutation relations

$$[a_m, \hat{a}] = \delta_{m,0} \qquad\qquad [a_m, a_n] = m\delta_{m,-n}\,. \tag{3.22}$$

The Heisenberg algebra acts on Fock spaces $\mathcal{F}^\lambda$ generated by a state $|\lambda\rangle,\ \lambda \in \mathbb{C}$

$$a_m|\lambda\rangle = \lambda\delta_{m,0}|\lambda\rangle \quad m \geq 0\,. \tag{3.23}$$

For the free field VOA V_L we restrict the charges λ of $\mathcal{F}^\lambda$ to a rescaled A_1 root lattice L and its dual $L^\vee$

$$L = \mathbb{Z}\alpha_+ \qquad\qquad L^\vee = \mathbb{Z}\frac{\alpha_-}{2}\,, \tag{3.24}$$

where $\alpha_+ = \sqrt{2p}$ and $\alpha_- = -\sqrt{\frac{2}{p}}$. The theory contains a single free bosonic field

$$\varphi(z) = \hat{a} + a_0 \log z + \sum_{n\neq 0} \frac{a_n}{-n} z^{-n} \tag{3.25}$$

that satisfies the OPE

$$\varphi(z)\varphi(w) \sim \log(z-w)\,. \tag{3.26}$$

The energy momentum tensor is given by

$$T(z) = \frac{1}{2} : (\partial\varphi(z))^2 : + \frac{\alpha_+ + \alpha_-}{2}\partial\varphi(z)\,, \tag{3.27}$$

where : : indicates normal ordering, *i.e.* arranging the Heisenberg operators in ascending order from left to right according to their index with $\hat{a}$ on the very left. Calculating the OPE of T with itself, one reproduces the the central charge

$$c_p = 1 - 6\frac{(p-1)^2}{p} \tag{3.28}$$

of $\mathcal{W}_p$. The primary fields are given by

$$V_\mu(z) =: e^{\mu\varphi(z)} :\,, \tag{3.29}$$

where $\mu \in L^\vee$ and the weight of $V_\mu(z)$ is

$$h_\mu = \frac{1}{2}\mu(\mu - (\alpha_+ + \alpha_-))\,. \tag{3.30}$$

The OPE of two primary fields is given by

$$V_\mu(z)V_\nu(w) = (z-w)^{\mu\cdot\nu} : V_\mu(z)V_\nu(w) : \ . \tag{3.31}$$

The VOA V_L contains the fields $\mathbb{1}$, $T(z)$ and $V_\mu(z)$ for $\mu \in L$ but not $V_\nu(z)$ for $\nu \in L^\vee \setminus L$. The representation category V_L-mod is semi-simple with $2p$ simple modules $\mathcal{V}_{[\lambda]}$, $[\lambda] \in L^\vee/L$. For later calculations it will prove useful to parametrise the the classes $[\lambda] \in L^\vee/L$ by

$$[r,s] =: [\alpha_{r,s}] = \left[\frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_-\right] \qquad r,s \in \mathbb{Z}\,, \tag{3.32}$$

where

$$[r+1, s+p] = [r,s]\,. \tag{3.33}$$

The V_L modules decompose into infinite sums of Fock spaces

$$\mathcal{V}_{[\alpha_{r,s}]} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^{\alpha_{r+2n,s}}\,. \tag{3.34}$$

The V_L-theory contains two weight 1 primary fields that can be used as screening operators

$$Q_+(z) = V_{\alpha_+}(z) \qquad\qquad Q_-(z) = V_{\alpha_-}(z)\,. \tag{3.35}$$

The $\mathcal{W}_p$ VOA is realised by screening with $Q_-(z)$

$$\mathcal{W}_p = \ker\left(\oint \mathrm{d}z\, Q_-(z) : \mathcal{V}_{[1,1]} \to \mathcal{V}_{[1,-1]}\right)\,. \tag{3.36}$$

As $\mathcal{W}_p$-modules the simple V_L-modules $\mathcal{V}_{[1,p]}$ and $\mathcal{V}_{[2,p]}$ are isomorphic to X_p^+ and X_p^- respectively. The remaining $2p-2$ simple V_L modules are reducible as $\mathcal{W}_p$ modules and form Felder complexes

$$\cdots \longrightarrow \mathcal{V}_{[\alpha_{1,s}]} \xrightarrow{Q_-^{(s)}} \mathcal{V}_{[\alpha_{2,p-s}]} \xrightarrow{Q_-^{(p-s)}} \mathcal{V}_{[\alpha_{1,s}]} \longrightarrow \cdots\,. \tag{3.37}$$

where $Q_-^{(a)}$ is the zero mode of a rather complicated a-fold product of $Q_-(z)$ whose details need not concern us here. The simple $\mathcal{W}_p$ modules $X_s^\pm$ for $1 \le s \le p-1$ are equivalent to the kernels and images of $Q^{(a)}$

$$\begin{aligned} X_s^+ &= \ker\left(Q_-^{(s)} : \mathcal{V}_{[1,s]} \to \mathcal{V}_{[2,p-s]}\right) = \mathrm{im}\left(Q_-^{(p-s)} : \mathcal{V}_{[2,p-s]} \to \mathcal{V}[1,s]\right) \\ X_{p-s}^- &= \ker\left(Q_-^{(p-s)} : \mathcal{V}_{[2,p-s]} \to \mathcal{V}_{[1,s]}\right) = \mathrm{im}\left(Q_-^{(s)} : \mathcal{V}_{[1,s]} \to \mathcal{V}[2,p-s]\right)\,. \end{aligned} \tag{3.38}$$

Proposition 32. *The screening operator maps $Q^{(a)}$ induce surjective $\mathcal{W}_p$-module maps*

$$\begin{aligned} \mathcal{V}_{[1,s]} &\to X^-_{p-s} & \qquad \mathcal{V}_{[2,s]} &\to X^+_{p-s} \\ |\alpha_{-1,s}\rangle &\mapsto v^+_{p-s} & \qquad |\alpha_{0,s}\rangle &\mapsto u_{p-s}\,, \end{aligned} \tag{3.39}$$

for $1 \leq s \leq p-1$ as well as injective $\mathcal{W}_p$-module maps

$$\begin{aligned} X^+_s &\to \mathcal{V}_{[1,s]} & \qquad X^-_s &\to \mathcal{V}_{[2,s]} \\ u_s &\mapsto |\alpha_{1,s}\rangle & \qquad v^-_s &\mapsto |\alpha_{2,s}\rangle\,, \end{aligned} \tag{3.40}$$

for $1 \leq s \leq p$.

3.3 The V_L fusion product

We recall some well known facts about the V_L-mod fusion product that will be relevant to our calculations below. The tuple $(V_L\text{−mod}, \dot{\otimes}_{V_L}, \mathcal{V}_{[1,1]})$ defines a braided monoidal category, *i.e.* there is a well defined fusion product of V_L-modules

$$\mathcal{V}_{[s_1,r_1]} \otimes_{V_L} \mathcal{V}_{[s_2,r_2]} = \mathcal{V}_{[s_1+s_2-1,r_1+r_2-1]}\,. \tag{3.41}$$

The free field VOA V_L contains another subVOA $(\mathcal{F}^{\alpha_{1,1}}, |0\rangle, T, Y)$ other than $\mathcal{W}_p$ called the Heisenberg VOA.[4] The current algebra $\mathcal{U}(\mathcal{F}^{\alpha_{1,1}})$ is given by the universal enveloping algebra $\mathcal{U}(\mathfrak{a})$ of the Heisenberg algebra, while the simple objects of $\mathcal{F}^{\alpha_{1,1}}$ are given by $\mathcal{F}^{\alpha_{r,s}}$

Proposition 33. *For $(r_1, s_1), (r_2, s_2) \in \mathbb{Z}^2$ the fusion product in $\mathcal{F}^{\alpha_{1,1}}$-mod is given by*

$$\begin{aligned} \mathcal{F}^{\alpha_{r_1,s_1}} \dot{\otimes}_{\mathcal{F}^{\alpha_{1,1}}} \mathcal{F}^{\alpha_{r_2,s_2}} &\overset{\cong}{\to} \mathcal{F}^{\alpha_{r_1+r_2-1,s_1+s_2-1}} \\ |\alpha_{r_1,s_1}\rangle \otimes |\alpha_{r_2,s_2}\rangle &\mapsto |\alpha_{r_1+r_2-1,s_1+s_2-1}\rangle \end{aligned} \tag{3.42}$$

and the following diagram commutes

$$\begin{tikzcd} \mathcal{F}^{\alpha_{r_1,s_1}} \dot{\otimes}_{\mathcal{F}^{\alpha_{1,1}}} \mathcal{F}^{\alpha_{r_2,s_2}} \arrow[r] \arrow[d] & \mathcal{F}^{\alpha_{r_1+r_2-1,s_1+s_2-1}} \arrow[d] \\ \mathcal{V}_{[r_1,s_1]} \dot{\otimes}_{V_L} \mathcal{V}_{[r_2,s_2]} \arrow[r] & \mathcal{V}_{[r_1+r_2-1,s_1+s_2-1]} \end{tikzcd} \tag{3.43}$$

where the vertical arrows are injective $\mathcal{F}^{\alpha_{1,1}}$-module maps and the horizontal arrows are a $\mathcal{F}^{\alpha_{1,1}}$ and a V_L-isomorphism respectively.

[4] This is the only appearance of a non-c_2-cofintie VOA in this paper.

4 Computing some fusion products in $\mathcal{W}_p$-mod

We now apply the apply the methods explained above to analyse the monoidal structure of $\mathcal{W}_p$-mod.

4.1 The fusion rules and rigidity of X_1^-

In this section we analyse the fusion products of X_1^- with simple modules and prove the rigidity of X_1^-.

Theorem 34. *The fusion product of X_1^- with itself is*

$$X_1^- \dot{\otimes} X_1^- = X_1^+ \,. \tag{4.1}$$

Also X_1^- is rigid and its dual is just X_1^- itself.

Sketch of the proof. We prove the the theorem in three steps.

1. We prove that there exists a surjection of $A_0(\mathcal{W}_P)$ modules

$$\mathcal{A}_0(X_1^+) \to \mathcal{A}_0(X_1^- \dot{\otimes} X_1^-) \,. \tag{4.2}$$

2. We prove that $\dim \mathcal{A}_1(X_1^- \dot{\otimes} X_1^-)[1] = 0$.

3. We prove the existence of a non-trivial $\mathcal{W}_p$-module map

$$X_1^- \dot{\otimes} X_1^- \to \mathcal{V}_{[2,p-1]} \,. \tag{4.3}$$

Step 1 implies that $X_1^- \dot{\otimes} X_1^-$ is a (possibly trivial) highest weight module generated by a state of conformal weight 0. Since $h_{p-1}^- = 1$ step 2 excludes the possibility of X_{p-1}^- being a submodule of $X_1^- \dot{\otimes} X_1^-$. Step 3 implies that $X_1^- \dot{\otimes} X_1^-$ is non-trivial and since the only non-trivial submodule of $\mathcal{V}_{[2,p-1]}$, generated by a state of conformal weight 0 is X_1^+, it follows that $X_1^- \dot{\otimes} X_1^- = X_1^+$.

The rigidity of X_1^- follows by choosing

$$\begin{aligned} \mathrm{id}_{X_1^+} &= e_{X_1^-} : X_1^- \dot{\otimes} X_1^- \to X_1^+ \\ \mathrm{id}_{X_1^+} &= i_{X_1^-} : X_1^+ \to X_1^- \dot{\otimes} X_1^- \end{aligned} \tag{4.4}$$

and the fact that therefore all the maps appearing in definition 17 are isomorphisms. □

Proof of step 1. As in proposition 30 we choose

$$X_1^{-s} = \bigoplus_{j=0}^{1} \bigoplus_{\varepsilon=\pm} \mathbb{C} L_{-1}^j v_1^\varepsilon \tag{4.5}$$

and as in proposition 29 we choose

$$X_1^{-0} = \bigoplus_{\varepsilon=\pm} \mathbb{C} v_1^\varepsilon \,. \tag{4.6}$$

Using the formulae (2.95) as well as the null vector in proposition 30 we can compute the action of L_0 on the classes represented by the elements of $(X_1^-)^s \otimes (X_1^-)^0$.

$$\begin{aligned} [v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2}] &\mapsto 2h_1^-[v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2}] + [L_{-1} v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2}] \\ [L_{-1} v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2}] &\mapsto (2h_1^- + 1)[v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2}] + [L_{-1}^2 v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2}] \\ &= p h_1^-[v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2}] + (2h_1^- + 1 - p)[L_{-1} v_{\varepsilon_1} \otimes v_1^{\varepsilon_2}] \,. \end{aligned} \tag{4.7}$$

Thus for each pair $\varepsilon_1, \varepsilon_2$ we can represent L_0 by

$$L_0 \cong \begin{pmatrix} 2h_1^- & p h_1^- \\ 1 & 2h_1^- + 1 - p \end{pmatrix} \tag{4.8}$$

on the basis $L_{-1}^j v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2},\ j = 0, 1$. The eigenvalues of this matrix are $h_1^+ = 0$ and $2p - 1$.

Next we will determine a lower bound on the dimension of the kernel of the surjection

$$(X_1^-)^s \otimes (X_1^-)^0 \to \mathcal{A}_0(X_1^- \dot{\otimes} X_1^-) \,. \tag{4.9}$$

Because $A_0(\mathcal{W}_p)$-mod does not contain any module with eigenvalue $2p - 1$ eigenvectors, the eigenvectors

$$v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2} + \tfrac{2}{3p-2} L_{-1} v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2} \,, \tag{4.10}$$

corresponding to the eigenvalue $2p-1$, must lie in the kernel of the surjection (4.9).

From formula (3.11) illustrating the action of W-field modes on X_1^- we know

$$L_{-1} v_1^\varepsilon = C_\varepsilon \cdot W_{-1}^\varepsilon v_1^{-\varepsilon} \qquad \varepsilon = \pm \tag{4.11}$$

for some constant C. This implies that $L_{-1}v_1^\varepsilon \otimes v_1^\varepsilon,\ \varepsilon = \pm$ lies in the kernel of (4.9), because

$$\begin{aligned}[L_{-1}v_1^\varepsilon \otimes v_1^\varepsilon] &= C_\varepsilon \cdot [W_{-1}^\varepsilon v_1^{-\varepsilon} \otimes v_1^\varepsilon] \\ &= C_\varepsilon \cdot \sum_{j=0}^{2p-3} \binom{2p-3}{j} (-1)^j [v_1^{-\varepsilon} \otimes W_{j-(2p-2)}^\varepsilon v_1^\varepsilon] = 0\,.\end{aligned} \tag{4.12}$$

Finally

$$\begin{aligned}[L_{-1}v_1^\varepsilon \otimes v_1^{-\varepsilon}] &= C_\varepsilon \cdot [W_{-1}^\varepsilon v_1^{-\varepsilon} \otimes v_1^{-\varepsilon}] \\ &= C_\varepsilon \cdot \sum_{j=0}^{2p-3} \binom{2p-3}{j} (-1)^j [v_1^{-\varepsilon} \otimes W_{j-(2p-2)}^\varepsilon v_1^{-\varepsilon}] \\ &= A_\varepsilon [L_{-1}v_1^{-\varepsilon} \otimes v_1^\varepsilon] + B_\varepsilon [v_1^{-\varepsilon} \otimes v_1^\varepsilon]\,,\end{aligned} \tag{4.13}$$

for some constants A_ε and B_ε, since by the action of the W-modes (3.11) $W_{j-(2p-2)}^\varepsilon v_1^{-\varepsilon} \in \mathcal{L}_{h_{2,1}}^\varepsilon$. This implies that some non-trivial linear combination of $[L_{-1}v_1^+ \otimes v_1^-]$ and $[L_{-1}v_1^- \otimes v_1^+]$ lies in the kernel of (4.9). Therefore the kernel of (4.9) is at least 7 dimensional and $\mathcal{A}_0(X_1^- \dot{\otimes} X_1^-)$ is at most one dimensional. If $\mathcal{A}_0(X_1^- \dot{\otimes} X_1^-)$ is indeed non-trivial, then the eigenvalue of L_0 is 0. □

Proof of step 2. We know that the image of the action of the W-field modes W_{-k}^μ on the highest weight vectors v_1^ε lies in the Virasoro submodules generated by v_1^+ and v_1^- for $k < 3p-1$. Therefore a spanning set of representatives for $\mathcal{A}_1(X_1^- \dot{\otimes} X_1^-)$ can be chosen from Virasoro descendants of $v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2},\ \varepsilon_1 = \pm, \varepsilon_2 = \pm$. Also since the relations (2.95) for Virasoro modes still hold for $n \geq 2$, we can restrict the spanning set of representatives for $\mathcal{A}_1(X_1^- \dot{\otimes} X_1^-)$ to L_{-1}-descendants of $v_1^{\varepsilon_1} \otimes v_1^{\varepsilon_2}$. Finally because of the null vector (3.17) at level 2 of X_1^- we have the following surjection of complex vector spaces

$$\bigoplus_{\substack{i=0\\j=0}}^{1} \bigoplus_{\substack{\varepsilon_1=\pm\\ \varepsilon_2=\pm}} \mathbb{C}[L_{-1}^i v_1^{\varepsilon_1} \otimes L_{-1}^j v_1^{\varepsilon_2}] \to \mathcal{A}_1(X_1^- \dot{\otimes} X_1^-)\,. \tag{4.14}$$

Because the image of the canonical Lie algebra homomorphism

$$\mathfrak{g}(\mathcal{W}_p) \to \mathcal{U}(\mathcal{W}_p) \tag{4.15}$$

is dense, we know that the image of L_{-1}^2 lies in $\mathfrak{g}_2(\mathcal{W}_p)(X_1^- \dot{\otimes} X_1^-)$ and that L_{-1}^2 therefore acts trivially on $\mathcal{A}_1(X_1^- \dot{\otimes} X_1^-)$ even if L_{-1} does not. This

implies the relation

$$
\begin{aligned}
&(j_{1,0}([T\otimes 1\cdot \mathrm{d}z^{-1}]))^2[v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] \\
&\quad = [L_{-1}v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] + 2[L_{-1}v_1^{\varepsilon_1}\otimes L_{-1}v_1^{\varepsilon_2}] + [v_1^{\varepsilon_1}\otimes L_{-1}v_1^{\varepsilon_2}] = 0
\end{aligned}
\tag{4.16}
$$

We therefore take $[v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}],\ [L_{-1}v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}]$ and $[v_1^{\varepsilon_1}\otimes L_{-1}v_1^{\varepsilon_2}]$ as a spanning set for $\mathcal{A}_1(X_1^{-1}\dot{\otimes}X_1^{-1})$ and compute the action of L_0

$$
\begin{aligned}
[v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] &\mapsto \left(\frac{3}{2}p-1\right)[v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] + [L_{-1}v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] \\
[L_{-1}v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] &\mapsto \left(\frac{3}{2}p-1\right)\frac{p}{2}[v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] + \frac{3}{2}p[L_{-1}v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] + p[v_1^{\varepsilon_1}\otimes L_{-1}v_1^{\varepsilon_2}] \\
[v_1^{\varepsilon_1}\otimes L_{-1}v_1^{\varepsilon_2}] &\mapsto -\left(\frac{3}{2}p-1\right)\frac{p}{2}[v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] + \frac{p}{2}[L_{-1}v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}] + p[v_1^{\varepsilon_1}\otimes L_{-1}v_1^{\varepsilon_2}]\,.
\end{aligned}
\tag{4.17}
$$

As a matrix L_0 is represented by

$$
\begin{pmatrix}
\left(\frac{3}{2}p-1\right) & \left(\frac{3}{2}p-1\right)\frac{p}{2} & -\left(\frac{3}{2}p-1\right)\frac{p}{2} \\
1 & \frac{3}{2}p & \frac{p}{2} \\
0 & p & p
\end{pmatrix}
\tag{4.18}
$$

on the basis $v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2},\ L_{-1}v_1^{\varepsilon_1}\otimes v_1^{\varepsilon_2}$ and $v_1^{\varepsilon_1}\otimes L_{-1}v_1^{\varepsilon_2}$ and the eigenvalues of this matrix are $0,\ 2p-1$ and $2p$, none of which are $h^-_{p-1}=1$. □

Proof of step 3. According to proposition 32 there exists an injective $\mathcal{W}_p$-module map

$$
\begin{aligned}
X_1^- &\to \mathcal{V}_{[2,1]} \\
v_1^- &\mapsto |\alpha_{2,1}\rangle\,.
\end{aligned}
\tag{4.19}
$$

By this map and proposition 24 there exits a non-trivial $\mathcal{W}_p$-module map

$$
\begin{aligned}
X_1^-\dot{\otimes}_{\mathcal{W}_p}X_1^- &\to \mathcal{V}_{[2,1]}\dot{\otimes}_{V_L}\mathcal{V}_{[2,1]} \\
v_1^-\otimes v_1^- &\mapsto |\alpha_{2,1}\rangle\otimes|\alpha_{2,1}\rangle\,.
\end{aligned}
\tag{4.20}
$$

By proposition 33 there exits a V_L-module isomorphism

$$
\begin{aligned}
&\mathcal{V}_{[2,1]}\dot{\otimes}_{V_L}\mathcal{V}_{[2,1]} \to \mathcal{V}_{[3,1]} \cong \mathcal{V}_{[1,1]} \\
&|\alpha_{2,1}\rangle\otimes|\alpha_{2,1}\rangle \mapsto |\alpha_{3,1}\rangle\,.
\end{aligned}
\tag{4.21}
$$

By concatenating these two maps we have constructed a non-trivial $\mathcal{W}_p$-module map

$$
X_1^-\dot{\otimes}_{\mathcal{W}_p}X_1^- \to \mathcal{V}_{[1,1]}\,.
\tag{4.22}
$$

□

Theorem 35. *The fusion rules of X_1^- with simple modules is given by*

$$X_1^- \dot{\otimes} X_s^{\varepsilon} = X_s^{-\varepsilon} \quad 1 \leq s \leq p, \ \varepsilon = \pm \,. \tag{4.23}$$

Sketch of proof. We prove the theorem in two steps

1. Let M be a simple module, then $X_1^- \dot{\otimes} M$ is also simple.
2. We prove the existence of a non-trivial $\mathcal{W}_p$-module map

$$X_1^- \dot{\otimes} X_s^+ \to \mathcal{V}_{[2,s]} \,. \tag{4.24}$$

The simplicity of $X_1^- \dot{\otimes} X_s^+$ implied by step 1 and the non-triviality of the map in step 2 implies that $X_1^- \dot{\otimes} X_s^+$ is a simple submodule of $\mathcal{V}_{[2,s]}$. Therefore $X_1^- \dot{\otimes} X_s^+ = X_s^-$. The theorem then follows by $X_1^- \dot{\otimes} X_1^- = X_1^+$. □

Proof of step 1. Proof by contradiction. Assume $X_1^- \dot{\otimes} M$ is not simple, then there exists an exact sequence

$$0 \to A \to X_1^- \dot{\otimes} M \to B \to 0 \,, \tag{4.25}$$

for some non-trivial $\mathcal{W}_p$ modules A and B. Because X_1^- is rigid, the sequence

$$0 \to X_1^- \dot{\otimes} A \to M \to X_1^- \dot{\otimes} B \to 0 \,, \tag{4.26}$$

must also be exact which is in contradiction to M being simple. □

Proof of step 2. According to proposition 32 there exist injective $\mathcal{W}_p$-module maps

$$\begin{aligned} X_1^- &\to \mathcal{V}_{[2,1]} \\ v_1^- &\mapsto |\alpha_{2,1}\rangle \end{aligned} \tag{4.27}$$

and for $1 \leq s \leq p$

$$\begin{aligned} X_s^+ &\to \mathcal{V}_{[1,s]} \\ u_s &\mapsto |\alpha_{1,s}\rangle \end{aligned} \tag{4.28}$$

By the above injective $\mathcal{W}_p$-module maps and proposition 24 there exits a non-trivial $\mathcal{W}_p$-module map

$$\begin{aligned} X_1^- \dot{\otimes}_{\mathcal{W}_p} X_s^+ &\to \mathcal{V}_{[2,1]} \dot{\otimes}_{V_L} \mathcal{V}_{[1,s]} \\ v_1^- \otimes u_s &\mapsto |\alpha_{2,1}\rangle \otimes |\alpha_{1,s}\rangle \,. \end{aligned} \tag{4.29}$$

By proposition 33 there exits a V_L-module isomorphism

$$\begin{aligned} \mathcal{V}_{[2,1]}\dot{\otimes}_{V_L}\mathcal{V}_{[1,s]} &\to \mathcal{V}_{[2,s]} \\ |\alpha_{2,1}\rangle \otimes |\alpha_{1,s}\rangle &\mapsto |\alpha_{2,s}\rangle \,. \end{aligned} \tag{4.30}$$

By concatenating these two maps we have constructed a non-trivial $\mathcal{W}_p$-module map

$$X_1^-\dot{\otimes}_{\mathcal{W}_p}X_s^+ \to \mathcal{V}_{[2,s]} \,. \tag{4.31}$$

□

4.2 The fusion rules and rigidity of X_2^+

In this section we analyse the fusion products of X_2^+ with simple modules and prove the rigidity of X_2^+.

Theorem 36. *The $\mathcal{W}_p$ module X_2^+ is rigid and he fusion rules of X_2^+ with simple modules is given by*

$$X_2^+\dot{\otimes}X_s^\varepsilon = \begin{cases} X_2^\varepsilon & s=1 \\ X_{s-1}^\varepsilon \oplus X_{s+1}^\varepsilon & 2\leq s\leq p-1 \\ P_{p-1}^\varepsilon & s=p \end{cases} \,. \tag{4.32}$$

Sketch of proof. We prove the theorem in a number of steps

1. We prove the existence of surjections of $\mathcal{A}_0$ modules

$$\left.\begin{array}{rr} s=1 & \mathcal{A}_0(X_2^-) \\ 1<s<p & \mathcal{A}_0(X_{s-1}^-)\oplus\mathcal{A}_0(X_{s+1}^-) \\ s=p & \mathcal{A}_0(P_{p-1}^-) \end{array}\right\} \to \mathcal{A}_0(X_2^+\dot{\otimes}X_s^-) \to 0\,.$$

2. We prove the existence of non-trivial $\mathcal{W}_p$-module maps

$$X_2^+\dot{\otimes}X_s^- \to \mathcal{V}_{[2,s+1]}\,, \tag{4.33}$$

   for $1\leq s\leq p-1$.

3. We prove the existence of non-trivial $\mathcal{W}_p$-module maps

$$X_2^+\dot{\otimes}\mathcal{V}_{[1,p-s]} \to X_{s-1}^-\,, \tag{4.34}$$

   for $2\leq s\leq p-1$.

4. We prove the existence of a surjective $\mathcal{W}_p$-module map

$$X_2^+ \dot{\otimes} X_p^- \to X_{p-1}^- \,. \tag{4.35}$$

5. We use the formalism outlined in section 2.4 to prove the rigidity of X_2^+.

Steps 1 through 3 prove

$$X_2^+ \dot{\otimes} X_s^- = \begin{cases} X_2^- & s = 1 \\ X_{s-1}^- \oplus X_{s+1}^- & 1 < s < p \end{cases} \,. \tag{4.36}$$

Step 5 implies that $X_2^+ \dot{\otimes} X_p^-$ is injective, since the product of a rigid and a projective module is again projective. The only projective module compatible with steps 1 and 4 is P_{p-1}^-, therefore

$$X_2^+ \dot{\otimes} X_p^- = P_{p-1}^- \,. \tag{4.37}$$

Finally the fusion products of the theorem follow by multiplying with X_1^- and the associativity of the fusion product. □

Proof of step 1. We choose the special subspace of X_2^+ to be given by

$$(X_2^+)^s = \bigoplus_{j=0}^{1} \mathbb{C} L_{-1}^j u_2 \oplus \bigoplus_{\varepsilon=0,\pm} \mathbb{C} W_{-2p+2}^{\varepsilon} u_2 \,, \tag{4.38}$$

as in proposition 31 and we chose the zero mode subspace of X_s^- to be given by

$$(X_s^-)^0 = \bigoplus_{\varepsilon=\pm} \mathbb{C} v_s^{\varepsilon} \,, \tag{4.39}$$

as in proposition 29. By proposition 23 there is a canonical surjection

$$(X_2^+)^s \otimes (X_1^-)^0 \to \mathcal{A}_0(X_2^+ \dot{\otimes} X_s^-) \,. \tag{4.40}$$

We first show that the spanning set

$$\mathrm{span}\{[L_{-1}^j u_2 \otimes v_s^{\varepsilon}],\ [W_{-2p+2}^{\mu} u_2 \otimes v_s^{\varepsilon}],\ j = 0, 1,\ \varepsilon = \pm,\ \mu = \pm, 0\} \tag{4.41}$$

is redundant. Consider

$$[W_{-2p+2}^{\mu} u_2 \otimes v_s^{\varepsilon}] = -[u_2 \otimes W_{-2p+2}^{\mu} v_s^{\varepsilon}] \,. \tag{4.42}$$

Because the highest weight of the Virasoro representations $\mathcal{L}_{h_{4,s}}$ in the decomposition (3.7) of X_s^- into Virasoro representation is $3p-1$ higher than the highest weight of $\mathcal{U}(\mathcal{L})v_s^\varepsilon$, $W^\mu_{-2p+2}v_s^\varepsilon$ must lie in $\mathcal{U}(\mathcal{L})v_s^\varepsilon$. Therefore $[W^\mu_{-2p+2}u_2 \otimes v_s^\varepsilon]$ depends linearly on $[L^j_{-1}u_2 \otimes v_s^\delta]$, $j=0,1$, $\delta=\pm$.

We therefore have a 4 dimensional spanning set for $\mathcal{A}_0(X_2^+ \dot{\otimes} X_s^-)$ on which we can compute the action of L_0.

$$\begin{aligned} [u_2 \otimes v_s^\pm] &\mapsto (h_2^+ + h_s^+)[u_2 \otimes v_s^\pm] + [L_{-1}u_2 \otimes v_s^\pm] \\ [L_{-1}u_2 \otimes v_s^\pm] &\mapsto \frac{h_s^+}{p}[u_2 \otimes v_s^\pm] + (h_2^+ + h_s^+ + 1 - \frac{1}{p})[L_{-1}u_2 \otimes v_s^\pm]\,. \end{aligned} \tag{4.43}$$

We can therefore represent L_0 by the matrix

$$\begin{pmatrix} h_2^+ + h_s^- & \frac{h_s^-}{p} \\ 1 & h_2^+ + h_s^- + 1 - \frac{1}{p} \end{pmatrix}$$

on the basis $L^j_{-1}u_2 \otimes v_s^\varepsilon$, $j=0,1$, $\varepsilon=\pm$. For $2 \le s \le p-1$ the eigenvalues of this matrix are h_{s-1}^- and h_{s+1}^- and for $s=p$ the eigenvalues of the above matrix are h_{p-1}^- and h_1^+. □

Proof of step 2. According to proposition 32 there exist injective $\mathcal{W}_p$-module maps

$$\begin{aligned} X_2^+ &\to \mathcal{V}_{[2,s]} \\ u_2 &\mapsto |\alpha_{1,s}\rangle\,. \end{aligned} \tag{4.44}$$

and for $1 \le s < p$

$$\begin{aligned} X_s^- &\to \mathcal{V}_{[2,s]} \\ v_s^- &\mapsto |\alpha_{2,s}\rangle\,. \end{aligned} \tag{4.45}$$

By these maps and proposition 24 there exits a non-trivial $\mathcal{W}_p$-module map

$$\begin{aligned} X_2^+ \dot{\otimes}_{\mathcal{W}_p} X_s^- &\to \mathcal{V}_{[1,2]} \dot{\otimes}_{V_L} \mathcal{V}_{[2,s]} \\ u_2 \otimes v_s^- &\mapsto |\alpha_{1,2}\rangle \otimes |\alpha_{2,s}\rangle\,. \end{aligned} \tag{4.46}$$

By proposition 33 there exits a V_L-module isomorphism

$$\begin{aligned} \mathcal{V}_{[1,2]} \dot{\otimes}_{V_L} \mathcal{V}_{[2,s]} &\to \mathcal{V}_{[2,s+1]} \\ |\alpha_{1,2}\rangle \otimes |\alpha_{2,s}\rangle &\mapsto |\alpha_{2,s+1}\rangle\,. \end{aligned} \tag{4.47}$$

By concatenating these two maps we have constructed a non-trivial $\mathcal{W}_p$-module map

$$X_2^+ \dot{\otimes}_{\mathcal{W}_p} X_s^- \to \mathcal{V}_{[2,s+1]}\,. \tag{4.48}$$

□

Proof of step 3. Before we begin with the proof we note that the proof of step one implies the existence of a surjective $\mathcal{W}_p$-module map

$$X_{s-1}^- \oplus X_{s+1}^- \to X_2^+ \dot{\otimes} X_s^- \tag{4.49}$$

for $1 < s < p$, *i.e.* the results for $\mathcal{A}_0(X_2^+ \dot{\otimes} X_s^-)$ allow for no modules larger than X_{s-1}^- or X_{s+1}^-. Therefore because X_1^- is rigid and the functor $X_1^- \dot{\otimes} -$ is exact, there exits a surjective $\mathcal{W}_p$-module map

$$X_{s-1}^+ \oplus X_{s+1}^+ \to X_2^+ \dot{\otimes} X_s^+ \,. \tag{4.50}$$

According to proposition 32 there exists an injective $\mathcal{W}_p$-module map

$$\begin{aligned} X_2^+ &\to \mathcal{V}_{[2,s]} \\ u_2 &\mapsto |\alpha_{1,s}\rangle \,. \end{aligned} \tag{4.51}$$

By this map and proposition 24 there exits a non-trivial $\mathcal{W}_p$-module map

$$\begin{aligned} X_2^+ \dot{\otimes}_{\mathcal{W}_p} \mathcal{V}_{[1,p-s]} &\to \mathcal{V}_{[1,2]} \dot{\otimes}_{V_L} \mathcal{V}_{[1,p-(s-1)]} \\ u_2 \otimes |\alpha_{-1,p-s}\rangle &\mapsto |\alpha_{1,2}\rangle \otimes |\alpha_{-1,p-s}\rangle \,. \end{aligned} \tag{4.52}$$

By proposition 33 there exits a V_L-module isomorphism

$$\begin{aligned} \mathcal{V}_{[1,2]} \dot{\otimes}_{V_L} \mathcal{V}_{[1,p-s]} &\to \mathcal{V}_{[1,p-(s-1)]} \\ |\alpha_{1,2}\rangle \otimes |\alpha_{-1,p-s}\rangle &\mapsto |\alpha_{-1,p-(s-1)}\rangle \,. \end{aligned} \tag{4.53}$$

By concatenating these two maps we have constructed a non-trivial $\mathcal{W}_p$-module map

$$\varphi : X_2^+ \dot{\otimes}_{\mathcal{W}_p} \mathcal{V}_{[1,p-s]} \to \mathcal{V}_{[1,p-(s-1)]} \,. \tag{4.54}$$

Also according to proposition 32 there exists a surjective $\mathcal{W}_p$-module map

$$\begin{aligned} \pi : \mathcal{V}_{[1,p-(s-1)]} &\to X_{s-1}^- \\ |\alpha_{-1,p-(s-1)}\rangle &\mapsto v_{s-1}^+ \,. \end{aligned} \tag{4.55}$$

The composition $\pi \circ \varphi$ is therefore a non-trivial $\mathcal{W}_p$-module map

$$\pi \circ \varphi : X_2^+ \dot{\otimes} \mathcal{V}_{[1,p-s]} \to X_{s-1}^- \,. \tag{4.56}$$

By the surjection (4.50) $X_2 \dot{\otimes} X_{p-s}^+$ must lie in the kernel of $\pi \circ \varphi$. Therefore there exists a non-trivial $\mathcal{W}_p$-module map

$$X_2^+ \dot{\otimes} X_s^- \to X_{s-1}^- \,. \tag{4.57}$$

□

Proof of step 4. According to proposition 32 there exists an injective $\mathcal{W}_p$-module map

$$\begin{aligned} X_2^+ &\to \mathcal{V}_{[1,2]} \\ u_2 &\mapsto |\alpha_{1,2}\rangle \end{aligned} \tag{4.58}$$

and a $\mathcal{W}_p$-module isomorphism $X_p^- \to \mathcal{V}_{[2,p]}$. By the above maps and proposition 24 there exits a non-trivial $\mathcal{W}_p$-module map

$$\begin{aligned} X_2^+ \dot{\otimes}_{\mathcal{W}_p} \mathcal{V}_{[2,p]} &\to \mathcal{V}_{[1,2]} \dot{\otimes}_{V_L} \mathcal{V}_{[2,p]} \\ u_2 \otimes |\alpha_{0,p}\rangle &\mapsto |\alpha_{1,2}\rangle \otimes |\alpha_{0,p}\rangle \,. \end{aligned} \tag{4.59}$$

By proposition 33 there exits a V_L-module isomorphism

$$\begin{aligned} \mathcal{V}_{[1,2]} \dot{\otimes}_{V_L} \mathcal{V}_{[2,p]} &\to \mathcal{V}_{[2,p+1]} = \mathcal{V}_{[1,1]} \\ |\alpha_{1,2}\rangle \otimes |\alpha_{0,p}\rangle &\mapsto |\alpha_{0,p+1}\rangle = |\alpha_{-1,1}\rangle \,. \end{aligned} \tag{4.60}$$

By concatenating these two maps we have constructed a non-trivial $\mathcal{W}_p$-module map

$$X_2^+ \dot{\otimes}_{\mathcal{W}_p} X_p^- \to \mathcal{V}_{[1,1]} \,. \tag{4.61}$$

Also according to proposition 32 there exists a surjective $\mathcal{W}_p$-module map

$$\begin{aligned} \mathcal{V}_{[1,1]} &\to X_{p-1}^- \\ |\alpha_{-1,1}\rangle &\mapsto v_{p-1}^+ \,. \end{aligned} \tag{4.62}$$

Therefore there exists a non-trivial $\mathcal{W}_p$-module map

$$X_2^+ \dot{\otimes}_{\mathcal{W}_p} X_p^- \to X_{p-1}^- \,. \tag{4.63}$$

□

Proof of step 5. In order to prove the rigidity of X_2^+, we need to consider three fold products of X_2^+, which at this stage we can only compute for $p \geq 4$. We will therefore explain in detail how the proof of rigidity can be reduced to analysing formal solutions of hypergeometric equations for $p \geq 4$. The advantage of this analysis is that it does not require us to explicitly know the three fold fusion product of X_2^+ and we can therefore also apply it to $p = 2, 3$ once we have discussed $p \geq 4$.

Until explicitly stated otherwise we will therefore assume that $p \geq 4$. Then we have proven that

$$X_2^+ \dot{\otimes} (X_2^+ \dot{\otimes} X_2^+) \cong (X_2^+ \dot{\otimes} X_2^+) \dot{\otimes} X_2^+ = 2 \cdot X_2^+ \oplus X_4^+ \,. \tag{4.64}$$

The rigidity of X_2^+ and self duality $X_2^{+\vee} = X_2^+$ requires the existence of $\mathcal{W}_p$-module maps $i : X_1^+ \to X_2^+ \dot{\otimes} X_2^+$ and $e : X_2^+ \dot{\otimes} X_2^+ \to X_1^+$, such that

$$\begin{tikzcd} X_2^+ \cong X_2^+ \dot{\otimes} X_1^+ \arrow[r, "\mathrm{id}\dot{\otimes} i"] \arrow[d, "f"'] & X_2^+ \dot{\otimes} (X_2^+ \dot{\otimes} X_2^+) \arrow[d, "\alpha_{X_2^+, X_2^+, X_2^+}"'] \\ X_2^+ \cong X_1^+ \dot{\otimes} X_2^+ & (X_2^+ \dot{\otimes} X_2^+) \dot{\otimes} X_2^+ \arrow[l, "e\dot{\otimes}\mathrm{id}"'] \end{tikzcd} \tag{4.65}$$

and

$$\begin{tikzcd} X_2^+ \cong X_1^+ \dot{\otimes} X_2^+ \arrow[r, "i\dot{\otimes}\mathrm{id}"] \arrow[d, "g"'] & (X_2^+ \dot{\otimes} X_2^+) \dot{\otimes} X_2^+ \arrow[d, "\alpha^{-1}_{X_2^+, X_2^+, X_2^+}"'] \\ X_2^+ \cong X_2^+ \dot{\otimes} X_1^+ & X_2^+ \dot{\otimes} (X_2^+ \dot{\otimes} X_2^+) \arrow[l, "\mathrm{id}\dot{\otimes} e"'] \end{tikzcd} \tag{4.66}$$

commute, where $f = \mu \cdot \mathrm{id}_{X_2^+}$, $g = \nu \cdot \mathrm{id}_{X_2^+}$ for two non zero constants μ and ν. We show that $\mu \neq 0$, the case of ν is similar so we omit the proof.

We fix highest weight vectors u_s of X_s^+ for $s = 1, 2, 3$, such that $X_s^+[h_s^+] = \mathbb{C}u_s$ and $u_1 = \Omega$ as in previous calculations. By the fusion products we have computed so far we know that the spaces of vertex operators

$$\binom{X_2^+}{X_2^+, X_s^+}, \quad \binom{X_2^+}{X_s^+, X_2^+}, \quad \binom{X_s^+}{X_2^+, X_2^+}, \tag{4.67}$$

are all one dimensional for $s = 1, 3$. We therefore fix non-trivial vertex operators

$${}_2\Psi_s^2 \in \binom{X_2^+}{X_2^+, X_s^+}, \quad {}_s\Psi_2^2 \in \binom{X_2^+}{X_s^+, X_2^+}, \quad {}_2\Psi_2^s \in \binom{X_s^+}{X_2^+, X_2^+}. \tag{4.68}$$

These vertex operators can be formally expanded as

$${}_b\Psi_c^a(\ ; z) = \sum_{n \in \mathbb{Z}} {}_b\Psi_{c;n}^a(\ ) z^{-n-(h_a^+ + h_b^+ - h_c^+)}, \tag{4.69}$$

for appropriate choices of a, b and c, where

$${}_b\Psi_{c;n}^a \in \bigoplus_{k,\ell \geq 0} \mathrm{Hom}_{\mathbb{C}}(X_a^+[h_a^+ + k] \otimes X_b^+[h_b^+ + \ell], X_c^+[h_c^+ + k + \ell - n]). \tag{4.70}$$

This allows us to define four power series

$$\Phi_s^{(1)}(z_4,z_3,z_2,z_1) = \langle\Omega|_1\Psi_2^2(u_2;z_4)_2\Psi_s^2(u_2;z_3)_s\Psi_2^2(u_2;z_2)_2\Psi_1^2(u_2;z_1)\Omega\rangle \tag{4.71}$$

$$\Phi_s^{(2)}(z_4,z_3,z_2,z_1) = \langle\Omega|_1\Psi_2^2(u_2;z_4)_2\Psi_2^s({}_s\Psi_2^2(u_2;z_2-z_3)u_2;z_3)_2\Psi_1^2(u_2;z_1)\Omega\rangle\,,$$

for $s = 1,3$. The power series $\Phi_s^{(1)}$ and $\Phi_s^{(2)}$ converge absolutely on the domains

$$U^{(1)} = \{(z_4,z_3,z_2,z_1)\in(\mathbb{C}^*)^4\,|\,|z_4|>|z_3|>|z_2|>|z_1|>0\}\,, \tag{4.72}$$
$$U^{(2)} = \{(z_4,z_3,z_2,z_1)\in(\mathbb{C}^*)^4\,|\,|z_4|>|z_3|>|z_1|>0, |z_3|>|z_2-z_3|>0\}\,,$$

respectively and satisfy the partial differential equations

1. for $n=-1,0,1$

$$\sum_{a=1}^{4} z_a^n\left(z_a\frac{\partial}{\partial z_a}+(n+1)h_2^+\right)\Phi = 0\,, \tag{4.73}$$

2. for $a=1,2,3,4$

$$\left(\frac{\partial^2}{\partial z_a^2}-\frac{1}{p}\sum_{\substack{b=1\\b\neq a}}^{4}\left(\frac{h_2^+}{(z_b-z_a)^2}-\frac{1}{z_b-z_a}\frac{\partial}{\partial z_b}\right)\right)\Phi = 0\,. \tag{4.74}$$

The solution space of these two sets of differential equations is two dimensional and the solutions define multivalued holomorphic functions on $(\mathbb{P})^4\setminus$ diagonals. Therefore $\Phi_s^{(1)}$ and $\Phi_s^{(2)}$ define bases of the solution space of the above differential equations on the two domains $U^{(1)}$ and $U^{(2)}$ and it is possible to analytically continue $\Phi_s^{(1)}$ to $U^{(2)}$ and vice versa. For a given path γ from $U^{(1)}$ to $U^{(2)}$, $\Phi_s^{(1)}$ can be written as a linear combination of $\Phi_1^{(2)}$ and $\Phi_3^{(2)}$. This defines a connection matrix

$$\begin{pmatrix}\Phi_1^{(1)}\\ \Phi_3^{(1)}\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\Phi_1^{(2)}\\ \Phi_3^{(2)}\end{pmatrix}\,. \tag{4.75}$$

Going along the path γ in the opposite direction one can express Φ_s^2 as a linear combination of $\Phi_1^{(1)}$ and $\Phi_3^{(1)}$ with the inverse of the connection matrix above

$$\begin{pmatrix}\Phi_1^{(2)}\\ \Phi_3^{(2)}\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1}\begin{pmatrix}\Phi_1^{(1)}\\ \Phi_3^{(1)}\end{pmatrix}\,. \tag{4.76}$$

The constant μ, in $f = \mu \cdot \mathrm{id}_{X_2^+}$ of diagram (4.65), being non-zero is equivalent to $\Phi_1^{(1)}$ having non-vanishing contributions from $\Phi_1^{(2)}$, *i.e.* a being non-zero. Similarly ν is non-vanishing if $\Phi_1^{(2)}$ has non-trivial contributions from $\Phi_1^{(1)}$, which is the case when d is non-zero.

The first set of differential equations (4.73) guarantees the covariance of $\Phi_s^{(a)}$ with respect to Möbius transformations. Since Möbius transformations act transitively on triples of pair wise distinct elements of $\mathbb{P}$, we can fix three of the arguments of $\Phi_s^{(a)}$ to uniquely determine

$$\Phi_s^{(a)} = \prod_{1 \leq i < j \leq 4} (z_i - z_j)^{\frac{2p-3}{6p}} x^{\frac{1}{3}} (1-x)^{\frac{1}{3}} H_s^{(a)}(x) \tag{4.77}$$

up to a function $H_s^{(a)}(x)$ of the Möbius invariant cross ratio

$$x = \frac{z_4 - z_3}{z_4 - z_2} \frac{z_1 - z_2}{z_1 - z_3} \,. \tag{4.78}$$

See [33] for a wealth of examples regarding such computations. The functions $H_s^{(1)}(x)$ and $H_s^{(2)}(x)$ are absolutely convergent on $1 > |x| > 0$ and $1 > |1-x| > 0$ respectively. The second set of differential equations (4.74) arise from the fact that the vertex operators above vanish upon inserting the null vector

$$(L_{-1}^2 - \tfrac{1}{p} L_{-2}) u_2 \,. \tag{4.79}$$

The prefactors of $H_s^{(a)}(x)$ in equation (4.77) have been chosen such that the differential equation for $H_s^{(a)}(x)$ induce by (4.73) is particularly simple. Namely the well known hypergeometric equations

$$x(1-x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} H_s^{(a)}(x) + \tfrac{2}{p}(1-2x) \frac{\mathrm{d}}{\mathrm{d}x} H_s^{(a)}(x) - \tfrac{3-p}{p^2} H_s^{(a)}(x) = 0 \,. \tag{4.80}$$

For a detailed list of solutions and all formulae we will be using see [37]. For $\Phi_s^{(1)}$ which converges on $U^{(1)}$, $H_s^{(1)}(x)$ is a power series in x, while for $\Phi_s^{(2)}$ which converges on $U^{(2)}$, $H_s^{(2)}(x)$ is a power series in $1-x$

$$\begin{aligned} H_1^{(1)}(x) &= {}_2F_1\left(\tfrac{1}{p}, \tfrac{3-p}{p}; \tfrac{2}{p}; x\right), \\ H_3^{(1)}(x) &= x^{\frac{p-2}{p}} {}_2F_1\left(\tfrac{p-1}{p}, \tfrac{1}{p}; \tfrac{2p-2}{p}; x\right), \\ H_1^{(2)}(x) &= {}_2F_1\left(\tfrac{1}{p}, \tfrac{3-p}{p}; \tfrac{2}{p}; 1-x\right), \\ H_3^{(2)}(x) &= (1-x)^{\frac{p-2}{p}} {}_2F_1\left(\tfrac{1}{p}, \tfrac{p-1}{p}; \tfrac{2p-2}{p}; 1-x\right). \end{aligned} \tag{4.81}$$

To prove that $\Phi_1^{(1)}$ has non-vanishing contributions from $\Phi_1^{(2)}$ we continue $H_1^{(1)}(x)$ along the path from 0 to 1 on the real line. The well known connection formula for hypergeometric functions then yields

$$H_1^{(1)}(x) = \frac{1}{2\cos\frac{\pi}{p}} H_1^{(2)}(x) + \frac{3-p}{2-p} \frac{\Gamma(\frac{2}{p})^2}{\Gamma(\frac{1}{p})\Gamma(\frac{3}{p})} H_3^{(2)}(x)\,, \tag{4.82}$$
$$H_1^{(2)}(x) = \frac{1}{2\cos\frac{\pi}{p}} H_1^{(1)}(x) + \frac{3-p}{2-p} \frac{\Gamma(\frac{2}{p})^2}{\Gamma(\frac{1}{p})\Gamma(\frac{3}{p})} H_3^{(1)}(x)\,.$$

And thus the rigidity of X_2^+ for $p \geq 4$ follows.

For $p = 2, 3$ the analysis is exactly the same. Specifying the domains and codomains of the vertex operators is just a bit trickier. The resulting differential equations are analogous however. For $p = 3$ we have shown so far that

$$X_2^+ \dot{\otimes} X_2^+ \dot{\otimes} X_2^+ = X_2^+ \oplus (X_2^+ \dot{\otimes} X_3^+) \tag{4.83}$$

and that there exits a surjective $\mathcal{W}_3$-module map

$$X_2^+ \dot{\otimes} X_3^+ \to X_2^+\,. \tag{4.84}$$

Therefore the right exactness of the fusion product implies the existence of a surjective $\mathcal{W}_3$-module map

$$X_2^+ \dot{\otimes} (X_2^+ \dot{\otimes} X_3^+) \to X_1^+ \oplus X_3^+\,. \tag{4.85}$$

The analysis above can therefore be repeated for $p = 3$ without any modifications. We consider the differential equations (4.73) and (4.74), Which can again be simplified to the hypergeometric equation (4.80). Analysing the connection formulae for $p = 3$ yields

$$H_1^{(1)}(x) = H_1^{(2)}(x)\,, \tag{4.86}$$

thus proving the rigidity of X_2^+ for $p = 3$.

For $p = 2$ the space of solutions for the hypergeometric equation

$$x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} H_s^{(a)}(x) + (1-2x)\frac{\mathrm{d}}{\mathrm{d}x} H_s^{(a)}(x) - \tfrac{1}{2} H_s^{(a)}(x) = 0 \tag{4.87}$$

is slightly more complicated than in the previous examples, because the poles encountered at $x = 0$ and $x = 1$ are logarithmic. This implies that vertex operators involved also contain logarithms. We will omit the details however

since they are not important for solving the above differential equation. The solutions $H_s^{(a)}$ are given by

$$\begin{aligned}
H_1^{(1)}(x) &= {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;x\right), \\
H_3^{(1)}(x) &= {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;x\right)\log(x) + G(x), \\
H_1^{(2)}(x) &= {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;1-x\right), \\
H_3^{(2)}(x) &= {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;1-x\right)\log(1-x) + G(1-x),
\end{aligned} \tag{4.88}$$

where $G(x)$ is a power series with vanishing constant term that converges for $1 > |x|$. The connection formulae for $p = 2$ yield

$$H_1^{(1)} = \frac{\log(4)}{\pi} H_1^{(2)}(x) - \frac{1}{\pi} H_3^{(2)}(x), \tag{4.89}$$

thus proving the rigidity of X_2^+ for $p = 2$. □

5 The rigidity of $(\mathcal{W}_p\text{-mod}, \dot{\otimes})$

In the previous sections we proved that X_1^- and X_2^+ are rigid self dual objects in $\mathcal{W}_p$-mod. In this section we will exploit this fact to compute the fusion product of X_1^- and X_2^+ with the projective modules P_s^ε, $1 \le s < p$, $\varepsilon = \pm$, allowing us to prove the rigidity of $\mathcal{W}_p$-mod and ultimately compute the fusion product on the set of all simple and all projective modules.

5.1 Fusion products between X_1^- and X_2^+ and projective modules

At first we prepare some more notation. For any object in $\mathcal{W}_p$-mod we denote by $[Z : X_s^\varepsilon]$ the multiplicity of X_s^ε in quotients $M_{i+1}(Z)/M_i(Z)$ of the Jordan-Hölder series (2.30) of Z Then we have

$$[Z : X_s^\varepsilon] = \dim_{\mathbb{C}} \operatorname{Hom}(P_s^\varepsilon, Z). \tag{5.1}$$

We have established that

$$\begin{aligned}
X_2^+ \dot{\otimes} X_s^\varepsilon &= \begin{cases} X_2^\varepsilon, & s = 1 \\ X_{s-1}^\varepsilon \oplus X_{s+1}^\varepsilon, & 2 \le s \le p-1 \\ P_{p-1}^\varepsilon, & s = p \end{cases} \\
X_1^- \dot{\otimes} X_s^\varepsilon &= X_s^{-\varepsilon}, \qquad 1 \le s \le p.
\end{aligned} \tag{5.2}$$

and that X_1^- and X_2^+ are self dual rigid objects. From the Jordan-Hölder series of the projective modules we also know that

$$\begin{aligned}
&[P_s^\varepsilon : X_t^\sigma] = 2\delta_{(s,\varepsilon),(t,\sigma)} + 2\delta_{(s,\varepsilon),(p-t,-\sigma)}\,, \quad 1 \le s < p \\
&[P_p^\varepsilon : X_t^\sigma] = \delta_{(p,\varepsilon),(t,\sigma)}\,.
\end{aligned} \tag{5.3}$$

So we have the following proposition.

Proposition 37. *The fusion rules of X_2^+ and X_1^- with projective modules are given by*

$$\begin{aligned}
X_2^+ \dot{\otimes} P_s^\mu &= \begin{cases} P_2^\mu \oplus 2\cdot P_p^{-\mu}\,, & s = 1 \\ P_{s-1}^\mu \oplus P_{s+1}^\mu\,, & 1 < s < p-1 \\ P_{p-2}^\mu \oplus 2\cdot P_p^\mu\,, & s = p-1 \end{cases} \\
X_1^- \dot{\otimes} P_s^\mu &= P_s^{-\mu}\,, \qquad 1 \le s \le p\,.
\end{aligned} \tag{5.4}$$

Proof. Because X_1^- and X_2^+ are rigid, their product with P_t^δ is projective. The most general ansatz for such a product is therefore

$$X \dot{\otimes} P_t^\delta = \bigoplus_{m=1}^{p} \bigoplus_{\mu=\pm} N_{m,\mu} \cdot P_m^\mu\,, \tag{5.5}$$

where X is either X_1^- or X_2^+ and $N_{m,\mu} \in \mathbb{Z}$ is the multiplicity of P_m^μ in $X \dot{\otimes} P_t^\delta$. We can determine $N_{m,\mu}$ by recalling that a rigid object X and two arbitrary objects A and B satisfy the relation

$$\mathrm{Hom}(A, X \dot{\otimes} B) \cong \mathrm{Hom}(X^* \dot{\otimes} A, B)\,. \tag{5.6}$$

Setting A to P_t^δ and C to X_m^μ and calculating the dimensions of the spaces of $\mathcal{W}_p$-module maps in the equation above, we are lead to

$$N_{m,\mu} = \dim \mathrm{Hom}(P_t^\sigma, X \dot{\otimes} X_m^\mu) = [X \dot{\otimes} X_m^\mu : X_t^\sigma]\,. \tag{5.7}$$

We can easily calculate the multiplicities $[X \dot{\otimes} X_m^\mu : X_t^\sigma]$ for $X = X_1^-,\ X_2^+$ by considering the fusion products 35 and 36

$$\begin{aligned}
&[X_1^- \dot{\otimes} X_m^\mu : X_t^\delta] = \delta_{(t,\delta),(m,-\mu)} \\
&[X_2^+ \dot{\otimes} X_m^\mu : X_t^\delta] = \begin{cases} \delta_{(2,+),(m,\mu)} + 2\delta_{(p,\delta),(m,-\mu)} & t = 1 \\ \delta_{(t-1,\delta),(m,\mu)} + \delta_{(t+1,\delta),(m,\mu)} & 2 \le t \le p-2 \\ \delta_{(p-2,\delta),(m,\mu)} + 2\delta_{(p,\delta),(m,\mu)} & t = p-1 \\ \delta_{(p-1,\delta),(m,\mu)} & t = p \end{cases}\,.
\end{aligned} \tag{5.8}$$

The proposition then follows directly by plugging in the multiplicities. □

5.2 Proving rigidity

We apply point 4 of proposition 19 to $\mathcal{W}_p$-mod. Since all simple and all projective $\mathcal{W}_p$-modules appear in the fusion products of X_1^- and X_2^+ the following proposition follows.

Proposition 38. *For $1 \leq s \leq p$, $\varepsilon = \pm$ the simple modules X_s^ε and the projective modules P_s^ε are self-dual rigid objects in $\mathcal{W}_p$-mod.*

In $\mathcal{W}_p$-mod all indecomposable objects M except the simple objects and the projective objects satisfy exact sequences

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{5.9}$$

such that L and M are direct sums of simple objects. So finally we obtain the rigidity of $\mathcal{W}_p$-mod by applying point 5 of proposition 19.

Theorem 39. *The weakly rigid monoidal category $(\mathcal{W}_p, \dot{\otimes}, X_1^+)$ is rigid. For any object M in $\mathcal{W}_p$-mod the dual $M^\vee$ is given by the contragredient M^*, i.e. $M^\vee = M^*$.*

5.3 The ring structure of $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$

We see in theorems 35 and 36, that the fusion products of X_1^- and X_2^+ with simple modules are direct sums of simple and projective modules. Because all simple modules appear as direct summands of products of X_1^- and X_2^+, the product of any two simple modules must also be a product of simple and projective modules. Therefore because all the simple modules are rigid, the fusion product closes on the set of all simple and all projective modules. In this section we will compute the fusion product on this set.

We introduce the free abelian group $P(\mathcal{W}_p)$ of rank $4p-2$ generated by all projective and all simple modules

$$P(\mathcal{W}_p) = \bigoplus_{s=1}^{p} \bigoplus_{\varepsilon=\pm} \mathbb{Z}[X_s^\varepsilon]_P \oplus \bigoplus_{s=1}^{p-1} \bigoplus_{\varepsilon=\pm} \mathbb{Z}[P_s^\varepsilon]_P \tag{5.10}$$

and the rank $2p$ Grothendieck group[5]

$$K(\mathcal{W}_p) = \bigoplus_{s=1}^{p} \bigoplus_{\varepsilon=\pm} \mathbb{Z}[X_s^\varepsilon]_K \tag{5.11}$$

[5]The Grothendieck group $K(\mathcal{C})$ can be defined for any ablian category $\mathcal{C}$. It is given by free abelian group generated by all objects of $\mathcal{C}$ modulo the subgroup generated by all formal differences $M - L - N$ where L, M, N satisfy an exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$. If the number of simple objects in the abelian category $\mathcal{C}$ is finite, then $K(\mathcal{C})$ is just the finite rank free abelian group generated by all simple objects.

By the rigidity of $\mathcal{W}_p$-mod and and the closure of the fusion product on simple and projective modules

1. $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$ have the structure of commutative rings.
2. The canonical projection $\pi : P(\mathcal{W}_p) \to K(\mathcal{W}_p)$ is a ring homomorphism.

By the above arguments the two operators

$$X = X_2^+ \dot{\otimes} - \qquad\qquad Y = X_1^- \dot{\otimes} - \,, \tag{5.12}$$

define $\mathbb{Z}$-linear endomorphisms of $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$. Because the fusion product is commutative, the two operators X and Y must also commute. Thus by the two operators X and Y the polynomial ring $\mathbb{Z}[X,Y]$ acts on $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$, *i.e.* $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$ are modules over $\mathbb{Z}[X,Y]$ and the canonical projection π is a $\mathbb{Z}[X,Y]$-module map.

Before we begin analysing the action of $\mathbb{Z}[X,Y]$ on $P(\mathcal{W}_p)$ we recall some elementary facts about Chebyshev polynomials that will prove helpful.

Definition 40. *We define elements $U_n(X),\ n = 0, 1, \ldots$ in $\mathbb{Z}[X]$ recursively*

$$\begin{aligned} U_0(x) &= 1\,, \qquad U_1(x) = x\,, \\ U_{n+1}(x) &= xU_n(x) - U_{n-1}(x)\,, \end{aligned} \tag{5.13}$$

Remark 41. *1. The coefficient of the leading order of $U_n(X)$ is 1, i.e.*

$$U_n(X) = X^n + \cdots \in \mathbb{Z}[X],\ m = 0, 1, 2, \ldots \tag{5.14}$$

so we have

$$\mathbb{Z}[X] = \bigoplus_{n=0}^{\infty} \mathbb{Z}U_n(X)\,. \tag{5.15}$$

2. The initial conditions and recursion relations of the polynomials $U_n(X)$ are those of the Chebyshev polynomials of the second kind, though with a non-standard choice of normalisation.

We define the $\mathbb{Z}[X,Y]$-module maps

$$\begin{aligned} \psi : \mathbb{Z}[X,Y] &\to P(\mathcal{W}_p) \\ f(X,Y) &\mapsto f(X,Y) \cdot [X_1^+]_P \\ \varphi : \mathbb{Z}[X,Y] &\to K(\mathcal{W}_p) \\ f(X,Y) &\mapsto f(X,Y) \cdot [X_1^+]_K \end{aligned} \tag{5.16}$$

Theorem 42. *The maps ψ and φ are surjective homomorphisms of commutative rings and the kernels are given by the ideals*

$$\begin{aligned} \ker\psi &= \left\langle Y^2-1, U_{2p-1}(X)-2YU_{p-1}(X)\right\rangle \\ \ker\varphi &= \left\langle Y^2-1, U_p(X)-U_{p-2}(X)-2Y\right\rangle . \end{aligned} \tag{5.17}$$

Proof. Consider the fusion products

$$\begin{aligned} X_2^+\dot{\otimes}X_1^+ &= X_2^+ \\ X_2^+\dot{\otimes}X_s^+ &= X_{s-1}^+\oplus X_{s+1}^+ , \end{aligned} \tag{5.18}$$

for $1<s<p$. Formally this looks exactly like the recursion relations and initial conditions (5.13) if one were to substitute X_2^+ with X and X_s^+ with $U_{s-1}(X)$. We can therefore write the generators of $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$ corresponding to the simple modules $X_s^+,\ 1\le s\le p$ as

$$[X_s^+]_P = U_{s-1}(X)[X_1^+]_P\,, \qquad [X_s^+]_K = U_{s-1}(X)[X_1^+]_K\,. \tag{5.19}$$

Since the remaining simple modules X_s^- can be written as $X_1^-\dot{\otimes}X_s^+$ their corresponding generators in $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$ can be written as

$$[X_s^-]_P = YU_{s-1}(X)[X_1^+]_P\,, \qquad [X_s^-]_K = YU_{s-1}(X)[X_1^+]_K\,. \tag{5.20}$$

Thus as a module over $\mathbb{Z}[X,Y]$, $K(\mathcal{W}_p)$ is generated by $[X_1^+]_K$ and φ is therefore a surjective $\mathbb{Z}[X,Y]$-homomorphism.

Next we consider the fusion products

$$\begin{aligned} X_2\dot{\otimes}X_p^+ &= P_{p-1}^+ \\ X_2^+\dot{\otimes}P_s^+ &= P_{s-1}^+\oplus P_{s+1}^+ , \end{aligned} \tag{5.21}$$

for $1<s<p$. These imply that the generators of $P(\mathcal{W}_p)$ corresponding to the projective modules $P_s^+,\ 1\le s<p$ can be written as

$$[P_s^+]_P = (U_{2p-1-s}(X)+U_{s-1}(X))[X_1^+]_P\,. \tag{5.22}$$

Since the remaining projective modules P_s^- can be written as $X_1^-\dot{\otimes}P_s^+$ their corresponding generators in $P(\mathcal{W}_p)$ can be written as

$$[P_s^-]_P = Y(U_{2p-1-s}(X)+U_{s-1}(X))[X_1^+]_P\,. \tag{5.23}$$

Thus as a module over $\mathbb{Z}[X,Y]$, $P(\mathcal{W}_p)$ is generated by $[X_1^+]_P$ and ψ is therefore a surjective $\mathbb{Z}[X,Y]$-homomorphism.

We verify that the two ideals

$$\begin{aligned} I &= \left\langle Y^2 - 1, U_{2p-1}(X) - 2YU_{p-1}(X) \right\rangle \\ J &= \left\langle Y^2 - 1, U_p(X) - U_{p-2}(X) - 2Y \right\rangle \,. \end{aligned} \tag{5.24}$$

are indeed the kernels of ψ and φ by showing that I and J lie in the kernels and that the ranks of $\mathbb{Z}(X,Y)/I$ and $\mathbb{Z}(X,Y)/J$ are equal to the ranks of $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$. From the fusion product $X_1^- \dot{\otimes} X_1^- = X_1^+$ it follows that

$$(Y^2 - 1)[X_1^+]_P = 0 \qquad\qquad (Y^2 - 1)[X_1^+]_K = 0\,. \tag{5.25}$$

The left and right hand sides of $X_2^+ \dot{\otimes} X_p^+ = P_{p-1}^+$ are given by the left and right hand sides of

$$XU_{p-1}(X)[X_1^+]_K = 2(U_{p-1}(X) + Y)[X_1^+]_K \tag{5.26}$$

respectively in $K(\mathcal{W}_p)$. By the recursion relations for Chebyshev polynomials it therefore follows that

$$(U_p(X) - U_{p-2}(X) - 2Y)[X_1^+]_K = 0 \tag{5.27}$$

Lastly by the left and right hand sides of the product $X_2^+ \dot{\otimes} P_1^+ = P_2^+ \oplus 2X_p^-$ are given by the left and right hand sides of

$$X(U_{2p-2}(X) + U_0(X))[X_1^+]_P = (U_{2p-3} + U_1(X) + 2YU_{p-1}(X))[X_1^+]_P \tag{5.28}$$

respectively in $P(\mathcal{W}_p)$. By the recursion relations for Chebyshev polynomials it therefore follows that

$$(U_{2p-1}(X) - 2YU_{p-1}(X))[X_1^+]_P = 0\,. \tag{5.29}$$

We write $\mathbb{Z}[X,Y]/I$ and $\mathbb{Z}[X,Y]/J$ out as free abelian groups to compute their rank

$$\begin{aligned} \frac{\mathbb{Z}[X,Y]}{I} &= \frac{\mathbb{Z}[X] \oplus \mathbb{Z}[X]Y}{\langle U_{2p-1}(X) - 2YU_{p-1}(X)\rangle} = \bigoplus_{i=0}^{2p-2} \mathbb{Z}X^i \oplus \bigoplus_{i=0}^{2p-2} \mathbb{Z}X^iY \\ \frac{\mathbb{Z}[X,Y]}{J} &= \frac{\mathbb{Z}[X] \oplus \mathbb{Z}[X]Y}{\langle U_p(X) - U_{p-2}(X) - 2Y\rangle} = \bigoplus_{i=0}^{p-1} \mathbb{Z}X^i \oplus \bigoplus_{i=0}^{p-1} \mathbb{Z}X^iY \end{aligned} \tag{5.30}$$

and see that the ranks are $4p-2$ and $2p$ respectively. □

Theorem 43. *The fusion products for all simple and all projective $\mathcal{W}_p$-modules are given by*

$$X_s^\mu \dot{\otimes} X_t^\nu = \bigoplus_{i=|s-t|+1;2}^{min\{s+t-1,2p-1-s-t\}} X_i^{\mu\nu} \oplus \bigoplus_{i=2p+1-s-t;2}^{m} P_i^{\mu\nu} \tag{5.31}$$

$$X_s^\mu \dot{\otimes} P_t^\nu = \bigoplus_{i=|s-t|+1;2}^{min\{s+t-1,2p-1-s-t\}} P_i^{\mu\nu} \oplus \bigoplus_{i=2p+1-(s+t));2}^{m} 2 \cdot P_i^{\mu\nu} \oplus \bigoplus_{i=p+1-(s-t));2}^{m} 2 \cdot P_i^{-\mu\nu}$$

$$P_s^\mu \dot{\otimes} P_t^\nu = 2 \cdot X_s^+ \dot{\otimes} P_t^+ \oplus 2 \cdot X_{p-s}^- \dot{\otimes} P_t^+ \,,$$

where ";2" indicates that the summation variable is incremented in steps of 2 and

$$m = \begin{cases} p & for\ p-i\ \ even \\ p-1 & for\ p-i\ \ odd \end{cases} . \tag{5.32}$$

The product on the Grothendieck group induced by the fusion product is given by

$$[X_s^\mu]_K \cdot [X_t^\nu]_K = \sum_{i=|s-t|+1;2}^{min\{s+t-1,2p+1-s-t\}} [X_i^{\mu\nu}]_K + \sum_{i=2p+3-s-t;2}^{\tilde{m}} 2([X_i^{\mu\nu}]_K + [X_{p-i}^{-\mu\nu}]_K)\,, \tag{5.33}$$

where

$$m = \begin{cases} p-2 & for\ p-i\ \ even \\ p-1 & for\ p-i\ \ odd \end{cases} . \tag{5.34}$$

Proof. The above fusion rules can be computed directly in $\mathbb{Z}[X,Y]$ by using multiplication formula for Chebyshev polynomials

$$U_k(x)U_j(x) = \sum_{i=|k-j|;2}^{k+j} U_i(x) \tag{5.35}$$

and subsequently projecting onto $P(\mathcal{W}_p)$ or $K(\mathcal{W}_p)$. Note that the ";2" in the subscript of the sum indicates that the summation variable k is incremented in steps of 2. □

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