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SPLIT NOTES (ON NON-COMMUTATIVE MIRROR SYMMETRY)

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Abstract. This text has non-empty intersection with my talk *Non-commutative mirror symmetry* on July 13, 2011 at *Homological Mirror Symmetry and Category Theory* workshop in Split. I'll discuss some consequences of positivity and convexity.

1. Laurent polynomials and random walks

Let N be a lattice with dual $M = \mathrm{Hom}(M, \mathbb{Z})$.Let u be coordinates on affine space $M \otimes_{\mathbb{Z}} \mathbb{A}^1$ and $x = e^u$ be coordinates on the torus $T = \mathrm{Hom}(N, \mathcal{G}_m) = M \otimes_{\mathbb{Z}} \mathcal{G}_m$. Note that map exp is an isomorphism between domains where all u are real and all x are real positive, further we denote this domain by V.

Consider a Laurent polynomial W with complex coefficients

$$W = \sum_{l \in N} a_l x^l = \sum_{l \in N} a_l e^{un} \tag{1.1}$$

Denote $A = \sum |a_l|$. Note that all coefficients a_l are real and non-negative $\iff$ $A = W(1)$. In this case we can consider restriction of W to V as a function of real positive argument x and A as its particular value. Furthermore in case $A = 1$ one may interpret Laurent polynomial W as a random walk in the lattice M: coefficient a_l is probability to go in direction m and each next step is independent of the past. In case $A \neq 1$ function W is simply a rescale of the probabilistic one.

Assume additionally that the origin $0 \in N$ lies in the interior of Newton polytope of W (so random walker has some chance to come back to the origin).

Remark 1.2. I consider properties of positivity and convexity (and their corollaries discussed below) as an Archimedean counterpart to p-crystal properties.

Definition 1.3. We say that point $x_0 \in (\mathbb{C}^*)^n$ is a usual *critical point* of W if $dW|_{x=x_0} = 0$. We say that $c \in \mathbb{C}$ is a usual *critical value* of W if $c = W(x_0)$ for some usual critical point x_0.

Lemma 1.4. *Function W has a unique critical point on V and it is the global minimum.*

Proof. Since 0 is contained in the interior of the Newton polytope for every direction $|u| \to \infty$ one of the monomials of W also goes to $+\infty$. Since all coefficients are positive W goes to $+\infty$ as well. This implies W has at least one minimum on V.

Note that W is a convex function in coordinates u_i: each monomial e^{um} is convex, so sum of monomials with positive coefficients is also convex. Since convex functions has at most one critical point we are done. $\square$

Let W_{min} be the minimum of W on real positive part i.e. the value of W at the unique critical point with real positive coordinates.

Proposition 1.5. *For $W_0 > W_{min}$ the fibers $W^{-1}(W_0) \subset V$ are diffeomorphic to $(n-1)$-dimensional sphere S^{n-1}.*

Proof. Once we know the lemma above apply the standard argument from Morse theory. $\square$

We point that uniqueness of real positive critical point will also follow from the arguments below, where we give an estimate of the respective critical value W_{min}.

Consider n-cycle $\Gamma = \{|x| = 1\}$ and a holomorphic volume form $\omega = \frac{1}{(2\pi i)^n} \prod \frac{dx_i}{x_i} = \frac{1}{(2\pi i)^n} \prod dt_i$ on n-dimensional complex torus $(\mathbb{C}^*)^n$.

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Definition 1.6. For Laurent polynomial W denote by $Tr(W)$ its constant term $\int_\Gamma W$. Let $Traces(W)$ be the set of natural numbers k such that $Tr(W^k) \neq 0$ ($k = 0$ is included), and *index* $r(W)$ be the greatest common divisor of all elements in $Traces(W)$. Define G-function and $\hat{G}$-function as (exponential) generating function for $Tr(W^k)$:

$$\hat{G}_W(t) = \sum_{k\geqslant 0} Tr(W^k)t^k = \int_\Gamma \frac{\omega}{1-tW} \tag{1.7}$$

$$G_W = \sum_{k\geqslant 0} Tr(W^k)\frac{t^k}{k!} = \int_\Gamma e^{tW}\omega \tag{1.8}$$

The formula for $\hat{G}_W(t)$ holds for $t < \frac{1}{W(1)}$.

Number $Tr(W^k)$ can be interpreted as probability to be back at the origin after k independent steps, so functions $\hat{G}_W$ and G_W are generating functions for these probabilities.

Proposition 1.9. *For positive W_1, W_2 we have $Tr(W_1 \cdot W_2) \geqslant Tr(W_1) \cdot Tr(W_2)$.*

Corollary 1.10. *The set $Traces(W)$ is an additive monoid: if $a, b \in Traces(W)$ then $(a + b) \in Traces(W)$. This implies that there is some number n_0 such that for all $n \geqslant n_0$ number $r(W)n$ belongs to $Traces(W)$.*

Denote by $R = R_W$ radius of convergence of $\hat{G}_W$, and define invariant $T = T_W = T(W) = \frac{1}{R}$.

Lemma 1.11. *Power series $G_W(t)$ exponentially converge everywhere on complex line and power series $\hat{G}_W(t)$ have positive radius of convergence R and $T \leqslant \sum |a_l|$.*

Proof. In case all coefficients a_l are real non-negative one can use " *probability is bounded by* 1" argument: since W is a Laurent polynomial with real positive coefficients, W^k is also a Laurent polynomial with real positive coefficients, so W^k equals to sum of positive monomials which are positive evaluated at any positive point, and $Tr(W^k)$ is one particular term, so it is bounded by the sum which is A^k. This implies $T = \lim_{k\to\infty} Tr(W^k)^{\frac{1}{k}} \leqslant \lim_{k\to\infty}(A^k)^{\frac{1}{k}} = A$. Case where some coefficients have non-zero argument can be absolutely bounded by the positive case. Exponential convergence of G_W immideately follows from convergence of $\hat{G}_W$. □

Lemma 1.12. *If $\hat{G}_W = \sum_{n\geqslant 0} g_n t^n$ then $\hat{G}_{W^k} = \sum_{n\geqslant 0} g_{kn} t^n$.*

Clearly $\hat{G}_W$ is Laplace transform of G_W.

Lemma 1.13 (Dutch trick)**.** *Function $\hat{G}_W$ is a period for the family $1 - tW = 0$ of hypersurfaces in the torus $(\mathbb{C}^*)^n$.*

Definition 1.14. Define spectrum of W as the set of inverses of critical points of $\hat{G}_W$.

Lemma 1.15. *Spectrum of Laurent polynomial W in the sense of definition 1.14 contains all usual critical values of W in the sense of definition 1.3.*

Proof. □

Lemma 1.13 implies that T is the maximal absolute value of all critical values of W (in a broad sense - including values at critical points outside the torus $(\mathbb{C}^*)^n$).

By Cauchy-Hadamard formula $T = \lim_{k\to\infty} Tr(W^k)^{\frac{1}{k}}$.

Proposition 1.16. *There is an upper bound $T \leqslant W(\alpha)$ for any real positive α.*

Proof. Lemma 1.11 implies $T \leqslant A = W(1)$. On the other hand for any real positive α Laurent polynomial $W'(x) = W(\alpha x)$ has the same $\hat{G}$-function (since Jacobian of the coordinate change equals one). However $W'(1) = W(\alpha)$. So same argument as above shows that $T \leqslant W(\alpha)$ for any real positive n-tuple α. □

Proposition above shows the inequality

$$T \leqslant W_{min} \tag{1.17}$$

On the other hand T equals to maximal absolute value of all complex critical values of W. In particular this implies there are no other critical points of W on $\mathbb{R}^n$ except for the global minimum.

Remark 1.18. The arguments above can be upgraded to answer positively the question of Ostrover and Tyomkin [17]: quantum cohomology algebra of any toric Fano manifold has a field as direct summand, see [8] and [10].

This implies that positive Laurent polynomials have a unique "canonical" real positive coordinates – namely coordinates where the unique real positive critical point is $(1,1,\dots,1)$. Indeed this fixes a "translational" symmetry of the torus ($t_i \to t_i + b_i$ or $x_i \to x_i \times a_i$), however there is still some "rotational" symmetry possible (which preserves the set of coefficients of W), and in fact these symmetries can be further exploited (see below).

Definition 1.19. In case a critical point of positive Laurent polynomial W is $t_i = 0$ we say that W is a *balanced* Laurent polynomial and t_i are balanced coordinated.

Definition 1.20. For Laurent polynomial $W = \sum a_n x^m$ define its *Obro vector* as $Obro(W) = \sum_{m \in M} a_n \cdot m$. Probabilistic interpretation of Obro vector is the average drift of random walker per one step. And the third interpretation: Obro vector is proportional to the centre of mass of a system of point particles positioned at the lattice points m with respective masses a_n.

Proposition 1.21. *Positive Laurent polynomial W is balanced $\iff$ its Obro vector vanishes $Obro(W) = 0 \in M$ $\iff$ $T(W) = W(1)$.*

Proof. Note that Obro vector equals to de Rham differential of W evaluated at $x = 1$ under natural isomorphism $T^*_{(1,\dots,1)}\mathcal{G}_m(\mathbb{R}) \simeq M \otimes \mathbb{R} : Obro(W) = dW|_{x=1}$. □

This immediately implies that

Proposition 1.22. *All balanced (maybe non-positive) Laurent polynomials form a subalgebra in algebra of all Laurent polynomials.*

Proof. Indeed the map $W \to dW|_{x=1}$ is linear and product of balanced polynomials is balanced by Leibniz rule. In fact balanced polynomials satisfying $W(1) = 0$ form an ideal in the ring of balanced polynomials and this ideal is the square of the ideal of Laurent polynomials vanishing at 1. □

Corollary 1.23. *Map $W \to T(W)$ restricted to balanced positive Laurent polynomials coincides with homomorphism of rings $W \to W(1)$. So if W_1 and W_2 are balanced Laurent polynomials and α_1, α_2 are positive numbers then $T(\alpha_1 W_1 + \alpha_2 W_2) = \alpha_1 T(W_1) + \alpha_2 T(W_2)$ and $T(W_1 W_2) = T(W_1) \cdot T(W_2)$.*

Definition 1.24. Define *index* $r(W)$ as the greatest common divisor of natural numbers n such that $Tr(W^k) \neq 0$ (index of constant function is defined to be ∞).

Remark 1.25. Index $r(W)$ can be also characterized as the greatest number r such that $\hat{G}_W(e^{\frac{2\pi i}{r}} t) = \hat{G}_W(t)$. or in other words $\hat{G}_W(t) = \hat{G}_{W^r}(t^r)$. From wandering drunkard's point of view this means that is return to the origin is possible only in number of steps divisible by r.

Lemma 1.26. *Indices of W and its powers are related as follows: $r(W^k) = \frac{r(W)}{gcd(k, r(W))}$. In particular $r(W^{r(W)}) = 1$.*

Theorem 1.27. *Complex number T' such that $|T'| = T_W$ is an element of spectrum of W $\iff$ $T'^r = T^r$ i.e. $T' = T \cdot e^{\frac{2\pi i p}{r(W)}}$ for some integer p.*

Proof. Since the spectrum is invariant of $\hat{G}_X$ the the inclusion statement follows from definition of index. Let us prove that other points on the circle of radius T do not lie in the spectrum. Lemmas 1.26 and **??** applied to W and $W^{r(W)}$ reduces the problem to the case $r(W) = 1$.

□

To be continued...

2. Invariant T of Fano varieties and mirror symmetry

For Fano manifold X denote by J_X its Givental's J-function $(ev_1)_* \frac{z}{z - \psi_1}$ restricted to anticanonical direction $\mathbf{t} = tc_1(X)$ and $z = 1$. Consider $G_X = \int_{[X]} J_X \cup [pt]$ and its Fourier-Laplace transform $\hat{G}_X$.

Definition 2.1 (See [9, 13])**.** Define *spectra* $Spectra(X)$ [1] as the collection of inverses of all critical points of the function $\hat{G}_X$. Equivalently spectra of Fano manifold is the collection of roots of its *quantum characteristic polynomial* (characteristic polynomial of the operator of quantum multiplication by $c_1(X)$).

Definition 2.2. Define $T(X)$ as inverse of radius convergence of $\hat{G}_X$. Equivalently, $T(X)$ is maximal absolute value of elements in the spectrum of X.

[1] Anticanonical spectrum in notations of [13].

Definition 2.3 (See [4] for detailed discussion)**.** We say that Laurent polynonial W *is a mirror image of* Fano manifold X if $G_W = G_X$ (or, equivalently, $\hat{G}_W = \hat{G}_X$).

Example 2.4. Let X be a toric Fano manifold and v_i — primitive generators on the rays of its fan. Then Laurent polynomial $W = \sum x^{v_i}$ is a mirror image of X by results of Givental [12]. Further we call this function W as the standard mirror image for toric Fano manifold X.

Question 2.5. *We are going to address the following questions:*

1. *What are the possible values of number $T(X)$ for Fano varieties X.*
2. *In particular, what are the bounds?*
3. *How they depend on dimension?*
4. *What are the values of $T(X)$ for toric Fano varieties and how they differ from generic?*

Theorem 2.6. *For toric Fano manifold X there is an upper bound*

$$T(X) \leqslant \dim X + \rho(X) \leqslant 3 \dim X.$$

Proof. Consider the standard mirror image W from example 2.4. We have $T(X) = T(W)$, by 1.16 $T(W) \leqslant W(1)$, finally $W(1)$ equals to number of vertices in the fan of toric manifold and this number equals $\dim X + \rho(X)$.

Inequality $\rho(X) \leqslant 2\dim X$ is proven in [3], and we'll reproduce much simpler proof of Obro from [16]. Consider *special facet* F — any facet whose cone contains Obro vector, and let f_F be a linear function on M that equals 1 on F. Note that $\sum_{v \in Vertices(X)} f_F(v) = f_F(Obro(X)) \geqslant 0$. Any vertex v of X falls in one of three categories by sign of $f_F(v)$. If $f_F(v) > 0$ then v is one of d vertices of facet F, so $f_F(v) = 1$ and $\sum_{v: f_F(v) \geqslant 0} f_F(v) = d$. Number of v with $f_F(v) < 0$ is bounded by d since $f_F(Obro(X)) \geqslant 0$. Number of vertices v with $f_F(v) = 0$ is also bounded by d due to combinatorial reasons. Altogether number of vertices is bounded by $d + d + d = 3d$. □

Below we list value of T_X for del Pezzo surfaces. Here $p(t)$ is the minimal polynomial for algebraic integer T_W.

X	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^1 \times \mathbb{P}^1$	S_8	S_7	S_6	S_5	S_4	S_3	S_2	S_1
$\lvert T_X \rvert$	2	3	4	4	5	6	9	12	21	52	372

X	$\mathbb{P}^3$	Q	B_1	B_2	B_3	B_4	B_5	V_2	V_4	V_6	V_8	V_{10}	V_{12}	V_{14}	V_{16}	V_{18}	V_{22}
$\lvert T \rvert$	4	5	42	16	11	8	7	1608	232	96	56	39	29	23	20	17	13

3. Reconstructions

Definition 3.1. *Auroux manifold* Y is a complement in a Fano manifold F to its smooth anti-canonical hypersurface $A \in |-K_F|$: $Y \simeq F - A$.

Lemma 3.2. *If Y is Auroux manifold then $H^0(Y, \mathcal{O}_Y^*) = \mathbb{C}$.*

Proof. Any function $f \in H^0(Y, \mathcal{O}_Y^*)$ can be considered as a rational function on F. Since functions f and $\frac{1}{f}$ are regular on Y their divisors of poles should be supported at A. However divisor of poles of $\frac{1}{f}$ is divisor of zeroes of f, so since A is irreducible function f doesn't have any poles or zeroes, so its a regular function on projective variety F that is a constant. □

Theorem 3.3. *The respective Fano manifold F and its anti-canonical Calabi–Yau section A can be uniquely reconstructed from Auroux manifold Y.*

Proof. Given Y consider its model, that is a smooth projective $M \supset Y$, such that $D = M \backslash Y$ is a divisor with simple normal crossings. There is at least one such M, that aditionally has a property that divisor D is irreducible, namely we can take (M, D) to be equal to (F, A). So irreducible divisor D gives a divisorial evaluation $v_D : \mathbb{C}(Y)^* \to \mathbb{Z}$ on the field of rational functions on Y. We claim that for any model (M, D) with irreducible D the respective divisorial evaluation v_D is exactly the same as the evaluation v_A in the model (F, A). Indeed, since A is a smooth Calabi–Yau manifold it is not uniruled, hence it has no Mori fiberations to a variety of smaller dimension, thus in *any* model M at least one of the irreducible components of the complementary divisor D is birational to A (consider an direct image of strict preimage in any Hironaka house). Finally, given Y and evaluation v, consider a filtered ring $R = \Gamma(Y, \mathcal{O}_Y)$, with filtration by order of pole at A: $F_v^k R = \{a \in R, v(a) \geqslant -k\}$. Then the associated graded ring equals to $Gr_{F_{v_A}} R = \sum_{n \geqslant 0} H^0(F, nA)$, and the variety F can be recovered as the projective spectre $F = \operatorname{Proj} Gr_{F_{v_A}} R$. □

Definition 3.4. Quasi-projective variety U is called *semi-projective*, if the natural map $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is projective. If aditionally, $\operatorname{Spec} \Gamma(X, \mathcal{O}_X) \simeq \mathbb{A}^1$ and canonical line bundle is trivial $\omega_U = \mathcal{O}_U$, then we call U *onaf manifold*. Equivalently, onaf manifold is a smooth quasi-projective variety U with trivial canonical bundle, equipped with a projective map $\pi : U \to \mathbb{A}^1$ with connected fibers.

4. Three incarnations and three levels

All this story has 3 incarnations: C (for commutative or classical), Q (for quantized) and NC (non-commutative [15]).

4.1. **Potentials.** All potentials are elements of the group rings of different kinds of groups.

C-potentials (of the usual commutative theory) are just usual Laurent polynomials. We may consider Laurent polynomial as an element of a group ring of a free abelian group.

Q-potentials (or *quantized Laurent polynomials*) are elements of the quantum torus i.e. group ring of Heisenberg group. Sometimes it is more convenient to work with a double central extension of Heisenberg group. Let q be the generator of the center of Heisenberg group.

Finally, NC-potentials are the noncommutative Laurent polynomials e.g. elements of a group ring of a free group.

4.2. **G-functions.** One defines [1, 15] the *trace* of a potential $W = \sum c_g x^g$ as its constant term $Tr(W) = c_1$ (where 1 is the identity element in the respective group).

In quantized setup one can also define the *central trace* as the sum of all central monomials $Tr(\sum c_g x^g) = \sum_{n \in \mathbb{Z}} c_{q^n} q^n$.

The name trace is partially motivated by the fact it vanishes on commutators i.e. $Tr(ab) = Tr(ba)$.

G-functions are defined as various generating functions for traces of powers of W, and can also be thought as generalized characteristic polynomials. As well these generating series count the probabilities to come in n steps to the origin for random walker on a respective group.

4.3. **Coordinate change formulae.** Assume we have some coordinate transformation (automorphism of (skew-)field of fractions of group ring of group G). Additionally assume a Laurent phenomenon: some potential W is mapped into another potential W'.

Under which conditions the G-functions are preserved i.e. traces of powers of W remain the same.

The commutative case is served by a coordinate change formula in integral: the Jacobian of the transformation is identity $\iff$ the holomorphic volume form ω on torus is mapped into itself.

It is a delightful gift of quantization that in Q case any coordinate change that preserves q is fine.

I don't know under what conditions (if any) G-functions are preserved in NC setting.

5. What is not yet covered in this survey

- Expand Laurent phenomenon [11, 5, 7],
- quantized and non-commutative random walks [6],
- Futaki–Mabuchi polynomial, degenerations, stabilities and Kähler–Einstein metrics,
- ...

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